Journal of Advanced Research 25 (2020) 39–48

Contents lists available at ScienceDirect

Journal of Advanced Research

journal homepage: www.elsevier.com/locate/jare

Research Article

Stable numerical results to a class of time-space fractional partial differential equations via spectral method

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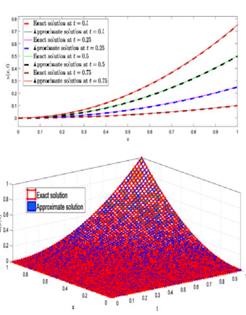
GRAPHICAL ABSTRACT



ARTICLE INFO

Article history:
Received 6 April 2020
Revised 2 May 2020
Accepted 20 May 2020
Available online 19 June 2020

Keywords:
Fractional partial differential equations
Caputo fractional derivative
Shifted Jacobin polynomials
Operational matrices
Numerical solution
Stability

ABSTRACT

In this paper, we are concerned with finding numerical solutions to the class of time–space fractional partial differential equations:

$$D_t^p u(t,x) + \kappa D_x^p u(t,x) + \tau u(t,x) = g(t,x), 1 < p < 2, (t,x) \in [0,1] \times [0,1],$$

under the initial conditions.

$$u(0,x) = \theta(x),\ u_t(0,x) = \phi(x),$$

and the mixed boundary conditions.

$$u(t,0) = u_x(t,0) = 0,$$

where D_t^p is the arbitrary derivative in Caputo sense of order p corresponding to the variable time t. Further, D_x^p is the arbitrary derivative in Caputo sense with order p corresponding to the variable space x. Using shifted Jacobin polynomial basis and via some operational matrices of fractional order

☆ Peer review under responsibility of Cairo University.

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https://doi.org/10.1016/j.jare.2020.05.022
2090-1232/

integration and differentiation, the considered problem is reduced to solve a system of linear equations. The used method doesn't need discretization. A test problem is presented in order to validate the method. Moreover, it is shown by some numerical tests that the suggested method is stable with respect to a small perturbation of the source data $g(t,x)$. Further the exact and numerical solutions are compared via 3D graphs which shows that both the solutions coincides very well.

Introduction

In present time significant attention has been given to study non-integer order partial differential equations. In fact, it was shown that in many situations, derivatives of non-integer order are very effective for the description of many physical phenomena (see, for example [3,17,20,23]). The current article is devoted to find numerical solutions to the following class of time–space fractional partial differential equations:

$$D_t^p u(t,x) + \kappa D_x^p u(t,x) + \tau u(t,x) = g(t,x), 1 < p < 2, \quad (t,x) \in [0,1] \times [0,1], \tag{1}$$

under the initial conditions

$$u(0,x) = \theta(x),\ u_t(0,x) = \phi(x), \tag{2}$$

and the mixed boundary conditions

$$u(t,0) = u_x(t,0) = 0, \tag{3}$$

where $(\tau,\kappa) \in R^* \times R, D_t^p$ denotes the Caputo fractional derivative of order p with respect to the variable time t, D_x^p denotes the Caputo fractional derivative of order p with respect to the variable space x, u_t is the derivative of u with respect to the variable time t, u_x is the derivative of u with respect to the variable space x, and $\theta,\ \phi : [0,1] \to R, g : [0,1] \times [0,1] \to R$ are given functions.

The modeling of some real world problems by using differential equations is a warm area of research in last many years. Here we, remark that partial differential equations have important applications in many branches of science and engineering. For instance heat transfer is a very important branch of mechanical and aerospace engineering analyses because many machines and devices in both these engineering disciplines are vulnerable to heat. An engineer can predict about with possible shape changes of the plate in vibrations from the simulation results of the aforesaid equations. Many engineering problems fall into such category by nature, and the use of numerical methods will to find their solutions are important for engineers. In particularly time–space one dimensional equation has many applications. For concerned applications detail, we recommend few article as [9–11].

Conventionally, numerous techniques were developed to find approximate solutions to different classes of fractional partial differential equations such as homotopy analysis method [8], He's variation iteration method [5], Adomian decomposition method [7], homotopy perturbation method [1], Fourier transform method [29], Laplace and natural transform methods [26,27]. But all these method have their own advantages and disadvantages in application point of view. For example, homotopy methods depend on a small parameter which restricted these methods. Similarly the methods that are involving integral transform also are limited in applications. In last few decades, some interesting numerical schemes based on radial basis functions (RBFs) and meshless techniques were introduced. These methods require collocation and (RBFs) to solve fractional partial differential equations [22,28]. Recently, numerical schemes based on operational matrices have attracted the attention of many researches. The mentioned techniques provide highly accurate numerical solutions to both linear and nonlinear ordinary as well as partial differential equations of classical and fractional order. In the mentioned schemes, some operational matrices of fractional order integration and differentiation are constructed, which play central roles to find approximate solutions for the considered problems. In the most existing works, the mentioned matrices are obtained using a certain polynomial basis and a Tau-collocation method (see, for example [4,12,13,21,24,25,30]). However, in these methods, discretization is required, which needs extra memory. Further for discretization and collocation extra amount of memory should be utilized. To overcome this difficulty, in [14,15], the authors constructed the operational matrices without discretizing the data and omitting collocation method to compute numerical solutions for both ordinary as well as partial fractional differential equations.

Motivated by the above cited works, in this paper, a numerical solution to (1)–(3) is computed using shifted Jacobin polynomial basis and some operational matrices of fractional order integration and differentiation without actually discretizing the problem. The Jacobi polynomials are more general polynomials and including "Legendre polynomials, Gegenbauer polynomials, Zernike and Chebyshev polynomials" as special cases. The concerned polynomials have numerous applications in Quantum physics, fluid mechanics and solitary theory of waves, see detail [16].

The used method reduces (1)–(3) to a system of linear algebraic equations of the form given by

$$\mathbf{H}_{k^2}^T A = B,$$

where the matrix $\mathbf{H}$ is the unknown which may be determined while the other matrices $A,\ B$ are known coefficient matrices of dimension $k^2 \times k^2$ and $1 \times k^2$ respectively. Here it is remarkable that the obtained system of algebraic equations is then solved by Gauss elimination method through Matlab for the unknown matrix $\mathbf{H}$. Further we demonstrate that by computational software, the solution is easily obtained up to better accuracy. The computations in our work are performed using Matlab-16."The paper is organized as follows. In Section 2, we recall briefly some necessary definitions and mathematical preliminaries about fractional calculus. In Section 3, we recall some basic properties on Jacobi polynomials, which are required for establishing main results. In Section 4, The shifted Jacobi operational matrices of fractional derivatives and fractional integrals are obtained. Section 5 is devoted to the numerical scheme, which is based on operational matrices. In Section 6, numerical experiments are presented. Also in the same section, we study the stability of the method with respect to a perturbation of the source data. Conclusion is made in Section 7."

Basic materials

Some fundamentals notions, definitions and results are recalled here from [6,17,18].

Definition 1. A real function $f(x), x > 0$ is said to be in space $C_\mu, \mu \in R$, if and only if, there exists a real number $m > \mu$ such that

$$f(x) = x^m g(x), x > 0,$$

where $g \in C(0,1)$.

Definition 2. A real function $f(x), x > 0$ is said to be in space $C_p^n,\ \mu \in R, n \in N_0 = N \cup 0$, if and only if, $f^n \in C_\mu$.

Definition 3. Corresponding to arbitrary order $p > 0$, for a function $f \in C_\mu, \mu \geqslant -1$, arbitrary order integral is recalled as

$$\begin{aligned} I^p f(x) = &\ \tfrac{1}{\Gamma(p)} \int_0^x (x-\mu)^{p-1} f(\mu) d\mu, \\ I^0 f(x) = &\ f(x). \end{aligned}$$

For $f \in C_\mu, \mu \geqslant -1, p, q \geqslant 0$and $\gamma > -1$, we have

$$I^p I^q f(x) = I^{p+q} f(x),$$

and

$$I^p x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(p+\gamma+1)} x^{p+\gamma}. \tag{4}$$

Definition 4. Let function $f \in C_{-1}^n$, then corresponding to order $p \in (n-1, n], n = [p]+1$, the arbitrary derivative in Caputo sense is provided by

$$(D^p f)(x) = I^{n-p} f^{(n)}(x).$$

For a power function for order $p \in (n-1, n], n = [p]+1$, the arbitrary derivative in Caputo sense, one has

$$D^p x^k = \begin{cases} 0 \quad if \ \ 0 \leqslant k \leqslant [p], \\ \frac{\Gamma(k+1)}{\Gamma(1+k-p)} x^{p+\gamma} \quad \text{if} \quad k \geqslant [p]+1, \end{cases} \tag{5}$$

where $k \in N_0$.

We have the following properties.

Lemma 1. Let $n-1 < p \leqslant n,\ n \in N,\ \mu \geqslant -1$ and $f \in C_\mu^n$. Then

$$D^p I^p f(x) = f(x),$$

and

$$I^p D^p f(x) = f(x) - \sum_{i=0}^{n-1} f^{(i)}(0^+) \frac{x^i}{i!}, \quad x \geqslant 0. \tag{6}$$

Derivation of Shifted Jacobi polynomials from fundamental Jacobi Polynomials

Here we provide fundamental characteristic of the Jacobi polynomials. The famous Jacobi polynomials $\mathscr{P}_i^{(\varpi,\omega)}(y)$ are defined over the interval $[-1, 1]$ as

$$\begin{aligned} \mathscr{P}_i^{(\varpi,\omega)}(y) = &\ \tfrac{(\varpi+\omega+2i-1)[\varpi^2-\omega^2+y(\varpi+\omega+2i-2)(\varpi+\omega+2i-2)]}{2i(\varpi+\omega+i)(\varpi+\omega+2i-2)} P_{i-1}^{(\varpi,\omega)}(y) \\ &- \tfrac{(\varpi+i-1)(\omega+i-1)(\varpi+\omega+2i)}{i(\varpi+\omega+i)(\varpi+\omega+2i-2)} P_{i-2}^{(\varpi,\omega)}(y),\ i = 2, 3, \ldots, \end{aligned} \tag{7}$$

where $\mathscr{P}_0^{(\varpi,\omega)}(y) = 1,\ \mathscr{P}_1^{(\varpi,\omega)}(y) = \frac{\varpi+\omega+2}{2} y + \frac{\varpi-\omega}{2}$.

By means of the substitution $\frac{y+1}{2} = \frac{t}{L}$, we get a revised version of the concerned polynomials called the shifted Jacobi polynomials over the interval $[0, L]$. A general term $\mathscr{Q}_{L,i}^{(\varpi,\omega)}(t)$ of degree i of the suggested polynomials on $[0, L]$, with $\varpi > -1,\ \omega > -1$ is as:

$$\mathscr{Q}_{L,i}^{(\varpi,\omega)}(t) = \sum_{n=0}^{i} \frac{(-1)^{i-n}\Gamma(i+\omega+1)\Gamma(i+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(i+\varpi+\omega+1)(i-n)!n!L^n} t^n, \tag{8}$$

where

$$\mathscr{Q}_{L,i}^{(\varpi,\omega)}(0) = (-1)^i \frac{\Gamma(i+\omega+1)}{\Gamma(\omega+1)i!},$$

and

$$\mathscr{Q}_{L,i}^{(\varpi,\omega)}(L) = \frac{\Gamma(i+\varpi+1)}{\Gamma(\varpi+1)i!}.$$

Result regarding orthogonality of the said polynomials is

$$\int_0^L \mathscr{Q}_{L,i}^{(\varpi,\omega)}(t)\mathscr{Q}_{L,j}^{(\varpi,\omega)}(t)\mathscr{W}_L^{(\varpi,\omega)}(t)dt = \Re_{L,j}^{(\varpi,\omega)} \tag{9}$$

and $\omega_L^{(\varpi,\omega)}(t) = (L-t)^\varpi t^\omega$ is the weight function, and

$$\Re_{L,j}^{(\varpi,\omega)} = \begin{cases} \theta_i \ \text{ if } \ i = j, \\ 0 \ \text{ if } \ i \neq j, \end{cases}$$

such that

$$\theta_i = \frac{L^{\varpi+\omega+1}\Gamma(i+\varpi+1)\Gamma(i+\omega+1)}{(2i+\varpi+\omega+1)i!\Gamma(i+\varpi+\omega+1)}. \tag{10}$$

Here for the readers we provide few special cases from shifted Jacobi basis as:

(i) $\mathscr{L}_{L,i}(t) = \mathscr{Q}_{L,i}^{(0,0)}(t)$ is the shifted Legendre polynomials by sitting $\varpi = \omega = 0$ in (8).

(ii) $\mathscr{T}_{L,i}(t) = \frac{\Gamma(i+1)\Gamma(\frac{1}{2})}{\Gamma(i+\frac{1}{2})} \mathscr{Q}_{L,i}^{(\frac{-1}{2},\frac{-1}{2})}(t)$, is called shifted Chebyshev polynomials by sitting $\varpi = \omega = \frac{-1}{2}$ in (8).

(iii) In same line one has $\mathscr{U}_{L,i}(t) = \frac{\Gamma(i+2)\Gamma(\frac{1}{2})}{\Gamma(i+\frac{3}{2})} \mathscr{Q}_{L,i}^{(\frac{1}{2},\frac{1}{2})}(t)$, is known as Chebyshev polynomials of second kind when $\varpi = \omega = \frac{1}{2}$ in (8).

(iv) Also if we sit $\varpi = \omega$ in (8) we get shifted Gegenbauer (Ultraspherical) polynomials as

$$\mathscr{G}_{L,i}^{\varpi}(t) = \frac{\Gamma(i+1)\Gamma(a+\frac{1}{2})}{\Gamma(i+a+\frac{1}{2})} \mathscr{Q}_{L,i}^{(\varpi-\frac{1}{2},\omega-\frac{1}{2})}(t).$$

(v) Further if one sit $\varpi = \frac{1}{2},\ \omega = \frac{-1}{2}$ in (8), we get third kind shifted Chebyshev polynomials as

$$\mathscr{V}_{L,i}(t) = \frac{(\Gamma(2i+1))!}{(\Gamma(2i-1))!} \mathscr{Q}_{L,i}^{(\frac{1}{2},\frac{-1}{2})}(t).$$

(vi) sitting $\varpi = \frac{-1}{2},\ \omega = \frac{1}{2}$ in (8), we have fourth kinds shifted Chebyshev polynomials as

$$\mathscr{W}_{L,i}(t) = \frac{(\Gamma(2i+1))!}{(\Gamma(2i-1))!} \mathscr{Q}_{L,i}^{(\frac{-1}{2},\frac{1}{2})}(t).$$

Here we claim that performing numerical computation with shifted Jacobi polynomials means that the above special cases are also considered. Some time the shifted jacobi polynomials are also called hypergeometric polynomials which constitute a big class of orthogonal polynomials. These polynomials are orthogonal with respect to some weight function, for more detail (see [19]).

Assume that $U(t)$ is a square integrable function with respect to the weight function $\omega_L^{(\varpi,\omega)}$ on $[0, L]$. Then it can be expressed in terms of shifted Jacobi polynomials as

$$U(t) = \sum_{j=0}^{\infty} D_j \mathscr{Q}_{L,j}^{(\varpi,\omega)}(t),$$

from which the coefficients D_j can be computed easily using the orthogonality condition (9). Onward we are switching over to shifted Jacobi polynomials of two variable instead of one (see [2]).

Definition 5. Let $\{\mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\}_{i=0}^{\infty}$ be the sequence of one variable shifted Jacobi polynomials on $[0,L]$. The notions $\{\mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)\}_{i,j=0}^{\infty}$ for two variable shifted Jacobi polynomials which are defined on $[0,L]\times[0,L]$ by

$$\mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)=\mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\mathscr{Q}^{(\varpi,\omega)}_{L,j}(x),\quad (t,x)\in[0,L]\times[0,L].$$

The family $\{\mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)\}_{i,j=0}^{\infty}$ is orthogonal with respect to the weighted function

$$\mathscr{W}^{(\varpi,\omega)}_{L}(t,x)=\mathscr{W}^{(\varpi,\omega)}_{L}(t)\mathscr{W}^{(\varpi,\omega)}_{L}(x),\quad (t,x)\in[0,L]\times[0,L].$$

Indeed, from (9), we have

$$\begin{aligned}&\int_0^L\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)\mathscr{Q}^{(\varpi,\omega)}_{L,k,l}(t,x)\mathscr{W}^{(\varpi,\omega)}_{L}(t,x)dtdx\\ &=\int_0^L\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\mathscr{Q}^{(\varpi,\omega)}_{L,j}(x)\mathscr{Q}^{(\varpi,\omega)}_{L,k}(t)\mathscr{Q}^{(\varpi,\omega)}_{L,l}(x)\mathscr{W}^{(\varpi,\omega)}_{L}(t)\mathscr{W}^{(\varpi,\omega)}_{L}(x)dtdx\\ &=\left(\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\mathscr{Q}^{(\varpi,\omega)}_{L,k}(t)\mathscr{W}^{(\varpi,\omega)}_{L}(t)dt\right)\times\left(\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,j}(x)\mathscr{Q}^{(\varpi,\omega)}_{L,l}(x)\mathscr{W}^{(\varpi,\omega)}_{L}(x)dx\right)\\ &=\mathfrak{R}^{(\varpi,\omega)}_{L,k}\mathfrak{R}^{(\varpi,\omega)}_{L,l},\end{aligned}$$

where

$$\mathfrak{R}^{(\varpi,\omega)}_{L,k}\mathfrak{R}^{(\varpi,\omega)}_{L,l}=\begin{cases}\theta_i\theta_j & \text{if}\quad (i,j)=(k,l),\\ 0 & \text{otherwise}.\end{cases}$$

assume that a square integrable function $U(t,x)$ with respect to the weight function $\mathscr{W}^{(\varpi,\omega)}_{L}$ on $[0,L]\times[0,L]$ is expressed in terms of the considered polynomials as

$$U(t,x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}D_{i,j}\mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x),\quad (t,x)\in[0,L]\times[0,L], \tag{11}$$

where the notions $D_{i,j}$ are Jacobi coefficients provided by

$$D_{i,j}=\frac{1}{\theta_i\theta_j}\int_0^L\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)U(t,x)\mathscr{W}^{(\varpi,\omega)}_{L}(t,x)dtdx. \tag{12}$$

Truncated the series (11) up to their K-terms which can be expressed as:

$$U(t,x)\simeq U_k(x,y)=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}D_{i,j}\mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)=\mathbf{H}^T_{k^2}\Phi_{k^2}(t,x),$$

where

$$\mathbf{H}^T_{k^2}=(D_{0,0},D_{0,1},\ldots,D_{0,k-1},\ldots,D_{k-1,0},D_{k-1,1},\ldots,D_{k-1,k-1})$$

and

$$\begin{aligned}\Phi_{k^2}(t,x)=\Big(&\mathscr{Q}^{(\varpi,\omega)}_{L,0,0}(t,x),\mathscr{Q}^{(\varpi,\omega)}_{L,0,1}(t,x),\ldots,\mathscr{Q}^{(\varpi,\omega)}_{L,0,k-1}(t,x),\ldots,\\ &\mathscr{Q}^{(\varpi,\omega)}_{L,k-1,0}(t,x),\mathscr{Q}^{(\varpi,\omega)}_{L,k-1,1}(t,x),\ldots,\mathscr{Q}^{(\varpi,\omega)}_{L,k-1,k-1}(t,x)\Big)^T.\end{aligned} \tag{13}$$

Construction of required matrices corresponding to arbitrary order derivatives and integrals

Here in this part, let $\Xi=\{0,1,\ldots,k-1\}$, some results are: For $p>0$ and $i,j,a,b\in\Xi$, let

$$\delta_{j,\,b}=\begin{cases}1 & \text{if}\quad b=j,\\ 0 & \text{if}\quad b\neq j\end{cases}$$

and

$$\mathscr{W}_{a,b}(i,j)=um^a_{n=0}\Delta_{a,n,p}G_{i,j,b},$$

where

$$\Delta_{a,n,p}=\frac{(-1)^{a-n}\Gamma(a+\omega+1)\Gamma(a+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(a+\varpi+\omega+1)(a-n)!\Gamma(p+n+1)}$$

and

$$G_{i,j,b}=\delta_{i,b}\sum_{l=0}^{i}\frac{(-1)^{i-l}\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)(i-l)}\frac{\Gamma(n+p+l+\omega+1)}{\Gamma(n+p+l+\omega+\varpi+2)}\times\frac{(2i+\varpi+\omega+1)i!L^p}{\Gamma(i+\varpi+1)}.$$

Keeping in mind the above definitions, notions, one has the results presented here as:

Lemma 2. From vector function given in (13) as $\Phi_{k^2}(t,x)$, we have

$$I^p_t(\Phi_{k^2}(t,x))\simeq M^p_{k^2\times k^2}\Phi_{k^2}(t,x),\quad (t,x)\in[0,L]\times[0,L], \tag{14}$$

where I^p_t is the Riemann–Liouville fractional integral of order $p>0$ with respect to the variable time t, and $M^p_{k^2\times k^2}$ is the square matrix of size k^2, given by

$$M^p_{k^2\times k^2}=(M^p_{v,r})_{1\leqslant v,r\leqslant k^2},$$

with

$$M^p_{v,r}=\mathscr{W}_{a,b}(i,j),\quad v=ka+b+1,\ r=ki+j+1,\ \ i,j,a,b\in\Xi.$$

Proof.

Let (a,b) be a fixed pair of positive integers such that $a,b\in\Xi$. Then

$$I^p_t\mathscr{Q}^{(\varpi,\omega)}_{L,a,b}(t,x)=\left(I^p_t\mathscr{Q}^{(\varpi,\omega)}_{L,a}(t)\right)\mathscr{Q}^{(\varpi,\omega)}_{L,b}(x).$$

On the other hand, we have

$$I^p_t\mathscr{Q}^{(\varpi,\omega)}_{L,a}(t)=\sum_{n=0}^{a}\frac{(-1)^{a-n}\Gamma(a+\omega+1)\Gamma(a+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(a+\varpi+\omega+1)(a-n)!n!L^n}.$$

From Property (4), we obtain

$$I^p_t t^n=\frac{n!}{\Gamma(p+n+1)}t^{n+p},$$

which yields

$$I^p_t\mathscr{Q}^{(\varpi,\omega)}_{L,a}(t)=\sum_{n=0}^{a}\frac{(-1)^{a-n}\Gamma(a+\omega+1)\Gamma(a+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(a+\varpi+\omega+1)(a-n)!L^n\Gamma(p+n+1)}t^{n+p}.$$

Therefore, we have

$$\mathscr{Q}^{(\varpi,\omega)}_{L,a,b}(t,x)=\sum_{n=0}^{a}\frac{\Delta_{a,n,p}}{L^n}t^{n+p}\mathscr{Q}^{(\varpi,\omega)}_{L,b}(x). \tag{15}$$

Approximating $t^{n+p}\mathscr{Q}^{(\varpi,\omega)}_{L,b}(x)$ in terms of said polynomials, one has

$$t^{n+p}\mathscr{Q}^{(\varpi,\omega)}_{L,b}(x)\simeq\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}S_{i,j,b}\mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\mathscr{Q}^{(\varpi,\omega)}_{L,j}(x), \tag{16}$$

where

$$S_{i,j,b}=\frac{1}{\theta_i\theta_j}\int_0^L\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,i,j}(t,x)t^{n+p}\mathscr{Q}^{(\varpi,\omega)}_{L,b}(x)\mathscr{W}^{(\varpi,\omega)}_{L}(t,x)dtdx.$$

On the other hand, we have

$$S_{i,j,b}=\frac{1}{\theta_i\theta_j}\left(\int_0^L t^{n+p}\mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\mathscr{W}^{(\varpi,\omega)}_{L}(t)dt\right)\left(\int_0^L \mathscr{Q}^{(\varpi,\omega)}_{L,j}(x)\mathscr{Q}^{(\varpi,\omega)}_{L,b}(x)\mathscr{W}^{(\varpi,\omega)}_{L}(x)dx\right)$$

Therefore, using the orthogonality condition (9), we obtain

$$S_{i,j,b}=\frac{\delta_{j,b}}{\theta_i}\left(\int_0^L t^{n+p}\mathscr{Q}^{(\varpi,\omega)}_{L,i}(t)\mathscr{W}^{(\varpi,\omega)}_{L}(t)dt\right).$$

Further, we have

$$\int_0^L t^{n+p}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{W}_L^{(\varpi,\omega)}(t)dt = \sum_{l=0}^{i}\frac{(-1)^{i-l}\Gamma(i+\omega+1)\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)\Gamma(i+\varpi+\omega+1)(i-l)!l!L^l} \times \int_0^L t^{n+p+l+\omega}(L-t)^{\varpi}dt.$$

Using the change of variable $s=\frac{t}{L}$, we obtain

$$\begin{aligned}\int_0^L t^{n+p+l+\omega}(L-t)^{\varpi}dt &= L^{n+p+l+\omega+1}\int_0^L s^{(n+p+l+\omega+\varpi+1)-1}(1-s)^{(\varpi+1)-1}ds\\ &= L^{n+p+l+\omega+\varpi+1}B(n+p+l+\omega+1,\varpi+1),\end{aligned}$$

where B is the beta function. Next, using the property

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad x>0,\ \ y>0,$$

we obtain

$$\int_0^L t^{n+p+l+\omega}(L-t)^{\varpi}dt = L^{n+p+l+\omega+\varpi+1} \times \frac{\Gamma(n+p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n+p+l+\omega+\varpi+2)}.$$

Hence,

$$\int_0^L t^{n+p}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{W}_L^{(\varpi,\omega)}(t)dt = \sum_{l=0}^{i}\frac{(-1)^{i-l}\Gamma(i+\omega+1)\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)\Gamma(i+\varpi+\omega+1)(i-l)!l!}\frac{\Gamma(n+p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n+p+l+\omega+\varpi+2)}L^{n+p+l+\omega+\varpi+1},$$

which yields

$$S_{i,j,b} = \frac{\delta_{i,b}}{\theta_i}\sum_{l=0}^{i}\frac{(-1)^{i-l}\Gamma(i+\omega+1)\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)\Gamma(i+\varpi+\omega+1)(i-l)!l!}\frac{\Gamma(n+p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n+p+l+\omega+\varpi+2)}L^{n+p+l+\omega+\varpi+1}.$$

Using (10), we obtain

$$S_{i,j,b}=L^nG_{i,j,b}.$$

On the other hand, from (15) and (16), we obtain

$$I_t^p\mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x)\simeq\sum_{n=0}^{a}\Delta_{a,n,p}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}G_{i,j,b}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{Q}_{L,j}^{(\varpi,\omega)}(x),$$

that is,

$$I_t^p\mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x)\simeq\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\Omega_{a,b}(i,j)\mathcal{Q}_{L,i,j}^{(\varpi,\omega)}(t,x),$$

which yields (16).

For $p>0$ and $i,j,a,b\in\Xi$, let

$$\delta_{i,\,a}=\begin{cases}1 & \text{if}\quad a=i,\\ 0 & \text{if}\quad a\neq i\end{cases}$$

and

$$\Omega^*_{a,b}(i,j)=\sum_{n=0}^{b}\Delta_{b,n,p}G^*_{i,j,a},$$

where

$$\Delta_{b,n,p}=\frac{(-1)^{b-n}\Gamma(b+\omega+1)\Gamma(b+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(b+\varpi+\omega+1)(b-n)!\Gamma(p+n+1)}$$

and

$$G^*_{i,j,a} = \delta_{i,a}\sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)(j-l)!l!}\frac{\Gamma(n+p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n+p+l+\omega+\varpi+2)}\frac{(2j+\varpi+\omega+1)j!L^p}{\Gamma(j+\varpi+1)}.$$

Following the same arguments used in the proof of Lemma 2, we obtain the following result.

Lemma 3. Let $\Phi_{k^2}(t,x)$ be the vectorial function defined by (13). Then

$$I_x^p(\Phi_{k^2}(t,x))\simeq N^p_{k_2\times k_2}\Phi_{k^2}(t,x),\quad (t,x)\in[0,L]\times[0,L], \tag{17}$$

where I_x^p is the Riemann–Liouville fractional integral of order $p>0$ with respect to the variable time x, and $N^p_{k^2\times k^2}$ is the square matrix of size k^2, given by

$$N^p_{k^2\times k^2}=(N^p_{v,r})_{1\leqslant v,r\leqslant k^2},$$

with

$$N^p_{v,r}=\Omega^*_{a,b}(i,j),\quad v=ka+b+1,\ r=ki+j+1,\ \ 0\leqslant i,j,a,b\leqslant k-1.$$

For $p>0$ and $i,j,a,b\in\Xi$, let

$$\mathcal{W}_{a,\,b}(i,j)=\begin{cases}0 & \text{if}\quad a=0,1,\ldots,[p],\\ \sum\limits_{n=[p]+1}^{a} & \text{if}\quad a=[p]+1,[p]+2,\ldots,k-1,\end{cases}$$

where

$$\Delta_{a,n,-p}=\frac{(-1)^{a-n}\Gamma(a+\omega+1)\Gamma(a+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(a+\varpi+\omega+1)(a-n)!\Gamma(1+n-p)}$$

and

$$\mathfrak{I}_{i,j,b} = \delta_{j,b}\sum_{l=0}^{i}\frac{(-1)^{i-l}\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)(i-l)!l!}\frac{\Gamma(n-p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n-p+l+\omega+\varpi+2)}\frac{(2i+\varpi+\omega+1)i!}{\Gamma(i+\varpi+1)L^p}.$$

The following result holds.

Lemma 4. Let $\Phi_{k^2}(t,x)$ be the vectorial function defined by (13). Then

$$D_t^p(\Phi_{k^2}(t,x))\simeq R^p_{k_2\times k_2}\Phi_{k^2}(t,x),\quad (t,x)\in[0,L]\times[0,L], \tag{18}$$

where $R^p_{k^2\times k^2}$ is the square matrix of size k^2, given by

$$R^p_{k^2\times k^2}=(R^p_{v,r})_{1\leqslant v,r\leqslant k^2},$$

with

$$R^p_{v,r}=\mathcal{W}_{a,b}(i,j),\quad v=ka+b+1,\ r=ki+j+1,\ \ 0\leqslant i,j,a,b\leqslant k-1.$$

Proof.

Let (a,b) be a fixed pair of positive integers such that $a,b\in\{0,1,\ldots,k-1\}$. Then

$$D_t^p\mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x)=\left(D_t^p\mathcal{Q}_{L,a}^{(\varpi,\omega)}(t)\right)\mathcal{Q}_{L,b}^{(\varpi,\omega)}(x).$$

On the other hand, we have

$$D_t^p\mathcal{Q}_{L,a}^{(\varpi,\omega)}(t)=\sum_{n=0}^{a}\frac{(-1)^{a-n}\Gamma(a+\omega+1)\Gamma(a+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(a+\varpi+\omega+1)(a-n)!n!L^n}D_t^pt^n.$$

We consider two cases.

Case.1 $a=0,1,\ldots,[p]$. In this case, from (1), we have

$$D_t^pt^n=0,\quad n=0,1,2,3,\ldots,a.$$

Therefore,

$$D_t^p \mathcal{Q}_{L,a}^{(\varpi,\omega)}(t,x) = 0. \tag{19}$$

Case.2 $a = [p]+1, [p]+2, \ldots, k-1$. In this case, from (1), we have

$$D_t^p t^n = 0, \quad n = 0,1,2,3,\ldots,[p]$$

and

$$D_t^p t^n = \frac{\Gamma(n+1)}{\Gamma(1+n-p)} t^{n-p}, \quad n = [p]+1, [p]+2, \ldots, a.$$

Therefore,

$$D_t^p \mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x)$$
$$= \sum_{n=[p]+1}^{a} \frac{(-1)^{a-n}\Gamma(a+\omega+1)\Gamma(a+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(a+\varpi+\omega+1)(a-n)!\Gamma(1+n-p)L^n} t^{n-p} \mathcal{Q}_{L,b}^{(\varpi,\omega)}(x).$$

Then, we obtain

$$D_t^p \mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x) = \sum_{n=[p]+1}^{a} \frac{\Delta_{a,n,-p}}{L^n} t^{n-p} \mathcal{Q}_{L,b}^{(\varpi,\omega)}(x). \tag{20}$$

Approximating $t^{n-p}\mathcal{Q}_{L,b}^{(\varpi,\omega)}(x)$ in terms of the considered polynomials, one has

$$t^{n-p} \mathcal{Q}_{L,b}^{(\varpi,\omega)}(x) \simeq \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} S_{i,j,b} \mathcal{Q}_{L,i}^{(\varpi,\omega)}(t) \mathcal{Q}_{L,j}^{(\varpi,\omega)}(x), \tag{21}$$

where

$$S_{i,j,b} = \frac{1}{\theta_i \theta_j} \int_0^L \int_0^L \mathcal{Q}_{L,i,j}^{(\varpi,\omega)}(t,x) t^{n-p} \mathcal{Q}_{L,b}^{(\varpi,\omega)}(x) \mathcal{W}_L^{(\varpi,\omega)}(t,x)\,dt\,dx.$$

On the other hand, we have

$$S_{i,j,b} = \frac{1}{\theta_i\theta_j}\left(\int_0^L t^{n-p}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{W}_L^{(\varpi,\omega)}(t)dt\right)\left(\int_0^L \mathcal{Q}_{L,j}^{(\varpi,\omega)}(x)\mathcal{Q}_{L,b}^{(\varpi,\omega)}(x)\mathcal{W}_L^{(\varpi,\omega)}(x)dx\right)$$

Due to orthogonality condition (9), one has

$$S_{i,j,b} = \frac{\delta_{j,b}}{\theta_i}\left(\int_0^L t^{n-p}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{W}_L^{(\varpi,\omega)}(t)dt\right).$$

Further, we have

$$\int_0^L t^{n-p}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{W}_L^{(\varpi,\omega)}(t)dt = \sum_{l=0}^{i} \frac{(-1)^{i-l}\Gamma(i+\omega+1)\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)\Gamma(i+\varpi+\omega+1)(i-l)!l!L^l} \times \int_0^L t^{n-p+l+\omega}(L-t)^{\varpi}dt.$$

Upon substitution $s = \frac{t}{L}$, one has

$$\int_0^L t^{n-p+l+\omega}(L-t)^{\varpi}dt = L^{n-p+l+\omega+\varpi+1} \times \frac{\Gamma(n-p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n-p+l+\omega+\varpi+2)}.$$

Hence,

$$\int_0^L t^{n-p}\mathcal{Q}_{L,i}^{(\varpi,\omega)}(t)\mathcal{W}_L^{(\varpi,\omega)}(t)dt = \sum_{l=0}^{i} \frac{(-1)^{i-l}\Gamma(i+\omega+1)\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)\Gamma(i+\varpi+\omega+1)(i-l)!l!} \frac{\Gamma(n-p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n-p+l+\omega+\varpi+2)} L^{n-p+l+\omega+\varpi+1},$$

which yields

$$S_{i,j,b} = \frac{\delta_{i,b}}{\theta_i}\sum_{l=0}^{i} \frac{(-1)^{i-l}\Gamma(i+\omega+1)\Gamma(i+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)\Gamma(i+\varpi+\omega+1)(i-l)!l!} \frac{\Gamma(n-p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n-p+l+\omega+\varpi+2)} L^{n-p+l+\omega+\varpi+1}.$$

Using (10), we obtain

$$S_{i,j,b} = L^n \mathfrak{I}_{i,j,b}.$$

On the other hand, from (20) and (21), we obtain

$$D_t^p \mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x) \simeq \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{n=[p]+1}^{a} \Delta_{a,n,-p}\mathfrak{I}_{i,j,b}\mathcal{Q}_{L,i,j}^{(\varpi,\omega)}(t,x),$$

that is,

$$D_t^p \mathcal{Q}_{L,a,b}^{(\varpi,\omega)}(t,x) \simeq \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \mathcal{W}_{a,b}(i,j)\mathcal{Q}_{L,i,j}^{(\varpi,\omega)}(t,x). \tag{22}$$

Finally, (19) and (22) yield (18).

For $p > 0$ and $i,j,a,b \in \Xi$, let

$$\mu_{a,b}(i,j) = \begin{cases} 0 & \text{if} \quad b = 0,1,\ldots,[p], \\ \sum_{n=[p]}^{b} \Delta_{b,n,-p}\mathfrak{I}_{j,i,a} & \text{if} \quad b = [p]+1,[p]+2,\ldots,k-1, \end{cases}$$

where

$$\Delta_{b,n,-p} = \frac{(-1)^{b-n}\Gamma(b+\omega+1)\Gamma(b+n+\varpi+\omega+1)}{\Gamma(n+\omega+1)\Gamma(b+\varpi+\omega+1)(b-n)!\Gamma(1+n-p)}$$

and

$$\mathfrak{I}_{i,j,a} = \delta_{i,a}\sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+\varpi+\omega+1)}{\Gamma(l+\omega+1)(j-l)!l!} \frac{\Gamma(n-p+l+\omega+1)\Gamma(\varpi+1)}{\Gamma(n-p+l+\omega+\varpi+2)} \frac{(2j+\varpi+\omega+1)j!}{\Gamma(j+\varpi+1)L^p}.$$

Following the same arguments used in the proof of Lemma, we obtain the following result.

Lemma 5. Let $\Phi_{k^2}(t,x)$ be the vectorial function defined by (13). Then

$$D_x^p(\Phi_{k^2}(t,x)) \simeq S_{k_2\times k_2}^p \Phi_{k^2}(t,x), \quad (t,x) \in [0,L]\times[0,L], \tag{23}$$

where $S_{k^2\times k^2}^p$ is the square matrix of size k^2, given by

$$S_{k^2\times k^2}^p = (S_{v,r}^p)_{1\leqslant v,r\leqslant k^2},$$

with

$$S_{v,r}^p = \mu_{a,b}(i,j), \quad v = ka+b+1,\ r = ki+j+1,\ 0 \leqslant i,j,a,b \leqslant k-1.$$

General algorithm for numerical results

In this section, using the previous obtained results, the problem of finding an approximate solution to (1)–(3) is reduced to solving a certain algebraic equation. Let $1 < p < 2$. We write $D_t^p u(t,x)$ in the form:

$$D_t^p u(t,x) = \mathbf{H}_{k^2}^T \Phi_{k^2}(t,x), \tag{24}$$

where function vector $\Phi_{k^2}(t,x)$ is given in (13) and unknown matrix $\mathbf{H}_{k^2}^T$ with size $1\times k^2$. Thus one has

$$I_t^p(D_t^p u(t,x)) = \mathbf{H}_{k^2}^T I_t^p(\Phi_{k^2}(t,x)).$$

Using (6) and Lemma 4, we obtain

$$u(t,x) = u(0,x) + tu_t(0,x) + \mathbf{H}_{k^2}^T M_{k^2\times k^2}^p \Phi_{k^2}(t,x),$$

which yields from the considered initial conditions (2)

$$u(t,x) = \theta(x) + t\phi(x) + \mathbf{H}_{k^2}^T M_{k^2\times k^2}^p \Phi_{k^2}(t,x).$$

On the other hand, using (11), we may write $\theta(x) + t\phi(x)$ in the form:

$$\theta(x) + t\phi(x) = Z^T_{k^2} M_{k^2},$$

where $Z^T_{k^2}$ is a matrix of size $1 \times k^2$. The coefficients of the matrix $Z^T_{k^2}$ can be computed using (29). Therefore, we obtain

$$u(t,x) = (\mathbf{H}^T_{k^2} M^p_{k^2 \times k^2} + Z^T_{k^2})\Phi_{k^2}(t,x). \quad (25)$$

Similarly, we may write $g(t,x)$ in the form:

$$g(t,x) = Q^T_{k^2}\Phi_{k^2}(t,x), \quad (26)$$

where $Q^T_{k^2}$ is a matrix of size $1 \times k^2$ that can be computed using (29). Now, using (1), (24), (25), and (26), we obtain

$$D^p_x u(t,x) = \frac{1}{\tau}\left[Q^T_{k^2}\Phi_{k^2}(t,x) - \kappa\left(\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2}\right)\Phi_{k^2}(t,x) - \mathbf{H}^T_{k^2}\Phi_{k^2}(t,x)\right],$$

that is,

$$D^p_x u(t,x) = \frac{1}{\tau}\left[Q^T_{k^2} - \kappa\left(\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2}\right) - \mathbf{H}^T_{k^2}\right]\Phi_{k^2}(t,x),$$

Next, we obtain

$$I^p_x(D^p_x u(t,x)) = \frac{1}{\tau}\left[Q^T_{k^2} - \kappa\left(\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2}\right) - \mathbf{H}^T_{k^2}\right] I^p_x(\Phi_{k^2}(t,x))$$

Using (6) and Lemma 3, we obtain

$$u(t,x) = u(t,0) + u_x(t,0)x + \frac{1}{\tau}\left[Q^T_{k^2} - \kappa\left(\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2}\right) - \mathbf{H}^T_{k^2}\right] N^p_{k^2\times k^2}\Phi_{k^2}(t,x),$$

which yields from the boundary conditions (3)

$$u(t,x) = \frac{1}{\tau}\left[Q^T_{k^2} - \kappa\left(\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2}\right) - \mathbf{H}^T_{k^2}\right] N^p_{k^2\times k^2}\Phi_{k^2}(t,x). \quad (27)$$

Using (25) and (27), by identification, we obtain

$$\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2} = \frac{1}{\tau}\left[Q^T_{k^2} - \kappa\left(\mathbf{H}^T_{k^2} M^p_{k^2\times k^2} + Z^T_{k^2}\right) - \mathbf{H}^T_{k^2}\right] N^p_{k^2\times k^2},$$

which yields the algebraic equation

$$\mathbf{H}^T_{k^2} A = B, \quad (28)$$

where A is the square matrix of order k^2 given by

$$A = \left[M^p_{k^2\times k^2} + \frac{1}{\tau}(\kappa M^p_{k^2\times k^2} + I_{k^2\times k^2})N^p_{k^2\times k^2}\right]$$

and B is the matrix of size $1 \times k^2$ given by

$$B = \frac{1}{\tau}(Q^T_{k^2} - \kappa Z^T_{k^2})N^p_{k^2\times k^2} - Z^T_{k^2}.$$

Here, $I_{k^2\times k^2}$ denotes the identity matrix of size k^2. The algebraic Eq. (28) is equivalent to a system of k^2 linear equations with k^2 variables, which can be solved using Matlab. Finally, after solving (28), the numerical solution to (1)–(3) can be computed using (25).

Numerical experiments

This portion is devoted to present a test problem. Therefore, consider the given problem as

$$D^{1.5}_t u(t,x) + D^{1.5}_x u(t,x) = g(t,x), (t,x) \in [0,1]\times[0,1], \quad (29)$$

under the initial conditions

$$u(0,x) = u_t(0,x) = 0 \quad (30)$$

and the mixed boundary conditions

$$u(t,0) = u_x(t,0) = 0, \quad (31)$$

where the source term $g(t,x)$ is given by

$$g(t,x) = \frac{2}{\Gamma(1.5)}(x^2\sqrt{t} + t^2\sqrt{x}), (t,x) \in [0,1]\times[0,1]. \quad (32)$$

The exact solution (29)–(31) is given by

$$u^*(t,x) = t^2x^2, (t,x) \in [0,1]\times[0,1].$$

For $(t,x) \in [0,1]\times[0,1]$, we denote by $E(t,x)$ the absolute error at the point (t,x), that is,

$$E(t,x) = |u^*(t,x) - u(t,x)|, (t,x) \in [0,1]\times[0,1].$$

The absolute errors at different points (t,x) in the case $k = 4$ and $(\varpi,\omega) = (0,0)$ are shown in Table 1.

The absolute errors at different points (t,x) in the case $k = 4$ and $(\varpi,\omega) = (0.5,1)$are shown in Table 2. Observe that in both cases, at every pair of point (t,x), the computed approximate solution is equal to the exact solution with a negligible amount of absolute error.

Next, we fix $(\varpi,\omega,k) = (0,0,4)$, we compare our result with the exact solution at some fixed values of t, i.e. $t = 0.1, t = 0.25, t = 0.5, t = 0.75$, and display the result in Fig. 1. We repeat the same experience with $(\varpi,\omega,k) = (0,0.1,4)$. As it is shown by Fig. 2, the obtained result is satisfactory.

Now, in order to check the stability of the approximated solution, a perturbation term is introduced in the source function $g(t,x)$. More precisely, we consider problem (29)–(31) with the perturbed source $g_\epsilon(t,x)$ given by

$$g_\epsilon(t,x) = g(t,x) + \epsilon t x, (t,x) \in [0,1]\times[0,1], \quad (33)$$

Table 1
Absolute errors in the case $(\varpi,\omega,k) = (0,0,4)$.

(t,x)	$u^*(t,x)$	$u(t,x)$	$E(t,x)$
(0.25, 0.25)	0.00390625	0.00385100	0.00005525
(0.25, 0.50)	0.01562500	0.01548100	0.00014400
(0.25, 0.75)	0.03515625	0.03481800	0.00033825
(0.25, 1.00)	0.06250000	0.06174500	0.00075500
(0.50, 0.25)	0.01562500	0.01548000	0.00014500
(0.50, 0.50)	0.06250000	0.06256000	0.00006000
(0.50, 0.75)	0.14062500	0.14080000	0.00017500
(0.50, 1.00)	0.25000000	0.24943000	0.00057000
(0.75, 0.25)	0.03515625	0.03482000	0.00033625
(0.75, 0.50)	0.14062500	0.14080000	0.00017500
(0.75, 0.75)	0.31640625	0.31733000	0.00092375
(0.75, 1.00)	0.56250000	0.56302000	0.00052000
(1.00, 0.25)	0.06250000	0.06170000	0.00080000
(1.00, 0.50)	0.25000000	0.24940000	0.00060000
(1.00, 0.75)	0.56250000	0.56300000	0.00050000
(1.00, 1.00)	1.00000000	1.00110000	0.00110000

Table 2
Absolute errors in the case $(\varpi,\omega,k) = (0.5,1,4)$.

(t,x)	$u^*(t,x)$	$u(t,x)$	$E(t,x)$
(0.25, 0.25)	0.00390625	0.00422600	0.00031975
(0.25, 0.50)	0.01562500	0.01579700	0.00017200
(0.25, 0.75)	0.03515625	0.03562900	0.00047275
(0.25, 1.00)	0.06250000	0.06510300	0.00260300
(0.50, 0.25)	0.01562500	0.01580000	0.00017500
(0.50, 0.50)	0.06250000	0.06279000	0.00029000
(0.50, 0.75)	0.14062500	0.14089000	0.00026500
(0.50, 1.00)	0.25000000	0.24954000	0.00046000
(0.75, 0.25)	0.03515625	0.03563000	0.00047375
(0.75, 0.5)	0.14062500	0.14089000	0.00026500
(0.75, 0.75)	0.31640625	0.31685000	0.00044375
(0.75, 1.00)	0.56250000	0.56231000	0.00019000
(1.00, 0.25)	0.06250000	0.06510000	0.00260000
(1.00, 0.50)	0.25000000	0.24950000	0.00050000
(1.00, 0.75)	0.56250000	0.56230000	0.00020000
(1.00, 1.00)	1.00000000	1.00930000	0.00930000

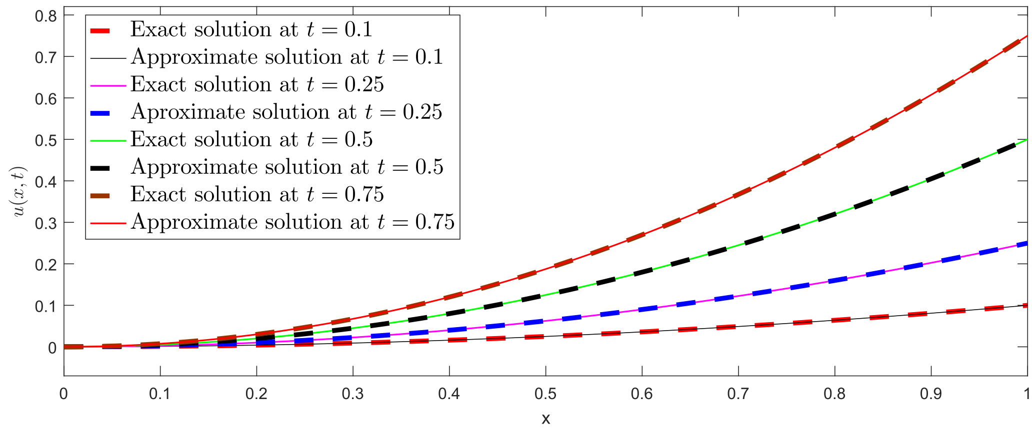

Fig. 1. Exact and approximate solutions at different values of t that is $(t = 0.1, t = 0.25, t = 0.5, t = 0.75)$ in the case $(\varpi, \omega, k) = (0; 0; 4)$.

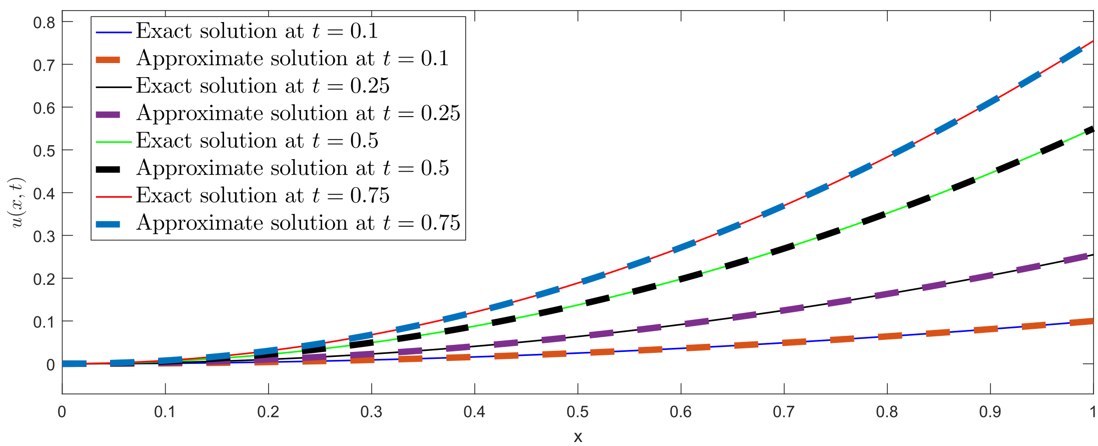

Fig. 2. Exact and approximate solutions at different values of t that is $(t = 0.1, t = 0.25, t = 0.5, t = 0.75)$ in the case $(\varpi, \omega, k) = (0; 0.1; 4)$.

where $\epsilon > 0$. We denote by u_ϵ the numerical solution of the perturbed problem. For $(t,x) \in [0,1] \times [0,1]$, we denote by $E_\epsilon(t,x)$ the absolute error at the point (t,x), that is

$$E_\epsilon(t,x) = |u^*(t,x) - u_\epsilon(t,x)|, (t,x) \in [0,1] \times [0,1],$$

where u is the approximate solution without noise (the approximate solution for $\epsilon = 0$).

The absolute errors $E_\epsilon(t,x)$ for $\epsilon = 0, 0.1$ at different points (t,x) in the case $k = 4$ and $(\varpi, \omega) = (0,0)$ are shown in Table 3. The absolute errors $E_\epsilon(t,x)$ for $\epsilon = 0, 0.05$ at different points (t,x) in the case $k = 4$ and $(\varpi, \omega) = (0,0)$ are shown in Table 4.

We observe from Tables 3 and 4 that at almost every pair of points (t,x), we have $E_\epsilon(t,x) < \epsilon$, which confirms the stability of the method with respect to a perturbation of the source data.

Graphical presentations are given in Fig. 1 for exact and approximate solutions at different values of t that is $(t = 0.1, t = 0.25, t = 0.5, t = 0.75)$ in the case $(\varpi, \omega, k) = (0; 0; 4)$. Similarly in Fig. 2, the exact and approximate solutions at different values of t that is $(t = 0.1, t = 0.25, t = 0.5, t = 0.75)$ in the case

Table 3
Absolute errors in the case $(\varpi, \omega, k, \epsilon) = (0, 0, 4, 0.01)$.

(t,x)	$u^*(t,x)$	$u_\epsilon(t,x)$	$E_\epsilon(t,x)$
(0.25, 0.25)	0.00385100	0.00390000	0.00000490
(0.25, 0.50)	0.01548100	0.01550000	0.00001900
(0.25, 0.75)	0.03481800	0.03490000	0.00008200
(0.25, 1.00)	0.06174500	0.06170000	0.00004500
(0.50, 0.25)	0.01548000	0.01550000	0.00002000
(0.50, 0.50)	0.06256000	0.06280000	0.00024000
(0.50, 0.75)	0.14080000	0.14130000	0.00050000
(0.50, 1.00)	0.24943000	0.24990000	0.00047000
(0.75, 0.25)	0.03482000	0.03490000	0.00008000
(0.75, 0.50)	0.14080000	0.14130000	0.00050000
(0.75, 0.75)	0.31733000	0.31850000	0.00117000
(0.75, 1.00)	0.56302000	0.56480000	0.00178000
(1.00, 0.25)	0.06170000	0.06170000	0.00000000
(1.00, 0.50)	0.24940000	0.24990000	0.00050000
(1.00, 0.75)	0.56300000	0.56480000	0.00180000
(1.00, 1.00)	1.00110000	1.00540000	0.00430000

Table 4
Absolute errors in the case $(\varpi, \omega, k, \epsilon) = (0, 0, 4, 0.05)$.

(t,x)	$u^*(t,x)$	$u_\epsilon(t,x)$	$E_\epsilon(t,x)$
(0.25, 0.25)	0.00385100	0.00400000	0.00014900
(0.25, 0.50)	0.01548100	0.01570000	0.00021900
(0.25, 0.75)	0.03481800	0.03500000	0.00018200
(0.25, 1.00)	0.06174500	0.06180000	0.00005500
(0.5, 0.25)	0.01548000	0.01570000	0.00022000
(0.50, 0.50)	0.06256000	0.06390000	0.00134000
(0.50, 0.75)	0.14080000	0.14320000	0.00240000
(0.50, 1.00)	0.24943000	0.25170000	0.00227000
(0.75, 0.25)	0.03482000	0.03500000	0.00018000
(0.75, 0.50)	0.14080000	0.14320000	0.00240000
(0.75, 0.75)	0.31733000	0.32310000	0.00577000
(0.75, 1.00)	0.56302000	0.57200000	0.00898000
(1.00, 0.25)	0.06170000	0.06180000	0.00010000
(1.00, 0.50)	0.24940000	0.25170000	0.00230000
(1.00, 0.75)	0.56300000	0.57200000	0.00900000
(1.00, 1.00)	1.00110000	1.02260000	0.02150000

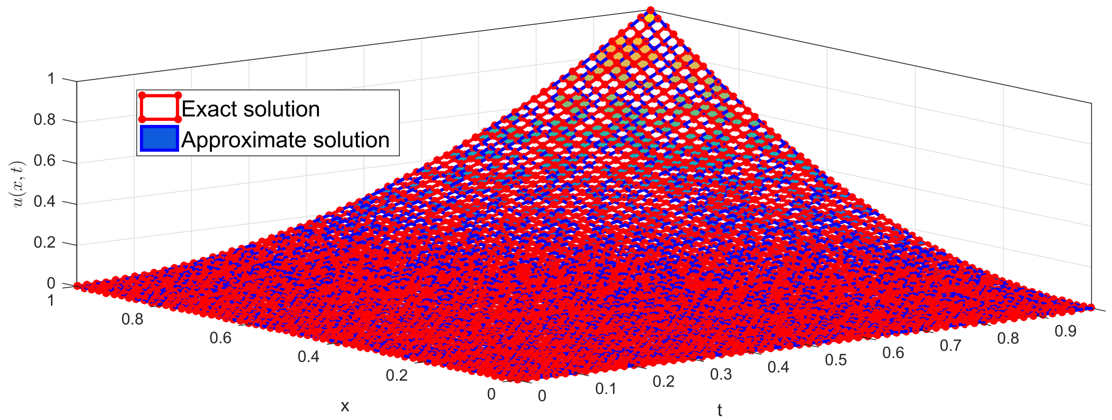

Fig. 3. Comparison between exact and approximate solutions over the square $(t,x) \in [0,1] \times [0,1]$ and taking $(\varpi, \omega, k) = (0; 0.1; 4)$.

$(\varpi, \omega, k) = (0; 0.1; 4)$. are presented. In both cases the effect of time and the parameters values have testified. At taking $(\varpi, \omega) = (0, 0)$ for parameters, we get the solution more precise as compare to $(\varpi, \omega) = (0; 0.1)$ at same scale $k = 4$. Further for more explanation, we give comparison between exact and approximate solution in Fig. 3 by using $(\varpi, \omega) = (0; 0.1)$ at same scale $k = 4$, to the given problem. We see that both surfaces coincide very well which illustrate the accuracy of the considered method.

Conclusion

The suggested method provides an easy way to solve numerically the class of fractional partial differential Eqs. (1)–(3). Using shifted Jacobi polynomial basis, the considered problem is reduced to a system of linear algebraic equations which has been solved by Matlab using Gauss elimination method for the unknown coefficient matrix which then used to obtained the required numerical solution of the considered problem. Moreover, from numerical experiments, we observed that the method is stable with respect to a perturbation of the source data. In future, the method can be easily extended to solve other types of fractional partial differential equations from physics and other fields of science.

Compliance with Ethics Requirements

Our research work does not contain any studies with human or animal subjects.

Declaration of Competing Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

We are thankful to the reviewer for their nice suggestions which improved this paper very well.

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