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On a system of fractional coupled hybrid Hadamard differential equations with terminal conditions

Karthikeyan Buvaneswari¹, Panjaiyan Karthikeyan² and Dumitru Baleanu^{3,4,5*} 

*Correspondence:

dumitru@cankaya.edu.tr

³Department of Mathematics, Cankaya University, Ankara, Turkey

⁴Institute of Space Sciences, 06530

Magurele-Bucharest, Romania
Full list of author information is available at the end of the article

Abstract

In this manuscript, we study the existence of solutions for a coupled system of nonlinear hybrid differential equations of fractional order involving Hadamard derivative with nonlocal boundary conditions. By using suitable fixed point theorems we establish sufficient conditions for the existence result. An example is provided to illustrate our main result.

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1 Introduction

Differential Equations of Fractional Order (DEFO) gain huge attention among scientists due to the applications which are not possible with integer order ordinary or partial differential equations. In recent days, researchers in various fields pay more attention on the existence of solutions of the aforesaid equations. In this regard, many literature surveys, some notable developments (refer [13, 14]), and significant studies are available in the theory of coupled systems with coupled boundary conditions in FDEs. For further references, one can refer to [1, 3, 6–9, 12, 15, 16, 19]. This is because the aforesaid equations include the perturbations of original differential equations in nonidentical ways.

The nonlinear hybrid Hadamard differential equations with initial conditions were discussed by Ahmad and Ntouyas in [4]. In 2015, Wang [22] proved the existence results for the impulsive systems with Hadamard derivative. For more details, the interested readers can refer to Wang et al. [21], where the authors used some classical methods to analyze the Ulam–Hyers stability and proved existence results for Hadamard fractional differential equations. Further, Samko et al. [20] proved the existence and uniqueness of the solution of Cauchy problems for FDE involving Hadamard derivatives in a nonsequential setting. In [17], the author studied the Hadamard-type fractional calculus.

We believe that only few literature studies are available which investigate coupled systems of FDEs of Hadamard type; see [23, 24] and the references therein. This paper will definitely fill up the gap between the theories and applications of Hadamard FDEs.

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In[4], Ahmad et al. studied

$$\begin{cases} {}_H D^{\alpha_1} \left(\frac{v(t)}{f_1(t, v(t))} \right) = g_1(t, v(t)), & 1 \leq t \leq T, 0 < \alpha_1 \leq 1, \\ {}_H J^{1-\alpha_1} v(t)|_{t=1} = \zeta, \end{cases} \tag{1}$$

In 2015, the authors of [5] investigated the nonlocal boundary value problems for hybrid FDEs and inclusions of Hadamard type as given below:

$$\begin{cases} D^{\alpha_1} \left(\frac{v(t)}{f_1(t, v(t))} \right) = g_1(t, v(t)), & 1 \leq t \leq e, 1 < \alpha_1 \leq 2, \\ v(1) = 0, \quad v(e) = m(v), \end{cases} \tag{2}$$

Ahmad et al. [2] in 2017 focused on the issues related to initial and boundary value problems for fractional differential equations of Hadamard type and studied mixed initial value problems involving Hadamard derivatives.

The above mentioned works motivated us to study the following coupled Hadamard FDEs with the boundary conditions:

$$\begin{cases} D^{\sigma_1} \left(\frac{\vartheta(t)}{f_1(t, \vartheta(t), v(t))} \right) = h_1(t, \vartheta(t), v(t)), & 1 \leq t \leq e, 1 < \sigma_1 \leq 2, \\ D^{\delta_1} \left(\frac{v(t)}{f_2(t, \vartheta(t), v(t))} \right) = h_2(t, \vartheta(t), v(t)), & 1 \leq t \leq e, 1 < \delta_1 \leq 2, \\ \vartheta(1) = 0, \quad \vartheta(e) = m(\vartheta), \\ v(1) = 0, \quad v(e) = n(v), \end{cases} \tag{3}$$

where $D^{\sigma_1}, D^{\delta_1}$ are the Hadamard fractional derivatives of order σ_1 and δ_1 , respectively; $f_i \in \mathcal{C}([1, e] \times \mathfrak{R} \times \mathfrak{R}, \mathfrak{R} \setminus \{0\})$ and $h_i \in \mathcal{C}([1, e] \times \mathfrak{R} \times \mathfrak{R}, \mathfrak{R}), i = 1, 2, m, n : \mathcal{C}([1, e], \mathfrak{R}) \rightarrow \mathfrak{R}$ are continuous functions.

The rest of the paper is organized as follows. In Sect. 2, some basic definitions and lemmas related to Hadamard fractional derivatives are presented. In Sect. 3, we establish sufficient conditions for the existence of the coupled solutions to (3), and in Sect. 4, we present a numerical example to illustrate our main result.

2 Auxiliary results

In this section we list some basic definitions and lemmas of fractional calculus.

We define the nonhomogeneous boundary value problems of coupled hybrid fractional differential equations of Hadamard type of the function space $\mathcal{X} = \mathcal{C}([1, e], \mathfrak{R})$ of continuous real valued functions $f_1 : [1, e] \rightarrow \mathfrak{R}$. Clearly, $\mathcal{X} = \mathcal{C}([1, e], \mathfrak{R})$ is a Banach space under the supremum norm

$$\|\vartheta\| = \sup\{|\vartheta(t)| : t \in [1, e]\} \tag{4}$$

which is again a Banach algebra with respect to the multiplication “.” defined by

$$(\vartheta \cdot v)(t) = \vartheta(t) \cdot v(t). \tag{5}$$

The product space $\mathcal{E} = \mathcal{X} \times \mathcal{X}$ is a Banach space under the norm

$$\|(\vartheta, v)\| = \|\vartheta\| + \|v\|. \tag{6}$$

Then the normed linear space $(\mathcal{E}, \|\cdot, \cdot\|)$ is a Banach space which further becomes a Banach algebra with respect to the multiplication “ \cdot ” defined by

$$((\vartheta, \nu) \cdot (\bar{u}, \bar{v}))(t) = (\vartheta, \nu)(t) \cdot (\bar{u}, \bar{v})(t) = (\vartheta(t)\bar{u}(t), \nu(t)\bar{v}(t)) \tag{7}$$

for all $t \in [1, e]$, where $(\vartheta, \nu), (\bar{u}, \bar{v}) \in \mathcal{X} \times \mathcal{X} = \mathcal{E}$. The following result concerning this fact of algebraic structure of the product space $\mathcal{E} = \mathcal{X} \times \mathcal{X}$ is proved by Dhage in [11]. But first we need

Definition 2.1 ([18]) The Hadamard fractional derivative of order β_1 for a continuous function $h_1 : [1, \infty) \rightarrow \mathfrak{R}$ is defined as

$$D^{\beta_1} h_1(t) = \frac{1}{\Gamma(n - \beta_1)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\ln \frac{t}{s} \right)^{n - \beta_1 - 1} \frac{h_1(s)}{s} ds, \quad n - 1 < \beta_1 < n,$$

where $n = [\beta_1] + 1$, $[\beta_1]$ denotes the integer part of the real number β_1 and $\ln(\cdot) = \log_e(\cdot)$.

Definition 2.2 ([18]) The Hadamard fractional integral of order β_1 for a function $h_1 : [1, \infty) \rightarrow \mathfrak{R}$ is defined as

$$I^{\beta_1} h_1(t) = \frac{1}{\Gamma(\beta_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\beta_1 - 1} \frac{h_1(s)}{s} ds, \quad \beta_1 > 0, t > 1,$$

provided the integral exists.

Definition 2.3 (Dhage [11]) Let \mathcal{E} be a Banach space. An operator $\mathcal{T} : \mathcal{E} \rightarrow \mathcal{E}$ is called Lipschitz if there exists a constant $L_{\mathcal{T}} > 0$ such that

$$\|\mathcal{T}(\vartheta) - \mathcal{T}(\nu)\| \leq L_{\mathcal{T}} \|\vartheta - \nu\|$$

for all elements $\vartheta, \nu \in \mathcal{X}$.

Lemma 2.4 (Dhage [11]) *The product space \mathcal{E} is a Banach algebra with respect to the norm $\|\cdot, \cdot\|$ and the multiplication “ \cdot ” defined by (6) and (7), respectively.*

Proof Let (ϑ, ν) and (\bar{u}, \bar{v}) be any two elements of \mathcal{E} then, by definition of the norm $\|\cdot, \cdot\|$ and the multiplication “ \cdot ” in \mathcal{E} , we obtain

$$\|(\vartheta, \nu) \cdot (\bar{u}, \bar{v})\| = \|(\vartheta\bar{u}, \nu\bar{v})\| = \|\vartheta\bar{u}\| + \|\nu\bar{v}\| \leq \|\vartheta\| \|\bar{u}\| + \|\nu\| \|\bar{v}\| \tag{8}$$

and

$$\|(\vartheta, \nu)\| \|(\bar{u}, \bar{v})\| = [\|\vartheta\| + \|\nu\|][\|\bar{u}\| + \|\bar{v}\|]. \tag{9}$$

Now from (8) and (9), it follows that

$$\|((\vartheta, \nu)(\bar{u}, \bar{v}))\| \leq \|(\vartheta, \nu)\| \|(\bar{u}, \bar{v})\|.$$

This shows that $(\mathcal{E}, \|(\cdot, \cdot)\|)$ is a Banach algebra and the proof of lemma is complete. \square

We employ the following fixed point theorem as a basic tool for proving our main existence result for a coupled system of the proposed problem (3).

Theorem 2.5 (Dhage [10]) *Let S be a nonempty subset of a Banach algebra, which is bounded, convex and closed. Assume also that the operators A and B are such that $A: X \rightarrow X$ and $B: S \rightarrow X$, and satisfy the following conditions:*

- (a₁) A is Lipschitzian with L_A as a Lipschitz constant,
- (a₂) B is completely continuous,
- (a₃) $\vartheta = A\vartheta Bv \implies \forall v \in S \implies \vartheta \in S$, and
- (a₄) $M_B < 1$, where $M_B = \|B(S)\| = \sup\{\|B\vartheta\| : \vartheta \in S\}$.

Then the operator equation $\vartheta = A\vartheta B\vartheta$ has a solution in S .

3 Existence result

To prove the main existence result for a mild coupled solution of problem (3), we need to prove the following lemma.

Lemma 3.1 *Given $\psi_1, \psi_2 \in \mathcal{C}([1, e] \times \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R})$, the integral solution of the boundary value problem*

$$\begin{cases} D^{\omega_1} \left(\frac{\vartheta(t)}{\psi_1(t, \vartheta(t), v(t))} \right) = \psi_2(t), & 1 \leq t \leq e, 1 < \omega_1 \leq 2, \\ \vartheta(1) = 0, & \vartheta(e) = m(\vartheta) \end{cases} \tag{10}$$

is given by

$$\begin{aligned} \vartheta(t) = \psi_1(t, \vartheta(t), v(t)) \times & \left[\frac{1}{\Gamma(\omega_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\omega_1-1} \frac{\psi_2(s)}{s} ds \right. \\ & \left. + (\ln t)^{\omega_1-1} \left(\frac{m(\vartheta)}{\psi_1(e, m(\vartheta), v(e))} - \frac{1}{\Gamma(\omega_1)} \int_1^e \left(\ln \frac{e}{s} \right)^{\omega_1-1} \frac{\psi_2(s)}{s} ds \right) \right]. \end{aligned} \tag{11}$$

Proof The solution of equation (10) can be written as

$$\begin{aligned} \vartheta(t) = \psi_1(t, \vartheta(t), v(t)) & \left[\frac{1}{\Gamma(\omega_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\omega_1-1} \frac{\psi_2(s)}{s} ds + c_1 (\ln t)^{\omega_1-1} \right. \\ & \left. + c_2 (\ln t)^{\omega_1-2} \right] \end{aligned} \tag{12}$$

where the arbitrary constants $c_1, c_2 \in \mathfrak{R}$ and the boundary conditions given in (10) give $c_2 = 0$ and

$$c_1 = (\ln t)^{\omega_1-1} \left(\frac{m(\vartheta)}{\psi_1(e, m(\vartheta), v(e))} - \frac{1}{\Gamma(\omega_1)} \int_1^e \left(\ln \frac{e}{s} \right)^{\omega_1-1} \frac{\psi_2(s)}{s} ds \right).$$

Substituting these values of c_1 and c_2 into (12), we get

$$\begin{aligned} \vartheta(t) = & \psi_1(t, \vartheta(t), \nu(t)) \times \left[\frac{1}{\Gamma(\omega_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\omega_1-1} \frac{\psi_2(s)}{s} ds \right. \\ & \left. + (\ln t)^{\omega_1-1} \left(\frac{m(\vartheta)}{\psi_1(e, m(\vartheta), \nu(e))} - \frac{1}{\Gamma(\omega_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\omega_1-1} \frac{\psi_2(s)}{s} ds \right) \right]. \end{aligned}$$

Now, the proof is finished. □

Definition 3.2 A solution $\vartheta \in \mathcal{C}([1, e], \mathfrak{R}, \mathfrak{R})$ of the hybrid Hadamard fractional differential equation in (11) is called a mild solution of the FDE (10) defined on $[1, e]$. Similarly, a mild coupled solution $(\vartheta, \nu) \in \mathcal{C}([1, e], \mathfrak{R}) \times \mathcal{C}([1, e], \mathfrak{R})$ of a system of the coupled Hadamard differential equation (3) is defined on $[1, e]$.

Theorem 3.3 Consider the following hypotheses (H1) to (H4):

(H1) For the continuous functions $f_i : [1, e] \times \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R} \setminus \{0\}$, $i = 1, 2$ with constants $L_{f_1} > 0$ and $L_{f_2} > 0$, we have

$$|f_1(t, \vartheta, \nu - f_1(t, \bar{\vartheta}, \bar{\nu})| \leq L_{f_1} (|\vartheta - \bar{\vartheta}| + |\nu - \bar{\nu}|),$$

and

$$|f_2(t, \vartheta, \nu) - f_2(t, \bar{\vartheta}, \bar{\nu})| \leq L_{f_2} (|\vartheta - \bar{\vartheta}| + |\nu - \bar{\nu}|),$$

for all $t \in [1, e]$ and $\vartheta, \bar{\vartheta}, \nu, \bar{\nu} \in \mathfrak{R}$.

(H2) There exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\left| \frac{m(\vartheta)}{f_1(e, m(\vartheta), n(\nu))} \right| \leq M_1 \quad \text{and} \quad \left| \frac{n(\nu)}{f_2(e, m(\vartheta), n(\nu))} \right| \leq M_2$$

and

$$F_0 = \sup_{t \in [1, e]} |f_1(t, 0, 0)|, \quad G_0 = \sup_{t \in [1, e]} |f_2(t, 0, 0)|.$$

(H3) There exist constants $M_{h_i} > 0$ such that

$$|h_i(t, \vartheta, \nu)| \leq M_{h_i}$$

for all $(t, \vartheta, \nu) \in [1, e] \times \mathfrak{R} \times \mathfrak{R}$ and $i = 1, 2$.

(H4) The constants in hypotheses (H1) through (H3) satisfy the following conditions:

$$\Omega = L_{f_1} \left(M_1 + \frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} \right) + L_{f_2} \left(M_2 + \frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} \right) < 1.$$

If these conditions hold, then the coupled system (3) has a coupled solution defined on $[1, e] \times [1, e]$.

Proof Using Lemma 3.1 for (3), we get

$$\begin{aligned} \vartheta(t) = & f_1(t, \vartheta(t), \nu(t)) \times \left[\frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\sigma_1-1} \frac{h_1(s)}{s} ds \right. \\ & \left. + (\ln t)^{\sigma_1-1} \left(\frac{m(\vartheta)}{f_1(e, m(\vartheta), n(\nu))} - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\sigma_1-1} \frac{h_1(s)}{s} ds \right) \right] \end{aligned} \tag{13}$$

and

$$\begin{aligned} z(t) = & f_2(t, \vartheta(t), \nu(t)) \times \left[\frac{1}{\Gamma(\delta_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\delta_1-1} \frac{h_2(s)}{s} ds \right. \\ & \left. + (\ln t)^{\delta_1-1} \left(\frac{n(\nu)}{f_2(e, m(\vartheta), n(\nu))} - \frac{1}{\Gamma(\delta_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\delta_1-1} \frac{h_2(s)}{s} ds \right) \right]. \end{aligned} \tag{14}$$

Choose

$$\rho \geq \frac{F_0(M_1 + \frac{2M_{h_1}}{\Gamma(\sigma_1+1)}) + G_0(M_2 + \frac{2M_{h_2}}{\Gamma(\delta_1+1)})}{1 - \Omega}$$

and define a subset \mathcal{S} of the Banach space $\mathcal{X} \times \mathcal{X}$ by

$$\mathcal{S} = \{(\vartheta, \nu) \in \mathcal{X} \times \mathcal{X} : \|(\vartheta, \nu)\| \leq \rho\}. \tag{15}$$

Clearly, \mathcal{S} is a closed, convex and bounded subset of the Banach space $\mathcal{E} = \mathcal{X} \times \mathcal{X}$. Define the operators $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) : \mathcal{E} \rightarrow \mathcal{E}$ and $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2) : \mathcal{S} \rightarrow \mathcal{E}$ by

$$\begin{cases} \mathcal{A}_1(\vartheta, \nu) = f_1(t, \vartheta(t), \nu(t)), & t \in [1, e], \\ \mathcal{A}_2(\vartheta, \nu) = f_2(t, \vartheta(t), \nu(t)), & t \in [1, e], \\ \mathcal{B}_1(\vartheta, \nu) = \left[\frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\sigma_1-1} \frac{h_1(s)}{s} ds \right. \\ \quad \left. + (\ln t)^{\sigma_1-1} \left(\frac{m(\vartheta)}{f_1(e, m(\vartheta), n(\nu))} - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\sigma_1-1} \frac{h_1(s)}{s} ds \right) \right], \\ \mathcal{B}_2(\vartheta, \nu) = \left[\frac{1}{\Gamma(\delta_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\delta_1-1} \frac{h_2(s)}{s} ds \right. \\ \quad \left. + (\ln t)^{\delta_1-1} \left(\frac{n(\nu)}{f_2(e, m(\vartheta), n(\nu))} - \frac{1}{\Gamma(\delta_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\delta_1-1} \frac{h_2(s)}{s} ds \right) \right]. \end{cases} \tag{16}$$

Now, the system of equations in (13) and (14) becomes

$$\mathcal{A}(\vartheta, \nu)(t)\mathcal{B}(\vartheta, \nu)(t) = (\vartheta, \nu)(t), \quad t \in [1, e], \tag{17}$$

which further, in view of the multiplication (7) of two elements in \mathcal{E} , yields

$$(\mathcal{A}_1(\vartheta, \nu)(t)\mathcal{B}_1(\vartheta, \nu)(t), \mathcal{A}_2(\vartheta, \nu)(t)\mathcal{B}_2(\vartheta, \nu)(t)) = (\vartheta, \nu)(t), \quad t \in [1, e]. \tag{18}$$

This further implies that

$$\begin{cases} \mathcal{A}_1(\vartheta, \nu)(t)\mathcal{B}_1(\vartheta, \nu)(t) = \vartheta(t), & t \in [1, e], \\ \mathcal{A}_2(\vartheta, \nu)(t)\mathcal{B}_2(\vartheta, \nu)(t) = \nu(t), & t \in [1, e]. \end{cases} \tag{19}$$

Now we prove that the operators \mathcal{A} and \mathcal{B} satisfy the conditions of Theorem 2.5 in a series of steps.

Step I. First we show that $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ is Lipschitz on \mathcal{E} with constant $(L_{f_1} + L_{f_2})$. Let $(\vartheta, \nu), (\bar{\vartheta}, \bar{\nu}) \in \mathcal{E}$ be arbitrary. Then, using (H1), we have

$$\begin{aligned} |\mathcal{A}_1(\vartheta, \nu)(t) - \mathcal{A}_1(\bar{\vartheta}, \bar{\nu})(t) &= |f_1(t, \vartheta(t), \nu(t)) - f_1(t, \bar{\vartheta}(t), \bar{\nu}(t))| \\ &\leq L_{f_1} (|\vartheta(t) - \bar{\vartheta}(t)| + |\nu(t) - \bar{\nu}(t)|) \\ &\leq L_{f_1} (\|\vartheta - \bar{\vartheta}\| + \|\nu - \bar{\nu}\|) \end{aligned}$$

for all $t \in [1, e]$. Taking the supremum over t , we obtain

$$\|\mathcal{A}_1(\vartheta, \nu) - \mathcal{A}_1(\bar{\vartheta}, \bar{\nu})\| \leq L_{f_1} (\|\vartheta - \bar{\vartheta}\| + \|\nu - \bar{\nu}\|)$$

for all $(\vartheta, \nu), (\bar{\vartheta}, \bar{\nu}) \in \mathcal{E}$. Similarly, we obtain

$$\|\mathcal{A}_2(\vartheta, \nu) - \mathcal{A}_2(\bar{\vartheta}, \bar{\nu})\| \leq L_{f_2} (\|\vartheta - \bar{\vartheta}\| + \|\nu - \bar{\nu}\|)$$

for all $(\vartheta, \nu), (\bar{\vartheta}, \bar{\nu}) \in \mathcal{E}$. Therefore, by definition of the operator \mathcal{A} , we obtain

$$\begin{aligned} \|\mathcal{A}(\vartheta, \nu) - \mathcal{A}(\bar{\vartheta}, \bar{\nu})\| &= \|(\mathcal{A}_1(\vartheta, \nu), \mathcal{A}_2(\vartheta, \nu)) - (\mathcal{A}_1(\bar{\vartheta}, \bar{\nu}), \mathcal{A}_2(\bar{\vartheta}, \bar{\nu}))\| \\ &= \|(\mathcal{A}_1(\vartheta, \nu) - \mathcal{A}_1(\bar{\vartheta}, \bar{\nu}), \mathcal{A}_2(\vartheta, \nu) - \mathcal{A}_2(\bar{\vartheta}, \bar{\nu}))\| \\ &\leq \|\mathcal{A}_1(\vartheta, \nu) - \mathcal{A}_1(\bar{\vartheta}, \bar{\nu})\| + \|\mathcal{A}_2(\vartheta, \nu) - \mathcal{A}_2(\bar{\vartheta}, \bar{\nu})\| \\ &\leq L_{f_1} (\|\vartheta - \bar{\vartheta}\| + \|\nu - \bar{\nu}\|) + L_{f_2} (\|\vartheta - \bar{\vartheta}\| + \|\nu - \bar{\nu}\|) \\ &= (L_{f_1} + L_{f_2})(\|\vartheta - \bar{\vartheta}\| + \|\nu - \bar{\nu}\|), \end{aligned}$$

for all $(\vartheta, \nu), (\bar{\vartheta}, \bar{\nu}) \in \mathcal{E}$, where $L_{\mathcal{A}} = (L_{f_1} + L_{f_2})$.

Step II. In this step we prove that $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$ from \mathcal{S} into \mathcal{E} is a compact and continuous operator. For continuity of \mathcal{B} , take any sequence of points (ϑ_n, ν_n) in \mathcal{S} converging to a point $(\vartheta, \nu) \in \mathcal{S}$. Then, by Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathcal{B}_1(\vartheta_n, \nu_n)(t) &= \lim_{n \rightarrow \infty} \left[\frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\sigma_1 - 1} \frac{h_1(s, \vartheta_n(s), \nu_n(s))}{s} ds \right. \\ &\quad \left. + (\ln t)^{\sigma_1 - 1} \left(\frac{m(\vartheta)}{f_1(e, m(x), n(\nu))} \right) \right. \\ &\quad \left. - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s} \right)^{\sigma_1 - 1} \frac{h_1(s, \vartheta_n(s), \nu_n(s))}{s} ds \right] \\ &= \frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{e}{s} \right)^{\sigma_1 - 1} \lim_{n \rightarrow \infty} \frac{h_1(s, \vartheta_n(s), \nu_n(s))}{s} ds \\ &\quad + (\ln t)^{\sigma_1 - 1} \left(\frac{m(\vartheta)}{f_1(e, m(x), n(\nu))} \right) \end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\sigma_1-1} \lim_{n \rightarrow \infty} \frac{h_1(s, \vartheta_n(s), \nu_n(s))}{s} ds \\
 & = \frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\sigma_1-1} \frac{h_1(s, \vartheta_n(s), \nu(s))}{s} ds \\
 & \quad + (\ln t)^{\sigma_1-1} \left(\frac{m(\vartheta)}{f_1(e, m(\vartheta), n(\nu))} \right) \\
 & \quad - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\sigma_1-1} \frac{h_1(s, \vartheta(s), \nu(s))}{s} ds \\
 & = \mathcal{B}_1(\vartheta, \nu)(t)
 \end{aligned}$$

for all $t \in [1, e]$. Similarly, we prove

$$\lim_{n \rightarrow \infty} \mathcal{B}_2(\vartheta_n, \nu_n)(t) = \mathcal{B}_2(\vartheta, \nu)(t)$$

for all $t \in [1, e]$. Hence $\mathcal{B}(\vartheta_n, \nu_n) = (\mathcal{B}_1(\vartheta_n, \nu_n); \mathcal{B}_2(\vartheta_n, \nu_n))$ converges to $\mathcal{B}(\vartheta, \nu)$ pointwise on $[1, e]$. This shows that \mathcal{B} is continuous.

Next, we prove \mathcal{B} is a compact operator on \mathcal{S} . Let $(\vartheta, \nu) \in \mathcal{S}$ be any point. Then, using (H4), we have

$$\begin{aligned}
 |\mathcal{B}_1(\vartheta, \nu)| & = \left| \frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\sigma_1-1} \frac{h_1(s, \vartheta(s), \nu(s))}{s} ds \right. \\
 & \quad \left. + (\ln t)^{\sigma_1-1} \left(\frac{m(\vartheta)}{f_1(e, m(\vartheta), n(\nu))} \right) \right. \\
 & \quad \left. - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\sigma_1-1} \frac{h_1(s, \vartheta(s), \nu(s))}{s} ds \right| \\
 & \leq \frac{M_{h_1}}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s}\right)^{\sigma_1-1} \frac{1}{s} ds + M_1 + \frac{M_{h_1}}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s}\right)^{\sigma_1-1} \frac{1}{s} ds \\
 & \leq \frac{2}{\Gamma(\sigma_1 + 1)} M_{h_1} + M_1
 \end{aligned}$$

for all $t \in [1, e]$.

Taking the supremum for the above inequality, we get

$$\|\mathcal{B}_1(\vartheta, \nu)\| \leq \frac{2}{\Gamma(\sigma_1 + 1)} M_{h_1} + M_1$$

for all $(\vartheta, \nu) \in \mathcal{E}$. Hence \mathcal{B}_1 is a uniformly bounded operator on \mathcal{S} . In a similar way, we can prove \mathcal{B}_2 is also uniformly bounded operator on \mathcal{S} . Hence \mathcal{B} is a uniformly bounded operator on \mathcal{S} .

Next to prove that $\{\mathcal{B}(\vartheta_n, \nu_n)\}$ is an equicontinuous sequence of functions in \mathcal{E} , assume $\tau_1, \tau_2 \in [1, e]$, provided $\tau_1 < \tau_2$, then

$$\begin{aligned}
 & |\mathcal{B}_1(\vartheta, \nu)(\tau_1) - \mathcal{B}_1(\vartheta, \nu)(\tau_2)| \\
 & \leq \frac{M_{h_1}}{\Gamma(\sigma_1)} \left| \int_1^{\tau_1} \left(\ln \frac{\tau_1}{s}\right)^{\sigma_1-1} \frac{1}{s} ds - \int_1^{\tau_2} \left(\ln \frac{\tau_2}{s}\right)^{\sigma_1-1} \frac{1}{s} ds \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{M_{h_1} + |(\ln \tau_2)^{\sigma_1-1} - (\ln \tau_1)^{\sigma_1-1}|}{\Gamma(\sigma_1)} \right) \int_1^e \left(\ln \frac{e}{s} \right)^{\sigma_1-1} \frac{1}{s} ds \\
 \leq & \frac{M_{h_1}}{\Gamma(\sigma_1)} \left| \int_1^{\tau_1} \left[\left(\ln \frac{\tau_1}{s} \right)^{\sigma_1-1} - \left(\ln \frac{\tau_2}{s} \right)^{\sigma_1-1} \right] \frac{1}{s} ds \right| \\
 & + \frac{M_{h_1}}{\Gamma(\sigma_1)} \left| \int_{\tau_1}^{\tau_2} \left(\ln \frac{\tau_2}{s} \right)^{\sigma_1-1} \frac{1}{s} ds \right| \\
 & + \frac{M_{h_1}}{\Gamma(\sigma_1)} |(\ln \tau_2)^{\sigma_1-1} - (\ln \tau_1)^{\sigma_1-1}| \int_1^e \left(\ln \frac{e}{s} \right)^{\sigma_1-1} \frac{1}{s} ds \\
 \rightarrow & 0 \quad \text{as } \tau_1 \rightarrow \tau_2,
 \end{aligned}$$

uniformly for all $(\vartheta, \nu) \in \mathcal{E}$. Similarly we have

$$|\mathcal{B}_2(\vartheta, \nu)(\tau_1) - \mathcal{B}_2(\vartheta, \nu)(\tau_2)| \rightarrow 0 \quad \text{as } \tau_1 \rightarrow \tau_2$$

uniformly for all $(\vartheta, \nu) \in S$. This shows that $\mathcal{B}(S) \subset \mathcal{E}$ is uniformly bounded, equicontinuous and compact, in view of Arzela–Ascoli theorem. Consequently, \mathcal{B} is a compact and continuous operator on \mathcal{E} .

Step III. Now we prove that the third condition (a_3) of Theorem 2.5 holds. Let (ϑ, ν) and $(\bar{x}, \bar{y}) \in \mathcal{E} = \mathcal{X} \times \mathcal{X}$ be such that

$$(\vartheta, \nu) = (\mathcal{A}_1(\vartheta, \nu)\mathcal{B}_1(\bar{x}, \bar{y}), \mathcal{A}_2(\vartheta, \nu)\mathcal{B}_2(\bar{x}, \bar{y})).$$

Then, we have

$$\begin{aligned}
 |\vartheta(t)| & = |\mathcal{A}_1(\vartheta, \nu)(t)\mathcal{B}_1(\bar{x}, \bar{y})(t)| \\
 & \leq |\mathcal{A}_1(\vartheta, \nu)(t)\mathcal{B}_1(\bar{x}, \bar{y})(t)| \\
 & \leq [|f_1(t, \vartheta, \nu) - f_1(t, 0, 0)| + |f_1(t, 0, 0)|] \\
 & \quad \times \left(\frac{1}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\sigma_1-1} \frac{h_1(s, \bar{x}(s), \bar{y}(s))}{s} ds \right. \\
 & \quad + (\ln t)^{\sigma_1-1} \left[\frac{m(\bar{x})}{f_2(e, m(\bar{x}), n(\bar{y}))} \right. \\
 & \quad \left. \left. - \frac{1}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s} \right)^{\sigma_1-1} \frac{h_1(s, \bar{x}(s), \bar{y}(s))}{s} ds \right] \right) \\
 & \leq [L_{f_1}(\|\vartheta\| + \|\nu\|) + F_0] \\
 & \quad \times \left(M_1 + \frac{M_{h_1}}{\Gamma(\sigma_1)} \int_1^t \left(\ln \frac{t}{s} \right)^{\sigma_1-1} \frac{1}{s} ds \right. \\
 & \quad \left. + \frac{M_{h_1}}{\Gamma(\sigma_1)} \int_1^e \left(\ln \frac{e}{s} \right)^{\sigma_1-1} \frac{1}{s} ds \right) \\
 & \leq [L_{f_1}(\|\vartheta\| + \|\nu\|) + F_0] \left(\frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 \right).
 \end{aligned}$$

Thus, by taking the supremum of the left-hand side, we get

$$\|\vartheta\| \leq [L_{f_1}(\|\vartheta\| + \|v\|) + F_0] \left(\frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 \right). \tag{20}$$

Similarly, proceeding as before, we obtain

$$\|v\| \leq [L_{f_2}(\|\vartheta\| + \|v\|) + G_0] \left(\frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} + M_2 \right). \tag{21}$$

Adding inequalities (20) and (21), we obtain

$$\begin{aligned} \|\vartheta(t)\| + \|v(t)\| &\leq [L_{f_1}(\|\vartheta\| + \|v\|) + F_0] \left(\frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 \right) \\ &\quad + [L_{f_2}(\|\vartheta\| + \|v\|) + G_0] \left(\frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} + M_2 \right) \\ &\leq \frac{F_0 \left(\frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 \right) + G_0 \left(\frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} + M_2 \right)}{1 - ([L_{f_1} \left(\frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 \right)] + [L_{f_2} \left(\frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} + M_2 \right)])} \\ &\leq \frac{F_0 \left(\frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 \right) + G_0 \left(\frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} + M_2 \right)}{1 - \Omega} \\ &\leq \rho. \end{aligned}$$

As $\|(\vartheta, v)\| = \|\vartheta\| + \|v\|$, we have that $\|(\vartheta, v)\| \leq \rho$ and so condition (a₃) of Theorem 2.5 holds.

Step IV. We now prove that $M_B < 1$. Consider

$$\begin{aligned} M_B &= \|\mathcal{B}(S)\| \\ &= \sup\{\|\mathcal{B}(\vartheta, v)\| : (\vartheta, v) \in S\} \\ &= \sup\{\|\mathcal{B}_1(\vartheta, v)\| + \|\mathcal{B}_2(\vartheta, v)\| : (\vartheta, v) \in S\} \\ &\leq \frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} + M_1 + \frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} + M_2 \\ &< 1. \end{aligned}$$

From the above estimate, we conclude that condition (a₄) of Theorem 2.5 also holds.

Thus, the operators \mathcal{A} and \mathcal{B} satisfy all the conditions of Theorem 2.5 and so, for the considered values of h_1 and h_2 , the coupled system of equations (3) has at least one solution on $[1, e]$. □

4 An example

Example 4.1 Consider the following coupled system of Hadamard fractional differential equations with boundary conditions:

$$\begin{aligned} D^{3/2} \left(\frac{|\vartheta(t)|}{\frac{1}{2}(1 + |\sin \vartheta(t)|)} \right) &= \frac{|\vartheta(t)|}{(24 + i + t)(1 + |\vartheta(t)| + |v(t)|)}, \quad 1 \leq t \leq e, \\ D^{3/2} \left(\frac{|v(t)|}{\frac{1}{2}(1 + |\cos \vartheta(t)|)} \right) &= \frac{|v(t)|}{(20 + i + t)|\vartheta(t)|}, \quad 1 \leq t \leq e, \end{aligned} \tag{22}$$

$$\begin{aligned}
 u(0) &= 0, & \vartheta(e) &= \frac{1}{16} \sin \vartheta(\eta), & \eta &\in (0, 1) \\
 v(0) &= 0, & v(e) &= \frac{1}{18} \cos v(\xi), & \xi &\in (0, 1).
 \end{aligned}
 \tag{23}$$

Here

$$\begin{aligned}
 L_{f_1} &= L_{f_2} = \frac{1}{2}, & M_1 &= \frac{1}{16}, \\
 M_2 &= \frac{1}{18}, & M_{h_1} &= \frac{1}{25}, & M_{h_2} &= \frac{1}{21}, \\
 L_{f_1} \left(M_1 + \frac{2M_{h_1}}{\Gamma(\sigma_1 + 1)} \right) &+ L_{f_2} \left(M_2 + \frac{2M_{h_2}}{\Gamma(\delta_1 + 1)} \right) &\simeq &0.124936655 < 1.
 \end{aligned}$$

Hence, all the conditions of Theorem 3.3 are satisfied; consequently, the problem (22) and (23) has a solution on $[1, e]$.

5 Conclusions

We have presented the existence result for boundary value problems of coupled hybrid differential systems involving the Hadamard fractional derivative. The proof of the existence result is based on the fixed point theorem due to Dhage [10]. The present work can be extended to coupled systems of nonlinear fractional differential equations involving Caputo–Hadamard fractional derivative.

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Author details

¹Department of Mathematics, Sona College of Technology, Salem, 636 005 Tamil Nadu, India. ²Department of Mathematics, Sri Vasavi College, Erode, 638 316 Tamil Nadu, India. ³Department of Mathematics, Cankaya University, Ankara, Turkey. ⁴Institute of Space Sciences, 06530 Magurele-Bucharest, Romania. ⁵Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

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