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Fourth order differential operators with distributional potentials

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Abstract: In this paper, regular and singular fourth order differential operators with distributional potentials are investigated. In particular, existence and uniqueness of solutions of the fourth order differential equations are proved, deficiency indices theory of the corresponding minimal symmetric operators are studied. These symmetric operators are considered as acting on the single and direct sum Hilbert spaces. The latter one consists of three Hilbert spaces such that a squarely integrable space and two spaces of complex numbers. Moreover all maximal self-adjoint, maximal dissipative and maximal accumulative extensions of the minimal symmetric operators including direct sum operators are given in the single and direct sum Hilbert spaces.

Key words: Distributional potentials, deficiency indices, extension theory, direct sum operator

1. Introduction

Weyl theory is an important tool to understand the nature of the solutions of an differential equation on an unbounded domain. In 1910, Weyl proved that following second order differential equation

$$-(py^{(1)})^{(1)} + qy = \lambda y, \quad x \in [0, \infty), \tag{1.1}$$

where p, q are real-valued and p^{-1}, q are locally integrable functions on $[0, \infty)$, has at least one solution that must be squarely integrable on $[0, \infty)$ [20]. Beside this, two linearly indpendent solutions of (1.1) and any combinations of them may be squarely integrable. These results are based on the nested property of the corresponding circles which are related with the regular boundary conditions. Indeed, if these circles converge to a circle at the singular point, then the equation (1.1) is said to be in limit-circle case. Otherwise, i.e., if these circles converge to a point, then the equation (1.1) is said to be in limit-point case. This classification has an alternative. Namely, one can find the number of the linearly independent solutions of (1.1) belonging to the squarely integrable space with the help of the deficiency indices of the corresponding minimal symmetric operator. Recall that the numbers (m, n) defined as

$$m = \dim N_i, \quad n = \dim N_{-i},$$

where

$$N_\lambda = H \ominus (L_0 - \lambda I), \quad N_{\overline{\lambda}} = H \ominus (L_0 - \overline{\lambda} I),$$

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are called the deficiency indices of the minimal symmetric operator L_0 acting on a Hilbert space H [17]. For a second order differential operator, $(1,1)$ is known as limit-point case and $(2,2)$ is known as limit-circle case.

In 2013, Eckhardt *et al.* [3] investigated the number of the squarely integrable solutions of the equation

$$-\left[p\left(y^{(1)}+sy\right)\right]^{(1)}+sp\left(y^{(1)}+sy\right)+qy=\lambda ry, \quad (a,b)\subseteq\mathbb{R}, \tag{1.2}$$

where p,q,r and s are real-valued, Lebesgue measurable functions on (a,b), p^{-1},q,r and s are locally integrable functions on (a,b), $r>0$ for almost all $x\in(a,b)$ and $y,p(y^{(1)}+sy)$ are locally absolutely continuous functions on (a,b). Such an investigation has been done with the aid of the deficiency indices theory. Clearly, for $s\equiv 0$ on (a,b) the differential equation (1.2) takes the form

$$-\left(py^{(1)}\right)^{(1)}+qy=\lambda ry, \quad (a,b)\subseteq\mathbb{R}, \tag{1.3}$$

which is the well-known second order Sturm-Liouville equation.

The equations being of the form (1.2) are called second order differential equation with distributional potentials. The readers may find some papers that are related with the differential equations with distributional potentials in [1], [5], [18], [19]. However, they are defined on the compact intervals.

In 2012, Maozhu *et al* investigated the number of the squarely integrable solutions of the equation

$$\begin{pmatrix} -\left(py^{(1)}\right)^{(1)}+qy \\ -\beta_1 y(a)+\beta_2(py^{(1)})(a) \end{pmatrix}=\lambda\begin{pmatrix} ry \\ \alpha_1 y(a)-\alpha_2(py^{(1)})(a) \end{pmatrix}, \tag{1.4}$$

where a is the regular point and b is the singular point for the equation (1.3) and $\alpha_1,\alpha_2,\beta_1,\beta_2$ are real numbers [16]. Moreover, they characterized self-adjoint extensions of the minimal symmetric differential operator generated by (1.4).

In this paper, we investigate the number of the squarely integrable solutions of the following fourth order differential equation

$$\begin{gathered}\left\{\left[q_2\left(y^{(2)}-s_1y^{(1)}+s_2y\right)\right]^{(1)}+q_2s_1\left(y^{(2)}-s_1y^{(1)}\right)-q_1(y^{(1)}+s_4y)+s_3y\right\}^{(1)} \\ +q_2s_2y^{(2)}-s_3y^{(1)}+q_1s_4\left(y^{(1)}+s_4y\right)+q_0y=\lambda wy\end{gathered} \tag{1.5}$$

defined on the interval $(a,b)\subseteq\mathbb{R}$. For $s_j\equiv 0,\ 1\leq j\leq 4$, equation (1.5) is reduced to

$$\left((q_2y^{(2)})^{(1)}-(q_1y^{(1)})\right)^{(1)}+q_0y=\lambda wy, \quad x\in[0,\infty) \tag{1.6}$$

or

$$(q_2y^{(2)})^{(2)}-(q_1y^{(1)})^{(1)}+q_0y=\lambda wy, \quad x\in[0,\infty) \tag{1.7}$$

provided that the first sum in (1.6) is differentiable. Clearly, (1.7) is the well-known fourth order Sturm-Liouville equation and it should be noted that the number of the squarely integrable solutions of (1.7) with $w\equiv 1$ was investigated by Everitt in 1963 [4]. However, this investigation was done with the help of the nested property of the corresponding surfaces. In this paper, we investigate the number of the squarerly integrable solutions of (1.5) with deficiency indices theory. Note that, such an investigation was done by Naimark for the equation (1.7) with $w \equiv 1$.

This paper is organized as follows. In section 2, we introduce the fourth order differential expression with distributional potentials and some results on the solutions or the fourth order equation. In section 3, regular and singular minimal symmetric differential operators are introduced and some results are given without proof because the methods are similar with the proofs of Naimark. In section 4, some expansions are introduced. In section 5, deficiency indices of the fourth order operators are investigated. Moreover some additional theorems are proved. The main results are given in sections 6-8. In particular, in section 6, direct sum Hilbert spaces and associated minimal and maximal operators are introduced and some results are given. In section 7, deficiency indices of the direct sum operators are investigated. In section 8, maximal self-adjoint, maximal dissipative and maximal accumulative extensions of the minimal operators defined on the single and direct sum Hilbert spaces are studied.

2. Basic results

Let us consider the differential expression

$$\begin{aligned}\tau[y] = \frac{1}{w}\Bigg\{ &\Big\{ \left[q_2\left(y^{(2)} - s_1 y^{(1)} + s_2 y\right)\right]^{(1)} + q_2 s_1\left(y^{(2)} - s_1 y^{(1)}\right) - \left[q_1(y^{(1)} + s_4 y)\right] + s_3 y\Big\}^{(1)} \\ &+ q_2 s_2 y^{(2)} - s_3 y^{(1)} + q_1 s_4\left(y^{(1)} + s_4 y\right) + q_0 y\Bigg\},\end{aligned}$$

on the interval $(a,b) \subseteq \mathbb{R}$. Throughout the paper, we assume that $q_0, q_1, q_2, s_1, ..., s_4$ are real-valued functions on (a,b), $q_0, q_1, q_2^{-1}, s_1, s_2, s_3, q_1 s_4, q_1 s_4^2, q_2 s_1 s_2, q_2 s_2^2$ and w are locally integrable functions on (a,b). Note that for $s_j \equiv 0,\ 1 \le j \le 4$, these are the ordinary assumptions on coefficients.

Now let us adopt the notations

$$\begin{aligned}&y^{[0]} = y, \\ &y^{[1]} = y^{(1)}, \\ &y^{[2]} = q_2\left(y^{(2)} - s_1 y^{(1)} + s_2 y\right), \\ &y^{[3]} = -\Big[q_2\left(y^{(2)} - s_1 y^{(1)} + s_2 y\right)\Big]^{(1)} - q_2 s_1\left(y^{(2)} - s_1 y^{(1)}\right) + q_1(y^{(1)} + s_4 y) - s_3 y.\end{aligned}$$

We shall call $y^{[r]}$ as the $r-$th *quasi-derivative* of y. Moreover we assume that $y^{[r]}$, $0 \le r \le 3$, are locally absolutely continuous functions on (a,b), i.e., $y^{[r]} \in AC_{loc}(a,b)$.

Consider the following set

$$D = \left\{ y \in AC_{loc}(a,b) \quad : \quad y^{[1]}, y^{[2]}, y^{[3]} \in AC_{loc}(a,b) \right\}.$$

Then we obtain the following theorem.

Theorem 2.1 *Let y belong to D, f be a measurable function on (a,b) and wf be locally integrable on (a,b). Then the equation*

$$\tau[y] - \lambda y = f \tag{2.1}$$

has one and only one solution $y(x,\lambda)$ *satisfying the conditions*

$$y^{[r]}(x_0,\lambda)=\alpha_r, \quad 0\le r\le 3, \tag{2.2}$$

where λ *is a complex number,* $x_0\in(a,b)$ *and* α_r *are arbitrary complex numbers.*

Proof Equation (2.1) with the initial conditions (2.2) can be handled as the following first order system

$$Y^{(1)}(x,\lambda)=A(x,\lambda)Y(x,\lambda)+F(x)$$

with

$$Y(x_0,\lambda)=a,$$

where

$$Y=\begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[3]} \end{pmatrix}, \qquad F=\begin{pmatrix} 0 \\ 0 \\ 0 \\ wf \end{pmatrix}, \qquad \alpha=\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix},$$

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -s_2 & s_1 & q_2^{-1} & 0 \\ q_1s_4-s_3+q_2s_1s_2 & q_1 & -s_1 & -1 \\ q_1s_4^2+q_0-q_2s_2^2-\lambda w & q_1s_4-s_3+q_2s_1s_2 & s_2 & 0 \end{pmatrix}.$$

Note that the elements of the matrix $A(x,\lambda)$ are measurable on the interval (a,b) and moreover $\|A(x,\lambda)\|$ and $\|F(x)\|$ are locally integrable on (a,b). Hence using the method of successive approximations we complete the proof. □

Definition 2.2 *A linearly independent system of solutions* $y_1,...,y_4$ *of the equation*

$$\tau[y]=\lambda y, \quad x\in(a,b)\subseteq\mathbb{R}, \tag{2.3}$$

is called a fundamental system.

Let us define the Wronskian of the set of functions $\{\psi_1(x),...,\psi_4(x)\}$ as follows

$$W_x[\psi_1,...,\psi_4]:=\det\begin{bmatrix} \psi_1^{[0]}(x) & \cdots & \psi_4^{[0]}(x) \\ \vdots & & \vdots \\ \psi_1^{[3]}(x) & \cdots & \psi_4^{[3]}(x) \end{bmatrix}.$$

Then following results follow from the results of [17].

Theorem 2.3 *(i) Let $y_1(x,\lambda),...,y_1(x,\lambda)$ be the solutions of (2.3). If $y_1(x,\lambda)$, $...,y_4(x,\lambda)$ are linearly dependent, then $W_x[y_1,...,y_4]$ vanishes identically in (a,b). Conversely, if $W_{x_0}[y_1,...,y_4]=0$ at a point $x_0\in(a,b)$, then $y_1(x,\lambda),...,y_4(x,\lambda)$ are linearly dependent;*
(ii) An arbitrary solution $\varphi(x,\lambda)$ of (2.3) is a linear combination of a fixed fundamental system;
(iii) The solutions of (2.3) form a linear space of dimension 4.

We denote by $[y,\chi]$ the *Lagrange form* of the functions y and χ defined as

$$[y,\chi]=y^{[0]}\chi^{[3]}-y^{[3]}\chi^{[0]}+y^{[1]}\chi^{[2]}-y^{[2]}\chi^{[1]}. \tag{2.4}$$

Let $L^2_w(a,b)$ be the Hilbert space consisting of all functions y satisfying

$$\int_a^b |y|^2\, wdx<\infty$$

with the inner product

$$(y,\chi)=\int_a^b y\overline{\chi}wdx.$$

Consider the set

$$\mathcal{D}=\Big\{y\in L^2_w(a,b)\quad:\quad y\in D,\tau[y]\in L^2_w(a,b)\Big\}.$$

For arbitrary two functions $y,\chi\in\mathcal{D}$ we have the following Green's formula

$$\int_a^b\Big\{\tau[y]\chi-y\tau[\chi]\Big\}wdx=[y,\chi](b)-[y,\chi](a).$$

Green's formula implies the fact that if $y(x,\lambda)$ and $\chi(x,\lambda)$ are the solutions of (2.3) corresponding to the same value of λ, then $[y,\chi]$ is independent of x and depends only on λ on (a,b). Moreover, in the case that one of the end points is singular or both of them are singular for τ, then the values $[y,\chi](a)$, $[y,\chi](b)$ and $[y,\overline{\chi}](a)$, $[y,\overline{\chi}](b)$ exist and are finite. Secondary values also follow from the Green's formula. In fact, it is sufficient to get the second factors with their complex conjugates.

3. Minimal and maximal differential operators

3.1. Regular case

At first we assume that a and b are regular points for τ. Then we have the following result [17].

Theorem 3.1 *Assume that a and b are regular points for τ and $f\in L^2_w(a,b)$. Then followings are equivalent:*
(i) y is a solution of $\tau[y]=f$ satisfying $y^{[r]}(a)=y^{[r]}(b)=0,\ 0\le r\le 3$.
(ii) f is orthogonal to all solutions of the equation $\tau[y]=0$.

Consider the following set

$$\mathcal{D}_0=\Big\{y\in\mathcal{D}:y^{[r]}(a)=y^{[r]}(b)=0\Big\},$$

where $0 \leq r \leq 3$. We define the operator T_0 the restriction of the operator T to the set $\mathcal{D}_0$, where $T_y = \tau[y]$, $y \in \mathcal{D}$.

In the case that a and b are regular points for τ, all solutions of the equation $\tau[y] = 0$ belong to $L^2_w(a,b)$. Therefore the set S consisting of all solutions of the equation $\tau[y] = 0$ is a subset of $L^2_w(a,b)$. From Theorem 2.3 (iii) we get that S is of dimension 4. Let R_0 denote the range of the operator T_0. A solution y of $\tau[y] = f$ satisfying $y^{[r]}(a) = y^{[r]}(b) = 0,\ 0 \leq r \leq 3$, is an element of $\mathcal{D}_0$. Therefore the existence of such a solution y implies that $f \in R_0$. Theorem 3.1 implies that $f \in L^2_w(a,b)$ lies in R_0 if and only if it is orthogonal to S.

Following results can be obtained using the method of Naimark [17].

Theorem 3.2 (i) *There is a function $y \in \mathcal{D}$ satisfying*

$$y^{[r]}(a) = c_r, \quad y^{[r]}(b) = d_r, \quad 0 \leq r \leq 3, \quad c_r, d_r \in \mathbb{C};$$

(ii) $\mathcal{D}_0$ *is dense in* $L^2_w(a,b)$;
(iii) $T = T_0^*$;
(iv) $T_0 = T^*$.

3.2. Singular case

Now we consider that a and b are singular points for τ. Let

$$\mathcal{B}_0' = \Big\{ y \in \mathcal{D} \quad : \quad y \text{ has compact support in } (a,b) \Big\}.$$

We denote by N_0' the restriction of T to the set $\mathcal{B}_0'$. One has the following results [17].

Theorem 3.3 (i) N_0' *is Hermitian;*
(ii) $\mathcal{B}_0'$ *is dense in* $L^2_w(a,b)$.

Theorem 3.3 (ii) implies that N_0' admits a closure. We denote it by N_0 with domain $\mathcal{B}_0$. Together with Theorem 3.3 (i) we get that N_0 is a closed, symmetric operator. Then we have the following [17].

Theorem 3.4 (i) $N_0^* = T$;
(ii) $\mathcal{B}_0 = \Big\{ y \in \mathcal{D} \quad : \quad [y, \overline{\chi}](b) - [y, \overline{\chi}](a) = 0, \quad \chi \in \mathcal{D} \Big\}$.

Now let us consider that the left end point a is regular and right end point b is singular for τ. Let

$$\mathcal{C}_0' = \Big\{ y \in \mathcal{D} \quad : \quad y \text{ has compact support in } (a,b) \Big\}.$$

We denote by M_0' the restriction of T to $\mathcal{C}_0'$. Then we obtain the following theorem [17].

Theorem 3.5 (i) M_0' *is Hermitian,*
(ii) $\mathcal{C}_0'$ *is dense in* $L^2_w(a,b)$.

We denote by M_0 the restriction of T to $\mathcal{C}_0$ the closure of the operator M_0'. Then following results are obtained.

Theorem 3.6 (i) $M_0^* = T$;

(ii) $\mathcal{C}_0 = \Big\{ y \in \mathcal{D} \quad : \quad [y,\overline{\chi}](b) - [y,\overline{\chi}](a) = 0, \quad \chi \in \mathcal{D} \Big\}$;

(iii) $\mathcal{C}_0 = \Big\{ y \in \mathcal{D} \quad : \quad y^{[r]}(a) = 0, \quad [y,\overline{\chi}](b) = 0, \quad \chi \in \mathcal{D} \Big\}, \qquad 0 \le r \le 3.$

If one chooses as a is singular point and b is regular point for τ, then following results are obtained for the corresponding operators M_0' and M_0.

Corollary 3.7 (i) $M_0^* = T$;

(ii) $\mathcal{C}_0 = \Big\{ y \in \mathcal{D} \quad : \quad [y,\overline{\chi}](b) - [y,\overline{\chi}](a) = 0, \quad \chi \in \mathcal{D} \Big\}$;

(iii) $\mathcal{C}_0 = \Big\{ y \in \mathcal{D} \quad : \quad [y,\overline{\chi}](a) = 0, \quad y^{[r]}(b) = 0, \quad \chi \in \mathcal{D} \Big\}, \qquad 0 \le r \le 3.$

4. Some identities

Following theorem describes the Wronskian in terms of the Lagrange forms. In fact, such an expansion for the ordinary Wronskian of solutions of arbitrary even order Sturm-Liouville equations was given by Kodaira [15]. However, it seems that there is not a proof. For the following theorem we give a proof.

Theorem 4.1 *Let $\varphi_r(x) \in D$, $1 \le r \le 4$. Then for $x \in (a,b)$*

$$W_x[\varphi_1,\varphi_2,\varphi_3,\varphi_4] = [\varphi_1,\varphi_2](x)[\varphi_3,\varphi_4](x) - [\varphi_1,\varphi_3](x)[\varphi_2,\varphi_4](x) + [\varphi_1,\varphi_4](x)[\varphi_2,\varphi_3](x). \tag{4.1}$$

Proof Let $\varphi_r \in D$, $1 \le r \le 4$. Then for $x \in (a,b)$ we get

$$\begin{aligned}
&\varphi_1^{[0]}\Big[\varphi_2^{[1]}\Big(\varphi_3^{[2]}\varphi_4^{[3]} - \varphi_3^{[3]}\varphi_4^{[2]}\Big) - \varphi_3^{[1]}\Big(\varphi_2^{[2]}\varphi_4^{[3]} - \varphi_2^{[3]}\varphi_4^{[2]}\Big) + \varphi_4^{[1]}\Big(\varphi_2^{[2]}\varphi_3^{[3]} - \varphi_2^{[3]}\varphi_3^{[2]}\Big)\Big] \\
&-\varphi_2^{[0]}\Big[\varphi_1^{[1]}\Big(\varphi_3^{[2]}\varphi_4^{[3]} - \varphi_3^{[3]}\varphi_4^{[2]}\Big) - \varphi_3^{[1]}\Big(\varphi_1^{[2]}\varphi_4^{[3]} - \varphi_1^{[3]}\varphi_4^{[2]}\Big) + \varphi_4^{[1]}\Big(\varphi_1^{[2]}\varphi_3^{[3]} - \varphi_1^{[3]}\varphi_3^{[2]}\Big)\Big] \\
&+\varphi_3^{[0]}\Big[\varphi_1^{[1]}\Big(\varphi_2^{[2]}\varphi_4^{[3]} - \varphi_2^{[3]}\varphi_4^{[2]}\Big) - \varphi_2^{[1]}\Big(\varphi_1^{[2]}\varphi_4^{[3]} - \varphi_1^{[3]}\varphi_4^{[2]}\Big) + \varphi_4^{[1]}\Big(\varphi_1^{[2]}\varphi_2^{[3]} - \varphi_1^{[3]}\varphi_2^{[2]}\Big)\Big] \\
&-\varphi_4^{[0]}\Big[\varphi_1^{[1]}\Big(\varphi_2^{[2]}\varphi_3^{[3]} - \varphi_2^{[3]}\varphi_3^{[2]}\Big) - \varphi_2^{[1]}\Big(\varphi_1^{[2]}\varphi_3^{[3]} - \varphi_1^{[3]}\varphi_3^{[2]}\Big) + \varphi_3^{[1]}\Big(\varphi_1^{[2]}\varphi_2^{[3]} - \varphi_1^{[3]}\varphi_2^{[2]}\Big)\Big].
\end{aligned} \tag{4.2}$$

On the other side, (4.2) is the Wronskian of $\{\varphi_1,...,\varphi_4\}$. Therefore the proof is completed. $\square$

Theorem 4.1 gives the following corollary.

Corollary 4.2 *Let $\varphi_r(x,\lambda)$, $r = \overline{1,4}$, be the solutions of (2.3). Then the Wronskian $W_x[\varphi_1,...,\varphi_4]$ is independent of x and depends only on λ.*

Let us consider the solutions $u_r(x)\,,\ 1 \le r \le 4,$ of the equation $\tau[y]=0,\ x\in(a,b),$ satisfying

$$\begin{pmatrix} u_1^{[0]}(c) & u_2^{[0]}(c) & u_3^{[0]}(c) & u_4^{[0]}(c) \\ u_1^{[1]}(c) & u_2^{[1]}(c) & u_3^{[1]}(c) & u_4^{[1]}(c) \\ u_1^{[2]}(c) & u_2^{[2]}(c) & u_3^{[2]}(c) & u_4^{[2]}(c) \\ u_1^{[3]}(c) & u_2^{[3]}(c) & u_3^{[3]}(c) & u_4^{[3]}(c) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.3}$$

where c$\in(a,b)$. Then one can immediately get that

$$[u_1,u_4]=[u_2,u_3]=1, \quad [u_r,u_r]=0, \quad 1\le r\le 4,$$
$$[u_1,u_2]=[u_1,u_3]=[u_2,u_4]=[u_3,u_4]=0.$$

Following Fulton's idea [6] let us associate the function $y\in\mathcal{D}$ with Y as follows

$$y \leftrightarrow Y = \begin{pmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{pmatrix}.$$

Note that $[y,z]=Z^tJY,$ where

$$J = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Consider the following association

$$u_r \leftrightarrow U_r, \quad 1\le r\le 4,$$

and let us construct the matrix U_0 as follows

$$U_0 = (U_1,U_2,U_3,U_4)\,.$$

A direct calculation shows that

$$U_0^t(x)JU_0(x)=J.$$

Let us define the transformation

$$SY=U_0^{-1}Y.$$

Since $U_0.SY = Y,$ we obtain from Cramer's rule that

$$SY = \begin{pmatrix} W_x[y, u_2, u_3, u_4] \\ W_x[u_1, y, u_3, u_4] \\ W_x[u_1, u_2, y, u_4] \\ W_x[u_1, u_2, u_3, y] \end{pmatrix}.$$

Using (4.1) and (4.3) we obtain the transformation SY in the form

$$SY = \begin{pmatrix} [y, u_4](x) \\ [y, u_3](x) \\ -[y, u_2](x) \\ -[y, u_1](x) \end{pmatrix}.$$

Finally, with the aid of the equation [6]

$$(SZ)^t J(SZ) = Z^t JY$$

we get the following theorem.

Theorem 4.3 *For $y, \chi \in \mathcal{D}$ we have*

$$[y, \chi] = [y, u_1][\chi, u_3] - [y, u_3][\chi, u_1] + [y, u_2][\chi, u_4] - [y, u_4][\chi, u_2]. \tag{4.4}$$

5. Deficiency indices of the fourth order operators

In this section we investigate the deficiency indices of the minimal symmetric operators using the idea of Naimark [17].

Theorem 5.1 (i) *Let a and b be regular points for τ. Then T_0 is a closed, symmetric operator with deficiency indices* $(4, 4)$;
(ii) *Let a and b be singular points for τ. Then N_0 is a closed, symmetric operator and has the deficiency indices of the form (n, n), where $0 \le n \le 4$;*
(iii) *Let a be regular point and b be singular point for τ. Then M_0 is a closed, symmetric operator and has the deficiency indices of the form (n, n), where $2 \le n \le 4$;*
(iv) *Let b be regular point and a be singular point for τ. Then M_0 is a closed, symmetric operator and has the deficiency indices of the form (n, n), where $2 \le n \le 4$.*

In the case that the deficiency indices of M_0 are $(4, 4)$, the set $\mathcal{C}_0$ can be described with the help of the real solutions of $\tau[y] = 0$ satisfying (4.3) as follows.

Theorem 5.2 *Let a be regular point and b be singular point for τ, the deficiency indices of M_0 be $(4, 4)$ and $u_r(x)$, $1 \le r \le 4$, be the solutions of $\tau[y] = 0$ satisfying (4.3). Then $\mathcal{C}_0$ can be described as follows*

$$\mathcal{C}_0 = \Big\{ y \in \mathcal{D} \quad : \quad y^{[r-1]}(a) = 0, \quad [y, u_r](b) = 0 \Big\}.$$

Proof Since the deficiency indices of N_0 are $(4,4)$, u_r belong to $L^2_w(a,b)$ and $\mathcal{D}$. Using (4.4), we obtain that

$$[y,\overline{\chi}](b)=[y,u_1](b)[\overline{\chi},u_3](b)-[y,u_3](b)[\overline{\chi},u_1](b)+[y,u_2](b)[\overline{\chi},u_4](b)-[y,u_4](b)[\overline{\chi},u_2](b)=0.$$

Since $[\overline{\chi},u_r](b)$, $1\le r\le 4$, can be chosen as arbitrarily, the proof follows from Theorem 3.6 (iii). □

Corollary 5.3 *Let a be singular point and b be regular point for τ, the deficiency indices of M_0 be $(4,4)$ and $u_r(x)$, $1\le r\le 4$, be the solutions of $\tau[y]=0$ satisfying (4.3). Then $\mathcal{C}_0$ can be described as follows*

$$\mathcal{C}_0=\Big\{y\in\mathcal{D}\quad:\quad [y,u_r](a)=0,\quad y^{[r-1]}(b)=0,\quad r=\overline{1,4}\Big\}.$$

Theorem 5.4 *Let the deficiency indices of N_0 be $(4,4)$ and let $u_r(x)$, $1\le r\le 4$, be the solutions of $\tau[y]=0$ satisfying (4.3). Then $\mathcal{B}_0$ can be described as follows*

$$\mathcal{B}_0=\Big\{y\in\mathcal{D}\quad:\quad [y,u_r](a)=[y,u_r](b)=0,\quad r=\overline{1,4}\Big\}.$$

Proof From Theorem 3.4 (ii), $\mathcal{B}_0$ consists of those functions y such that

$$[y,\overline{\chi}](b)-[y,\overline{\chi}](a)=0 \tag{5.1}$$

for arbitrary $\chi\in\mathcal{D}$. Since the deficiency indices of N_0 are $(4,4)$, u_r belong to $L^2_w(a,b)$ and $\mathcal{D}$. Therefore using (4.4), (5.1) can be written as

$$\begin{aligned}[y,\overline{\chi}](b)-[y,\overline{\chi}](a)&=[y,u_1](b)[\overline{\chi},u_3](b)-[y,u_3](b)[\overline{\chi},u_1](b)+[y,u_2](b)[\overline{\chi},u_4](b)\\&-[y,u_4](b)[\overline{\chi},u_2](b)-[y,u_1](a)[\overline{\chi},u_3](a)+[y,u_3](a)[\overline{\chi},u_1](a)\\&-[y,u_2](a)[\overline{\chi},u_4](a)+[y,u_4](a)[\overline{\chi},u_2](a)=0.\end{aligned} \tag{5.2}$$

Since $[\overline{\chi},u_r](b)$ and $[\overline{\chi},u_r](a)$ can be chosen arbitrarily, (5.2) is satisfied only if $[y,u_r](b)=[y,u_r](a)=0$. Therefore the proof is completed. □

In the case that the deficiency indices of M_0 are $(2,2)$, following theorem is obtained.

Theorem 5.5 (i) *Let a be regular point and b be singular point for τ and let the deficiency indices of M_0 be $(2,2)$. Then for $y,\chi\in\mathcal{D}$, $[y,\overline{\chi}](b)=0$;*
(ii) *Let a be singular point and b be regular point for τ and let the deficiency indices of M_0 be $(2,2)$. Then for $y,\chi\in\mathcal{D}$, $[y,\overline{\chi}](a)=0$.*

Now consider that a and b are singular points for τ and let ξ be any number in (a,b). In this case the operator N_0 can be handled as the direct sum of the operators M_0^- and M_0^+ generated by τ in the intervals (a,ξ) and (ξ,b), respectively. Following theorem is obtained [17].

Theorem 5.6 *Let n,n_- and n_+ be the deficiency indices of N_0,M_0^- and M_0^+, respectively. Then*

$$n=n_-+n_+-4.$$

We shall give some results associated with the deficiency indices $(4,4)$ of the corresponding operators. Note that we construct the real solutions $u_r,\ 1 \le r \le 4,$ defined on the corresponding intervals satisfying (4.3).

Theorem 5.7 *Let a be regular point and b be singular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then for $y \in \mathcal{D}$ there exists complex numbers $c_r, d_r,\ 1 \le r \le 4,$ such that*

$$y^{[r-1]}(a) = c_r, \quad [y, u_r](b) = d_r.$$

Proof Let $f \in L^2_w(a,b)$ and consider the equation $\tau[y] = f,$ where $y \in \mathcal{D}$ satisfying

$$y^{[r-1]}(a) = c_r, \quad r = \overline{1,4}, \tag{5.3}$$

and c_r are arbitrary constants. Moreover assume that following equalities hold

$$\begin{aligned} (f, u_1) &= d_1 + c_4, \quad (f, u_2) = d_2 + c_3, \\ (f, u_3) &= d_3 - c_2, \quad (f, u_4) = d_4 - c_1, \end{aligned} \tag{5.4}$$

where d_k are complex numbers. Note that u_r belong to $L^2_w(a,b)$ and $\mathcal{D}$ since the deficiency indices of M_0 are $(4,4)$. Since $\tau[u_r] = 0,\ 1 \le r \le 4,$ we have

$$(f, u_r) = (\tau[y], u_r) = [y, u_r](b) - [y, u_r](a).$$

Since a is regular point for τ we have

$$[y, u_1](a) = -c_4, \quad [y, u_2](a) = -c_3, \quad [y, u_3](a) = c_2, \quad [y, u_1](a) = c_1.$$

Therefore

$$\begin{aligned} (f, u_1) &= [y, u_1](b) + c_4, \quad (f, u_2) = [y, u_2](b) + c_3, \\ (f, u_3) &= [y, u_3](b) - c_2, \quad (f, u_4) = [y, u_4](b) - c_1. \end{aligned} \tag{5.5}$$

Comparing (5.4) and (5.5) we have

$$[y, u_1](b) = d_1, \quad [y, u_2](b) = d_2, \quad [y, u_3](b) = d_3, \quad [y, u_4](b) = d_4. \tag{5.6}$$

Therefore there exits a function $y \in \mathcal{D}$ satisfying (5.3) and (5.6). This completes the proof. □

Corollary 5.8 *Let a be singular point and b be regular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then for $y \in \mathcal{D}$ there exists complex numbers $c_r, d_r,\ 1 \le r \le 4,$ such that*

$$[y, u_r](a) = c_r, \quad y^{[r-1]}(b) = d_r.$$

Theorem 5.9 *Let a and b be singular points for τ and let the deficiency indices of N_0 be $(4,4)$. Then for $y \in \mathcal{D}$ there exists complex numbers $c_r, d_r,\ 1 \le r \le 4,$ such that*

$$[y, u_r](a) = c_r, \quad [y, u_r](b) = d_r.$$

Proof It is better to note that u_r belong to $L^2_w(a,b)$ and $\mathcal{D}$ since the deficiency indices of N_0 are $(4,4)$. Let $\zeta \in (a,b)$ and τ^- (τ^+) be the restriction of τ to the interval $(a,\zeta]$ $([\zeta,b))$. Then from Corollary 5.8 there exists a function $y^- \in \mathcal{D}^-$, where

$$\mathcal{D}^- = \left\{ y^- \in L^2_w(a,\zeta) \quad : \quad y^{-^{[k]}} \in AC_{loc}(a,\zeta), \quad \tau^-[y^-] \in L^2_w(a,\zeta) \right\},$$

$1 \le k \le 3,$ satisfying

$$[y^-,u_r](a) = c_r, \quad y^{[r-1]}(\zeta) = e_r.$$

Here c_r and e_r are complex numbers. Similarly there exists a function $y^+ \in \mathcal{D}^+$, where

$$\mathcal{D}^+ = \left\{ y^+ \in L^2_w(\zeta,b) \quad : \quad y^{+^{[k]}} \in AC_{loc}(\zeta,b), \quad \tau^+[y^+] \in L^2_w(\zeta,b) \right\},$$

$1 \le k \le 3,$ satisfying

$$y^{[r-1]}(\zeta) = e_r, \quad [y^+,u_r](b) = d_r.$$

Here e_r and d_r are complex numbers. Let

$$y(x) = \begin{cases} y^-(x), & x \in (a,\zeta], \\ y^+(x), & x \in [\zeta,b). \end{cases}$$

Since y is continuous at $\zeta,$ we obtain that there exists a function $y \in \mathcal{D}$ satisfying

$$[y,u_r](a) = c_r, \quad [y,u_r](b) = d_r.$$

□

6. Direct sum Hilbert spaces

In this section, our main aim is to decribe the corresponding direct sum operators defined on the vectors. These operators are useful when studying eigenparameter dependent boundary value problems.

Denote by $H = L^2_w(a,b) \oplus \mathbb{C} \oplus \mathbb{C}$ being the Hilbert space with the inner product

$$\langle \mathcal{Y}, \mathcal{Z} \rangle = \int_a^b y\overline{z}w dx + \frac{1}{\alpha} y_1 \overline{z}_1 + \frac{1}{\beta} y_2 \overline{z}_2,$$

where $(a,b) \subseteq \mathbb{R},$ $\alpha, \beta > 0$ and

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}, \quad \mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in H.$$

It can be considered that the points a and b are regular or a or b or both of them are singular for $\tau.$ Hence we shall investigate for these four cases.

6.1. Regular end points a and b for τ

Let a and b be regular points for τ. Consider the following set

$$\boldsymbol{D} = \left\{ \mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in H \quad : \quad y \in \mathcal{D}, \qquad \begin{array}{c} y_1 = \alpha_1 y^{[0]}(a) - \alpha_2 y^{[3]}(a) \\ \\ y_2 = \beta_1 y^{[1]}(a) - \beta_2 y^{[2]}(a) \end{array} \right\},$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers.

We define the operator $\mathbf{T}$ on $\boldsymbol{D}$ as

$$\mathbf{T} \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Ty \\ y_1' \\ y_2' \end{pmatrix},$$

where $Ty = \tau[y]$, $y \in \mathcal{D}$, $x \in (a,b)$, $y_1' = \alpha_1' y^{[0]}(a) - \alpha_2' y^{[3]}(a)$, $y_2' = \beta_1' y^{[1]}(a) - \beta_2' y^{[2]}(a)$, $\alpha_1', \alpha_2', \beta_1', \beta_2'$ are real numbers satisfying

$$\alpha := \alpha_1 \alpha_2' - \alpha_1' \alpha_2 > 0,$$

$$\beta := \beta_1 \beta_2' - \beta_1' \beta_2 > 0.$$

Let

$$\boldsymbol{D}_0 = \left\{ \mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{D} \quad : \quad y^{[r]}(b) = 0, \quad r = \overline{0,3} \right\}$$

and $\mathbf{T}_0$ be the operator the restriction of $\mathbf{T}$ to $\boldsymbol{D}_0$.

Theorem 6.1 *$\boldsymbol{D}_0$ is dense in H.*

Proof The domain $\mathcal{D}_0$ of the operator T_0 consists of those functions $y \in \mathcal{D}$ satisfying $y^{[r]}(a) = y^{[r]}(b) = 0$, $0 \le r \le 3$. Therefore for $y \in \mathcal{D}_0$ we get that $y_1 = y_2 = 0$ and

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{D}_0.$$

Hence for

$$\mathcal{G} = \begin{pmatrix} g \\ g_1 \\ g_2 \end{pmatrix} \in H,$$

which is orthogonal to $\boldsymbol{D}_0$ we have

$$\langle \mathcal{Y}, \mathcal{G} \rangle = \int_a^b y \bar{g} w dx = 0,$$

and this implies that $g = 0$ a.e. on (a,b). Thus

$$\mathcal{G} = \begin{pmatrix} 0 \\ g_1 \\ g_2 \end{pmatrix} \in H.$$

Let

$$\mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{D}_0.$$

Then $\mathcal{Z} \perp \mathcal{G}$ and

$$\langle \mathcal{Z}, \mathcal{G} \rangle = \frac{1}{\alpha} z_1 \overline{g}_1 + \frac{1}{\beta} z_2 \overline{g}_2 = 0. \tag{6.1}$$

Since $z_1 = \alpha_1 z^{[0]}(a) - \alpha_2 z^{[3]}(a)$ and $z_2 = \beta_1 z^{[1]}(a) - \beta_2 z^{[2]}(a)$ can be chosen as arbitrarily, (6.1) is satisfied only if $g_1 = g_2 = 0$. This implies that $\boldsymbol{D}_0$ is dense in H. □

Theorem 6.2 (i) $\mathbf{T}_0$ *is symmetric,*
(ii) $\mathbf{T}_0^* = \mathbf{T}$.

Proof For

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{D}_0, \quad \mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{D}$$

we get that

$$\langle \mathbf{T}_0 \mathcal{Y}, \mathcal{Z} \rangle - \langle \mathcal{Y}, \mathbf{T} \mathcal{Z} \rangle = -[y, z](a) + \frac{1}{\alpha}\left(y_1' \overline{z}_1 - y_1 \overline{z}_1'\right) + \frac{1}{\beta}\left(y_2' \overline{z}_2 - y_2 \overline{z}_2'\right). \tag{6.2}$$

On the other hand direct calculations give

$$\begin{aligned} y_1' \overline{z}_1 - y_1 \overline{z}_1' &= \alpha\Big(y^{[0]}(a)\overline{z}^{[3]}(a) - y^{[3]}(a)\overline{z}^{[0]}(a)\Big), \\ y_2' \overline{z}_2 - y_2 \overline{z}_2' &= \beta\Big(y^{[1]}(a)\overline{z}^{[2]}(a) - y^{[2]}(a)\overline{z}^{[1]}(a)\Big). \end{aligned} \tag{6.3}$$

Substituting (6.3) in (6.2) and using (2.4) we obtain that

$$\langle \mathbf{T}_0 \mathcal{Y}, \mathcal{Z} \rangle = \langle \mathcal{Y}, \mathbf{T} \mathcal{Z} \rangle$$

and therefore $\mathbf{T} \subset \mathbf{T}_0^*$. In particular the last equation implies that $\mathbf{T}_0$ is symmetric. In fact, it is sufficient to get $\mathcal{Z} \in \boldsymbol{D}_0$ (hence $\mathcal{Z} \in \boldsymbol{D}$). This proves (i).

Now for

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{D}_0, \quad \mathcal{Z} = \begin{pmatrix} z \\ \widetilde{z}_1 \\ \widetilde{z}_2 \end{pmatrix} \in \boldsymbol{D}_0^*$$

we get that

$$\langle \mathbf{T}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \int_a^b (\widehat{T}^* z)\overline{y} w dx + \frac{1}{\alpha}\widetilde{z}_1 \left(\alpha_1 \overline{y}^{[0]}(a) - \alpha_2 \overline{y}^{[3]}(a)\right) + \frac{1}{\beta}\widetilde{z}_2 \left(\beta_1 \overline{y}^{[1]}(a) - \beta_2 \overline{y}^{[2]}(a)\right). \tag{6.4}$$

Beside this we get that

$$\langle \mathbf{T}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Z}, \mathbf{T}_0 \mathcal{Y} \rangle = \int_a^b z\overline{Ty} w dx + \frac{1}{\alpha}\widetilde{z}_1 \left(\alpha_1' \overline{y}^{[0]}(a) - \alpha_2' \overline{y}^{[3]}(a)\right) + \frac{1}{\beta}\widetilde{z}_2 \left(\beta_1' \overline{y}^{[1]}(a) - \beta_2' \overline{y}^{[2]}(a)\right). \tag{6.5}$$

Let $y^{[r]}(a)=0,\ r=\overline{0,3}$. Then (6.4), (6.5) and Theorem 3.2 (iv) gives

$$(\widehat{T}^{*}z,y)=(z,T_0y)=(Tz,y)$$

and therefore $z\in\mathcal{D}$. From (6.4) and (6.5) we obtain that

$$\begin{aligned}\langle \mathbf{T}_0^{*}\mathcal{Z},\mathcal{Y}\rangle=\langle \mathcal{Z},\mathbf{T}_0\mathcal{Y}\rangle &=[z,\overline{y}](b)-[z,\overline{y}](a)\\ &+\frac{1}{\alpha}\widetilde{z}_1\Big(\alpha_1\overline{y}^{[0]}(a)-\alpha_2\overline{y}^{[3]}(a)\Big)-\frac{1}{\alpha}\widetilde{z}_1\Big(\alpha_1'\overline{y}^{[0]}(a)-\alpha_2'\overline{y}^{[3]}(a)\Big)\\ &+\frac{1}{\beta}\widetilde{z}_2\Big(\beta_1\overline{y}^{[1]}(a)-\beta_2\overline{y}^{[2]}(a)\Big)-\frac{1}{\beta}\widetilde{z}_2\Big(\beta_1'\overline{y}^{[1]}(a)-\beta_2'\overline{y}^{[2]}(a)\Big)=0.\end{aligned}\tag{6.6}$$

Since $y^{[r]}(b)=0,\ r=\overline{0,3},$ we obtain from (6.6) that

$$\begin{aligned}&-z^{[0]}(a)\overline{y}^{[3]}(a)+z^{[3]}(a)\overline{y}^{[0]}(a)-z^{[1]}(a)\overline{y}^{[2]}(a)+z^{[2]}(a)\overline{y}^{[1]}(a)+\frac{1}{\alpha}\widetilde{z}_1\overline{y}^{[0]}(a)\left(\alpha_1-\alpha_1'\right)\\ &-\frac{1}{\alpha}\widetilde{z}_1\overline{y}^{[3]}(a)\left(\alpha_2-\alpha_2'\right)+\frac{1}{\beta}\widetilde{z}_2\overline{y}^{[1]}(a)\left(\beta_1-\beta_1'\right)-\frac{1}{\beta}\widetilde{z}_2\overline{y}^{[2]}(a)\left(\beta_2-\beta_2'\right)=0.\end{aligned}\tag{6.7}$$

At first we assume that $\alpha_j-\alpha_j'\neq 0,\ \beta_j-\beta_j'\neq 0,\ j=1,2$. Let $y^{[0]}(a)=1,\ y^{[1]}(a)=y^{[2]}(a)=y^{[3]}(a)=0$. Then we have from (6.7) that

$$z^{[3]}(a)+\frac{1}{\alpha}\widetilde{z}_1\left(\alpha_1-\alpha_1'\right)=0.\tag{6.8}$$

Now consider that $y^{[3]}(a)=1,\ y^{[0]}(a)=y^{[1]}(a)=y^{[2]}(a)=0$. Then from (6.7) that

$$z^{[0]}(a)+\frac{1}{\alpha}\widetilde{z}_1\left(\alpha_2-\alpha_2'\right)=0.\tag{6.9}$$

Using (6.8) and (6.9) we obtain

$$\alpha_1z^{[0]}(a)-\alpha_2z^{[3]}(a)=\widetilde{z}_1.\tag{6.10}$$

Let $y^{[1]}(a)=1,\ y^{[0]}(a)=y^{[2]}(a)=y^{[3]}(a)=0$. Then we have from (6.7) that

$$z^{[2]}(a)+\frac{1}{\beta}\widetilde{z}_2\left(\beta_1-\beta_1'\right)=0.\tag{6.11}$$

Now consider that $y^{[2]}(a)=1,\ y^{[0]}(a)=y^{[1]}(a)=y^{[3]}(a)=0$. Hence (6.7) gives that

$$z^{[1]}(a)+\frac{1}{\beta}\widetilde{z}_2\left(\beta_2-\beta_2'\right)=0.\tag{6.12}$$

Therefore using (6.11) and (6.12) we get that

$$\beta_1z^{[1]}(a)-\beta_2z^{[2]}(a)=\widetilde{z}_2.\tag{6.13}$$

Now let $\alpha_1-\alpha_1'=0,\ \alpha_2-\alpha_2'\neq 0,\ \beta_j-\beta_j'\neq 0,\ j=1,2,$ and $y^{[0]}(a)=1,\ y^{[1]}(a)=y^{[2]}(a)=y^{[3]}(a)=0$. Then from (6.7) we arrive at

$$z^{[3]}(a)=0.$$

Therefore (6.8) is true. If $y^{[3]}(a)=1,\ y^{[0]}(a)=y^{[1]}(a)=y^{[2]}(a)=0,$ then (6.7) gives that

$$z^{[0]}(a)+\frac{1}{\alpha}\widetilde{z}_1\left(\alpha_2-\alpha_2'\right)=0. \tag{6.14}$$

Hence from (6.8) and (6.14) we obtain the equality (6.10) and (6.13) still holds.

Similar arguments hold in the cases that

$$\begin{array}{llll}
\alpha_1-\alpha_1'=0, & \alpha_2-\alpha_2'\neq 0, & \beta_1-\beta_1'=0, & \beta_2-\beta_2'\neq 0,\\
\alpha_1-\alpha_1'=0, & \alpha_2-\alpha_2'\neq 0, & \beta_2-\beta_2'=0, & \beta_1-\beta_1'\neq 0,\\
\alpha_2-\alpha_2'=0, & \alpha_1-\alpha_1'\neq 0, & \beta_j-\beta_j'\neq 0, & \\
\alpha_2-\alpha_2'=0, & \alpha_1-\alpha_1'\neq 0, & \beta_1-\beta_1'=0, & \beta_2-\beta_2'\neq 0,\\
\alpha_2-\alpha_2'=0, & \alpha_1-\alpha_1'\neq 0, & \beta_2-\beta_2'=0, & \beta_1-\beta_1'\neq 0,\\
\beta_1-\beta_1'=0, & \beta_2-\beta_2'\neq 0, & \alpha_j-\alpha_j'\neq 0, & \\
\beta_2-\beta_2'=0, & \beta_1-\beta_1'\neq 0, & \alpha_j-\alpha_j'\neq 0. &
\end{array}$$

Note that since $\alpha>0$, the case that $\alpha_1-\alpha_1'=0$ and $\alpha_2-\alpha_2'=0$ can not occur at the same time. Similarly since $\beta>0,\ \beta_1-\beta_1'=0$ and $\beta_2-\beta_2'=0$ can not occur at the same time. Consequently $\mathcal{Z}\in\boldsymbol{D}$ and hence $\mathbf{T}_0^*\subset\mathbf{T}$. This completes the proof. □

Note that since $\boldsymbol{D}_0=\mathcal{D}_0\oplus F$, where F is a finite-dimensional space, $\mathbf{T}_0$ is a closed operator.

6.2. Regular end point a and singular end point b

In this section we consider that a is regular point and b is singular point for τ.

Let

$$\boldsymbol{C}_0=\left\{\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in\boldsymbol{D}\quad:\quad [y,\overline{z}](b)=0,\quad \mathcal{Z}=\begin{pmatrix} z\\ z_1\\ z_2\end{pmatrix}\in\boldsymbol{D}\right\}.$$

We denote by $\mathbf{M}_0$ the restriction of $\mathbf{T}$ to $\boldsymbol{C}_0$.

Theorem 6.3 *$\boldsymbol{C}_0$ is dense in H.*

Proof Let $y\in\mathcal{C}_0$. Then from Theorem 3.6 (iii) we have $y^{[r]}(a)=0,\ r=\overline{0,3},\ [y,\overline{z}](b)=0$ for arbitrary $z\in\mathcal{D}$, and therefore $y_1=y_2=0$. Hence

$$\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in\boldsymbol{C}_0.$$

Therefore for

$$\mathcal{G}=\begin{pmatrix} g\\ g_1\\ g_2\end{pmatrix}\in H$$

which is orthogonal to $\boldsymbol{C}_0$ we have $g=0$ a.e. on (a,b). Consequently we obtain for

$$\mathcal{Z}=\begin{pmatrix} z\\ z_1\\ z_2\end{pmatrix}\in\boldsymbol{C}_0$$

that $\mathcal{Z}\perp\mathcal{G}$ and $g_1=g_2=0$. □

Theorem 6.4 *(i)* $\mathbf{M}_0$ *is symmetric,*
(ii) $\mathbf{M}_0^* = \mathbf{T}$.

Proof For

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{C}_0, \quad \mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{D}$$

we have

$$\langle \mathbf{M}_0 \mathcal{Y}, \mathcal{Z} \rangle = \langle \mathcal{Y}, \mathbf{T}\mathcal{Z} \rangle$$

and therefore $\mathbf{T} \subset \mathbf{M}_0^*$. Hence $\mathbf{M}_0$ is symmetric. This proves (i).

Now let

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{C}_0, \quad \mathcal{Z} = \begin{pmatrix} z \\ \widetilde{z}_1 \\ \widetilde{z}_2 \end{pmatrix} \in \boldsymbol{C}_0^*.$$

Then we obtain that

$$\langle \mathbf{M}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \int_a^b (\widehat{M}^* z)\overline{y} w dx + \frac{1}{\alpha}\widetilde{z}_1 \left(\alpha_1 \overline{y}^{[0]}(a) - \alpha_2 \overline{y}^{[3]}(a) \right) + \frac{1}{\beta}\widetilde{z}_2 \left(\beta_1 \overline{y}^{[1]}(a) - \beta_2 \overline{y}^{[2]}(a) \right) \tag{6.15}$$

and

$$\langle \mathbf{M}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Z}, \mathbf{M}_0 \mathcal{Y} \rangle = \int_a^b z \overline{Ty} w dx + \frac{1}{\alpha}\widetilde{z}_1 \left(\alpha_1' \overline{y}^{[0]}(a) - \alpha_2' \overline{y}^{[3]}(a) \right) + \frac{1}{\beta}\widetilde{z}_2 \left(\beta_1' \overline{y}^{[1]}(a) - \beta_2' \overline{y}^{[2]}(a) \right). \tag{6.16}$$

Consider that $y^{[r]}(a) = 0, \ r = \overline{0,3}$. Then (6.15), (6.16), Theorem 3.6 (i) and (iii) imply that

$$(\widehat{M}^* z, y) = (z, M_0 y) = (Tz, y)$$

and therefore $z \in \mathcal{D}$. Hence using (6.15) and (6.16) we have

$$\begin{aligned} \langle \mathbf{M}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Z}, \mathbf{M}_0 \mathcal{Y} \rangle = & [z, \overline{y}](b) - [z, \overline{y}](a) \\ & + \frac{1}{\alpha}\widetilde{z}_1 \left(\alpha_1 \overline{y}^{[0]}(a) - \alpha_2 \overline{y}^{[3]}(a) \right) - \frac{1}{\alpha}\widetilde{z}_1 \left(\alpha_1' \overline{y}^{[0]}(a) - \alpha_2' \overline{y}^{[3]}(a) \right) \\ & + \frac{1}{\beta}\widetilde{z}_2 \left(\beta_1 \overline{y}^{[1]}(a) - \beta_2 \overline{y}^{[2]}(a) \right) - \frac{1}{\beta}\widetilde{z}_2 \left(\beta_1' \overline{y}^{[1]}(a) - \beta_2' \overline{y}^{[2]}(a) \right) = 0. \end{aligned} \tag{6.17}$$

Since $[y, \overline{z}](b) = 0$, for $z \in \mathcal{D}$, we obtain from (6.17) that

$$\begin{aligned} & -z^{[0]}(a)\overline{y}^{[3]}(a) + z^{[3]}(a)\overline{y}^{[0]}(a) - z^{[1]}(a)\overline{y}^{[2]}(a) + z^{[2]}(a)\overline{y}^{[1]}(a) + \frac{1}{\alpha}\widetilde{z}_1 \overline{y}^{[0]}(a)\left(\alpha_1 - \alpha_1'\right) \\ & -\frac{1}{\alpha}\widetilde{z}_1 \overline{y}^{[3]}(a)\left(\alpha_2 - \alpha_2'\right) + \frac{1}{\beta}\widetilde{z}_2 \overline{y}^{[1]}(a)\left(\beta_1 - \beta_1'\right) - \frac{1}{\beta}\widetilde{z}_2 \overline{y}^{[2]}(a)\left(\beta_2 - \beta_2'\right) = 0. \end{aligned} \tag{6.18}$$

(6.18) implies that $\mathcal{Z} \in \boldsymbol{D}$ and hence $\mathbf{M}_0^* \subset \mathbf{T}$. This completes the proof. □

Note that since $\boldsymbol{C}_0 = \mathcal{C}_0 \oplus F$, where F is a finite-dimensional space, $\mathbf{M}_0$ is a closed operator.

6.3. Singular end point a and regular end point b

In this section we consider that a is singular point and b is regular point for τ. Moreover let the deficiency indices of M_0 be $(4,4)$.

Now consider the set $\boldsymbol{E}$ as follows

$$\boldsymbol{E}=\left\{\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in H \quad : \quad y\in\mathcal{D}, \qquad \begin{array}{c} y_1=\alpha_1[y,u_1](a)-\alpha_2[y,u_3](a)\\ \\ y_2=\beta_1[y,u_2](a)-\beta_2[y,u_4](a)\end{array}\right\},$$

where $u_r(x),\ 1\le r\le 4,$ are the solutions of $\tau[y]=0,\ x\in(a,b),\ \alpha_1,\alpha_2,\beta_1,\beta_2$ are real numbers. Note that since the deficiency indices of M_0 are $(4,4)$, $u_r,\ 1\le r\le 4,$ belong to $L^2_w(a,b)$ and $\mathcal{D}$.

Define the operator $\mathbf{T}$ on $\boldsymbol{E}$ as

$$\mathbf{T}\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}=\begin{pmatrix} Ty\\ y_1'\\ y_2'\end{pmatrix},$$

where $Ty=\tau[y],\ y\in\mathcal{D},\ x\in(a,b),\ y_1'=\alpha_1'[y,u_1](a)-\alpha_2'[y,u_3](a),\ y_2'=\beta_1'[y,u_2](a)-\beta_2'[y,u_4](a),$ $\alpha_1',\alpha_2',\beta_1',\beta_2'$ are real numbers satisfying

$$\alpha:=\alpha_1\alpha_2'-\alpha_1'\alpha_2>0,$$

$$\beta:=\beta_1\beta_2'-\beta_1'\beta_2>0.$$

Consider the set

$$\boldsymbol{E}_0=\left\{\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in \boldsymbol{E} \quad : \quad y^{[r]}(b)=0, \quad r=\overline{0,3}\right\}$$

and define the operator $\mathbf{K}_0$ the restriction of $\mathbf{T}$ to $\boldsymbol{E}_0$.

Theorem 6.5 *$\boldsymbol{E}_0$ is dense in H.*

Proof Let y be an element of $\mathcal{C}_0$. Then from Corollary 3.7 (iii), we have for $y\in\mathcal{D}$ that $[y,u_r](a)=0$, $r=\overline{1,4}$, and $y^{[r-1]}(b)=0$. Therefore for $y\in\mathcal{D}$ we have $y_1=y_2=0$ and

$$\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in\boldsymbol{E}_0.$$

Hence for

$$\mathcal{G}=\begin{pmatrix} g\\ g_1\\ g_2\end{pmatrix}\in H$$

which is orthogonal to $\boldsymbol{E}_0$ we have

$$\langle\mathcal{Y},\mathcal{G}\rangle=\int_a^b y\overline{g}w\,dx=0$$

and this implies that $g=0$ a.e. on (a,b). Thus

$$\mathcal{G}=\begin{pmatrix} 0 \\ g_1 \\ g_2 \end{pmatrix} \in H.$$

Let

$$\mathcal{Z}=\begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{E}_0.$$

Then $\mathcal{Z} \perp \mathcal{G}$ and

$$\langle \mathcal{Z}, \mathcal{G} \rangle = \frac{1}{\alpha} z_1 \overline{g}_1 + \frac{1}{\beta} z_2 \overline{g}_2 = 0. \tag{6.19}$$

Since $z_1=\alpha_1[y,u_1](a)-\alpha_2[y,u_3](a)$, and $z_2=\beta_1[y,u_2](a)-\beta_2[y,u_4](a)$ can be chosen as arbitrarily, (6.19) is satisfied only if $g_1=g_2=0$. This implies that $\boldsymbol{E}_0$ is dense in H. □

Theorem 6.6 (i) $\mathbf{K}_0$ *is symmetric,*
(ii) $\mathbf{K}_0^*=\mathbf{T}$.

Proof For

$$\mathcal{Y}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{E}_0, \quad \mathcal{Z}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{E}$$

we have

$$\langle \mathbf{K}_0 \mathcal{Y}, \mathcal{Z} \rangle - \langle \mathcal{Y}, \mathbf{T} \mathcal{Z} \rangle = -[y,z](a) + \frac{1}{\alpha}\Big(y_1' \overline{z}_1 - y_1 \overline{z}_1'\Big) + \frac{1}{\beta}\Big(y_2' \overline{z}_2 - y_2 \overline{z}_2'\Big). \tag{6.20}$$

Direct calculations give

$$\begin{aligned} y_1'\overline{z}_1 - y_1\overline{z}_1' &= \alpha\Big([y,u_1](a)[\overline{z},u_3](a) - [y,u_3](a)[\overline{z},u_1](a)\Big), \\ y_2'\overline{z}_2 - y_2\overline{z}_2' &= \beta\Big([y,u_2](a)[\overline{z},u_4](a) - [y,u_4](a)[\overline{z},u_2](a)\Big). \end{aligned} \tag{6.21}$$

Using (2.4), (6.20) and (6.21) we obtain that

$$\langle \mathbf{K}_0 \mathcal{Y}, \mathcal{Z} \rangle = \langle \mathcal{Y}, \mathbf{T} \mathcal{Z} \rangle$$

and $\mathbf{T} \subset \mathbf{K}_0^*$. In particular the last equation implies that $\mathbf{K}_0$ is symmetric. In fact, it is sufficient to get $\mathcal{Z} \in \boldsymbol{E}_0$ (hence $\mathcal{Z} \in \boldsymbol{E}$). This proves (i).

Taking

$$\mathcal{Y}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{E}_0, \quad \mathcal{Z}=\begin{pmatrix} z \\ \widetilde{z}_1 \\ \widetilde{z}_2 \end{pmatrix} \in \boldsymbol{E}_0^*$$

we get that

$$\langle \mathbf{K}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \int_a^b (\widehat{T}^* z)\overline{y} w dx + \frac{1}{\alpha}\widetilde{z}_1\Big(\alpha_1[\overline{y},u_1](a) - \alpha_2[\overline{y},u_3](a)\Big) + \frac{1}{\beta}\widetilde{z}_2\Big(\beta_1[\overline{y},u_2](a) - \beta_2[\overline{y},u_4](a)\Big). \tag{6.22}$$

On the other side we have

$$\langle \mathbf{K}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Z}, \mathbf{K}_0 \mathcal{Y} \rangle = \int_a^b z\overline{Ty}wdx + \frac{1}{\alpha}\widetilde{z}_1\Big(\alpha_1'[\overline{y},u_1](a) - \alpha_2'[\overline{y},u_3](a)\Big) + \frac{1}{\beta}\widetilde{z}_2\Big(\beta_1'[\overline{y},u_2](a) - \beta_2'[\overline{y},u_4](a)\Big). \tag{6.23}$$

Let $[y,u_k](a) = 0,\ k = \overline{1,4}$. Then (6.22), (6.23), Corollary 3.7 (iii) and (i) give

$$(\widehat{T}^* z, y) = (z, M_0 y) = (Tz, y)$$

and therefore $z \in \mathcal{D}$. Using (6.22) and (6.23) we get that

$$\begin{aligned}\langle \mathbf{K}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Z}, \mathbf{K}_0 \mathcal{Y} \rangle &= [z,\overline{y}](b) - [z,\overline{y}](a) \\ &+\frac{1}{\alpha}\widetilde{z}_1\left(\alpha_1[\overline{y},u_1](a) - \alpha_2[\overline{y},u_3](a)\right) - \frac{1}{\alpha}\widetilde{z}_1\left(\alpha_1'[\overline{y},u_1](a) - \alpha_2'[\overline{y},u_3](a)\right) \\ &+\frac{1}{\beta}\widetilde{z}_2\left(\beta_1[\overline{y},u_2](a) - \beta_2[\overline{y},u_4](a)\right) - \frac{1}{\beta}\widetilde{z}_2\left(\beta_1'[\overline{y},u_2](a) - \beta_2'[\overline{y},u_4](a)\right) = 0.\end{aligned} \tag{6.24}$$

Since $[y,\overline{z}](b) = 0,$ we obtain from (2.4) and (6.24) that

$$\begin{aligned}&-[z,u_1](a)[\overline{y},u_3](a) + [z,u_3](a)[\overline{y},u_1](a) - [z,u_2](a)[\overline{y},u_4](a) + [z,u_4](a)[\overline{y},u_2](a) \\ &+\frac{1}{\alpha}\widetilde{z}_1[\overline{y},u_1](a)\left(\alpha_1-\alpha_1'\right) - \frac{1}{\alpha}\widetilde{z}_1[\overline{y},u_3](a)\left(\alpha_2-\alpha_2'\right) + \tfrac{1}{\beta}\widetilde{z}_2[\overline{y},u_2](a)\left(\beta_1-\beta_1'\right) \\ &-\frac{1}{\beta}\widetilde{z}_2[\overline{y},u_4](a)\left(\beta_2-\beta_2'\right) = 0.\end{aligned} \tag{6.25}$$

At first we assume that $\alpha_j - \alpha_j' \neq 0,\ \beta_j - \beta_j' \neq 0,\ j = 1,2.$ Let $[y,u_1](a) = 1,\ [y,u_2](a) = [y,u_3](a) = [y,u_4](a) = 0.$ Then we have from (6.25) that

$$[z,u_3](a) + \frac{1}{\alpha}\widetilde{z}_1\left(\alpha_1-\alpha_1'\right) = 0. \tag{6.26}$$

Now consider that $[y,u_3](a) = 1,\ [y,u_1](a) = [y,u_2](a) = [y,u_4](a) = 0.$ Then from (6.25) we obtain that

$$[z,u_1](a) + \frac{1}{\alpha}\widetilde{z}_1\left(\alpha_2-\alpha_2'\right) = 0. \tag{6.27}$$

Using (6.26) and (6.27) we arrive at

$$\alpha_1[z,u_1](a) - \alpha_2[z,u_3](a) = \widetilde{z}_1. \tag{6.28}$$

Now let $[y,u_2](a) = 1,\ [y,u_1](a) = [y,u_3](a) = [y,u_4](a) = 0.$ Then (6.25) gives

$$[z,u_4](a) + \frac{1}{\beta}\widetilde{z}_2\left(\beta_1-\beta_1'\right) = 0. \tag{6.29}$$

If $[y,u_4](a) = 1,\ [y,u_1](a) = [y,u_2](a) = [y,u_3](a) = 0,$ then we get from (6.25) that

$$[z,u_2](a) + \frac{1}{\beta}\widetilde{z}_2\left(\beta_2-\beta_2'\right) = 0. \tag{6.30}$$

Using (6.29) and (6.30) we get that

$$\beta_1[z,u_2](a)-\beta_2[z,u_4](a)=\widetilde{z}_2. \tag{6.31}$$

Let $\alpha_1-\alpha_1'=0,\ \alpha_2-\alpha_2'\neq 0,\ \beta_j-\beta_j'\neq 0,\ j=1,2,$ and $[y,u_1](a)=1,\ [y,u_2](a)=[y,u_3](a)=[y,u_4](a)=0.$ Then from (6.25) we obtain

$$[z,u_3](a)=0.$$

Therefore (6.26) is true. Now let $[y,u_3](a)=1,\ [y,u_1](a)=[y,u_2](a)=[y,u_4](a)=0.$ Then (6.25) gives

$$[z,u_1](a)+\frac{1}{\alpha}\widetilde{z}_1\left(\alpha_2-\alpha_2'\right)=0. \tag{6.32}$$

Hence we have from (6.25) and (6.32) that (6.28) is true and (6.31) still holds.

Similar arguments hold in the cases that

$$\begin{array}{llll}
\alpha_1-\alpha_1'=0, & \alpha_2-\alpha_2'\neq 0, & \beta_1-\beta_1'=0, & \beta_2-\beta_2'\neq 0,\\
\alpha_1-\alpha_1'=0, & \alpha_2-\alpha_2'\neq 0, & \beta_2-\beta_2'=0, & \beta_1-\beta_1'\neq 0,\\
\alpha_2-\alpha_2'=0, & \alpha_1-\alpha_1'\neq 0, & \beta_j-\beta_j'\neq 0, & \\
\alpha_2-\alpha_2'=0, & \alpha_1-\alpha_1'\neq 0, & \beta_1-\beta_1'=0, & \beta_2-\beta_2'\neq 0,\\
\alpha_2-\alpha_2'=0, & \alpha_1-\alpha_1'\neq 0, & \beta_2-\beta_2'=0, & \beta_1-\beta_1'\neq 0,\\
\beta_1-\beta_1'=0, & \beta_2-\beta_2'\neq 0, & \alpha_j-\alpha_j'\neq 0, & \\
\beta_2-\beta_2'=0, & \beta_1-\beta_1'\neq 0, & \alpha_j-\alpha_j'\neq 0. &
\end{array}$$

Note that since $\alpha>0$, the case that $\alpha_1-\alpha_1'=0$ and $\alpha_2-\alpha_2'=0$ can not occur at the same time. Similarly since $\beta>0,\ \beta_1-\beta_1'=0$ and $\beta_2-\beta_2'=0$ can not occur at the same time. Consequently $\mathcal{Z}\in\boldsymbol{E}$ and hence $\mathbf{K}_0^*\subset\mathbf{T}$. This completes the proof. □

Note that since $\boldsymbol{E}_0=\mathcal{C}_0\oplus F$, where F is a finite-dimensional space, $\mathbf{K}_0$ is a closed operator.

6.4. Singular end points a and b

Consider that a and b are singular points for τ and let the deficiency indices of N_0 be $(4,4)$.

Let

$$\boldsymbol{F}_0=\left\{\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in\boldsymbol{E}\quad:\quad [y,\overline{z}](b)=0,\quad \mathcal{Z}=\begin{pmatrix} z\\ z_1\\ z_2\end{pmatrix}\in\boldsymbol{E}\right\}$$

and $\mathbf{N}_0$ be the operator which is the restriction of $\mathbf{T}$ to $\boldsymbol{F}_0$.

The following theorems can be proved similar with the proofs given in section 6.3.

Theorem 6.7 $\boldsymbol{F}_0$ *is dense in* H.

Proof Let $y\in\mathcal{B}_0$. Then Theorem 5.4 implies for $y\in\mathcal{D}$ that $[y,u_r](a)=0$ and $[y,u_r](b)=0$. Therefore $y_1=y_2=0$ and for arbitrary $z\in\mathcal{D}$ using (2.4) we obtain that $[y,\overline{z}](b)=0$. Hence

$$\mathcal{Y}=\begin{pmatrix} y\\ y_1\\ y_2\end{pmatrix}\in\boldsymbol{F}_0.$$

Consequently for

$$\mathcal{G}=\begin{pmatrix} g \\ g_1 \\ g_2 \end{pmatrix}\in H$$

which is orthogonal to $\boldsymbol{F}_0$ we have $g=0$ a.e. on (a,b). Thus for

$$\mathcal{Z}=\begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix}\in \boldsymbol{F}_0$$

we have $\mathcal{Z}\perp\mathcal{G}$ and

$$\langle \mathcal{Z},\mathcal{G}\rangle=\frac{1}{\alpha}z_1\overline{g}_1+\frac{1}{\beta}z_2\overline{g}_2=0. \tag{6.33}$$

Since $z_1=\alpha_1[y,u_1](a)-\alpha_2[y,u_3](a)$, and $z_2=\beta_1[y,u_2](a)-\beta_2[y,u_4](a)$ can be chosen as arbitrarily, (6.33) is satisfied only if $g_1=g_2=0$. This implies that $\boldsymbol{F}_0$ is dense in H. □

Theorem 6.8 (i) $\mathbf{N}_0$ *is symmetric,*
(ii) $\mathbf{N}_0^*=\mathbf{T}$.

Proof For

$$\mathcal{Y}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}\in \boldsymbol{F}_0,\quad \mathcal{Z}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}\in \boldsymbol{E}$$

we have

$$\langle \mathbf{N}_0\mathcal{Y},\mathcal{Z}\rangle=\langle \mathcal{Y},\mathbf{T}\mathcal{Z}\rangle$$

and $\mathbf{T}\subset\mathbf{N}_0^*$. Hence $\mathbf{N}_0$ is symmetric. This proves (i).

Consider that

$$\mathcal{Y}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}\in \boldsymbol{F}_0,\quad \mathcal{Z}=\begin{pmatrix} z \\ \widetilde{z}_1 \\ \widetilde{z}_2 \end{pmatrix}\in \boldsymbol{F}_0^*.$$

Then one obtains that

$$\langle \mathbf{N}_0^*\mathcal{Z},\mathcal{Y}\rangle=\int_a^b(\widehat{N}^*z)\overline{y}w dx+\frac{1}{\alpha}\widetilde{z}_1\left(\alpha_1[\overline{y},u_1](a)-\alpha_2[\overline{y},u_3](a)\right)+\frac{1}{\beta}\widetilde{z}_2\left(\beta_1[\overline{y},u_2](a)-\beta_2[\overline{y},u_4](a)\right). \tag{6.34}$$

and

$$\langle \mathbf{N}_0^*\mathcal{Z},\mathcal{Y}\rangle=\langle \mathcal{Z},\mathbf{N}_0\mathcal{Y}\rangle=\int_a^b z\overline{Ty}w dx+\frac{1}{\alpha}\widetilde{z}_1\Big(\alpha_1'[\overline{y},u_1](a)-\alpha_2'[\overline{y},u_3](a)\Big)+\frac{1}{\beta}\widetilde{z}_2\Big(\beta_1'[\overline{y},u_2](a)-\beta_2'[\overline{y},u_4](a)\Big). \tag{6.35}$$

Let $[y,u_r](a)=0,\ r=\overline{1,4}$. Hence from (6.34), (6.35), Theorem 3.4 (i) and Theorem 5.4 we obtain that

$$(\widehat{N}^*z,y)=(z,N_0y)=(Tz,y)$$

and consequently $z \in \mathcal{D}$. Using (6.34) and (6.35) give that

$$\begin{aligned}\langle \mathbf{N}_0^* \mathcal{Z}, \mathcal{Y} \rangle = \langle \mathcal{Z}, \mathbf{N}_0 \mathcal{Y} \rangle = [z, \overline{y}](b) - [z, \overline{y}](a) \\ + \frac{1}{\alpha} \widetilde{z}_1 \Big(\alpha_1 [\overline{y}, u_1](a) - \alpha_2 [\overline{y}, u_3](a) \Big) - \frac{1}{\alpha} \widetilde{z}_1 \Big(\alpha_1' [\overline{y}, u_1](a) - \alpha_2' [\overline{y}, u_3](a) \Big) \\ + \frac{1}{\beta} \widetilde{z}_2 \Big(\beta_1 [\overline{y}, u_2](a) - \beta_2 [\overline{y}, u_4](a) \Big) - \frac{1}{\beta} \widetilde{z}_2 \Big(\beta_1' [\overline{y}, u_2](a) - \beta_2' [\overline{y}, u_4](a) \Big) = 0.\end{aligned} \tag{6.36}$$

Since $[y, \overline{z}](b) = 0,$ we obtain from (2.4) and (6.36) that

$$\begin{aligned}&-[z, u_1](a)[\overline{y}, u_3](a) + [z, u_3](a)[\overline{y}, u_1](a) - [z, u_2](a)[\overline{y}, u_4](a) + [z, u_4](a)[\overline{y}, u_2](a) \\ &+ \frac{1}{\alpha} \widetilde{z}_1 [\overline{y}, u_1](a) \, (\alpha_1 - \alpha_1') - \frac{1}{\alpha} \widetilde{z}_1 [\overline{y}, u_3](a) \, (\alpha_2 - \alpha_2') + \frac{1}{\beta} \widetilde{z}_2 [\overline{y}, u_2](a) \, (\beta_1 - \beta_1') \\ &- \frac{1}{\beta} \widetilde{z}_2 [\overline{y}, u_4](a) \, (\beta_2 - \beta_2') = 0.\end{aligned} \tag{6.37}$$

(6.37) implies that $\mathcal{Z} \in \boldsymbol{E}$ and hence $\mathbf{N}_0^* \subset \mathbf{T}$. This completes the proof. □

Note that since $\boldsymbol{F}_0 = \mathcal{B}_0 \oplus F,$ where F is a finite-dimensional space, $\mathbf{N}_0$ is a closed operator.

7. Deficiency indices of the direct sum operators

We shall remind that a boundary value for the operator L is a continuous linear functional on the Hilbert space $Dom(L^*)$ which vanishes on $Dom(L)$. Let L be a symmetric operator with deficiency indices (m, n). Then the space of boundary values for L is a Hilbert space of dimension $m + n$ (see [2], p. 1234).

Theorem 7.1 *Let a and b be regular points for τ. Then the deficiency indices of $\mathbf{T}_0$ are $(2, 2)$.*

Proof Consider the equation

$$\mathbf{T}_0^* \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} = i \begin{pmatrix} y \\ y_1' \\ y_2' \end{pmatrix} \tag{7.1}$$

or equivalently (see Theorem 6.2, (ii))

$$\begin{aligned}&Ty = iy, \\ &\alpha_1' y^{[0]}(a) - \alpha_2' y_2^{[3]}(a) = i \Big[\alpha_1 y^{[0]}(a) - \alpha_2 y_2^{[3]}(a) \Big], \\ &\beta_1' y^{[1]}(a) - \beta_2' y_2^{[2]}(a) = i \Big[\beta_1 y^{[1]}(a) - \beta_2 y_2^{[2]}(a) \Big].\end{aligned} \tag{7.2}$$

From Theorem 5.1 (i), the equation $Ty = iy$ has four linearly independent solutions belonging to $L_w^2(a, b)$. However only two of them satisfy the equation (7.2). Therefore the first deficiency index of $\mathbf{T}_0$ is 2. To obtain the second one, it is sufficient to get $-i$ instead of i in (7.1). Therefore the second deficiency index is 2. □

Theorem 7.2 *Let a be regular point and b be singular point for τ.*
(i) If the deficiency indices of M_0 are $(4,4)$, then the deficiency indices of $\mathbf{M}_0$ are $(2,2)$.
(ii) If the deficiency indices of M_0 are $(3,3)$, then the deficiency indices of $\mathbf{M}_0$ are $(1,1)$.
(iii) If the deficiency indices of M_0 are $(2,2)$, then the deficiency indices of $\mathbf{M}_0$ are $(0,0)$.

Proof Consider the equation

$$\mathbf{M}_0^* \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} = i \begin{pmatrix} y \\ y_1' \\ y_2' \end{pmatrix} \tag{7.3}$$

or equivalently (see Theorem 6.4, (ii))

$$\begin{aligned} &Ty = iy, \\ &\alpha_1' y^{[0]}(a) - \alpha_2' y_2^{[3]}(a) = i\left[\alpha_1 y^{[0]}(a) - \alpha_2 y_2^{[3]}(a)\right], \\ &\beta_1' y^{[1]}(a) - \beta_2' y_2^{[2]}(a) = i\left[\beta_1 y^{[1]}(a) - \beta_2 y_2^{[2]}(a)\right]. \end{aligned} \tag{7.4}$$

From Theorem 5.1 (iii), we have that the deficiency indices of M_0 may be $(4,4)$, $(3,3)$ or $(2,2)$. Therefore two, one or none of them satisfy the conditions given (7.4).

(i) If the deficiency indices of M_0 are $(4,4)$ then there are four linearly independent solutions of $Ty = iy$ but only two of them satisfy the equation (7.4). Therefore the first deficience index of $\mathbf{M}_0$ is 2. The second one follows from taking $-i$ instead of i in (7.3).

(ii) Let the deficiency indices of M_0 be $(3,3)$. This implies that there are six boundary values of M_0. Since a is regular for M_0, four of them occur at a. Hence only two of them are given at b. If the deficiency indices of $\mathbf{M}_0$ were $(2,2)$ then there would four boundary values at b. Similarly if the deficiency indices of $\mathbf{M}_0$ were $(0,0)$, then there would not be any boundary value at b. Consequently the deficiency indices of $\mathbf{M}_0$ are $(1,1)$.

(iii) Let the deficiency indices of M_0 be $(2,2)$. The result follows from (ii). □

Theorem 7.3 *Let a be singular point and b be regular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then the deficiency indices of $\mathbf{K}_0$ are $(2,2)$.*

Proof Consider the equation

$$\mathbf{K}_0^* \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} = i \begin{pmatrix} y \\ y_1' \\ y_2' \end{pmatrix} \tag{7.5}$$

or equivalently (see Theorem 6.6, (ii))

$$\begin{aligned} &Ty = iy, \\ &\alpha_1'[y,u_1](a) - \alpha_2'[y,u_3](a) = i\left[\alpha_1[y,u_1](a) - \alpha_2[y,u_3](a)\right], \\ &\beta_1'[y,u_2](a) - \beta_2'[y,u_4](a) = i\left[\beta_1[y,u_2](a) - \beta_2[y,u_4](a)\right]. \end{aligned} \tag{7.6}$$

Since the deficiency indices of M_0 are $(4,4)$, the equation $Ty = iy$ has four linearly independent solutions belonging to $L^2_w(a,b)$. However only two of them satisfy the conditions given in (7.6). Therefore the first deficiency index of $\mathbf{K}_0$ is 2. The second deficiency index follows from (7.5) but taking $-i$ instead of i in (7.5). Hence the second deficiency index of $\mathbf{K}_0$ is 2. $\square$

Theorem 7.4 *Let the deficiency indices of N_0 be $(4,4)$. Then the deficiency indices of $\mathbf{N}_0$ are $(2,2)$.*

Proof Consider the equation

$$\mathbf{N}_0^* \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} = i \begin{pmatrix} y \\ y_1' \\ y_2' \end{pmatrix} \tag{7.7}$$

or equivalently (see Theorem 6.8, (ii))

$$\begin{aligned} &Ty = iy, \\ &\alpha_1'[y,u_1](a) - \alpha_2'[y,u_3](a) = i\left[\alpha_1[y,u_1](a) - \alpha_2[y,u_3](a)\right], \\ &\beta_1'[y,u_2](a) - \beta_2'[y,u_4](a) = i\left[\beta_1[y,u_2](a) - \beta_2[y,u_4](a)\right]. \end{aligned} \tag{7.8}$$

Since the deficiency indices of N_0 are $(4,4)$, the equation $Ty = iy$ has four linearly independent solutions belonging to $L^2_w(a,b)$. However only two of them satisfy the conditions given in (7.8). Therefore the first deficiency index of $\mathbf{N}_0$ is 2. The second deficiency index follows from (7.7) but taking $-i$ instead of i in (7.7). Hence the second deficiency index of $\mathbf{N}_0$ is 2. $\square$

8. Extensions of the operators

In this section, we shall describe all the self-adjoint, dissipative and accumulative extensions of the corresponding minimal operators. Therefore we shall remind that a linear operator L acting on a Hilbert space H with domain $Dom(L)$ is called dissipative if

$$\Im(Ly,y)_H \geq 0, \quad \text{for all} \quad y \in Dom(L),$$

and is called accumulative if

$$\Im(Ly,y)_H \leq 0, \quad \text{for all} \quad y \in Dom(L).$$

To describe these (and self-adjoint) extensions, we shall use Gorbachuks' theorem. In fact, let Γ_1 and Γ_2 be the linear mappings of $D(A^*)$ into the Hilbert space S, where A is a closed symmetric operator with equal deficiency indices acts in the Hilbert space S_1. Then (S,Γ_1,Γ_2) is called a space of boundary values (SBV) of the operator A if

(i) for any $f,g \in D(A^*)$, $(A^*f,g)_{S_1} - (f,A^*g)_{S_1} = (\Gamma_1 f,\Gamma_2 g)_S - (\Gamma_2 f,\Gamma_1 g)_S$;

(ii) for every $F_1,F_2 \in S$, there exists a vector $f \in D(A^*)$ such that $\Gamma_1 f = F_1$ and $\Gamma_2 f = F_2$.

They have introduced the following theorem [7].

Theorem 8.1 *For any contraction K in S the restriction of the operator A^* to the set of functions $f \in D(A^*)$ satisfying the boundary conditions*

$$(K-I)\Gamma_1 f + i(K+I)\Gamma_2 f = 0, \tag{8.1}$$

or

$$(K-I)\Gamma_1 f - i(K+I)\Gamma_2 f = 0, \tag{8.2}$$

is respectively, a maximal dissipative or maximal accumulative extension of the operator A, where A is the restriction of the operator A^* to the domain $D(A)$. Conversely every maximal dissipative (maximal accumulative) extension of A is the restriction of A^* to the set of vectors $f \in D(A^*)$ satisfying (8.1) ((8.2)) and the contraction K is uniquely determined by the extension. These conditions give self-adjoint extension if K is unitary. In the latter case (8.1) and (8.2) are equivalent to the condition

$$(\cos C)\mathit{\Gamma}_1 f - (\sin C)\mathit{\Gamma}_2 f = 0,$$

where C is a self-adjoint operator on S. The general form of the dissipative and accumulative extensions of the operator A is given by the conditions

$$K(\mathit{\Gamma}_1 f + i\mathit{\Gamma}_2 f) = \mathit{\Gamma}_1 f - i\mathit{\Gamma}_2 f, \quad \mathit{\Gamma}_1 f + i\mathit{\Gamma}_2 f \in D(K), \tag{8.3}$$

$$K(\mathit{\Gamma}_1 f - i\mathit{\Gamma}_2 f) = \mathit{\Gamma}_1 f + i\mathit{\Gamma}_2 f, \quad \mathit{\Gamma}_1 f - i\mathit{\Gamma}_2 f \in D(K), \tag{8.4}$$

respectively, where K is a linear operator satisfying $\|Kf\| \le \|f\|$, $f \in D(K)$. The general form of symmetric extensions is given by the formula (8.3) and (8.4), where K is an isometric operator.

We should note that Ismailov and his colleagues studied the self-adjoint extensions of the minimal symmetric operators with equal deficiency indices (r,r), $0 \le r \le \infty$, generated by the ordinary differential expressions with operator coeffcients [8], [9], [10], [11], [12], [13], [14].

8.1. Extensions in the single Hilbert space

In this section, we shall give the extensions in the single Hilbert space.

Theorem 8.2 *Let a and b be regular points for τ and let $\Gamma_1 y = (y^{[3]}(a),\ y^{[2]}(a),\ y^{[0]}(b),\ y^{[1]}(b))$ and $\Gamma_2 y = (y^{[0]}(a),\ y^{[1]}(a),\ y^{[3]}(b),\ y^{[2]}(b))$, where $y \in \mathcal{D}$. Then the triplet $(\mathbb{C}^4, \Gamma_1, \Gamma_2)$ is a space of boundary values of T_0.*

Proof From Theorem 3.2 (i) we have that there exists a function $y \in \mathcal{D}$ satisfying

$$y^{[r]}(a) = c_r, \quad y^{[r]}(b) = d_r, \quad r = \overline{0,3}, \tag{8.5}$$

where c_r and d_r are arbitrary complex numbers. On the other hand for $y, \chi \in \mathcal{D}$, direct calculations give

$$(T_0^* y, \chi) - (y, T_0^* \chi) = [y, \overline{\chi}](b) - [y, \overline{\chi}](a) \tag{8.6}$$

and

$$(\Gamma_1 y, \Gamma_2 \chi) - (\Gamma_2 y, \Gamma_1 \chi) = [y, \overline{\chi}](b) - [y, \overline{\chi}](a). \tag{8.7}$$

Therefore (8.5)-(8.7) completes the proof. □

Theorem 8.3 *Let a and b be regular points for τ. Then for $y \in \mathcal{D}$, the boundary conditions*

$$y^{[0]}(a) + h_1 y^{[3]}(a) = 0,$$

$$y^{[1]}(a) + h_2 y^{[2]}(a) = 0,$$

$$y^{[3]}(b) + h_3 y^{[0]}(b) = 0,$$

$$y^{[2]}(b) + h_4 y^{[1]}(b) = 0,$$

where all $\Im h_r > 0$, all $\Im h_r < 0$ and all $\Im h_r = 0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator T_0.

Theorem 8.4 *Let a be regular point and b be singular point forτ and let the deficiency indices of M_0 be $(4,4)$. Moreover consider the mappings $\Gamma_1 y = (y^{[3]}(a), y^{[2]}(a), [y,u_1](b), [y,u_2](b))$ and $\Gamma_2 y = (y^{[0]}(a), y^{[1]}(a), [y,u_3](b), [y,u_4](b))$, where $y \in \mathcal{D}$. Then the triplet $(\mathbb{C}^4, \Gamma_1, \Gamma_2)$ is a space of boundary values of M_0.*

Proof From Theorem 5.7, there exists a function $y \in \mathcal{D}$ satisfying

$$y^{[r-1]}(a) = c_r, \quad [y,u_r](b) = d_r, \quad 1 \le r \le 4, \tag{8.8}$$

where c_r and d_r are complex numbers.

For $y, \chi \in D$, using Green's formula and (2.4) we have

$$(M_0^* y, \chi) - (y, M_0^* \chi) = [y, \overline{\chi}](b) - [y, \overline{\chi}](a) \tag{8.9}$$

and

$$(\Gamma_1 y, \Gamma_2 \chi) - (\Gamma_2 y, \Gamma_1 \chi) = [y, \overline{\chi}](b) - [y, \overline{\chi}](a). \tag{8.10}$$

Therefore (8.8)-(8.11) completes the proof. □

Corollary 8.5 *Let a be regular point and b be singular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then for $y \in \mathcal{D}$, the boundary conditions*

$$y^{[0]}(a) + h_1 y^{[3]}(a) = 0,$$

$$y^{[1]}(a) + h_2 y^{[2]}(a) = 0,$$

$$[y, u_3](b) + h_3 [y, u_1](b) = 0,$$

$$[y, u_4](b) + h_4 [y, u_2](b) = 0,$$

where all $\Im h_r > 0$, all $\Im h_r < 0$ and all $\Im h_r = 0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator M_0.

Using Corollary 5.8 we obtain the following.

Corollary 8.6 *Let a be singular point and b be regular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then for $y \in \mathcal{D}$, the boundary conditions*

$$[y,u_1](a)+h_1[y,u_3](a)=0,$$

$$[y,u_2](a)+h_2[y,u_4](a)=0,$$

$$y^{[3]}(b)+h_3y^{[0]}(b)=0,$$

$$y^{[2]}(b)+h_4y^{[1]}(b)=0,$$

where all $\Im h_r>0$, all $\Im h_r<0$ and all $\Im h_r=0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator M_0.

Theorem 8.7 *Let a be regular point and b be singular point for τ and let the deficiency indices of M_0 be $(2,2)$. Moreover let $\Gamma_1 y=\left(y^{[3]}(a),y^{[2]}(a)\right)$ and $\Gamma_2 y=\left(y^{[0]}(a),y^{[1]}(a)\right)$, where $y\in\mathcal{D}$. Then the triplet $\left(\mathbb{C}^2,\Gamma_1,\Gamma_2\right)$ is a space of boundary values of M_0.*

Proof From Theorem 2.1 there exists a function $y \in \mathcal{D}$ satisfying

$$y^{[r]}(a)=c_r, \quad 0\le r\le 4, \tag{8.11}$$

where c_r are complex numbers. Moreover for $y,\chi\in\mathcal{D}$ we have

$$(M_0^* y,\chi)-(y,M_0^*\chi)=-[y,\overline{\chi}](a), \tag{8.12}$$

since the deficiency indices of M_0 are $(2,2)$ and $[y,\overline{\chi}](b)=0$ from Theorem 5.5 (i). On the other side we get that

$$(\Gamma_1 y,\Gamma_2\chi)-(\Gamma_2 y,\Gamma_1\chi)=-[y,\overline{\chi}](a). \tag{8.13}$$

Therefore (8.11)-(8.13) completes the proof. □

Corollary 8.8 *Let a be regular point and b be singular point for τ and let the deficiency indices of M_0 be $(2,2)$. Then for $y\in\mathcal{D}$, the boundary conditions*

$$y^{[0]}(a)+h_1y^{[3]}(a)=0,$$

$$y^{[1]}(a)+h_2y^{[2]}(a)=0,$$

where all $\Im h_r>0$, all $\Im h_r<0$ and all $\Im h_r=0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator M_0.

Theorem 8.9 *Let a and b be singular points for τ and let the deficiency indices of N_0 be $(4,4)$. Moreover consider the mappings $\Gamma_1 y=\Big([y,u_3](a),[y,u_4](a),[y,u_1](b),[y,u_2](b)\Big)$ and $\Gamma_2 y=\Big([y,u_1](a),[y,u_2](a),[y,u_3](b),[y,u_4](b)\Big)$, where $y\in\mathcal{D}$. Then the triplet $\left(\mathbb{C}^4,\Gamma_1,\Gamma_2\right)$ is a space of boundary values of N_0.*

Proof From Theorem 5.9, there exists a function $y \in D$ satisfying

$$[y, u_r](a) = c_r, \quad [y, u_r](b) = d_r, \quad 1 \leq r \leq 4, \tag{8.14}$$

where c_r and d_r are complex numbers. Using (2.4), we have for $y, \chi \in D$ that

$$(M_0^* y, \chi) - (y, M_0^* \chi) = [y, \overline{\chi}](b) - [y, \overline{\chi}](a) \tag{8.15}$$

and

$$(\Gamma_1 y, \Gamma_2 \chi) - (\Gamma_2 y, \Gamma_1 \chi) = [y, \overline{\chi}](b) - [y, \overline{\chi}](a). \tag{8.16}$$

Therefore (8.14)-(8.16) completes the proof. □

Corollary 8.10 *Let a and b be singular points for τ and let the deficiency indices of N_0 be $(4, 4)$. Then for $y \in \mathcal{D}$, the boundary conditions*

$$[y, u_1](a) + h_1 [y, u_3](a) = 0,$$

$$[y, u_2](a) + h_2 [y, u_4](a) = 0,$$

$$[y, u_3](b) + h_3 [y, u_1](b) = 0,$$

$$[y, u_4](b) + h_4 [y, u_2](b) = 0,$$

where all $\Im h_r > 0$, all $\Im h_r < 0$ and all $\Im h_r = 0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator N_0.

8.2. Extensions in the direct sum Hilbert space

In this section, we shall construct the extensions in the direct sum Hilbert space.

Theorem 8.11 *Let a and b be regular points for τ and*

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{D}.$$

Consider the mappings $\Gamma_1 \mathcal{Y} = \left(y^{[0]}(b), y^{[1]}(b) \right)$ and $\Gamma_2 \mathcal{Y} = \left(y^{[3]}(b), y^{[2]}(b) \right)$. Then $\left(\mathbb{C}^2, \Gamma_1, \Gamma_2 \right)$ is a space of boundary values of $\mathbf{T}_0$.

Proof Γ_1 and Γ_2 are linear mappings from $\boldsymbol{D}$ into $\mathbb{C}^2$. Moreover for $y \in \mathcal{D}$ we have form Theorem 2.1 that

$$y^{[r-1]}(b) = d_r, \quad 1 \leq r \leq 4, \tag{8.17}$$

where d_r are complex numbers.We have for

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{D},$$

that

$$\langle \mathbf{T}_0^* \mathcal{Y}, \mathcal{Z} \rangle - \langle \mathcal{Y}, \mathbf{T}_0^* \mathcal{Z} \rangle = [y, \overline{z}](b). \tag{8.18}$$

On the other side we get that

$$(\Gamma_1 \mathcal{Y}, \Gamma_2 \mathcal{Z}) - (\Gamma_2 \mathcal{Y}, \Gamma_1 \mathcal{Z}) = [y, \overline{z}](b). \tag{8.19}$$

Therefore (8.17)-(8.19) completes the proof. □

Corollary 8.12 *Let a and b be regular points for τ. Then the boundary conditions*

$$y^{[3]}(b) + h_1 y^{[0]}(b) = 0,$$

$$y^{[2]}(b) + h_2 y^{[1]}(b) = 0,$$

where all $\Im h_r > 0$, all $\Im h_r < 0$ and all $\Im h_r = 0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator $\mathbf{T}_0$.

Theorem 8.13 *Let a be regular point and b be singular point for τ and let the deficiency indices of M_0 be $(4,4)$. Consider the mappings $\Gamma_1 \mathcal{Y} = ([y, u_1](b), [y, u_2](b))$ and $\Gamma_2 \mathcal{Y} = ([y, u_3](b), [y, u_4](b))$, where*

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix} \in \boldsymbol{D}.$$

Then $(\mathbb{C}^2, \Gamma_1, \Gamma_2)$ is a space of boundary values of $\mathbf{M}_0$.

Proof Γ_1 and Γ_2 are linear mappings from $\boldsymbol{D}$ into $\mathbb{C}^2$. From Theorem 5.7 we have for $y \in \mathcal{D}$ that the values

$$[y, u_r](b) = d_r, \quad r = \overline{1, 4}, \tag{8.20}$$

where d_r are complex numbers, exist. Moreover for

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{D},$$

using (2.4) we obtain

$$\langle \mathbf{M}_0^* \mathcal{Y}, \mathcal{Z} \rangle - \langle \mathcal{Y}, \mathbf{M}_0^* \mathcal{Z} \rangle = [y, \overline{z}](b) \tag{8.21}$$

and

$$(\Gamma_1 \mathcal{Y}, \Gamma_2 \mathcal{Z}) - (\Gamma_2 \mathcal{Y}, \Gamma_1 \mathcal{Z}) = [y, \overline{z}](b). \tag{8.22}$$

Therefore (8.20)-(8.22) completes the proof. □

Corollary 8.14 *Let a be regular and b be singular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then the boundary conditions*

$$[y, u_3](b) + h_1 [y, u_1](b) = 0,$$

$$[y, u_4](b) + h_2 [y, u_2](b) = 0,$$

Theorem 8.15 *Let a be singular point and b be regular point for τ and let the deficiency indices of M_0 be $(4,4)$. Consider the mappings $\Gamma_1\mathcal{Y} = \left(y^{[0]}(b), y^{[1]}(b)\right)$ and $\Gamma_2\mathcal{Y} = \left(y^{[3]}(b), y^{[2]}(b)\right)$, where*

$$\mathcal{Y}=\begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}\in\boldsymbol{E}.$$

Then $\left(\mathbb{C}^2, \Gamma_1, \Gamma_2\right)$ is a space of boundary values of $\mathbf{K}_0$.

Proof Γ_1 and Γ_2 are linear mappings from $\boldsymbol{E}$ into $\mathbb{C}^2$. From Corollary 5.8, we have for $y \in \mathcal{D}$ that the values

$$y^{[r-1]}(b) = d_r, \quad r = \overline{1,4}, \tag{8.23}$$

where d_r are complex numbers, exist. We obtain for

$$\mathcal{Y} = \begin{pmatrix} y \\ y_1 \\ y_2 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \boldsymbol{E},$$

that

$$\langle \mathbf{K}_0^*\mathcal{Y}, \mathcal{Z}\rangle - \langle \mathcal{Y}, \mathbf{K}_0^*\mathcal{Z}\rangle = [y, \overline{z}](b) \tag{8.24}$$

and

$$(\Gamma_1\mathcal{Y}, \Gamma_2\mathcal{Z}) - (\Gamma_2\mathcal{Y}, \Gamma_1\mathcal{Z}) = [y, \overline{z}](b). \tag{8.25}$$

Hence (8.23)-(8.25) completes the proof. □

Corollary 8.16 *Let a be singular and b be regular point for τ and let the deficiency indices of M_0 be $(4,4)$. Then the boundary conditions*

$$y^{[3]}(b)+h_1 y^{[0]}(b) = 0,$$

$$y^{[2]}(b) + h_2 y^{[1]}(b) = 0,$$

where all $\Im h_r > 0$, all $\Im h_r < 0$ and all $\Im h_r = 0$ describe all maximal dissipative, all maximal accumulative and all maximal self-adjoint extensions, respectively, of the operator $\mathbf{N}_0$.

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