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RESEARCH PAPER

A MODIFIED VARIATIONAL ITERATION METHOD FOR SOLVING FRACTIONAL RICCATI DIFFERENTIAL EQUATION BY ADOMIAN POLYNOMIALS

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Abstract

In this paper, we introduce a modified variational iteration method (MVIM) for solving Riccati differential equations. Also the fractional Riccati differential equation is solved by variational iteration method with considering Adomians polynomials for nonlinear terms. The main advantage of the MVIM is that it can enlarge the convergence region of iterative approximate solutions. Hence, the solutions obtained using the MVIM give good approximations for a larger interval. The numerical results show that the method is simple and effective.

MSC 2010: Primary 34A08; Secondary 34K28, 34K37

Key Words and Phrases: Riccati equation; fractional derivative; modified variational iteration method; Adomian polynomials

1. Introduction

The fractional calculus has a tremendous use in basic sciences and engineering, see e.g. [2, 12, 13, 14, 15, 18, 21, 22]. Recently, the applications have included solving various classes of nonlinear fractional differential equations numerically (see for example Refs. [2, 21] and the references therein). Daftardar-Gejji and Jafari [4, 7] have employed the Adomian

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decomposition method to solve the linear/nonlinear systems of fractional differential equations which gives numerical answers to any order of desired accuracy.

The variational iteration method (VIM) is one of the powerful methods within the exact and approximate analytical solutions for solving nonlinear equations. The method was first initiated by [6], and it was successfully used by various researchers to investigate the linear and nonlinear problems [6, 10]. We mention that Jafari et.al. applied the variational iteration method to the modified Camassa-Holm and Degasperis-Procesi equations and fractional Davey-Stewartson equations, [9, 10]. Momani and Odibat [17] has implemented the variational iteration method to solve nonlinear fractional differential equations. It was shown by several authors (see e.g. Wazwaz [24]) that this method is more powerful than existing techniques such as the Adomian decomposition method [4, 16], perturbation method, etc. Besides, the VIM gives rapidly convergent successive approximations of the exact solution if such a solution exists. Another important advantage is related to the fact that the VIM is capable of greatly reducing the size of calculation while still maintaining high accuracy of the numerical solution. Moreover, the power of the method gives it a huge applicability in handling a wide number of analytical and numerical applications.

The fractional Riccati differential with respect the time is governed by the equation given below

$$D_*^{\alpha} y(t) = A(t) + B(t)y + C(t)y^2, \tag{1.1}$$

where $A(t), B(t)$ and $C(t)$ denote given functions and α represents a parameter describing the order of the fractional derivative.

There are several definitions of a fractional derivative of order $\alpha > 0$. For example, the Riemann-Liouville integral operator of order α is defined by ([12])

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \ \alpha > 0, \ x > 0, \tag{1.2}$$

and its fractional derivative of order $\alpha \geq 0$ is:

$$D^{\alpha} f(x) = \frac{d^m}{dx^m}(I^{m-\alpha} f(x)), \quad \text{with a suitable integer } \ m. \tag{1.3}$$

The Riemann-Liouville integral operator plays an important role in the development of the theory of fractional derivatives and integrals, [12]. However, it has some disadvantages for treating fractional differential equations with initial and boundary conditions. Therefore, we adopt here the Caputo definition, which is a modification of the Riemann-Liouville definition ([2, 12, 21]):

$$D_*^{\alpha} f(x) = I^{m-\alpha}(\frac{d^m}{dx^m} f(x)), \tag{1.4}$$

where $m \in N$: $m-1 < \alpha \leq m$. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. We mention that the Riemann-Liouville fractional derivative is computed in the reverse order. We have chosen to use the Caputo fractional derivative because it allows traditional (integer order) initial and boundary conditions to be included in the formulation of the problem, but for homogeneous initial conditions assumption, these two operators coincide. For more details on the geometric and physical interpretation of fractional derivatives of both the Riemann-Liouville and Caputo types, see Podlubny [20].

The aim of this paper is to extend the variational iteration method proposed by [6] to solve nonlinear Riccati differential equation of fractional order.

The manuscript is organized as follows: In Section **2** the analysis of the method is presented. Section **3** deals with modified variational iteration method. Numerical methods and presented in Section **4**. Finally, the conclusions are illustrated in Section **5**.

2. Method Analysis

We consider the fractional differential equation

$$D_*^\alpha y(t) = A(t) + B(t)y + C(t)y^2, \; 0 < \alpha \leq 1, \tag{2.1}$$

with initial condition y(0) = 0, where $D_*^\alpha = \frac{d^\alpha}{dt^\alpha}$ is the Caputo derivative. According to the variational iteration method [6], we construct a correction functional for Eq. (2.1), which reads

$$y_{n+1} = y_n + I^\alpha \lambda(\xi)[\frac{d^\alpha y_n}{d\xi^\alpha} - A(\xi) - B(\xi)y_n - C(\xi)y_n^2]. \tag{2.2}$$

To identify the multiplier, we approximately write (2.2) in the from

$$y_{n+1} = y_n + \int_0^t \lambda(\xi)[\frac{d^\alpha y_n}{d\xi^\alpha} - A(\xi) - B(\xi)\tilde{y_n} - C(\xi)\tilde{y}_n^2]d\xi, \tag{2.3}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\widetilde{y}_n$ is a restricted variation, i.e., $\delta\widetilde{y}_n = 0$.

The successive approximation y_{n+1}, $n \geq 0$ of the solution $y(t)$ will be readily obtained upon using Lagrange's multiplier, and by using any selective function y_0. The initial value $y(0)$ and $y_t(0)$ are usually used for selecting the zeroth approximation y_0. To calculate the optimal value of λ, we have

$$\delta y_{n+1} = \delta y_n + \delta \int_0^t \lambda(\xi)\frac{dy_n}{d\xi}d\xi = 0. \tag{2.4}$$

This yields the stationary conditions $\lambda'(\xi) = 0$, and $1 + \lambda(\xi) = 0$, which gives

$$\lambda = -1. \tag{2.5}$$

Substituting this value of Lagrangian multipliers in Eq. (2.3), we get the following iteration formula

$$y_{n+1} = y_n - I^{\alpha}[\frac{d^{\alpha} y_n}{d\xi^{\alpha}} - A(\xi) - B(\xi) y_n - C(\xi) y_n^2]. \tag{2.6}$$

Here the nonlinear term of Eq. (2.6) can be decomposed into an infinite series of polynomials given by

$$N(u) = y^2 = \sum_{n=0}^{\infty} A_i(u_0, u_1, ..., u_i), \tag{2.7}$$

where A_i are the so-called the Adomian polynomials defined by

$$A_i = \frac{1}{n!} \frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} u_i \lambda^i)|_{\lambda=0}, \tag{2.8}$$

and $u_{n+1} = y_{n+1} - y_n$.

It is now well known that these polynomials can be generated for all classes of nonlinearity according to specific algorithms defined by (2.7). Recently, an alternative algorithm for constructing Adomian polynomials has been developed by Wazwaz [23].

Beginning with an initial approximation $y_0(t) = y(0)$, we obtain the following successive approximations

$$y_{n+1} = y_n - I^{\alpha}[\frac{d^{\alpha} y_n}{d\xi^{\alpha}} - A(\xi) - B(\xi) y_n - C(\xi) \sum_{i=0}^{n} A_i(\xi)], \tag{2.9}$$

and finally the exact solution is obtained by

$$y(t) = \lim_{n \longrightarrow \infty} y_n(t). \tag{2.10}$$

In other words, the correction functional (2.3) will give a sequence of approximation and the exact solution is obtained at the limit of the resulting successive approximations.

THEOREM **2.1**. (*Banach's Fixed Point Theorem*) *Assume that X is a Banach space and $A : X \to X$ is a nonlinear mapping, and suppose that*

$$\|A[u] - A[v]\| \leq \kappa \|u - v\|, \quad u, v \in X$$

for some constants $\kappa < 1$. Then A has a unique fixed point. Furthermore, the sequence $u_{n+1} = A[u_n]$, with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A.

According to Theorem **2.1**, for the nonlinear mapping

$$A[y(t)] = y(t) + I^{\alpha} \lambda(\xi) [\frac{d^{\alpha} y_n}{d\xi^{\alpha}} - A(\xi) - B(\xi) y_n - C(\xi) y_n^2],$$

a sufficient condition for convergence of the variational iteration method is strict contraction of A. Furthermore, the sequence (2.2) converges to the fixed point of A which is also the solution of problem (2.1).

3. The modified variational iteration method

The main drawback of the standard VIM is that the sequence of successive approximations of the solution obtained can be rapidly convergent only in a small region, which will greatly restrict the application area of such a method.

To enlarge the convergence region of the sequence of successive approximations obtained, [5] modified the VIM by introducing an auxiliary parameter.

For using for MVIM (2.1), we rewrite it as

$$D_*^\alpha y(t) - D_*^\alpha y(t) + \gamma[D_*^\alpha y(t) - A(t) - B(t)y - C(t)y^2(t)] = 0,$$

where γ is an auxiliary parameter and $\gamma \neq 0$, which is used to adjust the convergence region of the following iterative formula. A correct functional for (2.1) can be written as

$$\begin{aligned} y_{n+1} &= y_n + I^\alpha \lambda(\xi)[D_*^\alpha y(\xi) - D_*^\alpha y(\xi) \\ &+ \gamma[D_*^\alpha y(\xi) - A(\xi) - B(\xi)y - C(\xi)y^2(\xi)]]. \end{aligned} \tag{3.1}$$

To identify the multiplier, we approximately write (3.1) in the form

$$\begin{aligned} y_{n+1} &= y_n + \int_0^t \lambda(\xi)[D_*^\alpha y_n(\xi) - D_*^\alpha \tilde{y}(\xi) \\ &+ \gamma[D_*^\alpha \tilde{y}_n(\xi) - A(\xi) - B(\xi)\tilde{y}_n - C(\xi)\tilde{y}_n^2(\xi)]]d\xi, \end{aligned} \tag{3.2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\widetilde{y}_n$ is a restricted variation, i.e., $\delta\widetilde{y}_n = 0$ and

$$\lambda = -1. \tag{3.3}$$

Substituting this value of Lagrangian multiplies in the Eq. (3.1) and according to the VIM, the following iteration formula can be obtained:

$$y_{n+1} = y_n - \gamma I^\alpha [D_*^\alpha y_n(\xi) - A(\xi) - B(\xi)y_n - C(\xi)y_n^2(\xi)], \tag{3.4}$$

From the convergence analysis in Section **2**, it is easy to see that the smaller the value of $|\gamma|$ is, the wider the convergence region of iterative sequence (3.4) is. In fact, iterative formula (3.4) gives us vast freedom of choice. For some strong nonlinear problems, one can choose a relatively small value of $|\gamma|$ (generally less than 1) to obtain a good approximation in a wider region. In addition, it should be especially pointed out that when $|\gamma| = 1$, (3.4) becomes the standard variational iteration formula (2.2).

THEOREM **3.1**. *Suppose that $y_0(t) = \alpha$ and the iterative sequence $\{y_n(t)\}$ obtained from (3.4) converges to $y(t)$; then $y(t)$ is the solution of Eq. (2.1).*

P r o o f. Taking limits in the iterative formula in (3.4), it follows that

$$\lim_{n\longrightarrow\infty} y_{n+1} = \lim_{n\longrightarrow\infty} y_n \\ -\gamma I^\alpha \lim_{n\longrightarrow\infty}[D_*^\alpha y_n(\xi) - A(\xi) - B(\xi)y_n(\xi) - C(\xi)y_n^2(\xi)], \tag{3.5}$$

and thus,

$$\gamma I^\alpha \lim_{n\longrightarrow\infty}[D_*^\alpha y_n(\xi) - A(\xi) - B(\xi)y_n(\xi) - C(\xi)y_n^2(\xi)] = 0. \tag{3.6}$$

Since $\gamma \neq 0$, it follows immediately that

$$\int_0^t \frac{(t-\xi)^{\alpha-1}}{\Gamma(\alpha)}[D_*^\alpha y_n(\xi) - A(\xi) - B(\xi)y_n(\xi) - C(\xi)y_n^2(\xi)]d\xi.$$

Then fractional differentiation of both sides with respect to t yields

$$D_*^\alpha y_n(t) = A(t) + B(t)y_n(t) + C(t)y_n^2(t) \tag{3.7}$$

Obviously, $y(t)$ satisfies Eq. (2.1). Also, $y_0(t) = \alpha$, since $y_n(0) = \alpha$.

Hence, $y(t)$ is the solution of Eq. (2.1) and the proof is complete.

According to Banach's fixed point theorem, it is easy to obtain the convergence condition for the sequence y_n obtained from (3.4). □

THEOREM **3.2**. *Define a nonlinear mapping*

$$T[y(t)] = y(t) - \gamma I^\alpha [D_*^\alpha y_n(t) - A(t) - B(t)y_n - C(t)y_n^2(t)]\,.$$

A sufficient condition for the convergence of the iterative sequence $\{y_n(t)\}$ obtained from (3.4) is strict contraction of the nonlinear mapping T. Furthermore, the sequence (3.4) converges to the fixed point of T which is also the solution of Eq. (2.1).

Therefore, according to (3.4), by choosing a proper and initial approximation $\{y_n(t)\}$, the successive approximations of the solution to (2.1) on the entire interval $[0, T]$ can be obtained.

4. Applications and numerical results

To give a clear overview of this method, we present the following illustrative examples.

EXAMPLE **4.1**. Consider the following fractional Riccati differential equation:

$$\frac{d^\alpha y}{dt^\alpha} = -y^2(t) + 1,\ 0 < \alpha \le 1, \tag{4.1}$$

subject to the initial condition $y(0) = 0$.

The exact solution of Eq. (4.1) is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$, when $\alpha = 1$.

In view of (2.9) the correction functional for (4.1) turns out to be:

$$y_{n+1}(t) = y_n(t) - I^\alpha[\frac{d^\alpha y_n}{d\xi^\alpha} + \sum_{i=0}^{n} A_i(\xi) - 1], \tag{4.2}$$

where the nonlinear term is $N(y) = y^2$, and we have

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2. \end{aligned}$$

Beginning with $y_0(t) = t$, by the iteration formulation (4.2), we can obtain directly the other components as:

$$\begin{aligned} y_1(t) &= \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)}, \\ y_2(t) &= \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)} \\ &- \frac{2\Gamma(2+\alpha)t^{1+2\alpha}}{\Gamma(1+\alpha)\Gamma(2+2\alpha)} + \frac{2\Gamma(3)\Gamma(4+\alpha)t^{3+2\alpha}}{\Gamma(3+\alpha)\Gamma(4+2\alpha)}, \\ y_3(t) &= \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)} - \frac{2\Gamma(2+\alpha)t^{1+2\alpha}}{\Gamma(1+\alpha)\Gamma(2+2\alpha)} \\ &+ \frac{2\Gamma(3)\Gamma(4+\alpha)t^{3+2\alpha}}{\Gamma(3+\alpha)\Gamma(4+2\alpha)} + \frac{6\Gamma(3)\Gamma(4+\alpha)t^{3+2\alpha}}{\Gamma(3+\alpha)\Gamma(4+2\alpha)} \\ &+ (\frac{\Gamma(3)}{\Gamma(3+\alpha)} + \frac{2\Gamma(2+\alpha)}{\Gamma(2+2\alpha)})\frac{2\Gamma(3+2\alpha)t^{2+3\alpha}}{\Gamma(1+\alpha)\Gamma(3+3\alpha)} \\ &- (\frac{\Gamma(3)}{\Gamma(3+\alpha)} + \frac{\Gamma(4+\alpha)}{\Gamma(4+2\alpha)})\frac{\Gamma(3)\Gamma(5+2\alpha)t^{4+3\alpha}}{\Gamma(3+\alpha)\Gamma(5+3\alpha)} \\ &- \frac{\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{2\Gamma(2+\alpha)t^{1+2\alpha}}{\Gamma(1+\alpha)\Gamma(2+2\alpha)}, \\ &\vdots \end{aligned}$$

So, we approximate the solution $y(t) = \lim_{n \longrightarrow \infty} y_n(t)$.

Follow up our discussion the Eq. (4.1) was solved by MVIM, in view of (3.4) the correction functional for (4.1) turns out to be:

$$y_{n+1}(t) = y_n(t) - \gamma I^\alpha[\frac{d^\alpha y_n}{d\xi^\alpha} + \sum_{i=0}^{n} A_i(\xi) - 1]. \tag{4.3}$$

Beginning with $y_0(t) = t$, by the iteration formulation (4.3), we can obtain directly the other components as

$$\begin{aligned} y_1(t) &= t - t\gamma + \gamma(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)}), \\ y_2(t) &= t - t\gamma - \gamma(t - t\gamma + \gamma(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)})) \\ &+ \gamma(\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{\Gamma(3)t^{2+\alpha}}{\Gamma(3+\alpha)}) + \gamma(\frac{3t^\alpha}{(2+3\alpha+\alpha^2)\Gamma(\alpha)} \\ &+ \frac{2t^\alpha}{\Gamma(3+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{\alpha^2 t^\alpha}{\Gamma(3+\alpha)} + \frac{4t^{2+\alpha}\gamma}{\Gamma(3+\alpha)} \\ &- \frac{4^{-\alpha}\sqrt{\pi}t^{1+2\alpha}\gamma}{\Gamma(\alpha)\Gamma(\frac{3}{2}+\alpha)} - \frac{2t^{1+2\alpha}\gamma}{\Gamma(2+2\alpha)} + \frac{12t^{3+2\alpha}\gamma}{\Gamma(4+2\alpha)} \\ &+ \frac{4t^{3+2\alpha}\gamma\Gamma(1+\alpha)}{\Gamma(\alpha)\Gamma(4+2\alpha)}) \\ &\vdots \end{aligned} \tag{4.4}$$

In Figure 1, approximate solution of variational iteration method for $\alpha = 0.98$ using the 4-term and the exact solution have been plotted. The plot presented in Figure 2 is the approximate solution of modified variational iteration method which considers $\gamma = 0.35$ and $\alpha = 0.98$.

Comments. This example has been solved using HAM, ADM, VIM and HPM in [3, 11, 16, 19]. It should be noted that these methods have given same result after applying the Padé approximants on $y(t)$.

Example **4.2**. Consider the following fractional Riccati differential equation:

$$\frac{d^\alpha y}{dt^\alpha} = 2y(t) - y^2(t) + 1, \ 0 < \alpha \leq 1, \tag{4.5}$$

subject to the initial condition $y(0) = 0$.

The exact solution of Eq. (4.5) is $y(t) = 1+\sqrt{2}\tanh(\sqrt{2}t+\frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$, when $\alpha = 1$.

Expanding $y(t)$ using the Taylor expansion about $t = 0$ gives

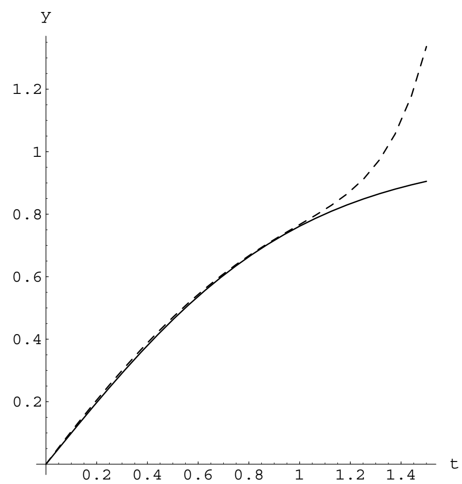

FIGURE 1. Comparison of $y(t)$ (solid line: Analytical, Dashed line: Approximate solution $y_4(t)$)

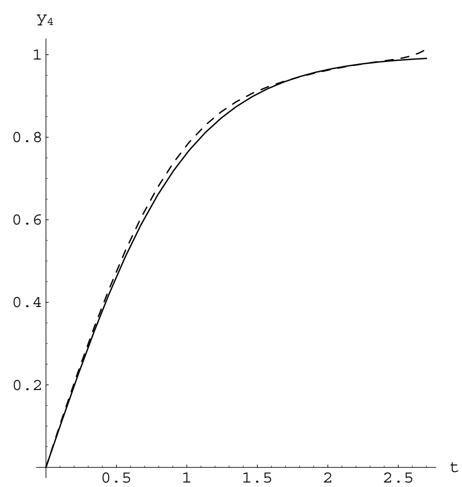

FIGURE 2. Comparison of the exact solution of (4.1) with the solution expression (4.4) (solid line: Exact solution, Dashed line: Approximate solution)

$$y(t) = t + t^2 + \frac{t^3}{3} - \frac{t^4}{3} - \frac{7t^5}{15} - \frac{7t^6}{45} + \frac{53t^7}{315} + \dots. \tag{4.6}$$

The correction functional for (4.5) turns out to be:

$$y_{n+1}(t) = y_n(t) + I^\alpha \lambda[\frac{d^\alpha y_n}{d\xi^\alpha} - 2y_n(\xi) + \sum_{i=0}^{n} A_i(\xi) - 1], \tag{4.7}$$

where the nonlinear term is $N(y) = y^2$ and we have

$$\begin{aligned} A_0 &= u_0^2, \\ A_1 &= 2u_0u_1, \\ A_2 &= 2u_0u_2 + u_1^2, \\ A_3 &= 2u_0u_3 + 2u_1u_2, \end{aligned}$$

and so on. Beginning with $y_0(t) = t$, by the iteration formulation (4.7), we can obtain directly the other components as

$$\begin{aligned} y_1(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+1}\Gamma(2)}{\Gamma(\alpha+2)} - \frac{t^{\alpha+2}\Gamma(3)}{\Gamma(\alpha+3)}, \\ y_2(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{\alpha+2}\Gamma(3)}{\Gamma(\alpha+3)} \\ &+ \frac{2t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{2\alpha+1}}{\Gamma(2\alpha+2)}(2\Gamma(2) - \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)}) \\ &- \frac{t^{2\alpha+2}}{\Gamma(2\alpha+3)}(2\Gamma(3) + \frac{4\Gamma(\alpha+3)\Gamma(2)}{\Gamma(\alpha+2)}) + \frac{2t^{2\alpha+3}\Gamma(\alpha+4)\Gamma(3)}{\Gamma(2\alpha+4)\Gamma(\alpha+3)}, \\ y_3(t) &= \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{2t^\alpha}{\Gamma(2\alpha+1)} + \frac{2t^{\alpha+2}\Gamma(3)}{\Gamma(\alpha+3)} \\ &- 4\frac{t^{\alpha+1}\Gamma(\alpha+2)}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} + \frac{2t^{2\alpha+2}}{\Gamma(2\alpha+3)}(\Gamma(3) \\ &- \frac{2\Gamma(\alpha+3)}{\Gamma(\alpha+2)\Gamma(\alpha+1)}) + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}(4 + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}) \\ &+ \frac{4t^{3\alpha+1}}{\Gamma(3\alpha+2)}(2\Gamma(2) - \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)} - \frac{\Gamma(2\alpha+2)}{\Gamma(2\alpha+1)} \\ &+ \frac{\Gamma(2\alpha+2)\Gamma(2)}{\Gamma(\alpha+2)\Gamma(\alpha+1)}) - \frac{2t^{3\alpha+2}}{\Gamma(3\alpha+3)}((2\Gamma(3) + \frac{4\Gamma(\alpha+3)\Gamma(2)}{\Gamma(\alpha+2)}) \\ &+ (2\frac{\Gamma(2\alpha+3)}{\Gamma(2\alpha+2)}(2\Gamma(2) - \frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1)})) - \frac{2\Gamma(2\alpha+3)\Gamma(2)^2}{\Gamma(\alpha+2)^2} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\Gamma(2\alpha+3)\Gamma(3)}{\Gamma(\alpha+1)\Gamma(\alpha+3)}) + \frac{2t^{3\alpha+3}}{\Gamma(3\alpha+4)}(\frac{2\Gamma(\alpha+4)\Gamma(3)}{\Gamma(\alpha+3)} \\
&+ \quad \frac{\Gamma(2\alpha+4)}{\Gamma(2\alpha+3)}(2\Gamma(3) + \frac{4\Gamma(\alpha+3)\Gamma(2)}{\Gamma(\alpha+2)}) \\
&- \quad \frac{2\Gamma(2\alpha+4)\Gamma(3)\Gamma(2)}{\Gamma(\alpha+2)\Gamma(\alpha+3)}) + \frac{t^{3\alpha+4}\Gamma(2\alpha+5)\Gamma(3)}{\Gamma(3\alpha+5)\Gamma(\alpha+3)} \\
&\quad\; (\frac{\Gamma(3)}{\Gamma(\alpha+3)} - \frac{4\Gamma(\alpha+4)}{\Gamma(2\alpha+4)}) \\
&\vdots
\end{aligned}
$$

For $\alpha = 1$ we get the same result as obtained by [1].

Follow up our discussion the Eq. 4.5 was solved by MVIM, in view of (3.4) the correction functional for (4.5) turns out to be:

$$y_{n+1}(t) = y_n(t) - \gamma I^{\alpha}[\frac{d^{\alpha} y_n}{d\xi^{\alpha}} - 2y_n(t) + \sum_{i=0}^{n} A_i(\xi) - 1]. \tag{4.8}$$

Beginning with $y_0(t) = t$, by the iteration formulation (4.8), we can obtain directly the other components as

$$
\begin{aligned}
y_1(t) \quad &= \quad t - t\gamma + \gamma(\frac{3t^{\alpha}}{(2+3\alpha+\alpha^2)\Gamma(\alpha)} + \frac{2t^{1+\alpha}}{(2+3\alpha+\alpha^2)\Gamma(\alpha)} \\
&+ \quad \frac{2t^{\alpha}}{\Gamma(3+\alpha)} + \frac{4t^{1+\alpha}}{\Gamma(3+\alpha)} - \frac{2t^{2+\alpha}}{\Gamma(3+\alpha)} + \frac{\alpha^2 t^{\alpha}}{\Gamma(3+\alpha)}) \\
&\vdots
\end{aligned}
$$

In Figure 3 approximate solution of VIM for $\alpha = 0.98$ using the 3-term and the exact solution have been plotted. The plot presented in Figure 4 is the approximate solution of modified VIM which considers $\gamma = 0.7$ and $\alpha = 0.98$.

5. Conclusion

In this manuscript, a MVIM has been presented for solving Riccati differential equations. Comparing with the variational iteration method for solving fractional Riccati differential equation by Adomian polynomials results, the results for numerical examples demonstrate that the present method can give a more accurate approximation in a larger region. This is also the main advantage of the present method. Therefore, the modification of the VIM can overcome the restriction of the application area of the VIM, and then expand its scope of application. However, generally, when the value of $|\gamma|$ chosen is small, the rate of convergence of the iterative formula

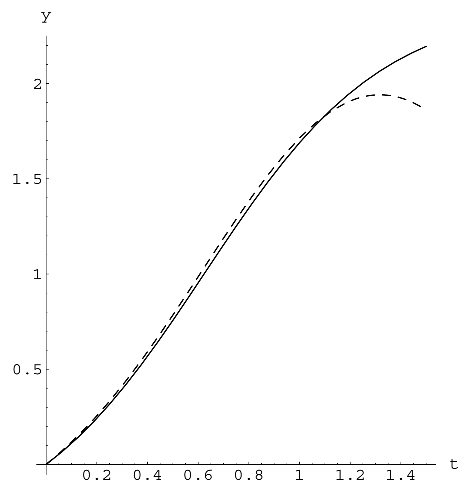

Figure 3. Comparison of $y(t)$ (solid line: Analytical, Dashed line: Approximate solution $y_3(t)$)

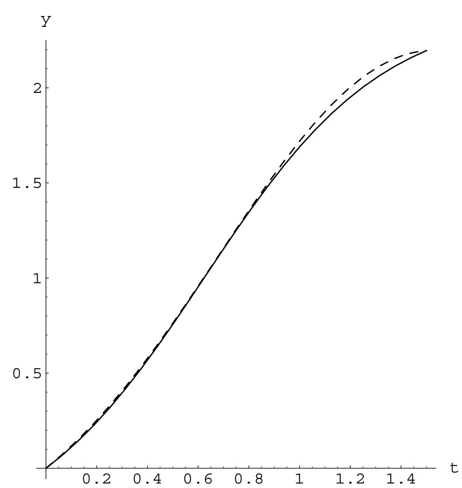

Figure 4. Comparison of the exact solution of (4.5) with the solution expression (4.8) (solid line: exact solution, Dashed line: Approximate solution)

is relatively slow, and so more iterative steps are required. *Mathematica* has been used for computations and programming in this manuscript.

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