ON GENERATING FUNCTIONS

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ABSTRACT

ON GENERATING FUNCTIONS

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The Generating Functions play very important role in many areas of Mathematics for they lead as to explore the properties of functions that they generate.
In this thesis, definitions of generating functions and their kinds and properties are presented. Their relations to some special functions are discussed. Moreover, and applications of generating functions in several areas of Mathematics are discussed.

Keywords: Generating Functions, Bernoulli Polynomials, Orthogonal Polynomials.

ÖZ

ÜZERİNDE ÜRETEN FONKSİYONLAR

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Üreten fonksiyonlar, ürettikleri fonksiyonların özelliklerinin araştırılmalarına yardımcı oldukları için, matematiğin her alanında çok önemli rol oynarlar.
Bu tezde üreten fonksiyonların tanımları, türleri ve özellikleri sunulmaktadır. Bazı özel fonksiyonlarla ilişkileri irdelenmiştir. Ayrıca, matematiğin bazı alanlarındaki üreten fonksiyonların uygulamaları tartışılmıştır.

Anahtar Kelimeler: Üreten Fonksiyonlar, Bernoulli Polinomları, Dik Polinomları.

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GF	Generating Functions
BP	Bernoulli Polynomials
EQ	Equations
ODE	Ordinary Differential Equations

CHAPTER 1

INTRODUCTION

1.1 Background

Generating functions are clotheslines on which we hang up a sequence of numbers for display. These functions provide a powerful tool for solving recursively defined and combinatorial problems. They were invented in 1718 by the French mathematician Abraham De Moivre (1667-1754), when he used them to solve the Fibonacci recurrence relation $F_n = F_{n-1} + F_{n-2}$. Then they were developed by Euler in 1748 in connection with partition problems, and were extensively treated in the late eighteenth century and early nineteenth century by Laplace. Who gave them their name and their first systematic treatment in the course of his great work on the theory of Probability. Since then, applications have been developed in many areas of mathematics and physical problems. [1]

These functions are bridges between discrete mathematics, on one hand, and continuous analysis on the other. They can be tools to solve discrete problems as well as differential equations [2]. They play an important role in the investigation of various useful properties of the sequence which they generate. They are used as Z-transforms in solving certain classes of difference equations which arise in a wide variety of problems in operations research (including for example, queuing theory and related stochastic process).[3]

Generating Functions admit a natural splitting into classes. The simplest is the class of rational functions. It is well studied and huge bunch of problems leading to rational generating functions is known. The classical orthogonal polynomials including, for example, Lengendre, Tchebychev, Laguerre and Hermite polynomials orthogonal polynomials were developed in the late 19th century and their various generalizations studied in recent years.

The generalized hyper-geometric polynomials of Bateman, Bedient, Brafman, Fasenmyer, Gould Hopper, Mittage-leffler, Rice, Shively, Sylvester and others, a familiar lagrange polynomials (which arise in certain problems in statistics) and so on [1].

The existence of generating function for a sequence $\{f_n\}$ of numbers or functions may be useful for finding $\sum_{n=0}^{\infty} f_n$ by such summability methods as those due to Abel and Cesaro [1]. Generating functions are used also to count selections and arrangements with limited repetition, solutions of linear equations, distributions, and partitions of an integer n. Ordinary generating functions are used to count distributions of identical objects and arrangements where the order is not important. On the other hand, exponential generating functions are used to count distributions of different objects and arrangements where the order is important [4].

Generating functions are also standard topic in most combinatorics. Without generating functions it is possible to turn in one of the following directions. More generally, the subject of generating functions belongs to the domain of operational methods which are widely used in the theory of differential and integral equations. Algebraic generating functions also appear frequently. In the beginning of 1960s Schützenberger showed that their non-commutative analogues arise naturally as language generated by unambiguous formal grammars. However, the class of algebraic functions (in contrast to that of rational ones) is not closed under the natural operation of the Hadamard product. Generally, the Hadamard product of two algebraic functions is an algebro-logarithmic function and the class of algebro-logarithmic functions, which is closed under the Hadamard products seems to be natural [5].

This thesis maybe considered as a survey on generating functions and some of their applications. To the extent of our knowledge, there is no work on generating function, which contains more knowledge on these functions than this work.

1.2. Organization of the Thesis

This thesis contains four chapters and conclusion. All the necessary information about the Generating Functions, definitions, theorems, examples, properties,some Special Functions and some applications on Generating Functions.

Chapter 1 is an introduction to the history and background to the Generating Functions.

In chapter 2, we will present some definitions and properties of generating functions.

In chapter 3, the relation between generating functions and some Special Functions are discussed.

In chapter 4, some applications of generating functions are mentioned.

Chapter 5 is devoted to the conclusion.

CHAPTER 2

DEFINITIONS AND PROPERTIES OF GENERATING FUNCTIONS

Definition 2.1 [5]

The generating function for the infinite sequence of real numbers < a_0,a_1,…>
is the power series

$$G(x)=a_0+a_1x+a_2x^2+\ldots=\sum_{n=0}^{\infty} a_n x^n \qquad \textbf{(2.1)}$$

Example 2.1.1. [3]

The famous Fibonacci sequence is defined by its first two terms f_0=f_1 $= 1$ and the relation

$$f_{n+2} = f_{n+1} + f_n. \qquad \textbf{(2.2)}$$

This relation allows one to easily produce the first few terms of the Fibonacci

1, 1, 2, 3, 5, 8, 13, 21, 34…;

Starting with f_2, each element of this sequence is the sum of the two preceding elements. To compute the generating function

$$\text{Fib}(s)=1+s+2s^2+3s^3 + 5s^4 + \cdots, \qquad \textbf{(2.3)}$$

Let us multiply both parts of Eq. **(2.3)** by s+s^2. We obtain

$$(s+s^2)\,\text{Fib}(s) = s+s^2 + 2s^3 + 3s^4 + 5s^5 + \cdots$$
$$+ s^2 + s^3 + 2s^4 + 3s^5 + \cdots$$
$$= s+2s^2 + 3s^3 + 5s^4 + 8s^5 + \cdots,$$

Or

$$(s+s^2)\ \mathrm{Fib}(s) = \mathrm{Fib}(s)-1,$$

Whence

$$\mathrm{Fib}(s) = \frac{1}{1-s-s^2}. \qquad \textbf{(2.4)}$$

For functions of two variables we have the following definition

Definition 2.2. [6]

Let $\{Y_k\ (t)\}$ be a sequence of (possibly constant) functions. If there is a function g(t,x) such that:

$$g(t,x) = \sum_{k=0}^{\infty} Y_k\ (t)x^k \qquad \textbf{(2.5)}$$

For all x in an open interval about zero, then g is called “generating function” for $\{Y_k\ (t)\}$.

Examples 2.2.1. [7]

The following are simple examples of generating functions

1) For each n$\in N^*$, let$\langle a_r \rangle$ be the sequence where

$$a_r = \begin{cases} 1, & if\ r = n \\ 0, & otherwise \end{cases}$$

That is,

$$\langle a_r \rangle = \langle 0, 0 \ldots, 0, 1, 0, 0, \ldots \rangle$$

Then, the generating function for $\langle a_r \rangle$ is x^n.

2) The generating function for the sequence$\langle \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, \ldots \rangle$ is

$$\sum_{r=0}^{n} \binom{n}{r} x^r = (1+x)^n$$

3) The generating function for the sequence $\langle 1,1,1, \ldots \rangle$ is

$$1+x+x^2+...=\frac{1}{1-x}$$

More generally function for the sequence $\langle 1, k, k^2, ..\rangle$, where k is an arbitrary constant, is

$$1+kx+k^2x^2+k^3x^3+...=\frac{1}{1-kx}$$

4) The generating function for the sequence $\langle 1,2,3,...\rangle$ is

$$1+2x+3x^2+...=\frac{1}{(1-x)^2}$$

5) The generating function for the sequence

$$\left\langle \binom{n-1}{0}, \binom{1+n-1}{1}, ..., \binom{r+n-1}{r}, ... \right\rangle \; is$$

$$\sum_{r=0}^{\infty} \binom{r+n-1}{r} x^r = \frac{1}{(1-x)^n}$$

Formula **(2, 3, 4, 5)** are very useful in finding the coefficients of generating functions.

6) Suppose that $a_k = \frac{1}{k!}, k = 0,1,2, ... then$

$$G(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

G(x)= e^x for all values of x.

7) Suppose that G(x)=x $\sin x^2$ is the ordinary generating function for the sequence $\langle a_k \rangle$. To find a_k,

$$G(x) = x\left[x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5\right] \Rightarrow G(x) = x^3 - \frac{1}{3!}x^7 + \frac{1}{5!}x^{11}. . .$$

Thus, we see that a_k is the k^{th} term of the sequence, $0,0,0,1,0,0,0 - \frac{1}{3!}, 0,0,0, \frac{1}{5!}, 0, ...$

8) There is available an unlimited number of pennies, nickels, quarters and generating function g(x) for the number $h_n$of ways of making n cents with these pieces? The number h_n equals the number of non-negative integral solutions of the equations e_1+5e_2+10e_3+25e_4+50e_5=n

The generating function is $g(x) = \frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}\frac{1}{1-x^{50}}$

9) $(1+s)^{\propto} = 1+\frac{\propto}{1!}s+\frac{\propto(\propto-1)}{2!}s^2+\frac{\propto(\propto-1)(\propto-2)}{3!}s^3+\cdots$. Where n!=1.2.3…n and $\propto$ is an arbitrary complex numbers. Coefficients in this generating function are called the binomial coefficient is denoted by $\binom{\propto}{n} = \frac{\propto(\propto-1)\ldots(\propto-n+1)}{n!}$.

10) $Ln\left(\frac{1}{1-s}\right) = s+\frac{1}{2}s^2\frac{1}{3}s^3+\cdots;$

11) Sins=$s-\frac{1}{3!}s^3+\frac{1}{5!}s^5+\cdots;$

12) Coss=$1-\frac{1}{2!}s^2+\frac{1}{4!}s^4$ _….

Theorem 2.3. (Operations on Generating Functions)

Let A(x) and B(x) be, respectively, the generating functions for the sequence (a_r) and (b_r).**[8]** Then

(i.) For any numbers $\propto$ and $\beta, \propto A(x)+\beta B(x)$ is the generating function for the sequence (cr), where cr $=\propto a_r+\beta\ br$, for all r;

(ii.) A(x)B(x) is the generating function for the sequence (c_r),,where, c_r=a_0b_r +a_1b_{r-1} +a_2b_{r-2} +…+$a_{r-1}b_1$+a_rb_0, for all r;

(iii.) $A^2(x)$ is the generating function for the sequence (c_r), where c_r=a_0a_r+a_1a_{r-1}+ a_2a_{r-2}+…+$a_{r-1}a_1$ + a_ra_0, for all r;

(iv.) x^m A (x) is the generating function for the sequence (c_r),

$$c_r = \begin{cases} 0 & if\ 0 \le r \le m-1 \\ a_{r-m} & if\ r \ge m; \end{cases}$$

(v.) A(kx), where k is a constant, is the generating function for the sequence (c_r), where

$$c_r = k^r a_r, for\ all\ r;$$

(vi.) (1-x)A(x) is the generating function for the sequence (c_r), where

c_0=a_0 and c_r=a_r-a_{r-1}, for all r ≥ 1;

i.e., (c_r)=(a_0, $a_1 - a_0$, $a_2 - a_1$,…);

(vii.) $\frac{\mathrm{A(x)}}{1-x}$ is the generating function for the sequence (c_r), where

c_r=a_0+a_1+…+a_r, for all r;

i.e,(c_r)=(a_0,a_0+a_1,a_0+a_1+a_2,…);

(viii.) A'(x) is the generating function for the sequence (c_r),, where c_r, =(r+1)a_{r+1}, for all r;

i.e., (c_r) = (a_1,2a_2,3a_3,…);

(ix.) $\mathrm{x}\mathrm{A}'(\mathrm{x})$ is the generating function for the sequence (c_r), where

c_r= ra_r, for all r;

i.e., (c_r) = (0,a_1,2a_2,3a_3,…);

(X.)$\int_0^x \mathrm{A(t)dt}$ is the generating function for the sequence (c_r), where

c_0=0 and $c_r = \frac{a_{r-1}}{r}, for\ all\ r \geq 1$;

i.e. (c_r) = (0,a_0,$\frac{a_1}{2}$,$\frac{a_2}{3}$, …)

proof:

(i),(ii) and(v) follow directly from the definition whereas (iii),(iv) and(vi) are special cases of (ii). Also, (viii), (ix) and(x) are straightforward. We shall prove (vii) only.

(vii) by $\frac{1}{1-x} = 1 + x + x^2 + \cdots$. Thus

$$\frac{\mathrm{A(x)}}{1-x} = (a_0 + a_1x + a_2x^2 + \cdots)(1 + x + x^2 + \cdots)$$

$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots.$$

Hence,

$$\frac{\mathrm{A(x)}}{1-x}$$

Is the generating function for the sequence (c_r) where

$$c_r = a_0 + a_1 + + \ldots + a_r + .$$

2.4. Permutations

2.4.1 The number of ways in this theorem

Theorem 2.4.2 **[7]** For all r, n $\in$ ℕ, with r$\leq$ n,

$$P(n, r) = n(n-1)(n-2)\ldots(n-r+1),$$

That is

$$P(n, r) = \frac{n!}{(n-r)!}. \qquad \textbf{(2.6)}$$

Proof:

When we choose r objects in order from n objects, the first object can be chosen in n ways, leaving n-1 choices for the second object, and so on. After r-1 objects have been chosen there remain n-(r-1) = n-r+1 objects from which to make the r^{th} choice. Hence, by the principle of multiplication of choices, the total number of ways of making these choices is

$$n \times (n-1) \times \ldots \times (n-r+1).$$

This number can be rewritten as

$$\frac{n(n-1)\ldots(n-r+1)(n-r)(n-r-1)\ldots 2\times 1}{(n-r)(n-r-1)\ldots 2\times 1},$$

And using the factorial notation, we can write this expression succinctly as

$$\frac{n!}{(n-r)!}.$$

By theorem 2.6 the number of ways of doing this is

$$P(n, n) = \frac{n!}{0!} = n!$$

So we have:

Theorem 2.4.3 [8] The number of permutations of n objects is $n!$.

the values of $n!$ grow very fast. Even for quite small values of n, $n!$ is very large. For example 10!=3628800, and $100!$ is larger than 10^{157}.

2.5. Combinations [8]

Algebraic proofs of the chief properties of the Binomial coefficients

Theorem 2.6. [8] For all $r, n \in \mathbb{N}, \text{with } r \leq n,$

$$c(n,r) = \frac{n!}{(n-r)!\,r!}. \quad \textbf{(2.7)}$$

Proof:

We have already discovered that the number of ways in which r objects can be chosen, in order, from n objects is given by P (n, r) = $\frac{n!}{(n-r)!}$. A set of r objects can be ordered in $r!$ different ways. Thus, P (n, r) gives the number of r-element subsets of n objects, when each r-element subset is counted $r!$ times. Hence the number of different r-element subset is

$$\frac{p(n,r)}{r!} = \frac{n!}{(n-r)!r!}.$$

The numbers C(n, r) are very well known. They are usually called binomial coefficients.

Theorem 2.7. [8, 9] For all $r, n \in \mathbb{N}, \text{with } r \leq n,$ we have

$$C(n,r) = C(n, n-r).$$

Proof:

Deciding which r objects to select from a set of n objects amounts to exactly the same thing as deciding which n-r objects not to select. Hence, the number of ways of choosing r objects from n is the same as the number of ways of choosing n-r objects from n.

2.8. The Binomial Theorem [5, 9, 10]

We begin with the following simplest form of the binomial theorem discovered by Issac Newton (1646-1727) in 1676.

Theorem 2.9. Let x, y be a real number and for any integer number non negative $\mathrm{n} \geq 0$, $(\mathrm{x}+\mathrm{y})^{n} = \binom{n}{0}x^{n} + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^{2} + \cdots + \binom{n}{n-1}x\,y^{n-1} + \binom{n}{n}y^{n}$

$$= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}\,y^{i} \qquad \textbf{(2.8)}$$

Proof:

By using mathematical induction for n. The theorem is true when n=0 because the left side equal $(x+y)^0=1$. Suppose veracity of the theorem when $\mathrm{n} = \mathrm{k} \geq 0$, so that :

$(\mathrm{x}+\mathrm{y})^{k} = \binom{k}{0}x^{k} + \binom{k}{1}x^{k-1}y + \binom{k}{2}x^{k-2}y^{2} + \cdots + \binom{k}{k}y^{k}$. We want to proof the veracity of the theorem when n=k+1. So we want to proof

$$(\mathrm{x}+\mathrm{y})^{k+1} = \binom{k+1}{0}x^{k+1} + \binom{k+1}{1}x^{k}y + \binom{k+1}{2}x^{k-1}y^{2} + \cdots + \binom{k+1}{k+1}y^{k+1}$$

notice that:

$$(x+y)^{k+1}=(x+y)(x+y)^{k}$$

And from the hypothesis induction

$$(\mathrm{x}+\mathrm{y})^{k+1} = (x+y)[\binom{k}{0}x^{k} + \binom{k}{1}x^{k-1}y + \binom{k}{2}x^{k-2}y^{2} + \cdots + \binom{k}{k}y^{k}]$$

$$= \binom{k}{0}x^{k+1} + \binom{k}{1}x^{k}y + \cdots + \binom{k}{k-1}x^{2}y^{k-1} + \binom{k}{k}xy^{k} + \binom{k}{0}x^{k}y + \binom{k}{1}x^{k-1}y^{2} + \cdots + \binom{k}{k-1}xy^{k} + \binom{k}{k}y^{k+1}$$

$$= \binom{k}{0}x^{k+1} + \left\{\binom{k}{1} + \binom{k}{0}\right\}x^{k}y + \left\{\binom{k}{2} + \binom{k}{1}\right\}x^{k-1}y^{2} + \cdots + \{\binom{k}{k-1} + \binom{k}{k}y^{k+11}$$

$$= \binom{k+1}{0}x^{k+1} + \binom{k+1}{1}x^k y + \binom{k+1}{2}x^{k-1}y^2 + \cdots + \binom{k+1}{k+1}y^{k+1}.$$

We get the final equal by using Pascal's identity.

Note that the number of different terms in factorial $(x+y)^n$ equal n+1. Called the series $1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)\ldots(k-n+1)}{n!}x^n + \cdots$. (The Binomial Series), and from calculus we know if was |x|<1 then for all real number $k \in \mathbb{R}$ it will be

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \cdots + \frac{k(k-1)\ldots(k-n+1)}{n!}x^n + \cdots$$

$$= \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

Such that n integer number and $\binom{k}{n} = \begin{cases} 0 & ,n < 0 \\ 1 & ,n = 0 \\ \frac{k(k-1)\ldots(k-n+1)}{n!} & ,n > 0 \end{cases}$

It is Generalized Binomial coefficients in purpose of using in generating functions. We find now factorial $(1-x)^{-m}$ such that is positive integer number as following:

$$(1-x)^{-m} = \sum_{n=0}^{\infty}\binom{-m}{n}(-x)^n = \sum_{n=0}^{\infty}(-1)^n\binom{-m}{n}x^n$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n\frac{(-m)(-m-1)\ldots(-m-n+1)}{n!}x^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(m)(m+1)\ldots(m+n-1)}{n!}x^n$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(m+n-1)\ldots(m+1)m}{n!}x^n$$

$$= 1 + \sum_{n=0}^{\infty}\binom{m+n-1}{n}x^n$$

$$= \sum_{n=0}^{\infty}\binom{m+n-1}{n}x^n$$

About Binomial coefficients leads to a very well –known method for calculating their values:

Theorem 2.10. [8] For all $n, r \in \mathbb{N}$, with $r \leq n$, we have

$$C(n+1, r) = C(n, r) + C(n, r-1).$$

Proof:

Again we emphasize that our aim is to give a proof of this formula. An algebraic proof, using the formula for the binomial coefficients is very straightforward, but hides the meaning of the formula.

Let X be a set containing n+1 objects. We count the number of ways of choosing a subset of r objects from X. let a be one particular fixed element of X. we divide the r-element subsets of X into two classes.

The first class consists of those r-element subsets of X which do not include a. such subsets are made up of r elements chosen from n-element set X\{a}, and hence there are C(n, r) sets in this class.

The second class consists of those r-element subsets which include a. Such subset consists of a and r-1 elements chosen from the n-element set X\{a}. Thus there are C(n,r-1) elements in this class.

These two classes between them include all the r-element subsets of X and no r-element subset is in both classes. Hence C(n+1,r), the number of r-element subsets of X, is given by

$$C(n+1,r) = C(n, r) + C(n,r-1).$$

Theorem 2.11. [5, 10](Pascal's Identity)

For any to integer numbers $1 \leq k \leq n$, then the following identities is fulfill

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. \quad \textbf{(2.9)}$$

Proof:

Let A={a_1,a_2,…,a_n} set there are numbers n. let B is a partial set from A their numbers k. we have two cases: either $a_n \in B$. According to the rule of sum that the number partial set from A capacity k equal number of partial set from A capacity k and which not containing a_n adding to its partial set numbers from A capacity k which contain a_n. Numbers of partial set from A capacity k, which not containing an equal numbers of partial set from $\{a_n\}$ A from capacity k. If equal $\binom{n-1}{k}$ numbers of partial set from A capacity k which containing an equal the numbers of partial set from $\{a_1,a_2,\ldots,a_{n-1}\}$ } from capacity k-1, if equal $\binom{n-1}{k-1}$ for that

$$\binom{n-1}{k}\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

In using Pascal identities may make Pascal's triangle, which containing from value$\binom{n}{k}$.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

(Figure.1 Pascal's triangle)

The binomial coefficients from an array, usually called Pascal's Triangle, after the seventeenth-century a French mathematician Blaise Pascal, although the triangle was known much earlier, occurring in Chinese manuscript. The first few rows of Pascal's Triangle are shown in figure 1.

2.12. Multinomial Theorem [5, 10]

If $x_1, x_2, \ldots, x_m$ real numbers and was n non negative integer number, then:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{r_1, r_2, \ldots, r_m} x_1^{r1} x_2^{r2} \ldots x_m^{rm}.$$

Therefore, the addition was taken for all non-negative integer numbers $r_1, r_2, \ldots, r_m$ in which fulfillment $r_1 + r_2 + \ldots + r_m = n$,

And where

$$\binom{n}{r_1, r_2, \ldots, r_m} = \frac{n!}{r_1! r_2! \ldots r_m!} \qquad \textbf{(2.10)}$$

Proof:

$(x_1 + x_2 + \cdots + x_m)^n = (x_1 + x_2 + \cdots + x_m) \ldots (x_1 + x_2 + \cdots + x_m),$

for that any term from factorial terms will be from the shape

$$x_1^{r1} x_2^{r2} \ldots . x_m^{rm},$$

Where $r_1, r_2, \ldots, r_m$ integer numbers non-negative fulfillment

$$r_1, + r_2 + \ldots + r_m = n.$$

The coefficients of this term is permutation number r_1 element from type x_1 and r_2 element from type x_2 and… and r_n element from type x_m from Binomial theorem will be the coefficient $x_1^{r1}\ x_2^{r2} \ldots x_m^{rm}$ it is $\binom{n}{r_1, r_2, \ldots, r_m}$. Note: if we put m=2 in multinomial theorem, we will get the Binomial theorem.

Theorem 2.13. (Vandermonde's Identity): [9, 10]

For all $m, n, r \in \mathbb{N}$, *we have*

$$\sum_{i=0}^{r} \binom{m}{i} \binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$

$$= \binom{m+n}{r} \qquad \textbf{(2.11)}$$

Proof 1.

Expanding the expression on both sides of the identity

$$(1+x)^{m+n} = (1+x)^m (1+x)^n$$

We have by the theorem (Binomial theorem)

$$\sum_{k=0}^{m+n} \binom{m+n}{k} x^k = \left(\sum_{i=0}^{m} \binom{m}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right)$$
$$= \binom{m}{0}\binom{n}{0} + \left\{\binom{m}{0}\binom{n}{1} + \binom{m}{1}\binom{n}{0}\right\}x + \left\{\binom{m}{0}\binom{n}{2} + \binom{m}{2}\binom{n}{1}\right\}x^2$$
$$+ \cdots + \binom{m}{m}\binom{n}{n} x^{m+n}.$$

Now, comparing the coefficients of x^r on both sides yields

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$

Proof 2.

Let x={a_1,a_2,…,a_m,b_1,b_2,…,b_n} be a set of m+n objects. We shall count the number of r-combinations A of x.

Assuming that A contains exactly i a's, where i=0,1,..,r, then the other r-i elements of A are b's; and in this case, the number of ways to form A is given by $\binom{m}{i}\binom{n}{r-i}$.

Thus, by (AP), we have

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}.$$

For a closer look on generating function, it proves the following important theorem.

Theorem 2.14. [12](About the inverse function).

Let a function

$$B(t)=b_1 t + b_2 t^2+b_3 t^3+\ldots, \qquad \textbf{(2.12)}$$

Be such that B(0)=b_0=0, and $b_1 \neq 0$. Then there exist functions

$$A(s) = a_1 s + a_2 s^2 + a_3 s^3 + \cdots, A(0) = 0,$$

and $C(u) = c_1 u + c_2 u^2 + c_3 u^3 + \cdots, C(0) = 0,$

Such that

$$A(B(t)) = t \text{ and } B(c(u)) = u.$$

Each of the functions A and C is a unique function possessing this property.
The function A is said to be left inverse and the function C is said to be right inverse to the function B.

Proof.

Let us proof the proof the existing and uniqueness of the left inverse function. For the right inverse function is similar. We compute the coefficients of the function A step by step. The coefficient a_1 is the solution of the equation $a_1 b_1$=1, whence

$$a_1 = \frac{1}{b_1}.$$

Now suppose the coefficients a_1, a_2,…,a_n are already known. Then the coefficient a_n+1 is the solution of the equation $a_n+1 b_1^{n+1} + \cdots = 0$, where dots denote some polynomial in $a_1, \dots, a_n$…and, $b_1, \dots, b_n$. Hence, the equation is a linear equation with respect to a_{n+1} and the coefficient of a_{n+1} is b_1^{n+1}. This coefficient is non-zero, therefore, the equation has a unique solution and the proof of the theorem is completed

CHAPTER 3

GENERATING FUNCTIONS AND SPECIAL FUNCTIONS

In this chapter we will give some important results on generating function in Bernoulli polynomial and special functions of classical orthogonal polynomial.

3.1 Bernoulli Polynomials [11]

In this section we will introduce exponential generating function, because it is the fundamental of Bernoulli Polynomial.

Definition 3.1

Let $\{Y_k^{(t)}\}$ be a sequence of (possibly constant) functions. If there is a function on h(t,x) so that

$$h(t,x) = \sum_{k=0}^{\infty} \frac{yk(t)x^k}{k!}.$$

For all x in an open interval about zero. Then h is called the "exponential generating function" for $\{Y_k(t)\}$.

Example 3.1.1. [11]

Let G_n (t) = $(u(r))^n$ for some function u(r). To compute the generating function for $G_n(r)$, we must the series

$$f(r,s) = \sum_{n=0}^{\infty} (u(r))^n s^n = \sum_{n=0}^{\infty} (u(r)s)^n.$$

By simply recognizing this to be a geometric series, we obtain the sum

$$\frac{1}{1-u(r)s} = f(r,s).$$

For |u(r)s|<1. This result can be used to find other generating function using differentiation,

$$\frac{\partial}{\partial s}\left(\frac{1}{1-u(r)s}\right) = \frac{\partial}{\partial s}\sum_{n=0}^{\infty}(u(r))^n s^n$$

$$\frac{u(r)}{(1-u(r)s)^2} = \sum_{n=0}^{\infty} n\,(u(r))^n s^{n-1},$$

$$\frac{s\,u(r)}{(1-u(r)s)^2} = \sum_{n=0}^{\infty} n\,\left(u(r)\right)^n s^n.\frac{1}{s}.(s)$$

So $\frac{s\,u(r)}{(1-u(r)s)^2}$ is the generating function for the sequence $\{n(u(r))^n\}$.

The exponential generating function for the sequence $\{(u(r))^n\}$ is

$$e^{u(r)s} = \sum_{n=0}^{\infty}\frac{(u(r))^n s^n}{n!}.$$

Definition 3.2. The "Bernoulli Polynomials" B_k(t) are defined by the equation

$$\frac{x\ e^{tx}}{e^x-1} = \sum_{k=0}^{\infty}\frac{B_k(t)}{k!}x^k; \qquad \textbf{(3.1)}$$

In other words, $\frac{xe^{tx}}{e^x-1}$ is the exponential generating function for the sequence B_k(t).

Theorem 3.3. The exponential generating function of Bernoulli Polynomials is given by

$$\sum_{k=0}^{\infty} B_k(t)\frac{x^k}{k!} = \frac{x\,e^{tx}}{e^x-1}. \qquad \textbf{(3.2)}$$

Proof:

Start with

$$\sum_{k=0}^{\infty}[B_k(t) - B_k]\frac{x^k}{k!} = \sum_{k=1}^{\infty}\frac{x^k}{(k-1)!}\sum_{n=1}^{t-1} n^{k-1}$$

$$= x\sum_{n=1}^{t-1}\sum_{k=0}^{\infty}\frac{(tn)^k}{k!}$$

$$= x.\frac{e^{tx}}{e^x-1}$$

$$= \frac{x.e^{tx}}{e^x-1}$$

Definition 3.4. The "Bernoulli numbers" B_k are given by B_k=B_k (0), the value of the k^{th} Bernoulli Polynomial at t=0. We could use equation

$$Y_k(t) = \frac{1}{k!}\ \frac{\partial k}{\partial x^k}\ g(t,0) = \frac{\partial k}{\partial x^k} h(t,0).$$

To compute the first few Bernoulli Polynomials, but it is easier to use the equation in **Definition 2** directly. First, multiply both sides of the equation by $\frac{e^x-1}{x}$

$$e^{tx} = \frac{e^x-1}{x} \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^k.$$

Then expand the exponential functions on each side in their Taylor series about zero and collect terms containing the same power of X:

$$1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots$$

$$= (1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots)(B_0(t) + \frac{B_1(t)}{1!}x + \frac{B_2(t)}{2!}x^2 + \cdots$$

$$= B_0(t) + \left(\frac{B_1(t)}{1!} + \frac{B_0(t)}{2!}\right)x + \left(\frac{B_2(t)}{2!} + \frac{B_1(t)}{2!1!} + \frac{B_0(t)}{3!}\right)x^2 + \cdots.$$

Equating coefficients of like powers of x, we have

$$B_0(t) = 1, B_1(t) = t - \frac{1}{2}, B_2(t) = t^2 - t + \frac{1}{6}, B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \ldots.$$

Then the first eight Bernoulli numbers are

B_0=1, B_1=$-\frac{1}{2}$, B_2=$\frac{1}{6}$, B_3=0, B_4=$-\frac{1}{3}$, B_5=0, B_6=$\frac{1}{42}$, B_7=0, B_8=$-\frac{1}{3}$.

The first one is easy to compute Bernoulli numbers:

$$B_0 = \lim_{x\to 0} \frac{x}{e^x-1}$$

$$= \lim_{x\to 0} \frac{x}{e^x-1}$$

and =1.

$$B_1 = \lim_{x\to 0} \frac{\partial}{\partial x} \left(\frac{x}{e^x - 1}\right)$$

$$= \lim_{x\to 0} \frac{(-1 + e^x - xe^x)}{(e^x - 1)^2}$$

$$= \lim_{x\to 0} \frac{-x}{2(e^x-1)}$$

$$= -\lim_{x\to 0} \frac{1}{2.e^x}$$

$$= -\frac{1}{2}$$

k	0	1	2	3	4	5	6	7	8
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{3}$	0	$\frac{1}{42}$	0	$-\frac{1}{3}$

(Table.1 Bernoulli Numbers)

And so forth. The first few Bernoulli Polynomials are given by

$$B_0(t) = 1, B_1(t) = t - \frac{1}{2}, B_2(t) = t^2 - t + \frac{1}{6}, B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \ldots.$$

We present several properties of Bernoulli polynomials in the following theorem

Theorem 3.5. [9]

a) $B'_r(s) = rB_{r-1}(s)$ $(r \geq 1)$. **(3.3)**

b) $\Delta s\, B_r(s) = r\, s^{r-1}$ $(r \geq 0)$. **(3.4)**

c) $B_r = B_r(0) = B_r(1)$ $(r \neq 1)$. **(3.5)**

d) $B_{2w+1} = 0$ $(w \geq 1)$. **(3.6)**

Proof:

a) $\frac{u^2 \quad e^{su}}{e^u - 1} = \sum_{r=0}^{\infty} \frac{B' r(s)}{r!} u^r \; \sum_{r=0}^{\infty} \frac{Br(s)}{r!} u^{r+1} = \sum_{r=0}^{\infty} \frac{B' r(s)}{r!} u^r.$

Now make change of index $r \to r - 1$ in the left- hand sum:

$$\sum_{r=1}^{\infty} \frac{B_{r-1}(s)}{(r-1)!} u^r = \sum_{r=0}^{\infty} \frac{B' r(s)}{r!} u^r.$$

Equating coefficients.

$$\sum_{r=0}^{\infty} \frac{\Delta s Br(s)}{r!} u^r = \frac{u}{e^u - 1}\left(e^{(s+1)u} - e^{su}\right)$$

$$= u\, e^{su}$$

$$= \sum_{r=0}^{\infty} \frac{s^r \; u^{r+1}}{r!}$$

$$= \sum_{r=1}^{\infty} \frac{s^{r-1} \; u^r}{(r-1)!}$$

b) Immediately by equating coefficients.

c) $B_r(0) = B_r(1) for\ r \neq 1?$

$$\Delta_s B_r(s) = rs^{r-1}, r \geq 0$$

$$B_r(s+1) - B_r(s) = r\ \ s^{r-1}$$

$$\text{for } s = 0: B_r\ (1) - B_r(0) = 0$$

$$B_r(1) = B_r(0) = B_r, (r \neq 1)$$

d) $\frac{u\ e^{su}}{e^{u-1}} = \sum_{r-0}^{\infty} \frac{B_r(s)}{r!}\ u^r$

S=0

$$\frac{u}{e^u - 1} = \sum_{r-0,r\neq 1}^{\infty} \frac{B_r}{r!}\ u^r + B_1(0)u$$

$$= \sum_{r-0,r\neq 1}^{\infty} \frac{Br}{r!} u^r - \frac{1}{2}u$$

$$\Rightarrow \sum_{r-0,r\neq 1} \frac{Br}{r!} u^r = \frac{u}{e^u - 1} + \frac{1}{2}u$$

Let $\text{h}(\text{u}) = \frac{u}{e^u - 1} + \frac{1}{2}u$

Then $\text{h}(-\text{u}) = \frac{-u}{e^u - 1} - \frac{1}{2}u$

$$= \frac{-u\ e^u}{1 - e^u} - \frac{1}{2}u$$

$$= -\frac{1}{2}u + \frac{u}{e^u - 1}$$

So,

$$\sum_{r-0,r\neq 1}^{\infty} \frac{B_r}{r!} u^r = \sum_{r-0,r\neq 1}^{\infty} \frac{B_r}{r!} (-u)^r$$

$$\sum_{r-0,r\neq 1}^{\infty} \frac{B_r}{r!} u^r = \sum_{r-0,r\neq 1}^{\infty} \frac{(-1)^r B_r}{r!} u^r$$

Comparing the coefficients, we have

B_r=$(-1)^r\ B_r, r \neq 1$

If $\text{r} = 2\text{n}, B_{2n}\ = (-1)^{2n} B_{2n}, n \geq 1$

If $r_{2n+1}, B_{2n+1} = (-1)^{2n+1} B_{2n+1} = -B_{2n+1} \Rightarrow 2B_{2n+1} = 0$

That is, B_{2n+1}=0, n $\geq$ 1.

Corollary 3.6.

If r=0,1,2,…, then

$$\sum s^r = \frac{1}{r+1} B_{r+1}(s) + c(s), when\ \Delta_c(s) = 0$$

Example 3.6.1. Use corollary to show that $(r \geq 1)$

$$\sum_{b=1}^{w-1} b^r = \frac{1}{r+1}[B_{r+1}(w) - B_{r+1}]$$

Solution: Let $\text{hb} = \sum b^r$, then by Thm.

$$\sum_{b=1}^{w-1} b^r = [hb]_1^w = h_w - h_1.$$

By corollary, we have

$$h_w = \sum w^r = \frac{1}{r+1} B_{r+1}(w) + c$$

$$h_1 = \sum 1^r = \frac{1}{r+1} B_{r+1}(1) + c$$

$$= \frac{1}{r+1} B_{r+1}$$

Then

$$\sum_{b=1}^{w-1} b^r = \frac{1}{r+1} B_{r+1}(w) - \frac{1}{r+1} B_{r+1}$$

$$= \frac{1}{r+1}(B_{r+1}(w) - B_{r+1}), r \geq 1$$

Example 3.6.2. The result of example 3.6 to compute $\sum_{b=1}^{w-1} b^2$

Solution: by **(3.6)**

$$\sum_{b=1}^{w-1} b^2 = \frac{1}{3}(B_3(w) - B_3 = \frac{1}{3} B_3(w))$$

$$B_3(w) = w^3 - \frac{3}{2} w^2 + \frac{1}{2} w$$

$$\sum_{b=1}^{w-1} b^2 = \frac{w^3}{3} - \frac{w^2}{2} + \frac{w}{6}$$

Theorem 3.7. (Euler Summation Formula)

Suppose that the $2b^{th}$ derivative of Y(h), $Y^{2b}(h)$, is continuous on [1,w] for some integers $b \geq 1$ and $w \geq 2$. Then

$$\sum_{r=1}^{w} Y(r) = \int_{1}^{w} Y(h)\, dh + \frac{Y(w)+Y(1)}{2} + \sum_{n=1}^{b} \frac{B_{2n}}{(2n)!}\left[Y^{(2n-1)}(w) - Y^{(2n-1)}(1)\right] - \frac{1}{(2b)!}\int_{1}^{w} Y^{(2b)}(h)\; B_{2b}\;\, (h - \lfloor h \rfloor) dh,$$

(3.7)

Where $\lfloor h \rfloor =$ the greatest integer less than or equal to h,

(called the "floor function" or greatest integer functions").

Proof:

Integration by parts gives each r,

$$\int_{r}^{r+1} (h - \lfloor h \rfloor - \tfrac{1}{2}) y'(h) dh = \int_{r}^{r+1} \left(t - r - \tfrac{1}{2}\right) y'(h) dh$$

$$= \frac{y(r+1)+y(r)}{2} - \int_{r}^{r+1} y(h) dh \qquad (3.8)$$

Note that in the first integral in Eq**(3.8)**

$$h - \lfloor h \rfloor - \tfrac{1}{2} = B_1(h - \lfloor h \rfloor).$$

Similarly, we have by **Theorem 3.6**(a) and (c)

$$\int_{r}^{r+1} B_n(h - \lfloor h \rfloor) y^{(n)}(h) dh = \int_{r}^{r+1} B \quad (h - r) y^{(n)}(h) dh$$

$$= \frac{B_{n+1}}{n+1}\left[y^{(n)}(r+1) - y^{(n)}(r)\right] - \frac{1}{1+n}\int_{r}^{r+1} B_{n+1}(h - \lfloor h \rfloor) y^{(n+1)}(h) dh \quad (3.9)$$

For n=1,… 2b-1. Summing Eq**(3.8)**and Eq**(3.9)** as r goes from 1 to n-1, we have, respectively,

$$\int_1^w B_1(h-\lfloor h\rfloor)y'(h)dh = \sum_{r=1}^{w} y(r) - \frac{1}{2}[y(1)+y(w)] \tag{3.10}$$

$$-\int_1^w y(h)dh,$$

$$\int_1^w B_n(h-\lfloor h\rfloor)y^{(n)}(h)dh = \frac{B_{n+1}}{n+1}\left[y^{(n)}(w) - y^{(n)}(1)\right] \tag{3.11}$$

$$-\frac{1}{n+1}\int_1^w B_{n+1}(h-\lfloor h\rfloor)y^{(n+1)}(h)\ dh.$$

Finally we begin with Eq**(3.10)** and use Eq**(3.11)** repeatedly to obtain

$$\sum_{r=1}^{w} y(r) - \frac{1}{2}(y(1)+y(w) - \int_1^w y(h)dh$$

$$= \int_1^w B_1(h-\lfloor h\rfloor)y'(h)dh$$

$$= \frac{B_2}{2}[y'(w)-y'(1)] - \frac{1}{2}\int_1^w B_2(h-\lfloor h\rfloor)y^{(2)}(h)dh$$

$$= \cdots$$

$$\sum_{n=1}^{b}\frac{B_{2n}}{(2n)!}[y^{(2n-1)}(w) - y^{(2n-1)}(1)] - \frac{1}{(2b)!}\int_1^w B_{2b}(h-\lfloor h\rfloor)y^{(2b)}(h)dh,$$

Where we have used theorem 3.6 (d). Rearrangement yields the Euler Summation Formula.

Example 3.7.1. Given an estimate for $\int_{r=1}^{400} r^{1/2}$.

Solution:

$$\sum_{r=1}^{400} r^{1/2} = \int_1^{400} h^{1/2}\, dh + \frac{w^{1/2}+1}{2} + \frac{B2}{2!}\left[\frac{1}{2\sqrt{w}} - \frac{1}{2}\right] + \frac{1}{8}\int_1^{400} h^{\frac{-3}{2}} B_2(h - \lfloor h \rfloor)\, dh.$$

$$= \int_1^{400} h^{1/2}\, dh + \frac{w^{1/2}+1}{2} + \frac{1}{24}\left(w^{-1/2} - 1\right) + \frac{1}{8}\left(\frac{h^{-1/2}}{-1/2}\right)\Big|_1^{400}$$

$$= \frac{(400)^{3/2}-1}{3/2} + \frac{(400)^{1/2}+1}{2} + \frac{1}{24}\left[(400)^{-1/2} - 1\right]$$

$$\frac{1}{48}(\mathrm{w}^{-1/2} - 1) \leq \int_1^{400} h^{-3/2} B_2(h - \lfloor h \rfloor)dh$$

$$\leq -\frac{1}{24}(w^{-1/2} - 1)$$

$$error \leq \frac{1}{24}$$

3.8. The Classical Orthogonal functions

In this section we will give some results about the classical orthogonal polynomials.

For more details of these polynomials, the reader might refer to **[1, 11, 12]**

Definition 3.8.1. (Orthogonal Functions) [1, 12]

Functions set $\{h_0(\mathrm{u}), h_1(\mathrm{u}), \ldots, h_r(\mathrm{u}), \ldots\}$ orthogonal on interval [f,j] with weight function quantity w(u) if $\int_f^j h_r(u)\, h_i(u)\; w(u)\; du = 0, r \neq i$

$$\int_f^j h_r(u)\, h_i(u)\; w(u)\; du \neq 0, r = i$$

Theorem 3.9. for all h, t: [f, j]→/R real functions

$< h,t > w = \int_f^j h(u)\ t(u)\ w(u)\ du$

Where

$w(u) \geq 0 \quad on\ [f,j]$ is inner product.

Proof:

For all h, t, y is real functions and $a \in /R$ we have

$$< h,t >_w = < h,t >_w$$

$$< h,t+y >_w = < h,t >_w + < h,y >_w$$

$$< ah,t >_w = a < h,t >_w$$

Remark : called scale function h and quantity $\sqrt{< h,h >_w}$ and we symbolize to him in symbol ||h||.

Some of Orthogonal Polynomials 3.10. [1, 13]

1) **Laguerre Polynomials** $(L_r^{\mu}(u))$

Definition 3.10.1. (Laguerre Polynomials)

$L_r^{\mu}\ (\mu \geq 0)$ Are known Polynomials unit from degree r and they are the solutions of the following differential equations:

$$u\ \frac{\partial^2 u}{\partial\ v^2} + (\mu + 1 - u)\frac{\partial u}{\partial\ v} + r\ v = 0 \qquad (3.12)$$

The following is the generating function of the Laguerre polynomials:

$$\sum_{r=0}^{\infty} L_r^{(\mu)}(v)k^r = (1-k)^{-\mu-1} \exp\left(\frac{vk}{k-1}\right), |k| < 1 \qquad (3.13)$$

Example 3.10.2. [9] Given that y(v,u)=(1-u)$^{-1}$ exp$\left(\frac{-uv}{1-u}\right) = \sum_{r=0}^{\infty} L_r(v)\ u^r$ is the generating function for the Laguerre Polynomials L_r(v), find L_r(v) for $0 \leq r \leq 3$?

<u>**Solution:**</u> $y(v,u) = (1-u)^{-1} e^{\left(\frac{-uv}{1-u}\right)}$

$$\frac{\partial}{\partial u}\ y(v,u) = -1\ (1-u)^{-2}(-1)\ e^{\left(\frac{-uv}{1-u}\right)} + \frac{(-v)}{(1-u)^2}\ e^{\frac{-uv}{1-u}}\ (1-u)^{-1}$$

$$\frac{\partial^2}{\partial u^2}\ y(v,u) = e^{\frac{-uv}{1-u}}((1-u)^{-2} - v(1-u)^{-3})$$

$$\frac{\partial^3}{\partial u^3}\ y(v,u) = e^{\frac{-uv}{1-u}}((2(1-u)^{-3} - 4v(1-u)^{-4} + v^2\ (1-u)^{-5})$$

$$\frac{\partial}{\partial u}\ y(v,u) = e^{\frac{-uv}{1-u}}(6(1-u)^{-4} - 18v(1-u)^{-5} + 9v^2(1-u)^{-6} - v^3(1-$$

$u)^{-7})$. Thus when u=0, y(v,o)=1, $\frac{\partial}{\partial u}$y(v,o)=1-v,

$$\frac{\partial^2}{\partial u^2}\ y(v,o) = v^2 - 4v + 2, \frac{\partial^3}{\partial u^3} = -v^3 + 9v^2 - 8v + 6$$

$$L_0(v) = y(v,o) = 1$$

$$L_1(v) = \frac{\partial}{\partial u}\ y(v,o) = 1 - v$$

$$L_2(v) = \frac{1}{2!}\frac{\partial^2}{\partial u^2}\ y(v,o) = \frac{1}{2}\ v^2 - 2v + 1$$

$$L_3(v) = \frac{1}{3!}\frac{\partial^3}{\partial u^3}\ y(v,o) = -\frac{1}{6}v^3 + \frac{3}{2}\ v^2 - 3v + 1$$

2) **Legendre Polynomials (P_r(u))**

Definition [1, 14] (Legendre polynomials) we say that P_r it is Legendre polynomials of degree r if we have:

$$P_r(1) = 1\begin{cases}\int_{-1}^{1} P_s(u)\ P_n(u)du = 0, & s \neq n\\ \int_{-1}^{1} P_s(u)P_n(u)du \neq 0, & s = n\end{cases} \qquad \textbf{(3.14)}$$

The following is the generating function of the Legendre polynomials:

$$\sum_{r=0}^{\infty} P_r\ (v)k^r\ = \frac{1}{\sqrt{1-2vk+k^2}} \quad ,-1<v<1,|k|<1 \qquad \textbf{(3.15)}$$

Example 3.10.3. [11] Given that $Y(v,u) = (1-2vu+u^2)^{-\frac{1}{2}}$ is the generating function for the Legendre polynomials $P_r(v), find\ P_r(v) for\ \ 0 \leq r \leq 3$?

Solution: $Y(v,u) = (1-2vu+u^2)^{-\frac{1}{2}}$

$$\frac{\partial}{\partial u}\ Y(v,u)\ = \frac{-1}{2}(1-2uv+u^2)^{-\frac{3}{2}}\ (-2v+2u)$$

$$\frac{\partial}{\partial u}Y(v,u) = (v-u)(1-2vu+u^2)^{\frac{-3}{2}}$$

$$\frac{\partial^2}{\partial u^2}Y(v,u) = -(1-2vu+u^2)^{\frac{-3}{2}} + 3(v-u)^2(1-2vu+u^2)^{\frac{-5}{2}}$$

$$\frac{\partial^3}{\partial u^3}Y(v,u) = -9(v-u)(1-2vu+u^2)^{\frac{-5}{2}} + 15(v-3)^3(1-2vu + u^2$$

$)^{\frac{-7}{2}}$

$$Y(v,0) = 1, \frac{\partial}{\partial u} Y(v,0) = 0, \frac{\partial^2}{\partial u^2} Y(v,0) = -1 + 3v^2,$$

$$\frac{\partial^3}{\partial u^3} Y(v,0) = -gv + 15v^3$$

$$P_0(v) = Y(v,0) = 1$$

$$P_1(v) = \frac{\partial}{\partial u} Y(v,0) = v$$

$$P_2(v) = \frac{1}{2!} \frac{\partial^2}{\partial u^2} Y(v,0) = \frac{3}{2} v^2 - \frac{1}{2}$$

$$P_3(v) = \frac{1}{3!} \frac{\partial^3}{\partial u^3} Y(v,0) = \frac{5}{2} v^3 - \frac{3}{2} v$$

3) **Hermite Polynomials $H_r(u)$**

Definition [1, 12] (Hermite Polynomials) Hermite functions H_r are defined follows:

$$Y(v,u) = e^{2vu-u^2} = \sum_{r\geq 0} \frac{H_r(v)}{r!} u^r \ where\ u \in/R \qquad (3.16)$$

Y(v,u) is called generating function for Hermite Polynomials.

Example 3.10.4. [11] The function $Y(v,u) = \exp(2vu - u^2)$ is the generating function for the Hermite Polynomials $H_r(v)$. Compute $H_r(v) for\ 0 \leq r \leq 3$

Solution:

$$Y(v,u) = e^{(2vu-u^2)}$$

$$\frac{\partial}{\partial u} Y(v,u) = (2v - 2u)e^{(2vu-u^2)}$$

$$\frac{\partial^2}{\partial u^2} Y(v,u) = (-2 + (2v - 2u)^2\ e^{(2vu-u^2)}$$

$$\frac{\partial^3}{\partial u^3} Y(v,u) = (-6 + (2v - 2u)^2)(2v - 2u)e^{(2vu-u^2)}$$

$$Y(v,0) = 1, \frac{\partial}{\partial u} Y(v,0) = 2v, \frac{\partial^2}{\partial u^2} Y(v,0) = 4v^2 - 2$$

$$\frac{\partial^3}{\partial u^3} Y(v,0) = 8v^3 - 12v$$

$$H_0(v) = Y(v,0) = 1$$

$$H_1(v) = \frac{\partial}{\partial u} Y(v,0) = 2v$$

$$H_2(v) = \frac{1}{2!}\frac{\partial^2}{\partial u^2} Y(v,0) = 2v^2 - 1$$

$$H_3(v) = \frac{1}{3!}\frac{\partial^3}{\partial u^3} Y(v,0) = -\frac{4}{3}v^3 - 2v$$

4) **Tchebychev Polynomials from first kind (T_r) and second kind (U_r)**

Definition: [1, 13] Tchebychev Polynomials is from first kind (T_r) and second kind (U_r) on interval I=[-1,1] as follows:

$$\begin{cases} T_r(v) = \cos(r\theta) \\ U_r(v) = \dfrac{\sin((r+1)\theta)}{\sin\theta} \, where\ \theta = \cos^{-1} u \end{cases}$$

The following is the generating function of the Tchebychev polynomials:

First type:

$$\sum_{r=0}^{\infty} T_r(v)k^r = (\frac{1-k^2}{1-2vk+k^2} + 1), -1<v<1, |k|<1 \quad \textbf{(3.17)}$$

$$\sum_{r=0}^{\infty} T_r(v)k^r = \frac{1}{\sqrt{1-2vk+k^2}}\left(1 - vk\sqrt{1-2vk+k^2}\right)^{-1/2}, -1<v<1, |k|<1 \quad \textbf{(3.18)}$$

$$\sum_{r=0}^{\infty} T_r(v)k^r = \frac{1-vk}{(1-2vk+k^2)}, -1<v<1, |k|<1 \quad \textbf{(3.19)}$$

Second type:

$$\sum_{r=0}^{\infty} U_r(v)k^r = \frac{1}{1-2VK+K^2}, -1<v<1, |k|<1 \quad \textbf{(3.20)}$$

$$\sum_{r=0}^{\infty} U_r(v)k^r = \frac{1}{\sqrt{1-2VK+K^2}}\left(1 - vk + \sqrt{1-2VK+K^2}\right)^{-1/2}, -1<v<1, |k|<1$$

(3.21)

Example 3.10.5. [11] Find the generating function for

$\lambda_\alpha(v) = \cos \alpha v.$ $(Hint\ write\ \cos\alpha v = Re\left(e^{i\alpha v}\right)).$

Solution:

$\lambda_{\partial}(v) = \cos \alpha v \, , \cos \alpha v = Re(e^{i\alpha v}) \;\; (e^{i\alpha v} = \cos(\alpha v) + i \sin(\alpha v))$

$$\sum_{\alpha=0}^{\infty} [\cos(\alpha v) u^{\alpha}] = Re\left(\sum_{\alpha=0}^{\infty} e^{i\alpha v} \, u^{\alpha}\right)$$

$$= Re\left[\sum_{\alpha=0}^{\infty} (e^{iv} \, u)^{\alpha}\right] \Rightarrow = Re\left(\frac{1}{1 - e^{iv} u}\right)$$

$$= Re\left[\frac{1}{1 - (\cos v + i \sin v) u}\right]$$

$$= Re\left[\frac{1}{(1 - (\cos v)u - i \, (\sin v) u)} \cdot \frac{(1 - (\cos v) \, u + i(\sin v) u)}{(1 - (\cos v) u + i \, (\sin v) u)}\right]$$

$$= Re\left[\frac{1 - (\cos v) u}{1 - 2(\cos v) u + u^2} + i \frac{(\sin v) u}{1 - 2(\cos v) u + u^2}\right]$$

$$= \frac{1 - (\cos v) u}{1 + u^2 - 2(\cos v) u}$$

$T_r(v) = so, \; Y(v, u) = \frac{1 - (\cos v) u}{1 - 2(\cos v) u + u^2}$ is the generating function for

$\lambda_{\alpha}(v) = \cos(\alpha v) = Re(e^{i\alpha v})$

CHAPTER 4

SOME APPLICATIONS OF GENERATING FUNCTIONS

4.1. The Method of Generating Functions. [3]

A recurrence formula that is to be solved by the method of generating functions.

1) Make sure that the set of values of the free variable (say n) for which the given recurrence relation is true, is clearly delineated.

2) Give a name to the generating function and look for it. And write out that function in terms of the unknown sequence (e.g., call it A(x), and define it to be $\sum_{n\geq 0} a_{nx^n}$).

3) Multiply both sides of the recurrence by x^n, and sum overall values of n for which the recurrence holds.

4) Express both sides of the resulting equation explicitly in terms of the generating function A(x).

5) Solve the resulting equation for the unknown generating function A(x).

6) The exact formula for the sequence that is defined by the given recurrence relation, then to get such a formula by expanding A(x) into a power series by any method you can think of. In particular, if A(x) a rational function (quotient of two polynomials), success will result from expanding in partial fractions and then handling each of the resulting terms separately.

Example 4.1.1. How many strings are there of n digits which do not contain consecutive zeroes?

A string of n digits is simply a sequence $d_1, d_2, \ldots, d_n$, where each d_i is one of the numbers 0,1,2,…,9. We let a_n be the number of such strings which do not contain consecutive zeros. We calculate a_n by considering how strings of n digits, not containing consecutive zeros, can be built up from shorter strings with this property. We can divide the strings of length n without consecutive zeros into two disjoint classes. The first class consists of all those strings which do not begin with a zero. The first digit can those be any of the other nine and the remaining n-1 digit must themselves from a string without consecutive zeros. There are a_{n-1} these. Hence there are all together $9a_{n-1}$ strings in this class. The second class consists of those strings which start with a zero. The initial zero must be followed by one of the other 9 digits, and the remaining numbers make up a string of n-2 digits not containing consecutive zeros. Thus, there are $9a_{n-2}$ strings in this class. It follows that for all n$\in lN^+, with\ n \geq 3,$

$$a_n = 9a_{n-1} + 9a_{n-2} \qquad (4.1).$$

This formula makes it quite straightforward to calculate the value of a_n for any particular value of n, given the additional facts that

$$\left.\begin{matrix} a_1=10 \\ a_2=99 \end{matrix}\right\} \qquad (4.2).$$

Which can easily be checked, for example, we can calculate a_5 as follows:

a_3=9a_2+9a_1

=891+90

=981

a_4=9a_3=+9a_2

=8829+891

=9720

a_5=9a_4+9a_3

=87480+8829

=96309.

As a first shot we could say that a recurrence relation has the form

$$a_n = f(a_{n-1}, a_{n-2}, \ldots) \qquad (4.3)$$

This can be defined by the recurrence relation

$$a_n = na_{n-1} \qquad (4.4).$$

Together with the initial value a_0=1 Thus, general recurrence relation has the form

$$a_n = f(n, a_{n-1}, a_{n-2}, \ldots) \qquad (4.5)$$

Where f is some given function. Again, our notation is deliberated vague about the number of terms of the sequence which are involved on the right-hand side of equation (4.5) as we wish to allow for such cases as

$$a_n = \sum_{k=1}^{n-1} a_k,$$

Where the number of terms of the sequence which are involved in the definition of a_n , can vary from the value of n.

Recurrence relations are classified according to the form of the function f which occurs in the relation. Recurrence relations can be solved by using the device of generating functions.

4.2. Generating Functions and Recurrence Relations [8]

The basic idea of generating function approach to recurrence relations is to translate the recurrence relation into equation involving the generating function of the sequence. If we can extract from this equation an explicit formula for the generating function, we may be able to use this to derive a formula for the coefficients in its power series. These coefficients are, of course, just the terms of the sequence In which we are interested.

Before discussing this method in general we illustrate it in relation to the particular recurrence relation of **(4.1),** as given by equation **(4.2)** and subject to the initial conditions **(4.2).**

We let A be the generating function for the sequence $\{a_n\}$. Thus,

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

If we multiply both sides of the recurrence relation (1) by x^n, and sum for all integers $n \geq 3$, for which the relation **(4.1)** holds, we obtain

$$\sum_{n=3}^{\infty} a_n x^n = 9\sum_{n=5}^{\infty} a_{n-1} x^n + 9\sum_{n=3}^{\infty} a_{n-2} x^n. \qquad (4.6)$$

The sum on the left-hand side of equation **(4.6)** is just the power series for A without the first two terms. Thus

$$\sum_{n=3}^{\infty} a_n x^n = A(x) - a_1 x - a_2 x^2$$
$$= \mathrm{A(x)} - 10\mathrm{x} - 99x^2.$$

To relate the term

$$9\sum_{n=3}^{\infty} a_{n-1} x^n.$$

Which occurs on the right-hand side of equation **(4.6)** to the generating function A we need to pull out a factor x so that a_{n-1} multiplies x^{n-1}, as it does in the series for A

$$9\sum_{n=3}^{\infty} a_{n-1} x^n = 9x\sum_{n=3}^{\infty} a_{n-1} x^{n-1}$$
$$= 9\mathrm{x}(\mathrm{A(x)} - 10\mathrm{x}) \qquad (4.7).$$

(Note that $\sum_{n=3}^{\infty} a_{n-1} x^n = a_2 x^2 a_3 x^3 + \cdots$, and so is the power series for A without the first term, that is, A(x)-a_1x, which is where we obtained equation**(4.7)** from).

In similar way,

$$9\sum_{n=3}^{\infty} a_{n-2} x^n = 9x^2\sum_{n=3}^{\infty} a_{n-2} x^{n-2}$$
$$= 9x^2 A(x).$$

Thus, we can reduce from the equation **(4.6)** that

$$\mathrm{A(x)} - 10\mathrm{x} - 99x^2 = 9x(A(x) - 10x) + 9x^2 A(x) \qquad (4.8).$$

It is now a straightforward matter to rearrange equation**(4.8)** to give

$$\mathrm{A(x)} = \frac{1+x}{1-9x-9x^2} - 1 \qquad (4.9).$$

Thus, we have now achieved the first stage of our objective. We have obtained an explicit formula for the generating function of the sequence $\{a_n\}$. More than one method can be used to derive from this a formula for the coefficients in the corresponding power series. Probably the most coefficients of these is to rewrite the formula for A(x) using the technique of partial fractions

$$1 - 9\mathrm{x} - 9x^2 = (1 - \propto x)(1 - \beta x),$$

Where $\propto$ and β are the reciprocals of the solutions of the equation

$$1 - 9x - 9x^2 = 0 \qquad (4.10).$$

Thus, $\propto$ and β are the solutions of the equation

$$y^2 - 9y - 9 = 0 \qquad (4.11).$$

$$A(x) = \frac{1}{\propto - \beta}\left(\frac{\propto + 1}{1 - \propto x} - \frac{\beta + 1}{1 - \beta x}\right) - 1 \qquad (4.12)$$

$$A(x) = \frac{1}{\propto - \beta}\left((\propto + 1)\sum_{n=0}^{\infty}(\propto x)^n - (\beta + 1)\sum_{n=0}^{\infty}(\beta x)^n\right) - 1,$$

And by equating coefficients we can deduce that, for $n \geq 1$,

$$a_n = \frac{1}{\propto - \beta}\left((\propto + 1) \propto^n - (\beta + 1)\beta^n\right) \qquad (4.13)$$

With $\propto$ and β as given above. We have those obtained an explicit formula for a_n.

4.3. Generating Functions for Combinations [15, 16]

We have seen that the polynomial (1+ax)(1+bx) is the ordinary generating function of the different ways to select the objects a, b and c.

a	ab	abc
b	ac	
c	bc	

Instead of the different ways of selection, we may only be interested in the number of ways of selection. By setting a=b=c=1, we have

$$(1+x)(1+x)(1+x) = (1+x)^3 = 1+3x+3x^2+x^3.$$

Clearly, we see that there is one way to select no objects from the three objects, C (3,0) three ways to select one object out of three, C (3,1), etc.

Usually, a generating function that gives the number of combinations or permutations is called an ordinary enumerator. This notion can be extended immediately. To find the number of combinations of n distinct objects, we have the ordinary enumerator

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\dots(n-r+1)}{r!}x^r + \cdots + x^n$$

$$= C(n,0) + C(n,1)x + C(n,2)x^2 + \cdots + C(n,r)x^r + \cdots + C(n,n)x^n.$$

In this expansion of $(1+x)^n$, the coefficients of the term x^r is the number of ways the term x^r can be formed by taking rx's and n-r1's among the n factors 1+x. It is for the reason that the C (n, r)'s are called the binomial coefficients. In binomial expansion, $\binom{n}{r}$ is a common alternative notation for C (n, r).

Examples 4.3.1. [16]

1) From

$$\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n = (1+x)^n$$

We have the identity $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{r} + \cdots + \binom{n}{n} = 2^n$

By setting x equal to 1. The combinatorial significant of this identity is that both sides give the number of ways of selecting none, or one, or two,.., or n objects out of n distinct objects. We also have the identity

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} \ldots + (-1)^r\binom{n}{r} + \cdots + (-1)^n\binom{n}{r} = 0,$$

by setting x equal to -1.

Writing this as

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots.$$

We see that the number of ways of selecting an even number of objects is equal to the number of ways of selecting an odd number of objects from n distinct objects.

2) The identity

$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{r}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$, can be provided in two ways.

Method 1:

We observe that the expression on the left-hand side is the constant term in $(1+x)^n(1+x^{-1})^n$.

Since $(1+x)^n(1+x^{-1})^n=(1+x)^n(1+x)^n\ x^{-n}$

$=x^{-n}(1+x)^{2n}$,

And the constant term in $x^{-n}(1+x)^{2n}$ is $\binom{2n}{n}$, we have proved the identity.

Method 2:

We rewrite the identity to be proved as

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{r}\binom{n}{n-r}+\cdots+\binom{n}{n}\binom{n}{0}=\binom{2n}{n},$$

and use a combinatorial argument. To select n objects out of 2n objects, we shall first divide them(in any arbitrary manner) into two piles with n objects from the first pile and $\binom{n}{n-i}$ ways to select n-I objects from the second pile to make up a selection of n objects. Therefore, the number of ways to make the selection is $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$ which is also equal to $\binom{2n}{n}$.

To see an application of this result, let us consider the problem of finding the number of 2n- digit binary sequences which are such that the number of 0's in the first n digits of a sequence is equals to the number of 0's in the last n digits of the sequence. Since the number of n-digit binary sequences containing r 0's is $\binom{n}{r}$, the number of 2n-digit binary sequences containing r 0's in the first n digits as well as in the last n digits is $\binom{n}{r}^2$. Therefore, the number of 2n-digit binary sequence which are such that the sequence is equal to the number of 0's in the last n-digits of the sequence is

$$\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\cdots+\binom{n}{r}^2+\cdots+\binom{n}{n}^2=\binom{2n}{n}$$

3) Prove the identity

$$\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+r\binom{n}{r}+\cdots+n\binom{n}{n}n2^{n-1},$$

4) Differentiation both sided of the identity

$$\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\cdots+\binom{n}{r}x^r+\cdots+\binom{n}{n}x^n=(1+x)^n.$$

We have

$$\binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + r\binom{n}{r}x^{r-1} + \cdots + n\binom{n}{n}x^{n-1} = n(1+x)^{n-1}.$$

The given identity is obtained by setting x equal to 1.

5) What is the coefficients of the term x^{23} in $(1+x^5+x^9)^{100}$?

Since $x^5\ x^9\ x^9=x^{23}$ is the only way to term x^{23} can be made up in the expansion of $(1+x^5+x^9)^{100}$ and there are C(100,2) ways to choose the two factors x^9 and then C(98,1) ways to choose the factor x^5, the coefficients of x^{23} is

$$C(100,2)\times C(98,1)=\frac{100\times 99}{2}\times 98 = 485.100.$$

6) Show the ordinary generating function of the sequence

$\binom{0}{0}, \binom{2}{1}, \binom{4}{2}, \binom{6}{3}, \ldots, \binom{2r}{r}, \ldots$, is $(1-4x)^{-1/2}$. According to the binomial theorem

$$(1-4x)^{-1/2} = 1 + \sum_{r=1}^{\infty} \frac{(-1/2)(-1/2-1)\ldots(-1/2-r+1)}{r!!}(-4rx)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{4^r(1/2)(3/2)(5/2)\ldots[(2r-1/2)]}{r!}x^r$$

The binomial theorem is

$$(1+x)^n = 1 + \sum_{r=1} \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{4^r[1\times 3\times 5\times \ldots\times(2r-1)]}{r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(2^r\times r!)[1\times 3\times 5\times \ldots\times(2r-1)]}{r!\ r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(2r)!}{r!\,r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \binom{2r}{r}x^r$$

As an application of this result we evaluate the sum

$\sum_{i=0}^{t}\binom{2i}{i}\binom{2t-2i}{t-i}$ for a given t.

Since $\binom{2i}{i}$ is the coefficients of the term x^i in $(1-4x)^{-1/2}$ and $\binom{2t-2i}{t-i}$ is the coefficients of the term x^{t-i} in $(1-4x)^{-1/2}$, $\sum_{i=0}^{t}\binom{2i}{i}\binom{2t-2i}{t-i}$ is the coefficients of the term x^t in $(1-4x)^{-1/2}(1-4x)^{-1/2}$.

Since $(1-4x)^{-1/2}(1-4x)^{-1/2} = (1-4x)^{-1}$

$$= 1 + 4x + (4x)^2 + (4x)^3 + \cdots + (4x)^r + \cdots$$

We have

$$\sum_{i=0}^{t}\binom{2i}{i}\binom{2t-2i}{t-i} = 4^t.$$

4.4. Generating Functions and Permutations [15, 16]

For permutations, the generating function is a little less inclusive; it is reduced to enumerator.. in the simplest case, for n distinct things and no repetition, the number of permutations k at a time is P(n, k), where

$$P(n, k)=k!\, C(n, k),$$

Since the positions of the objects in a combination of k may be permuted in k(k-1)…1=k! ways. Hence we have

$$(1+t)^n = \sum_{k=0}^{n} C(n,k)t^k$$

$$= \sum_{k=0}^{n} \frac{p(n,k)t^k}{k!}$$

The enumerator for permutations is an exponential generating function. When repetitions are allowed, the enumerator for any object is a series containing a term $t^k/_{k!}$ for each k in the specification for repetitions.

If an object may appear zero, one or two times, the enumerator is the polynomial

$$1 + t + \frac{t^2}{2!};$$

If unlimited repetition is specified, it is

$$1 + t + \frac{t^2}{2!} + \cdots = e^t; etc.$$

The following examples illustrate a few of the possibilities.

Example 4.4.1.

Consider permutations k at a time of n objects with repetition. The enumerator is $(e^t)^n = e^{nt}$, and

$$e^{nt} = \sum_{k=0}^{\infty} \frac{n^k t^k}{k!},$$

The number of permutation in equation is n^k, a result easily found by other means.

Example 4.4.2.

Consider again the permutations of example (3.7), with the added condition that each object must appear at least once. The enumerator is $(e^t-1)^n$,

$$(e^t-1)^n=\sum_{j=0}^{n} \binom{n}{j} (-1)^j\, e^{(n-j)t}$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^k$$

If E is the shift operator, Ef(n)=f(n+1) and $\Delta = E - 1$, then the inner sum may be written as $\Delta^n o^k$, which is equal to n! S(k,n), with S(k,n) astirling number of the second kind, an ubiquitous number in combinatorics. Thus

$$\frac{(e^t-1)^n}{n!} = \sum_{k=0}^{\infty} \frac{s(k,n)t^k}{k!}$$

is the exponential generating function for stirling numbers of the second kind.

4.5. Generating Functions and Finding Averages [17]

To finding means, standard deviations and other distributions, with lower work by power series generating functions are unusual.

How to find means (μ) by generating functions ?

$$\mu=\frac{1}{M}\sum_m m\, f(m) \qquad (4.14)$$

Suppose f(m) represents the number of objects. In a clear set T of M objects, that have properly m properties, for each m =0, 1, 2,…, with

$$\sum_m f(m) = M$$

Averages can be computed immediately from generating functions by object power series generating functions of the sequence $\{f(m)\}$, say F(u)$\overset{ops}{\longleftrightarrow}$ $\{f(m)\}$.

In the equation **(4.14)** way to express the mean μ. In terms of F. certain,

$$\mu = F'(1)/F(1)$$

How to find the standard deviation σ, of the distribution by generating functions?

$$\sigma^2 = \frac{1}{M}\sum_{h\in T}(m(h) - \mu)^2 \qquad (4.15)$$

h : represents an object in the set T.

m(h): is the number of properties that h has .

σ^2 : which is known as the variance of the distribution, is therefore the mean square of the difference between the number of properties that each object has and the mean number of properties μ.

$(m(h)-\mu)^2$: Every one of the f(m) objects h that has exactly m properties will contribute $(m-\mu)^2$ to the sum in (4.14), and therefore

$$\mu^2 = \frac{1}{M}\sum_m (m-\mu)^2 \text{ f (m)}$$

$$= \frac{1}{M}\sum_m (m^2 \text{-2 } \mu \text{ m} + \mu^2) \text{ f (m)}$$

$$= \frac{1}{M}\{(u\ D)^2 - 2\mu(u\ D) + \mu^2\}\ F(u)$$

$$= (F''\ (1) + (1\text{-}2\mu)\ F'(1) + \mu^2 F(1)/F(1)$$

$$= F''\ (1)\ /\ F\ (1) + F'\ (1)\ /\ F\ (1) - (F'\ (1)\ /\ F(1))^2$$

$$= \{(\log F)' + (\log F)''\}$$

Can also be calculated standard deviation in terms of the values of F and its first two derivatives at u=1.

In an exponential family F. The average number μ (m), of cards in a hand of weight m. If y (m, r) is the number of hands of weight m that have r cards, then the average is

$$\mu\ (m) = \frac{1}{y(m)}\sum_r r y(m,r) \qquad \textbf{(4.16)}$$

Now if we begin with exponential formula

$$\sum_{m,r} y(m,r)\frac{u^m}{m!}\ m^r = e^{wD(u)}$$

Apply the operator $\frac{\partial}{\partial w}$ and then set w=1.

The result is that $\sum \frac{u^m}{m!} \sum_r r y(m,r) = D(U)e^{D(u)}$ = D(u) y(u) **(4.17)**

Theorem 4.6.

In exponential family F, the average number of cards in hands of weight m is

$$\mu = (m) = \left[\frac{y(m)u^m}{m!}\right] \text{D (u) y(u)}$$

$$= \frac{1}{y(m)}\sum_i \binom{m}{i} \text{ di y(m-i)} \qquad \textbf{(4.18)}$$

Example 4.7. (Cycles of Permutations)

The averaging relations **(4.14)** are particularly if y (m) = m!, as in the family of all permutations. There, (4.14) becomes

$$\mu\ (m) = \frac{1}{m!}\sum_i \binom{m}{i}\ (i-1)\ !\ (m-i)\ !$$

$$= 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}\ .$$

Consequently, the average number of cycles in a permutation of m letters is the harmonic number y m.

What is the standard deviation? The function F(u) that appears in **(4.11)**, in the case of permutations, is, for m fixed,

$$F\ (u) = \sum_r y(m,r)\ u^r$$

$$= u(u+1)(u+2)\ldots(u+m-1),$$

By

$$\sum_r \begin{bmatrix} m \\ r \end{bmatrix} w^r = \begin{bmatrix} \frac{u^m}{m!} \end{bmatrix}\ (1-u)^{-w}.$$

After taking logarithms and differentiating, following **(4.11),** we find

$$F(1)= m!,\ (\log F)'\ (1) = Y_m,\ \text{and}$$

$$(\log F)''\ (1)= -1-1/4-1/9-1/16-\ldots-1/m^2.$$

If we substitute this into **(4.11),** we find that the variance of the distribution of cycles over permutations of n letters is

$$\sigma^2 = Y_m - 1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \cdots - 1/m^2$$

$$= \log m + \gamma - \frac{\pi^2}{6} + \sigma(1).$$

Where γ is Euler's constant.

Hence the average number of cycles is ~ log m with a standard deviation

$$\sigma \sim \sqrt{\log m}$$

4.8. Ordinary Differential Equations and Generating Functions. [18]

When deriving generating function for the Bernoulli and Euler sides of the Bernoulli-Euler triangle we had to solve ordinary differential equations satisfied by these functions.

The theorem below solves the problem of existence and uniqueness of a solution for a large class of ordinary differential equations containing both equations

$$B'(x)=B^2(x)+1 \qquad \textbf{(4.19)}$$

And

$$E'(y)=E(y)\,B(y) \qquad \textbf{(4.20)}$$

Theorem 4.9.

Consider the ordinary differential equation

$$f'(s) = F(s, f(s)) \qquad \textbf{(4.21)}$$

With respect to the generating function f(s), where F=F (s, t) is a generating function in two variables, Polynomial in t(i.e., having finite degree in t). Then for each f_0 Eq. **(4.21)** possesses a unique solution with the initial condition f(0)=f_0.

For equation **(4.19),** the function F is

$$F(s, t) = t^2+1,$$

While for equation **(4.20)** it is

$$F(s, t) = B(s)\, t.$$

Proof of the theorem:

The proof follows our usual pattern of finding the coefficients of the unknown function f one by one. Let n be the degree of F with respect to t and let

$$\begin{aligned} F(s, t) = {} & (F_{00}+F_{01}\,s+F_{02}\,s^2+\dots\dots.) \\ & + (F_{01}+F_{11}\,s+F_{21}\,s^2+\dots\dots.)t \\ & +\dots.+ \\ & + (F_{0n}+F_{1n}\,s+F_{2n}s^2+\dots.)t^n, \end{aligned}$$

$$f(s) = f_0+f_1\,s+f_2\,s^2+\dots$$

Equating the coefficients of s^0 on the left-and on the right-hand sides of Equation (4.21) we obtain

$$f_1= F_{00}+F_{01}f_0+\dots+F_{0n}f_0^n.$$

Similarly, the equation for the coefficients of s^1 yields

$$2\, f_2 = F_{10} + F_{01} f_1 + F_{11} f_0 + \ldots + F_{0n} f_0^{n-1} f_1 + F_{1n} f_0^{n}.$$

More generally, f_n is the root of the equation

$$n f_n = F_{(n-1)n}\, f_0^{n}. \qquad (4.22)$$

Where dots denote a Polynomial in coefficients of F and the coefficients $f_0, f_1, \ldots, f_{n-1}$ of f. For each n>0 Eq. **(4.22)** has a unique solution.

CHAPTER 5

CONCLUSION

The subject of generating functions belongs to the domain of operation methods which are widely used in many areas of Mathematics such as the theory of differential equations, difference equations, integral equations and Algebra.

These functions also appear in various fields of Sciences and Engineering. These functions are considered as a link between the discrete analysis and the continuous one. Such functions play an important role in searching for many useful properties of the sequence that they generate.

In this thesis, I presented some important definitions, theorems and elementary operations related to these functions and relations between these functions and some orthogonal polynomials were discussed some applications of these functions were considered.

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APPENDICES A

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