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# Computation of iterative solutions along with stability analysis to a coupled system of fractional order differential equations

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## Abstract

In this research article, we investigate sufficient results for the existence, uniqueness and stability analysis of iterative solutions to a coupled system of the nonlinear fractional differential equations (FDEs) with higher order boundary conditions. The foundation of these sufficient techniques is a combination of the scheme of lower and upper solutions together with the method of monotone iterative technique. With the help of the proposed procedure, the convergence criteria for extremal solutions are smoothly achieved. Furthermore, a major aspect is devoted to the investigation of Ulam–Hyers type stability analysis which is also established. For the verification of our work, we provide some suitable examples along with their graphical representation and errors estimates.

**Keywords:** Monotone iterative technique; Fractional differential equations; Extremal solutions; Ulam stability

## 1 Introduction

FDEs and their systems are the core purpose of most of the researchers because they have many important applications in various fields of advanced technology and science. There are many books and articles which have been written on the topic, see the works of Mainardi and Podlubny. Some applications where FDEs have been used are as follows:

- Modeling materials or systems which are dependent on past history (memory) such as nonlocal elasticity, biological tissues, propagation in complex medium, earth sediments, polymers, and expansion of the universe.
- Strong connection with fractal geometry.
- Can be derived even from probabilistic approaches.

Moreover, classical DEs are just discrete aspects of the whole spectrum of DEs, while FDEs give the total spectrum.

Although some phenomena have been produced in the form of integral order differential equations, in general, every phenomenon cannot be produced accurately in the form of integral order differential equations like the phenomena of electrochemistry, rheology, porous media, geology, electromagnetism, optics, medicine, bioscience, bioengineering, probability, and statistics. In this concern, most of the researchers prefer nonlinear FDEs, which are powerful tools, instead of the integral order differential equations for mathemat-

ical modeling of the aforesaid phenomena. Therefore, plenty of the research items have been written on nonlinear FDEs, see [1–8] to explore more of their features for modeling purposes. Numerous properties and useful features of mathematical modeling of FDEs have been established by many authors in almost every field of science and technology such as ecology, control theory, seepage flow in porous media, and nonlinear oscillations due to earthquake (for details, see [9–15]). Furthermore, many researchers have devoted their works to the uniqueness and existence of solutions to FDEs; for reading, see [16–20]. As far as we know, the aforesaid area of fractional calculus has been greatly developed. The field of a coupled system of fractional differential equations is the main research field of most of the authors. It is the cascading form of single fractional differential equation. The handling of a coupled system of FDEs is a difficult task as compared with single fractional differential equation. In concern to the establishment of various aspects of the coupled system of FDEs is the need of science and technology because some phenomena describe more than one required task (solution) to be found. These phenomena can be expressed as a system of FDEs instead of DE/FDE. In this regard, some authors have devoted their treasured work to basic features and properties of the coupled systems of FDEs; for reading, see [21–24].

Existence and uniqueness of differential equations involving fractional derivatives have been considered in many works. For details on these works, we refer the readers to [25–31] and the works cited therein. Recently, the existence and uniqueness of lower and upper solutions of FDEs have attracted the focus of most of researchers. Although some authors have well studied lower and upper solutions of integral order partial and ordinary differential equations using iterative techniques (see [32–42]), the iterative techniques have been rarely studied for the lower and upper solutions of FDEs as well as coupled systems of FDEs. It is a useful tool for the existence of approximation to the solutions of many applied problems of integral and differential equations of arbitrary order. The aforementioned scheme has been studied by some authors for FDEs. Quite recently, Ali [43] has considered this scheme (technique) for the following fractional differential equations:

$$\begin{cases} \mathbb{D}^\alpha \theta(\xi) + f(\xi, \theta(\xi)) = 0, & \xi \in (0, 1), 1 < \alpha < 2, \\ \theta'(0) = 0, & \theta(1) = \xi \theta(\rho), \end{cases}$$

where  $\xi, \rho \in (0, 1)$ .

As far as we know, the aforementioned scheme for the coupled system of FDEs is in its initial stage.

On the other hand, stability aspect is the most popular topic in the discussion among most of the researchers. It has been well studied in major research articles for the classical differential equations. Some types of stability analysis for both classical differential equations and FDEs, like Lyapunov, exponential and Ulam–Hyers type stability, etc., have been developed. Among them, Ulam–Hyers type stability is fundamental and very important stability analysis. It has been verified in many articles for the classical differential equations as well as fractional differential equations. Ulam–Hyers stability analysis has been well studied for the initial value problems of fractional differential equations. As far as we know, it has been rarely investigated for the boundary value problems of fractional differential equations; for reading, see [44–48].

The aim of our manuscript is to extend the scheme of Ali [43] to coupled systems of FDEs. We consider the following class of nonlinear fractional coupled system of higher order boundary conditions:

$$\begin{cases} \mathbb{D}^p \theta(\xi) + \Phi(\xi, \vartheta(\xi)) = 0, & \xi \in [0, 1], \\ \mathbb{D}^q \vartheta(\xi) + \Psi(\xi, \theta(\xi)) = 0, & \xi \in [0, 1], \\ \theta(1) = \theta'(0) = \theta''(0) = \dots = \theta^{(n-2)}(0) = \theta^{(n-1)}(0) = 0, \\ \vartheta(1) = \vartheta'(0) = \vartheta''(0) = \dots = \vartheta^{(n-2)}(0) = \vartheta^{(n-1)}(0) = 0, \end{cases} \tag{1}$$

where  $n = 2, 3, 4, \dots, n - 1 < q, p \leq n, \theta, \vartheta \in C[0, 1]$ .

$\mathbb{D}$  is considered in the sense of Caputo derivative assuming that  $\Phi, \Psi : I \times \mathbb{R} \rightarrow \mathbb{R}$ .

This manuscript has the following main contributions:

- To extend and develop the iterative scheme while combining the method of lower and upper solutions for the nonlinear fractional coupled system of higher order boundary conditions.
- Existence of lower and upper solutions of the nonlinear fractional coupled system of higher order boundary conditions.
- The investigation of Ulam–Hyers stability for the coupled system of nonlinear FDEs of higher order boundary conditions.
- Approximations of extremal solutions of the nonlinear fractional coupled system of higher order boundary conditions.
- Provision of maximum errors estimate of extremal solutions of the nonlinear fractional coupled system of higher order boundary conditions.
- The presentation of lower and upper solutions of the nonlinear fractional coupled system of higher order boundary conditions via plots using Matlab software.

Three illustrative examples are provided to demonstrate the obtained results. Further we remark that Caputo type fractional order derivative is considered throughout this paper.

## 2 Background materials

In the present section of our work, we mention supportive basic definitions with useful lemmas from fractional calculus as well as measure theory and functional analysis; for reading, see [2–5, 19, 20, 37].

**Definition 2.1** Let  $\theta(\xi) \in L([0, 1], \mathbb{R})$  be a function. Then the Riemann–Liouville integral of fractional order  $\delta \in \mathbb{R}_+$  of function  $\theta(\xi)$  is given by

$$I^\delta \theta(\xi) = \frac{1}{\Gamma(\delta)} \int_0^\xi (\xi - \sigma)^{\delta-1} \theta(\sigma) d\sigma,$$

provided that the integral is pointwise defined on the right-hand side.

**Definition 2.2** ([3]) The Caputo derivative of fractional order  $p$  of function  $\theta(\xi)$  is defined by

$$\mathbb{D}^p \theta(\xi) = \frac{1}{\Gamma(n - p)} \int_0^\xi (\xi - \sigma)^{n-p-1} \theta^{(n)}(\sigma) d\sigma,$$

provided that the integral on the right-hand side is pointwise defined on  $(0, \infty)$ , where  $n = [p] + 1$  and  $[p]$  denotes the integer part of the real number  $p$ .

**Definition 2.3** As in [21, 43], “Let  $\mathcal{U} = C[0, 1]$  be the Banach space endowed with  $\|\theta\| = \max_{\xi \in [0,1]} |\theta(\xi)|$  which satisfies the partial ordering, and let  $W = [\theta_1, \theta_2]$  with  $\theta_1 \leq \theta_2$  be a set such that  $W \subset \mathcal{U}$ , then the operator  $\mathbb{T} : W \rightarrow \mathcal{U}$  is known as increasing function if for each  $\theta_1, \theta_2 \in W$  and  $\theta_1 \leq \theta_2$  gives  $\mathbb{T}\theta_1 \leq \mathbb{T}\theta_2$ . The operator  $\mathbb{T}$  is known as decreasing function if for each  $\theta_1, \theta_2 \in W$  gives  $\mathbb{T}\theta_1 \geq \mathbb{T}\theta_2$ .”

**Definition 2.4** As in [21, 43], “Let  $I$  be an identity operator. If  $(I - \mathbb{T})\theta_1 \leq 0$ , then the function  $\theta_1 \in W$  is known as a minimal solution of  $(I - \mathbb{T})\theta = 0$  and if  $(I - \mathbb{T})\theta_2 \geq 0$ , then the function  $\theta_2 \in W$  is known as a maximal solution of  $(I - \mathbb{T})\theta = 0$ .”

**Lemma 2.5** ([20]) *For the Banach space  $\mathcal{U}$  which partially satisfies order with  $W \subset \mathcal{U}$  and  $\theta_n, \theta_n^* \in W$  such that  $\theta_n \leq \theta_n^*$ ,  $n \in \mathbb{Z}^+$ . If  $\theta_n \rightarrow \theta$  and  $\theta_n^* \rightarrow \theta^*$ , then  $\theta \leq \theta^*$ .*

**Lemma 2.6** ([3, 49]) *Let  $p > 0$ , then the FDE*

$$\mathbb{D}^p \theta(\xi) = 0$$

has the solution in the form of

$$\theta(\xi) = \sum_{i=0}^{[p]} \frac{\theta^{(i)}(0)}{i!} t^i.$$

**Lemma 2.7** ([3, 49]) *Let  $p > 0$ , then*

$$I^p [\mathbb{D}^p \theta(\xi)] = \theta(\xi) - \sum_{i=0}^{[p]} \frac{\theta^{(i)}(0)}{i!} t^i.$$

### 3 Iterative solutions

This section is committed to the existence theory, approximation and error estimates to the extremal solution of system (1). In the first attempt we transfer system (1) to the equivalent system of integral equations. It is to be noted that the integral representation of the coupled system is very useful for onward computations. Green’s function in the integral representation is also necessary to be computed for iterative solution. The next step after the integral representation is establishing sufficient conditions for the existence of extremal solutions to system (1). We also provide iterative approximation to extremal solution of the system by a monotone iterative scheme coupled with the lower and upper solutions method. Further, by the convergent property of monotonic sequence, we consequently compute the error estimates for the iterative solution to extremal solutions of system (1). For the aforesaid concerns and computations, the following assumptions are considered:

- (C<sub>1</sub>) The real-valued functions  $\Phi, \Psi : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$  satisfy the Carathéodory lemma (for reading, see the preliminary section of [20]).
- (C<sub>2</sub>) The functions  $\Phi(\xi, \theta)$  and  $\Psi(\xi, \vartheta)$  are increasing in  $\theta$  and  $\vartheta$  for every  $\xi \in [0, 1]$  respectively.

(C<sub>3</sub>) The existence of the constants  $\mathbb{A}, \mathbb{B} > 0$  such that  $0 \leq \Phi(\xi, \theta_1(\xi)) - \Phi(\xi, \theta_2(\xi)) \leq \mathbb{A}(\theta_1 - \theta_2)$  and  $0 \leq \Psi(\xi, \vartheta_1(\xi)) - \Psi(\xi, \vartheta_2(\xi)) \leq \mathbb{B}(\vartheta_1 - \vartheta_2)$ .

**Lemma 3.1** *In view of  $C_1$  with  $h \in C([0, 1], \mathbb{R})$ , then the fractional differential equation*

$$\begin{aligned} \mathbb{D}^p \theta(\xi) + h(\xi) &= 0, \quad \xi \in [0, 1], (n - 1) < p \leq n, \\ \theta(1) = \theta'(0) = \theta''(0) = \dots = \theta^{(n-2)}(0) &= \theta^{(n-1)}(0) = 0 \end{aligned} \tag{2}$$

has the solution defined by

$$\theta(\xi) = \int_0^1 \mathbb{G}_1(\xi, \sigma) h(\sigma) d\sigma, \tag{3}$$

where  $\mathbb{G}_1(\xi, \sigma)$  is given by

$$\mathbb{G}_1(\xi, \sigma) = \frac{1}{\Gamma(p)} \begin{cases} (1 - \sigma)^{p-1} - (\xi - \sigma)^{p-1}, & 0 \leq \sigma \leq \xi \leq 1, \\ (1 - \sigma)^{p-1}, & 0 \leq \xi \leq \sigma \leq 1. \end{cases} \tag{4}$$

$\mathbb{G}_1(\xi, \sigma)$  is known as Green's function.

*Proof* In view of Lemma 2.7 and derivative considered in the sense of Caputo derivative, the linear BVP (2) yields that

$$\theta(\xi) = C_1 + C_2 \xi + \dots + C_n \xi^{n-1} - I^p h(\xi). \tag{5}$$

We face singularity by using the conditions  $\theta'(0) = \theta''(0) = \dots = \theta^{(n-2)}(0) = \theta^{(n-1)}(0) = 0$ . Therefore to avoid singularity, here we have  $C_2 = C_3 = C_4 = \dots = C_n = 0$ . Further, as  $\theta(1) = 0$ , then we have  $C_1 = I^p h(1)$ . Therefore, we deduce from (5) that

$$\theta(\xi) = \int_0^1 \frac{(1 - \sigma)^{p-1}}{\Gamma(p)} h(\sigma) d\sigma - \int_0^\xi \frac{(\xi - \sigma)^{p-1}}{\Gamma(p)} h(\sigma) d\sigma. \tag{6}$$

Hence, we obtain the result from (6) that is

$$\theta(\xi) = \int_0^1 \mathbb{G}_1(\xi, \sigma) h(\sigma) d\sigma.$$

This is the end of the proof. □

**Lemma 3.2** *Let  $W = C[0, 1]$  be the Banach space with  $\|\theta\| = \max_{\xi \in [0,1]} |\theta(\xi)|$ . Then the coupled system (1) has the integral representation*

$$\begin{cases} \theta(\xi) = \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi(\sigma, \vartheta(\sigma)) d\sigma, & \xi \in [0, 1], \\ \vartheta(\xi) = \int_0^1 \mathbb{G}_2(\xi, \sigma) \Psi(\sigma, \theta(\sigma)) d\sigma, & \xi \in [0, 1], \end{cases} \tag{7}$$

where  $\mathbb{G}_1(\xi, \sigma)$  is given in (4) and

$$\mathbb{G}_2(\xi, \sigma) = \begin{cases} \frac{(1 - \sigma)^{q-1} - (\xi - \sigma)^{q-1}}{\Gamma(q)}, & 0 \leq \sigma \leq \xi \leq 1, \\ \frac{(1 - \sigma)^{q-1}}{\Gamma(q)}, & 0 \leq \xi \leq \sigma \leq 1. \end{cases}$$

Clearly,  $\mathbb{G}_1(\xi, \sigma) \geq 0, \mathbb{G}_2(\xi, \sigma) \geq 0$  for all  $\xi, \sigma \in [0, 1]$ .

**Lemma 3.3** *The functions  $\mathbb{G}_1, \mathbb{G}_2$  satisfy the following property:*

$$\int_0^1 |\mathbb{G}_1(\xi, \sigma)| d\sigma \leq \frac{1}{\Gamma(p+1)}, \quad \int_0^1 |\mathbb{G}_2(\xi, \sigma)| d\sigma \leq \frac{1}{\Gamma(q+1)} \quad \text{for all } \xi \in [0, 1].$$

*Proof* As

$$\frac{(1-\sigma)^{p-1} - (\xi-\sigma)^{p-1}}{\Gamma(p)} \leq \frac{(1-\sigma)^{p-1}}{\Gamma(p)}$$

implies that

$$\mathbb{G}_1(\xi, \sigma) \leq \frac{(1-\sigma)^{p-1}}{\Gamma(p)},$$

thus

$$|\mathbb{G}_1(\xi, \sigma)| \leq \frac{(1-\sigma)^{p-1}}{\Gamma(p)},$$

from which we have

$$\int_0^1 |\mathbb{G}_1(\xi, \sigma)| d\sigma \leq \frac{1}{\Gamma(p+1)}.$$

Similarly, we can prove that  $\int_0^1 |\mathbb{G}_2(\xi, \sigma)| d\sigma \leq \frac{1}{\Gamma(q+1)}$ .

Hence, the proof is completed. □

We write the coupled system (7) of integral equations as

$$\begin{aligned} \theta(\xi) &= \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi(\sigma, \vartheta(\sigma)) d\sigma \\ &= \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi\left(\sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho\right) d\sigma. \end{aligned} \tag{8}$$

Further, we define  $\mathbb{T} : W \rightarrow \mathcal{U}$  is an operator by

$$\mathbb{T}(\theta(\xi)) = \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi\left(\sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho\right) d\sigma. \tag{9}$$

Thanks to equation (8) and equation (9), we obtain the equation

$$[I - \mathbb{T}(\theta(\xi))] = 0, \quad \xi \in [0, 1]. \tag{10}$$

It is to be noted that equations (8) and (10) have the same solutions that are fixed points of  $\mathbb{T}$ . As in  $C_2$ , we have that, for  $\theta, \theta^* \in W$  with  $\theta \leq \theta^*$ , we obtain

$$\begin{aligned} \mathbb{T}(\theta(\xi)) &= \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho \right) d\sigma \\ &\leq \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta^*(\varrho)) d\varrho \right) d\sigma = \mathbb{T}(\theta^*(\xi)) \end{aligned} \tag{11}$$

implies that

$$\mathbb{T}(\theta(\xi)) \leq \mathbb{T}(\theta^*(\xi)),$$

thus  $\mathbb{T}$  is an increasing operator.

( $C_4$ ) Assume that the lower and upper solutions of (10) are  $\alpha$  and  $\beta \in W$ . Then the inequality  $\alpha \leq \beta$  on  $[0, 1]$  holds.

**Lemma 3.4** Consider all the assumptions ( $C_1$ ) to ( $C_4$ ). Then the solutions of linear problems are an iterative convergent sequence which converges to solution of the integral equation (8).

*Proof* Under assumptions ( $C_1$ ), ( $C_2$ ), and ( $C_4$ ), let  $\theta_i, \theta_j \in W \subset \mathcal{U}$ , where  $i, j = 1, 2, 3, \dots, n$

$$\begin{aligned} &|\mathbb{T}(\theta_i(\xi)) - \mathbb{T}(\theta_j(\xi))| \\ &= \left| \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta_i(\varrho)) d\varrho \right) d\sigma \right. \\ &\quad \left. - \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta_j(\varrho)) d\varrho \right) d\sigma \right| \\ &\leq \int_0^1 \mathbb{G}_1(\xi, \sigma) \left| \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta_i(\varrho)) d\varrho \right) \right. \\ &\quad \left. - \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta_j(\varrho)) d\varrho \right) \right| d\sigma \\ &\leq \int_0^1 \mathbb{G}_1(\xi, \sigma) \mathbb{A} \int_0^1 \mathbb{G}_2(\sigma, \varrho) |\Psi(\varrho, \theta_i(\varrho)) - \Psi(\varrho, \theta_j(\varrho))| d\varrho d\sigma \\ &\leq \frac{\mathbb{A}\mathbb{B}}{\Gamma(q+1)} \Gamma(p+1) \|\theta_i - \theta_j\| = \nabla \|\theta_i - \theta_j\|, \quad \text{where } \nabla = \frac{\mathbb{A}\mathbb{B}}{\Gamma(q+1)} \Gamma(p+1). \end{aligned} \tag{12}$$

Therefore, the operator  $\mathbb{T}$  is continuous. Further, it is obvious that  $\mathbb{T}$  is also equicontinuous and uniformly bounded. Arzelá–Ascoli theorem which states that “If  $M$  is a family (finite or infinite) of an equicontinuous, uniformly bounded real-valued functions  $\theta$  on an interval  $[0, 1]$ , then  $M$  contains a uniformly convergent sequence of functions  $\theta_n$ , converging to a function  $\theta$  in  $W$  as  $n \rightarrow \infty$ , where  $W$  denotes the space of all continuous bounded functions on  $[0, 1]$ . Thus any sequence in  $M$  contains a uniformly bounded convergent subsequence on  $[0, 1]$  and consequently  $M$  has a compact closure in  $W$ .” Hence, in view of the Arzelá–Ascoli theorem,  $\mathbb{T}$  is also a compact operator. Let the lower solution of integral equation (10) be  $\theta_0 = \alpha$ . Then, in view of condition ( $C_4$ ), we obtain

$$\theta_0 \leq \beta.$$

As  $\mathbb{T}$  is an increasing operator, then we obtain

$$\theta_0 \leq \mathbb{T}\theta_0 \leq \mathbb{T}\beta \leq \beta, \quad \text{that is, } \theta_0 \leq \theta_1 \leq \beta \text{ on the interval } [0, 1],$$

where  $\theta_1 = \mathbb{T}\theta_0$  is an iterative solution of integral equation (10). By applying  $\mathbb{T}$ , we obtain

$$\mathbb{T}\theta_0 \leq \mathbb{T}\theta_1 \leq \mathbb{T}\beta \leq \beta, \quad \text{that is, } \theta_1 \leq \theta_2 \leq \beta \text{ on } [0, 1],$$

where  $\theta_2 = \mathbb{T}\theta_1$ . Consequently, we obtain a bounded monotone sequence  $\{\theta_n\}$  which is

$$\theta_0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_{n-1} \leq \theta_n \leq \beta \quad \text{on } [0, 1], \tag{13}$$

where  $\theta_n = \mathbb{T}\theta_{n-1}$  is solution of equation (10). Thus, in view of the bounded monotone sequence  $\{\theta_n\}$ , there exists  $\theta \in W$  so that  $\theta_n \rightarrow \theta$  as  $n \rightarrow \infty$ . Therefore  $\theta = \mathbb{T}\theta$ , which is the solution of integral equation (8) defined by

$$\theta(\xi) = \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho \right) d\sigma, \quad \xi \in [0, 1].$$

Hence, in view of (13) and (12), we get

$$\begin{aligned} \|\theta_2 - \theta_1\| &= \|\mathbb{T}\theta_1 - \mathbb{T}\theta_0\| \leq \nabla e_1, \\ \|\theta_3 - \theta_2\| &= \|\mathbb{T}\theta_2 - \mathbb{T}\theta_1\| \leq \nabla^2 e_1, \\ \|\theta_4 - \theta_3\| &= \|\mathbb{T}\theta_3 - \mathbb{T}\theta_2\| \leq \nabla^3 e_1, \\ &\vdots \\ \|\theta_{n+1} - \theta_n\| &= \|\mathbb{T}\theta_n - \mathbb{T}\theta_{n-1}\| \leq \nabla^n e_1. \end{aligned}$$

Consequently, for positive integers  $m$  and  $n$ , we have

$$\begin{aligned} \|\theta_{m+n} - \theta_n\| &\leq \|\theta_{n+m} - \theta_{n+m-1}\| + \|\theta_{n+m-1} - \theta_{n+m-2}\| + \dots + \|\theta_{n+1} - \theta_n\| \\ &\leq \nabla^n \frac{1 - \nabla^m}{1 - \nabla} e_1. \end{aligned} \tag{14}$$

It is obvious that  $\nabla < 1$ , which implies that  $\|\theta_{m+n} - \theta_n\| \rightarrow 0$  when  $n \rightarrow \infty$ . Thus  $\theta_n$  is a Cauchy sequence in  $W$ . Let  $\theta^*(\xi) = \lim_{n \rightarrow \infty} \theta_n(\xi)$ , thus  $\mathbb{T}\theta^* = \theta^*$ . Therefore, if  $m \rightarrow \infty$  in (14), then error estimate for the lower solution is

$$e_n = \|\theta^* - \theta_n\| \leq \frac{\nabla^n}{1 - \nabla} e_1, \quad \text{where } e_1 = \|\theta_1 - \theta_0\|. \quad \square$$

*Remark* Let us choose  $\theta_0 = \beta$ , we have a sequence  $\{\theta_n\}$  so that

$$\theta_0 \geq \theta_1 \geq \theta_2 \geq \dots \geq \theta_{n-1} \geq \theta_n \geq \alpha \quad \text{on } [0, 1],$$

which converges to a solution of the integral equation (8). Therefore, we can obtain the estimate error for the maximal solution which is given by  $e_n^* = \|\theta_n^* - \bar{\theta}^*\| \leq \frac{\nabla^n}{1 - \nabla} e_1^*$ ,



where  $e_1^* = \|\theta_0^* - \theta_1^*\|$ . Further, we use Lemma 3.4, the iterative sequences for lower and upper solutions of system (1) are:

$$\begin{aligned} \theta_n(\xi) &= \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta_{n-1}(\varrho)) d\varrho \right) d\sigma, \quad n \geq 1, \\ \theta_n^*(\xi) &= \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\sigma, \varrho) \Psi(\varrho, \theta_{n-1}^*(\varrho)) d\varrho \right) d\sigma, \quad n \geq 1. \end{aligned}$$

Hence, we have

$$\begin{aligned} \theta^*(\xi) &= \lim_{n \rightarrow \infty} \theta_n(\xi), \quad \vartheta^*(\xi) = \int_0^1 \mathbb{G}_2(\xi, \sigma) \Psi(\sigma, \theta^*(\sigma)) d\sigma, \\ \bar{\theta}^*(\xi) &= \lim_{n \rightarrow \infty} \theta_n^*(\xi), \quad \bar{\vartheta}^*(\xi) = \int_0^1 \mathbb{G}_2(\xi, \sigma) \Psi(\sigma, \bar{\theta}^*(\sigma)) d\sigma. \end{aligned}$$

**Theorem 3.5** *Consider conditions  $(C_1)$ ,  $(C_2)$ , and  $(C_3)$  and  $\nabla < 1$ . Then the coupled system of BVPs (1) has unique minimal and maximal solutions respectively.*

*Proof* We prove that  $(\theta^*, \vartheta^*)$  and  $(\bar{\theta}^*, \bar{\vartheta}^*)$  are the minimal and maximal solutions of (7). Let for any  $u(\xi) \in W$  with  $\mathbb{T}u = u$  and  $\theta_n \leq u \leq \theta_n^*$ , due to the increasing behavior of  $\mathbb{T}$  and using Lemma 3.4, we have  $\theta^*(\xi) \leq u(\xi) \leq \theta^*(\xi)$  that  $\theta^*(\xi)$  and  $\theta^*(\xi)$  are the minimal and maximal fixed points of  $\mathbb{T}$  respectively. Therefore  $(\theta^*, \vartheta^*)$  and  $(\bar{\theta}^*, \bar{\vartheta}^*)$  are the minimal and maximal solution of (7) respectively.

Next, for the concern of uniqueness of minimal and maximal solutions of system (7), let  $\alpha, \beta \in W$  be the lower and upper solutions of  $\mathbb{T}\theta = \theta$  respectively. Then  $\alpha \leq \mathbb{T}\alpha, \beta \geq \mathbb{T}\beta, \xi \in [0, 1]$ . We use  $\alpha$  and  $\beta$  as initial iterations respectively so that  $\alpha_n \rightarrow \alpha^*$  and  $\beta_n \rightarrow \beta^*, n \rightarrow \infty$ . We also have  $\mathbb{T}\alpha^* = \alpha^*, \mathbb{T}\beta^* = \beta^*$ . To prove  $\alpha^* = \theta^*$ , observe that  $\theta_0 \leq \alpha^*$  and  $\mathbb{T}$  is increasing, so we have  $\theta_n = \mathbb{T}^n \theta_0 \leq \mathbb{T}^n \alpha^*$  for each  $n \in \{1, 2, 3, \dots\}$ . Then  $\theta_0 \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_n \leq \dots \leq \alpha^*$ . Therefore from (12) and mathematical induction, it is obvious that  $\|\alpha^* - \theta_n\| = \|\mathbb{T}^n \alpha^* - \mathbb{T}^n \theta_0\| \leq \nabla^n \|\alpha^* - \theta_0\| \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore  $\|\alpha^* - \theta^*\| \rightarrow 0$  as  $n \rightarrow \infty$ , which gives  $\theta^* = \alpha^*$ . In the same manner, we obtain that  $\bar{\theta}^* = \beta^*$ . Thus, the minimal and maximal solutions of the coupled system (7) are unique.  $\square$

#### 4 Generalized Ulam–Hyers stability of the solutions of BVPs (1)

This section is devoted to the investigation of more general stability analysis for the system of fractional differential equations. The stability analysis is an important aspect of fractional calculus for the ordinary and partial differential equations of fractional order. As far as we know, it is in the initial stage for a coupled system. Therefore the aim of this section is to investigate some sufficient conditions for a system of fractional differential equations. In this concern, we study Ulam–Hyers and generalized Ulam–Hyers stability for a coupled system of BVPs (1) of the nonlinear FDEs. For this concern, we require the following auxiliary definitions.

**Definition 4.1** The considered problem (1) is Ulam–Hyers stable if we can find a real number  $\hat{C}_{\phi, \psi} > 0$  with the property that, for every  $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$  with  $\varepsilon_1 > 0$  and

$\varepsilon_2 > 0$  and for every solution  $(\theta, \vartheta) \in C([0, 1], \mathbf{R}) \times C([0, 1], \mathbf{R})$  of the inequality

$$\begin{aligned} |\mathbb{D}^p \theta(\xi) - \Phi(\xi, \vartheta(\xi))| &\leq \varepsilon_1, \quad \xi \in [0, 1], \\ |\mathbb{D}^q \vartheta(\xi) - \Psi(\xi, \theta(\xi))| &\leq \varepsilon_2, \quad \xi \in [0, 1], \end{aligned} \tag{15}$$

there exists a unique solution  $(\tilde{\theta}, \tilde{\vartheta}) \in C([0, 1], \mathbf{R}) \times C([0, 1], \mathbf{R})$  of the proposed BVPs (1) with

$$\|(\theta, \vartheta) - (\tilde{\theta}, \tilde{\vartheta})\| \leq \hat{C}_{\Phi, \Psi} \varepsilon, \quad \xi \in [0, 1].$$

**Definition 4.2** The proposed BVPs (1) is called to be generalized Ulam–Hyers stable if we can find  $\Theta_{\Phi, \Psi} : (0, \infty) \rightarrow \mathbf{R}^+$  with  $\Theta_{\Phi, \Psi}(0) = 0$ , with the property that, for every  $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$  with  $\varepsilon_1 > 0$  and  $\varepsilon_2 > 0$  and for every solution  $(\theta, \vartheta) \in C([0, 1], \mathbf{R}) \times C([0, 1], \mathbf{R})$  of the inequality

$$\begin{aligned} |\mathbb{D}^p \theta(\xi) - \Phi(\xi, \vartheta(\xi))| &\leq \varepsilon_1, \quad \xi \in [0, 1], \\ |\mathbb{D}^q \vartheta(\xi) - \Psi(\xi, \theta(\xi))| &\leq \varepsilon_2, \quad \xi \in [0, 1], \end{aligned}$$

there exists a unique solution  $(\tilde{\theta}, \tilde{\vartheta}) \in C([0, 1], \mathbf{R}) \times C([0, 1], \mathbf{R})$  of the proposed BVPs (1) with

$$\|(\theta, \vartheta) - (\tilde{\theta}, \tilde{\vartheta})\| \leq \hat{C}_{\Phi, \Psi} \Theta_{\Phi, \Psi}(\varepsilon), \quad \xi \in [0, 1].$$

*Remark 4.3* The  $(\theta, \vartheta) \in C([0, 1], \mathbb{R})$  is said to be the solution of BVPs (1) and satisfy the inequality given in (15) if and only if we can find a function  $\alpha, \beta \in C([0, 1], \mathbb{R})$  depending only on  $(\theta, \vartheta)$  respectively, then

- (i)  $|\alpha(\xi)| \leq \varepsilon_1, |\beta(\xi)| \leq \varepsilon_2$ , for all  $\xi \in [0, 1]$ ;
- (ii)  $\mathbb{D}^p \theta(\xi) + \Phi(\xi, \vartheta(\xi)) = \alpha(\xi)$ , for all  $\xi \in [0, 1]$ ,  $\mathbb{D}^q \vartheta(\xi) + \Psi(\xi, \theta(\xi)) = \beta(\xi)$ , for all  $\xi \in [0, 1]$ .

Thanks to Remark 4.3 and assumption  $(C_3)$  for  $\xi \in ([0, 1])$ , the considered solution  $(\theta, \vartheta)$  of the problem

$$\begin{cases} \mathbb{D}^p \theta(\xi) + \Phi(\xi, \vartheta(\xi)) = \alpha(\xi), & \xi \in [0, 1], \\ \mathbb{D}^q \vartheta(\xi) + \Psi(\xi, \theta(\xi)) = \beta(\xi), & \xi \in [0, 1], \\ \theta(1) = \theta'(0) = \theta''(0) = \dots = \theta^{(n-2)}(0) = \theta^{(n-1)}(0) = 0, \\ \vartheta(1) = \vartheta'(0) = \vartheta''(0) = \dots = \vartheta^{(n-2)}(0) = \vartheta^{(n-1)}(0) = 0, \end{cases}$$

given by

$$\begin{aligned}
 \theta(\xi) &= \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\xi, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho + \int_0^1 K_2(\xi, \varrho) \beta(\varrho) d\varrho \right) d\sigma \\
 &\quad + \int_0^1 \mathbb{G}_1(\xi, \sigma) \alpha(\sigma) d\sigma, \quad \xi \in [0, 1], \\
 \vartheta(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \Psi \left( \sigma, \int_0^1 \mathbb{G}_1(\xi, \varrho) \Phi(\varrho, \vartheta(\varrho)) d\varrho + \int_0^1 K_1(\xi, \varrho) \alpha(\varrho) d\varrho \right) d\sigma \\
 &\quad + \int_0^1 \mathbb{G}_2(\xi, \sigma) \beta(\sigma) d\sigma, \quad \xi \in [0, 1],
 \end{aligned} \tag{16}$$

satisfies the following inequality:

$$\begin{aligned}
 &\left| \theta(\xi) - \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\xi, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho + \int_0^1 K_2(\xi, \varrho) \beta(\varrho) d\varrho \right) d\sigma \right| \\
 &\leq \frac{\varepsilon_2}{\Gamma(p+1)}, \quad \xi \in [0, 1],
 \end{aligned} \tag{17}$$

and

$$\begin{aligned}
 &\left| \vartheta(\xi) - \int_0^1 \mathbb{G}_2(\xi, \sigma) \Psi \left( \sigma, \int_0^1 \mathbb{G}_1(\xi, \varrho) \Phi(\varrho, \vartheta(\varrho)) d\varrho + \int_0^1 K_1(\xi, \varrho) \alpha(\varrho) d\varrho \right) d\sigma \right| \\
 &\leq \frac{\varepsilon_1}{\Gamma(q+1)}.
 \end{aligned} \tag{18}$$

**Theorem 4.4** *Let  $\nabla < 1$  and assumption  $(C_3)$  hold. Then solutions of the considered coupled system (1) are Ulam–Hyers stable and consequently generalized Ulam–Hyers stable.*

*Proof* Let  $(\theta, \vartheta)$  be any solution of BVPs (1) and satisfy inequality (15), and let  $(\bar{\theta}, \bar{\vartheta})$  be the unique solution of the considered BVPs problem, then consider

$$\begin{aligned}
 &|\theta - \bar{\theta}| \\
 &= \left| \theta - \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\xi, \varrho) \Psi(\varrho, \bar{\theta}(\varrho)) d\varrho + \int_0^1 K_2(\xi, \varrho) \beta(\varrho) d\varrho \right) d\sigma \right| \\
 &\leq \left| \theta - \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\xi, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho + \int_0^1 K_2(\xi, \varrho) \beta(\varrho) d\varrho \right) d\sigma \right| \\
 &\quad + \left| \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\xi, \varrho) \Psi(\varrho, \theta(\varrho)) d\varrho + \int_0^1 K_2(\xi, \varrho) \beta(\varrho) d\varrho \right) d\sigma \right. \\
 &\quad \left. - \int_0^1 \mathbb{G}_1(\xi, \sigma) \Phi \left( \sigma, \int_0^1 \mathbb{G}_2(\xi, \varrho) \Psi(\varrho, \bar{\theta}(\varrho)) d\varrho + \int_0^1 K_2(\xi, \varrho) \beta(\varrho) d\varrho \right) d\sigma \right|. \tag{19}
 \end{aligned}$$

Therefore from inequality (19)

$$\begin{aligned}
 \|\theta - \bar{\theta}\| &\leq \frac{\varepsilon_2}{\Gamma(p+1)} + \nabla \|\theta - \bar{\theta}\|, \\
 \|\theta - \bar{\theta}\| &\leq \varepsilon_2 \hat{C}_{\Phi, \Psi}, \quad \hat{C}_{\Phi, \Psi} = \frac{1}{(1 - \nabla)\Gamma(p+1)} > 0.
 \end{aligned} \tag{20}$$

In the same manner, it is easy to prove that

$$\|\vartheta - \bar{\vartheta}\| \leq \varepsilon_1 \hat{C}_\psi, \quad \hat{C}_\psi = \frac{1}{(1 - \nabla)\Gamma(q + 1)} > 0. \tag{21}$$

Therefore, from inequalities (20) and (21), we have

$$\|(\theta, \vartheta) - (\bar{\theta}, \bar{\vartheta})\| \leq \max\{\varepsilon_1 \hat{C}_\phi, \varepsilon_2 \hat{C}_\psi\} = \varepsilon \hat{C}_{\phi, \psi}. \tag{22}$$

Therefore the solutions of a coupled system of BVPs (1) are Ulam–Hyers stable. Further, if  $\Theta_{\phi, \psi}(\varepsilon) = \varepsilon$ , hence (22) can be expressed as

$$\|(\theta, \vartheta) - (\bar{\theta}, \bar{\vartheta})\| \leq \hat{C}_{\phi, \psi} \Theta_{\phi, \psi}(\varepsilon). \tag{23}$$

Therefore,  $\Theta_{\phi, \psi}(0) = 0$  in (23) holds. Thus, solutions of the system of BVP (1) are generalized Ulam–Hyers stable. Thus we have investigated sufficient condition for Ulam–Hyers stability generalized Ulam–Hyers stability, that is,  $\nabla < 1$ . □

### 5 Examples

In this section, we provide some examples of the coupled system (1). We compute the iterative approximation for the extremal solution of the corresponding examples. We also provide error estimates for the extremal solutions. Furthermore, the stability analysis under the sufficient conditions for the examples of system (1) is also the concern of this section. We also provide the plots of each example of system (1) in the frame figures.

*Example 5.1* Consider the following coupled system of BVPs of FDEs:

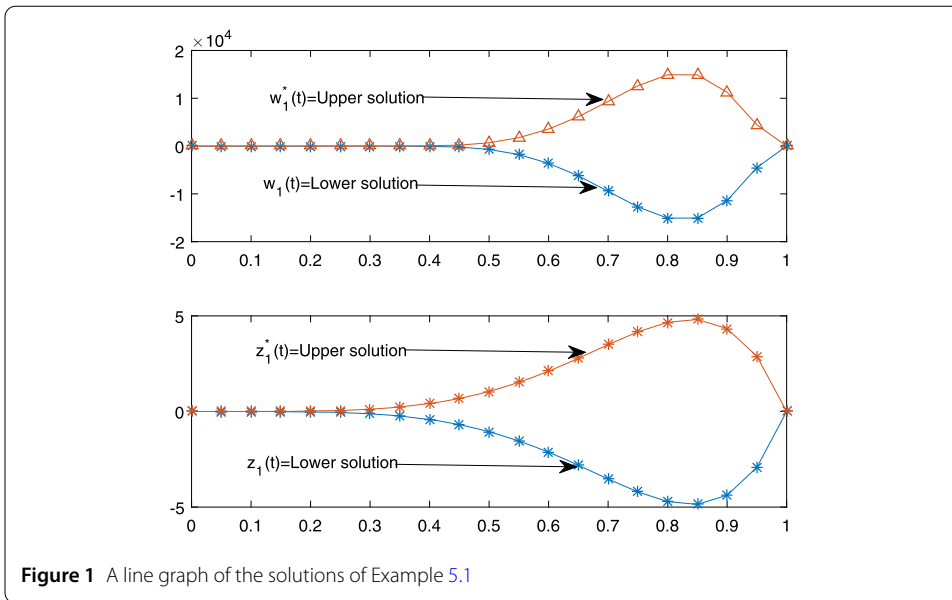
$$\begin{aligned} \mathbb{D}^{\frac{5}{2}}\theta(\xi) + \frac{t^{\frac{7}{2}}\vartheta(\xi)(56,754\vartheta^2(\xi) + 160)}{1 + \vartheta^2(\xi)} &= 0, \quad 0 \leq \xi \leq 1, \\ \mathbb{D}^{\frac{27}{10}}\vartheta(\xi) + \frac{t^{\frac{9}{2}}\theta(\xi)(\theta(\xi) + 1)(23,471e^{-\theta(\xi)} + 120)}{2 + \theta(\xi)} &= 0, \quad 0 \leq \xi \leq 1, \end{aligned} \tag{24}$$

under the boundary condition given by

$$\theta(1) = \theta'(0) = \theta''(0) = 0, \quad \vartheta(1) = \vartheta'(0) = \vartheta''(0) = 0. \tag{25}$$

As for  $\mathbb{A} = \mathbb{B} = 0.0001$ , we have  $\nabla = 7.2147e^{-10} < 1$ . When  $n = 3$  is large enough, iterative sequences for the approximate minimal and maximal solutions  $\theta_n$  and  $\bar{\theta}_n^*$  respectively are:

$$\begin{aligned} \theta^*(\xi) &= \theta_3(\xi), \\ \vartheta^*(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \frac{\sigma^{\frac{9}{2}}\theta_2(\sigma)(\theta_2(\sigma) + 1)(23,471e^{-\theta_2(\sigma)} + 120)}{1 + \theta_2(\sigma)} d\sigma, \\ \bar{\theta}^*(\xi) &= \theta_3^*(\xi), \\ \bar{\vartheta}^*(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \frac{\sigma^{\frac{9}{2}}\theta_2^*(\sigma)(\theta_2^*(\sigma) + 1)(23,471e^{-\theta_2^*(\sigma)} + 120)}{1 + \theta_2^*(\sigma)} d\sigma. \end{aligned} \tag{26}$$



Let  $(\theta_0, \vartheta_0) = (-0.01, -0.01)$ ,  $(\theta_0^*, \vartheta_0^*) = (0.01, 0.01)$  be the minimal and maximal solutions of Example 5.1 respectively, then we have their corresponding maximum error estimates:

$$\begin{aligned}
 e_3 &= \|\theta - \theta_3\| \leq \frac{\nabla^3}{1 - \nabla} \times e_1 \leq 3.7554292227e^{-28} \times \max_{\xi \in [0,1]} |\theta_1(\xi) + 0.01| \\
 &\simeq 5.66292592845e^{-24}, \\
 e_3^* &= \|\theta - \theta_3^*\| \leq \frac{\nabla^3}{1 - \nabla} \times e_1^* \leq 3.7554292227e^{-28} \times \max_{\xi \in [0,1]} |0.01 - \theta_1^*(\xi)| \\
 &\simeq 5.60233983913e^{-24}.
 \end{aligned}$$

The iterative solutions of minimal and maximal of Example 5.1 are plotted in Figs. 1–2 wherein for the simple execution of Matlab code, we have replaced  $(\theta, \vartheta)$  by  $(w, z)$  and  $(\theta^*, \vartheta^*)$  by  $(w^*, z^*)$ . Each plot in the figures has the demonstration of physical behavior of the approximate solutions.

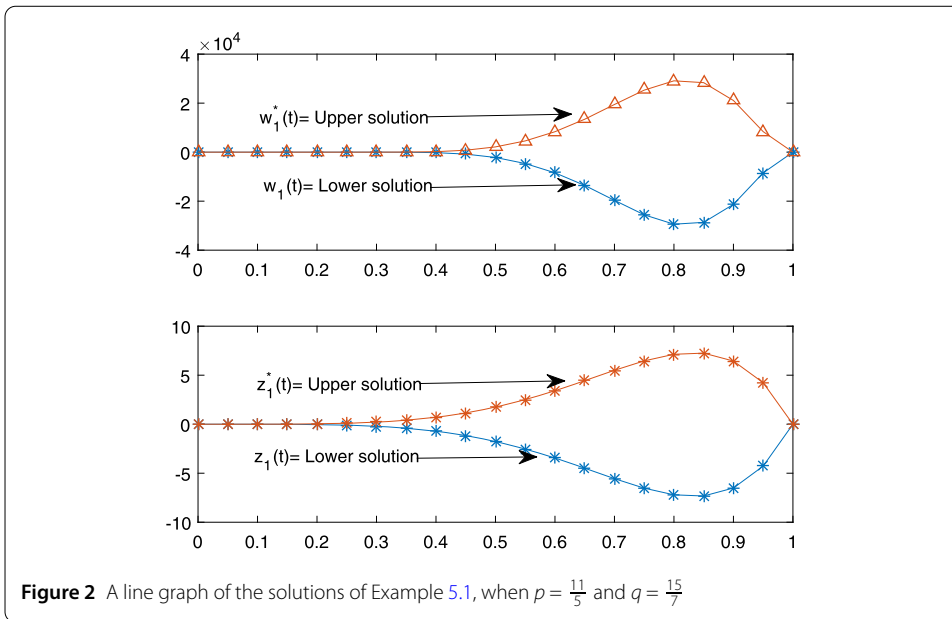
Further, as  $\nabla < 1$ , therefore, the Ulam–Hyers stability and generalized Ulam–Hyers stability are obvious. The stability of the iterative solutions can also be observed from Figs. 1–2.

**Example 5.2** To strengthen our results, we provide another example as follows:

$$\begin{aligned}
 \mathbb{D}^{\frac{19}{5}} \theta(\xi) + \sqrt{1 + \vartheta^2(\xi)} (\vartheta^{1.1}(\xi) + \vartheta^{2.3}(\xi)) &= 0, \quad 0 \leq \xi \leq 1, \\
 \mathbb{D}^{\frac{7}{2}} \vartheta(\xi) + \sqrt{\theta(\xi)} (\theta^{2.3}(\xi) + \theta^{1.1}(\xi)) &= 0, \quad 0 \leq \xi \leq 1,
 \end{aligned} \tag{27}$$

subject to boundary conditions given as

$$\theta(1) = \theta'(0) = \theta''(0) = \theta'''(0) = 0, \quad \vartheta(1) = \vartheta'(0) = \vartheta''(0) = \vartheta'''(0) = 0. \tag{28}$$



As for  $\mathbb{A} = \mathbb{B} = \frac{1}{64}$ , we have  $\nabla = 1.1767e^{-6} < 1$ . When  $n = 3$  is large enough, iterative sequences for the approximate minimal and maximal solutions  $\theta_n$  and  $\bar{\theta}_n^*$  respectively are:

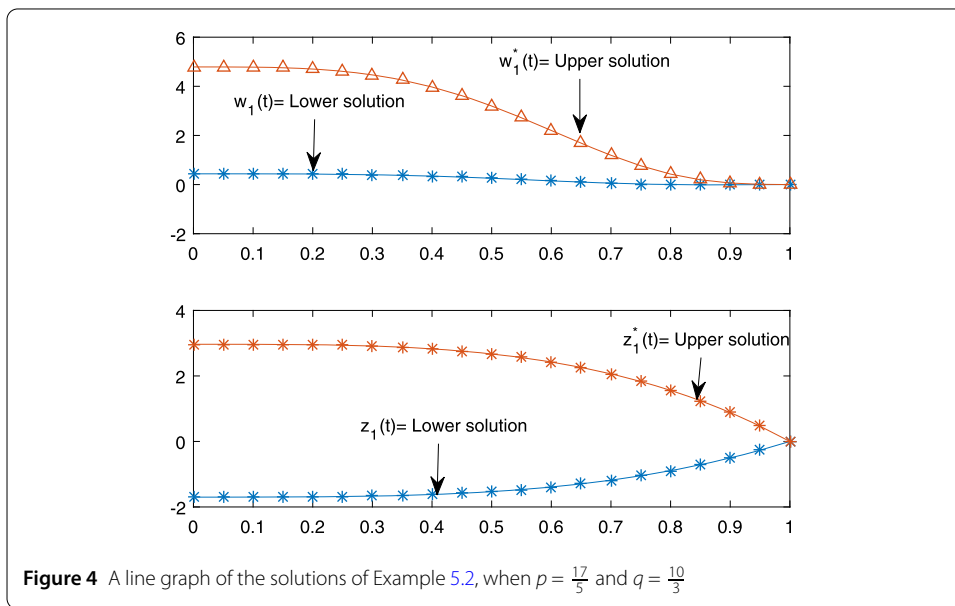
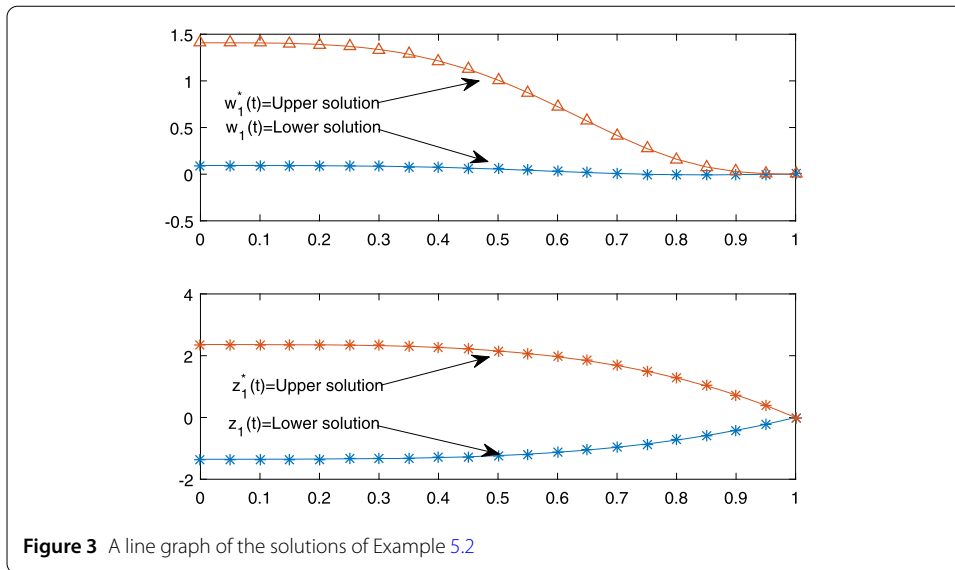
$$\begin{aligned} \theta^*(\xi) &= \theta_3(\xi), & \vartheta^*(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \sqrt{\theta_2(\sigma)} (\theta_2^{2.3}(\sigma) + \theta_2^{1.1}(\sigma)) d\sigma, \\ \bar{\theta}^*(\xi) &= \theta_3^*(\xi), & \bar{\vartheta}^*(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \sqrt{\theta_2^*(\sigma)} ((\theta_2^*(\sigma))^{2.3} + (\theta_2^*(\sigma))^{1.1}) d\sigma. \end{aligned} \tag{29}$$

Let  $(\theta_0, \vartheta_0) = (-3, -3)$ ,  $(\theta_0^*, \vartheta_0^*) = (3, 3)$  be the maximal and minimal solutions of Example 5.2 respectively, then the error estimates are:

$$\begin{aligned} e_3 &= \|\theta - \theta_3\| \leq \frac{\nabla^3}{1 - \nabla} \times e_1 \leq 1.62914414119e^{-18} \times \max_{\xi \in [0,1]} |\theta_1(\xi) + 3| \\ &\simeq 5.03862710112e^{-18}, \\ e_3^* &= \|\theta - \theta_3^*\| \leq \frac{\nabla^3}{1 - \nabla} \times e_1^* \leq 1.62914414119e^{-18} \times \max_{\xi \in [0,1]} |3 - \theta_1^*(\xi)| \\ &\simeq 4.88743242358e^{-18}. \end{aligned}$$

The iterative solutions of minimal and maximal of Example 5.2 are plotted in Figs. 3–4, wherein for the simple execution of Matlab code, we have replaced  $(\theta, \vartheta)$  by  $(w, z)$  and  $(\theta^*, \vartheta^*)$  by  $(w^*, z^*)$ . Each plot in the figures has the demonstration of physical behavior of the approximate solutions.

Further, as  $\nabla < 1$ , therefore, the Ulam–Hyers stability and generalized Ulam–Hyers stability are obvious. The stability of the iterative solutions can also be observed from Figs. 3–4.



**Example 5.3** Consider the following coupled system of BVPs of FDEs:

$$\begin{aligned} \mathbb{D}^{\frac{21}{5}}\theta(\xi) + \frac{t^{\frac{15}{4}}\vartheta(\xi)(\vartheta^2(\xi) + 160)}{(1 + \vartheta^2(\xi))^2} &= 0, \quad 0 \leq \xi \leq 1, \\ \mathbb{D}^{\frac{9}{2}}\vartheta(\xi) + \frac{t^{\frac{19}{4}}\theta(\xi)(\theta(\xi) + 1)(\theta(\xi) + 9)(e^{-2\theta(\xi)} + 120)}{(2 + \theta(\xi))^2} &= 0, \quad 0 \leq \xi \leq 1, \end{aligned} \tag{30}$$

with the given boundary condition

$$\begin{aligned} \theta(1) = \theta'(0) = \theta''(0) = \theta'''(0) = \theta''''(0) &= 0, \\ \vartheta(1) = \vartheta'(0) = \vartheta''(0) = \vartheta'''(0) = \vartheta''''(0) &= 0. \end{aligned} \tag{31}$$

Obviously,  $\mathbb{A} = \mathbb{B} = \frac{1}{64}$ , from which it is easy to see that  $\nabla = 1.4317e^{-7} < 1$ . For taking  $n = 3$  is large enough, we have the iterative sequences for approximate minimal and maximal solutions  $\theta_n$  and  $\bar{\theta}_n^*$  as given by

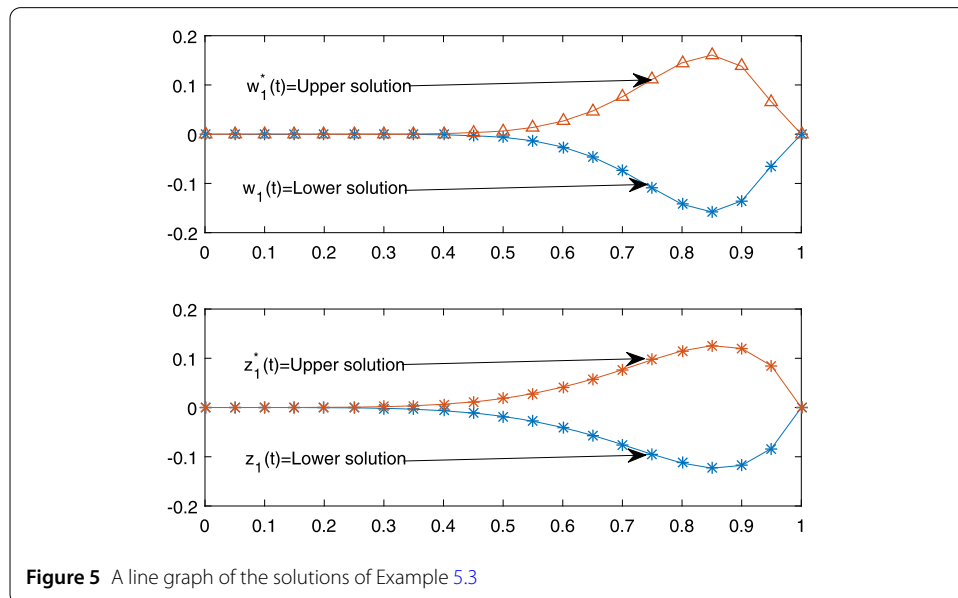
$$\begin{aligned} \theta^*(\xi) &= \theta_3(\xi), \\ \vartheta^*(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \frac{\sigma^{\frac{19}{4}} \theta_2(\sigma)(\theta_2(\sigma) + 1)(\theta_2(\sigma) + 9)(e^{-2\theta_2(\sigma)} + 120)}{(2 + \theta_2(\sigma))^2} d\sigma, \\ \bar{\theta}^*(\xi) &= \theta_3^*(\xi), \\ \bar{\vartheta}^*(\xi) &= \int_0^1 \mathbb{G}_2(\xi, \sigma) \frac{t^{\frac{19}{4}} \theta_2^*(\sigma)(\theta_2^*(\sigma) + 1)(\theta_2^*(\sigma) + 9)(e^{-2\theta_2^*(\sigma)} + 120)}{(2 + \theta_2^*(\sigma))^2} d\sigma, \end{aligned} \tag{32}$$

respectively. Let  $(\theta_0, \vartheta_0) = (-0.1, -0.1)$ ,  $(\theta_0^*, \vartheta_0^*) = (0.1, 0.1)$  be maximal and minimal solutions respectively of Example 5.3, then

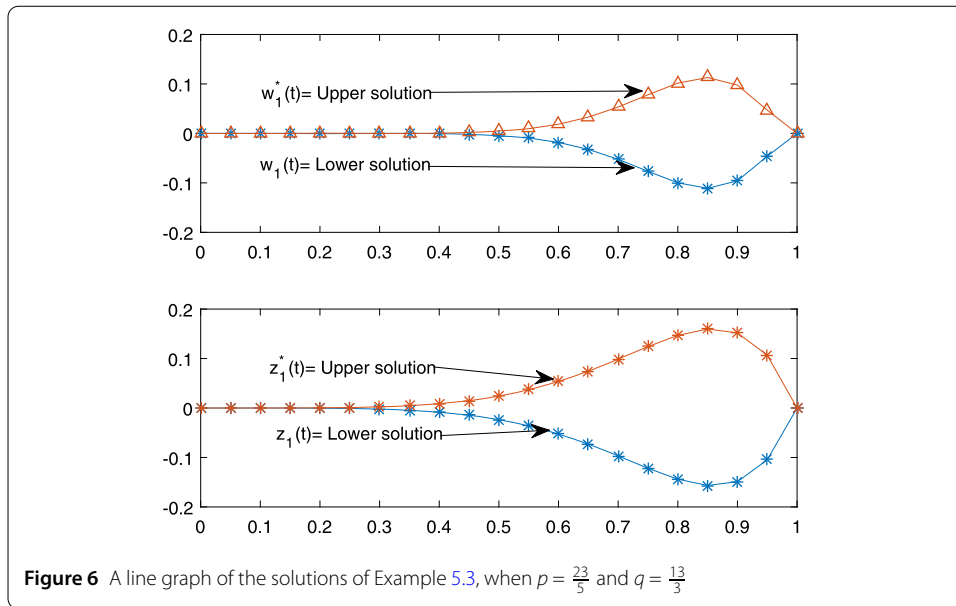
$$\begin{aligned} e_3 &= \|\theta - \theta_3\| \leq \frac{\nabla^3}{1 - \nabla} \times e_1 \leq 2.93476154975e^{-21} \times \max_{\xi \in [0,1]} |\theta_1(\xi) + 0.1| \\ &\simeq 2.93476154975e^{-22}, \\ e_3^* &= \|\theta - \theta_3^*\| \leq \frac{\nabla^3}{1 - \nabla} \times e_1^* \leq 2.93476154975e^{-21} \times \max_{\xi \in [0,1]} |0.1 - \theta_1^*(\xi)| \\ &\simeq 2.93476154975e^{-22}. \end{aligned}$$

Physical behavior of the approximate maximal and minimal solutions has also been demonstrated by providing Figs. 5–6, wherein, for the simple execution of Matlab code, we have replaced  $(\theta, \vartheta)$  by  $(w, z)$  and  $(\theta^*, \vartheta^*)$  by  $(w^*, z^*)$ .

It is easy to obtain the conditions of various types of Ulam–Hyers stabilities for the solutions. The stability of iterative solutions is also obvious from Figs. 5–6.







*Figures discussion:* We have plotted the iterative solutions of three considered examples of system (1). These plots are provided in Figs. 1–6. For the sketch of plots, we have used the subplot command in the Matlab software. Each plot of numerical results shows a meaningful behavior of the extremal solutions of the coupled system of fractional differential equations. The different visualizations of each upper and lower solutions are very clear.

### 6 Conclusion

We have developed sufficient conditions for the existence of extremal solutions to a coupled system of FDEs with higher order boundary conditions. We have successfully provided these sufficient conditions on the basis of combining the method of upper and lower solutions together with the method of monotone iterative technique. We have constructed two monotonic sequences corresponding to upper and lower solutions such that a monotonically increasing sequence converged to the maximal solutions and the monotonically decreasing sequence converged to the minimal solutions of the system under consideration. Finally, an error estimate has been established, which provided the efficiency of the method.

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#### Abbreviations

FDEs; BVPs

#### Availability of data and materials

Not applicable.

#### Competing interests

There does not exist any competing interest regarding this article.

### Authors' contributions

All authors have done equal contribution in this article. All authors read and approved the final manuscript.

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