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An iterative algorithm for robust simulation of the Sylvester matrix differential equations

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Abstract

This paper proposes a new effective pseudo-spectral approximation to solve the Sylvester and Lyapunov matrix differential equations. The properties of the Chebyshev basis operational matrix of derivative are applied to convert the main equation to the matrix equations. Afterwards, an iterative algorithm is examined for solving the obtained equations. Also, the error analysis of the propounded method is presented, which reveals the spectral rate of convergence. To illustrate the effectiveness of the proposed framework, several numerical examples are given.

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Keywords: Sylvester matrix differential equations; Iterative algorithm; Chebyshev polynomials; Coupled linear matrix equations; Collocation method

1 Introduction

For study on the dynamical systems, filtering, model reduction, image restoration, etc., many phenomena can be modeled more efficiently by the matrix differential equations [1–5]. We consider the subsequent Sylvester matrix differential equations

$$\begin{cases} P'(t) = A(t)P(t) + P(t)B(t) + Q(t), & t_0 \leq t \leq t_f, \\ P(t_0) = P_0, \end{cases} \quad (1)$$

where $P \in \mathbb{R}^{p \times q}$ is an unknown matrix, the matrices $P_0 \in \mathbb{R}^{p \times q}$, $A(t) : [t_0, t_f] \rightarrow \mathbb{R}^{p \times p}$, $B(t) : [t_0, t_f] \rightarrow \mathbb{R}^{q \times q}$, and $Q(t) : [t_0, t_f] \rightarrow \mathbb{R}^{p \times q}$ are given. We assume $A(t), B(t), Q(t) \in C^s[t_0, t_f]$, $s \geq 1$. In the particular case, where $B(t)$ is the transpose of $A(t)$, system (1) is called the Lyapunov differential equation. Such equations occur frequently in various fields of science and are widely used in control problems and theory of stability in time varying systems [1, 4]. The analytical and numerical approaches have been studied by several authors to solve the Sylvester equations [6–17].

In recent years spectral collocation methods have received attention of many researchers [18–20]. The ease of implying and the exponential rate of convergence are two main advantages of these methods [21, 22]. The main contribution of the current paper is to implement the Chebyshev collocation method to evaluate (1). With the aid of collocation points,

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we obtain a coupled linear matrix equations where their solution gives the unknown coefficients. In the literature, finding a solution to different kinds of linear and coupled matrix equations is a subject of interest and has been studied extensively. Several approaches have been established for solving the mentioned equations; for example, the idea of conjugate gradient method, gradient-based iterative algorithm, Paige algorithm, and Krylov subspace methods; for more details, see [13, 23–27] and the references therein. We propose an iterative algorithm based on Paige’s algorithm [28] to solve the obtained coupled matrix equations.

In Sect. 2, we first review some definitions and notations. Then, we review some of the necessary properties of the Chebyshev polynomials for our latest developments. In Sect. 3, we employ the Chebyshev basis to reduce problem (1) to the solution of coupled matrix equations. Then, a new iterative algorithm is presented to solve the obtained coupled matrix equations. Moreover, we give an error estimation of the proposed method. Section 4 is dedicated to numerical simulations. Finally, a conclusion is provided.

2 Preliminaries

We review Paige’s algorithm and some basic definitions and properties of matrix algebra and the Chebyshev polynomials.

The main idea behind of Paige’s algorithm [27–29] is using bidiagonalization algorithm as a basis for solution of

$$Ax = b, \quad A \in \mathbb{R}^{n \times m}, \quad B \in \mathbb{R}^n.$$

The solution generated by Algorithm 1 is the minimum Euclidean norm solution of the computational importance of the algorithms in their application to very large problems with sparse matrices. In this case the computation per step and storage is about as small as could be hoped. Moreover, theoretically the number of steps will be no greater than the minimum dimension of A .

Algorithm 1 Paige’s algorithm

- $$\tau_0 = 1; \quad \xi_0 = -1; \quad \theta_0 = 0;$$
- $$z_0; \quad \omega_0 = 0; \quad \beta_1 u_1 = b; \quad \alpha_1 v_1 = A^T u_1.$$
- (a) $\xi_i = -\xi_{i-1} \beta_i / \alpha_i; \quad z_i = z_{i-1} + \xi_i v_i;$
 - (b) $\theta_i = (\tau_{i-1} - \beta_i \theta_{i-1}) / \alpha_i; \quad \omega_i = \omega_{i-1} + \theta_i v_i;$
 - (c) $\beta_{i+1} u_{i+1} = A v_i - \alpha_i u_i;$
 - (d) $\tau_i = -\tau_{i-1} \alpha_i / \beta_{i+1};$
 - (e) $\alpha_{i+1} v_{i+1} = A^T u_{i+1} - \beta_{i+1} v_i;$
 - (f) $\gamma_i = \beta_{i+1} \xi_i / (\beta_{i+1} \theta_i - \tau_i);$
 - (g) $x_i = z_i - \gamma_i \omega_i.$
-

Definition 1 ([30]) Consider two matrix groups, $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_k)$ and $\tilde{\mathcal{X}} = (\tilde{\mathcal{X}}_1, \tilde{\mathcal{X}}_2, \dots, \tilde{\mathcal{X}}_k)$, where $\mathcal{X}_j, \tilde{\mathcal{X}}_j \in \mathbb{R}^{p \times q}$ for $j = 1, 2, \dots, k$. The inner product $\langle \cdot, \cdot \rangle$ is

$$\langle \mathcal{X}, \tilde{\mathcal{X}} \rangle := \sum_{j=1}^k \text{tr}(\mathcal{X}_j^T \tilde{\mathcal{X}}_j). \tag{2}$$

Remark 1 The norm of \mathcal{X} is $\|\mathcal{X}\|^2 := \sum_{j=1}^k \text{tr}(\mathcal{X}_j^T \mathcal{X}_j)$.

Definition 2 ([30]) Let $L^2_\omega[t_0, t_f]$ and $\omega(x)$ be the weight function, the inner product and norm in L^2_ω are defined as follows:

$$\begin{aligned} \langle f, g \rangle_{L^2_\omega} &= \int_{t_0}^{t_f} f(x)g(x)\omega(x) dx, \quad \forall f, g \in L^2_\omega[t_0, t_f], \\ \|f\|_{L^2_\omega}^2 &= \langle f, f \rangle_{L^2_\omega} = \int_{t_0}^{t_f} f^2(x)\omega(x) dx, \quad \forall f \in L^2_\omega[t_0, t_f]. \end{aligned}$$

Definition 3 ([30]) A function $f : [t_0, t_f] \rightarrow \mathbb{R}$, belongs to the Sobolev space $H^{k,2}_\omega$ if its j th weak derivative lies in $L^2_\omega[t_0, t_f]$ for $0 \leq j \leq k$ with the norm

$$\|f(x)\|_{H^{k,2}_\omega(t_0, t_f)} = \left(\sum_{j=0}^k \|f^{(j)}(x)\|_{L^2_\omega}^2 \right)^{\frac{1}{2}}.$$

The Chebyshev polynomials are defined on the interval $[-1, 1]$ and determined by recurrence formulae

$$\begin{aligned} T_0(t) &= 1, & T_1(t) &= t, \\ T_{i+1}(t) &= 2tT_i(t) - T_{i-1}(t), & i &= 1, 2, 3, \dots \end{aligned}$$

These polynomials are orthogonal with $\omega(t) = \frac{1}{\sqrt{1-t^2}}$. With change of variable

$$t = \frac{2(x - t_0) - h}{h}, \quad t_0 \leq x \leq t_f,$$

where $h = t_f - t_0$, the shifted Chebyshev polynomials in x on the interval $[t_0, t_f]$ are obtained as follows:

$$\begin{aligned} \phi_0(x) &= 1, \\ \phi_1(x) &= \frac{2(x - t_0) - h}{h}, \end{aligned}$$

and for $i = 1, 2, 3, \dots$,

$$\phi_{i+1}(x) = \frac{2(2(x - t_0) - h)}{h} \phi_i(x) - \phi_{i-1}(x).$$

The orthogonality condition for the shifted Chebyshev basis functions is

$$\int_{t_0}^{t_f} \omega(x)\phi_i(x)\phi_j(x) dx = \frac{h}{2} \begin{cases} \pi, & \text{for } i = j = 0, \\ \frac{\pi}{2}, & \text{for } i = j \neq 0, \\ 0, & \text{for } i \neq j, \end{cases}$$

where $\omega(x) = \frac{1}{\sqrt{1 - (\frac{2(x-t_0)-h}{h})^2}}$.

It is well known that $Y = \text{span}\{\phi_0, \phi_1, \dots, \phi_m\}$ is a complete subspace of $H = L^2_\omega[t_0, t_f]$. So, $f \in H$ has unique best approximation such as y_m , that is,

$$f(x) \approx y_m(x) = \sum_{j=0}^m c_j \phi_j(x) = C^T \Phi(x),$$

where $C^T = (c_0, c_1, \dots, c_m)$ such that c_j s are uniquely calculated by $c_j = \frac{(f, \phi_j)_{L^2_\omega}}{\|\phi_j\|^2_{L^2_\omega}}$ and

$$\Phi^T(x) = (\phi_0(x), \phi_1(x), \dots, \phi_m(x)). \tag{3}$$

For more details, see [31].

Proposition 1 ([22]) *Assume that $f \in H^{k,2}_\omega(t_0, t_f)$ and $\mathcal{P}_m f = \sum_{i=0}^m c_i \phi_i$ is the truncated orthogonal Chebyshev series of f and $\mathcal{I}_m f$ is the interpolation of f in the Chebyshev–Gauss points. Then*

$$\|f - \mathcal{P}_m f\|_{L^2_\omega(t_0, t_f)} \leq Ch^{\min(k,m)} m^{-k} |f|_{H^{k,2}_\omega(t_0, t_f)}, \tag{4}$$

$$\|f - \mathcal{I}_m f\|_{L^2_\omega(t_0, t_f)} \leq \tilde{C}h^{\min(k,m)} m^{-k} |f|_{H^{k,2}_\omega(t_0, t_f)}, \tag{5}$$

where

$$|f|_{H^{k,2}_\omega(t_0, t_f)} = \left(\sum_{j=\min(k,m+1)}^k \|f^{(j)}\|^2_{L^2_\omega(t_0, t_f)} \right)^{1/2},$$

$h = t_f - t_0$ and C, \tilde{C} are constants independent of m and f .

The derivative of $\Phi(x)$, defined in (3), can be given by

$$\frac{d\Phi(x)}{dx} \approx D\Phi(x),$$

where the $(m + 1) \times (m + 1)$ matrix D is called the operational matrix of derivative for the Chebyshev polynomials in $[t_0, t_f]$. Straightforward computations show that each entry of $D = [d_{ij}]_{(m+1) \times (m+1)}$ for $i, j = 1, \dots, m + 1$ is defined as follows [32]:

$$d_{ij} = \frac{1}{h} \begin{cases} 4(i - 1), & \text{if } i + j \text{ is odd and } 1 < j < i, \\ 2(i - 1), & \text{if } j = 1 \text{ and } i \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

3 Main results

Let us approximate each entry of $P(t) = [p_{ij}(t)]_{p \times q}$ in (1) by the Chebyshev polynomials. Consequently, we have

$$p_{ij}(t) \approx \mathcal{A}_{ij}\Phi(t), \quad i = 1, \dots, p, j = 1, \dots, q, \tag{6}$$

where $\mathcal{A}_{ij} \in \mathbb{R}^{1 \times (m+1)}$ are the unknown row vectors and m is the order of the Chebyshev polynomial. We can write

$$P(t) \approx \overline{\mathcal{A}}(I_q \otimes \Phi(t)), \tag{7}$$

where notation \otimes denotes the Kronecker product of two matrixes, $\overline{\mathcal{A}} \in \mathbb{R}^{p \times q(m+1)}$ and

$$\overline{\mathcal{A}} = (\mathcal{A}_{ij}), \quad i = 1, \dots, p, j = 1, \dots, q.$$

Thus

$$P'(t) \approx \overline{\mathcal{A}}(I_q \otimes D\Phi(t)). \tag{8}$$

By substituting equations (7) and (8) in (1), we derive

$$\overline{\mathcal{A}}(I_q \otimes D\Phi(t)) = A(t)\overline{\mathcal{A}}(I_q \otimes \Phi(t)) + \overline{\mathcal{A}}(I_q \otimes \Phi(t))B(t) + Q(t) + R_m(t). \tag{9}$$

We discretize the above equation at m points ξ_i ($1 \leq i \leq m$) such that $R_m(\xi_i) = 0_{p \times q}$. The selected collocation points are the roots of $\phi_m(t)$ (the Chebyshev–Gauss nodes in $[t_0, t_f]$). These m roots that we use as the collocation nodes are defined by

$$\xi_i = \frac{t_f - t_0}{2} \left(\cos \frac{(2i - 1)\pi}{2m} + 1 \right) + t_0, \quad i = 1, \dots, m,$$

which are all in $[t_0, t_f]$. By replacing ξ_i nodes in (9), we obtain the coupled equations

$$\overline{\mathcal{A}}C_i = \mathcal{D}_i\overline{\mathcal{A}}\mathcal{E}_i + \overline{\mathcal{A}}\mathcal{F}_i + \mathcal{G}_i, \quad i = 1, 2, \dots, m,$$

where $C_i = I_q \otimes D\Phi(\xi_i)$, $\mathcal{D}_i = A(\xi_i)$, $\mathcal{E}_i = I_q \otimes \Phi(\xi_i)$, $\mathcal{F}_i = (I_q \otimes \Phi(\xi_i))B(\xi_i)$, and $\mathcal{G}_i = Q(\xi_i)$. Moreover, from the initial condition we set $\overline{\mathcal{A}}(I_q \otimes \Phi(t_0)) = P(t_0)$ and define $\xi_0 = t_0$, $C_0 = 0_{q(m+1) \times q}$, $\mathcal{D}_0 = I_p$, $\mathcal{E}_0 = I_q \otimes \Phi(t_0)$, $\mathcal{F}_0 = 0_{q(m+1) \times q}$, and $\mathcal{G}_0 = -P(t_0)$. Therefore, we may solve the coupled equations

$$X\mathcal{H}_i - \mathcal{D}_iX\mathcal{E}_i = \mathcal{G}_i, \quad i = 0, 1, \dots, m, \tag{10}$$

where $\mathcal{H}_i = C_i - \mathcal{F}_i$, \mathcal{G}_i , \mathcal{D}_i and $X := \overline{\mathcal{A}}$. Using the following relation [2]

$$(A\vec{X}B) = (B^T \otimes A)(\vec{X})$$

and the operator “vec” which transforms a matrix A of size $m \times s$ to a vector $a = \text{vec}(A)$ of size $ms \times 1$ by stacking the columns of A , equations (10) are equivalent to the linear

system $Ax = b$ with the following parameters:

$$\begin{cases} A_i = \mathcal{H}_i^T \otimes I_p - \mathcal{E}_i^T \otimes \mathcal{D}_i \in \mathbb{R}^{pq \times pq(m+1)}, \\ x = \text{vec}(X) \in \mathbb{R}^{pq(m+1) \times 1}, \quad b_i = \text{vec}(\mathcal{G}_i) \in \mathbb{R}^{pq \times 1}, \end{cases} \quad \text{for } i = 0, 1, \dots, m,$$

where A_i and b_i denote the rows of A and b , respectively, and I_p is the identity matrix of order p . We can solve the above linear system by classical schemes such as Krylov subspace methods, but the size of the coefficient matrix may be too large. So, it is more advisable to apply an iterative scheme to solve the coupled linear matrix equations rather than the linear system.

3.1 Solving the coupled matrix equations

We propose a new iterative algorithm to solve (10), which is essentially based on Paige’s algorithm. Recently, Paige’s algorithm has been extended to find the bisymmetric minimum norm solution of the coupled linear matrix equations [27]

$$\begin{aligned} A_1XB_1 &= C_1, \\ A_2XB_2 &= C_2. \end{aligned}$$

Using the “ $\text{vec}(\cdot)$ ” operator, the authors elaborated on how Paige’s algorithm can be derived. The reported results reveal the superior convergence properties of their algorithm in comparison to the algorithms derived via the extension of the conjugate gradient method [29], which was presented in the literature for solving different types of coupled linear matrix equations. This motivates us to generalize Paige’s algorithm to resolve (10).

For simplicity, let us consider the linear operator \mathcal{M} defined as

$$\begin{aligned} \mathcal{M} : \mathbb{R}^{p \times q(m+1)} &\rightarrow \underbrace{\mathbb{R}^{p \times q} \times \dots \times \mathbb{R}^{p \times q}}_{(m+1)\text{-fold}}, \\ X &\mapsto \mathcal{M}(X) = (M_0(X), M_1(X), \dots, M_m(X)), \end{aligned}$$

where $M_i(X) = X\mathcal{H}_i - \mathcal{D}_iX\mathcal{E}_i$, $i = 0, 1, \dots, m$. Using the above operator for equation (10), we have

$$\mathcal{M}(X) = \mathcal{G},$$

where $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1, \dots, \mathcal{G}_m)$ and $\mathcal{G}_i \in \mathbb{R}^{p \times q}$, $i = 0, 1, \dots, m$. Suppose that the linear operator \mathcal{D} is defined as

$$\begin{aligned} \mathcal{D} : \underbrace{\mathbb{R}^{p \times q} \times \dots \times \mathbb{R}^{p \times q}}_{(m+1)\text{-fold}} &\rightarrow \mathbb{R}^{p \times q(m+1)}, \\ Y = (Y_0, Y_1, \dots, Y_m) &\mapsto \mathcal{D}(Y), \end{aligned}$$

where $\mathcal{D}(Y) = \sum_{i=0}^m (Y_i\mathcal{H}_i^T - \mathcal{D}_i^T Y_i\mathcal{E}_i^T)$.

In Algorithm 2, ϵ is the given small tolerance to compute the unique minimum Frobenius norm, and for computational purposes, we choose $\epsilon = 10^{-14}$.

Algorithm 2 The process of solving matrix equations (10)

1. $\tau_0 = 1; \xi_0 = -1; \theta_0 = 0; Z_0 = 0_{p \times (m+1)q}; W_0 = Z_0;$

$$\beta_1 = \|\mathcal{G}\|; \quad U^{(1)} = \mathcal{G}/\beta_1;$$

$$\bar{V}_1 = \mathcal{D}(U^{(1)}); \quad \alpha_1 = \|\bar{V}_1\|; \quad V_1 = \bar{V}_1/\alpha_1;$$

2. For $i = 1, 2, \dots$, compute:

- $\xi_i = -\xi_{i-1}\beta_i/\alpha_i; Z_i = Z_{i-1} + \xi_i V_i;$
- $\theta_i = (\tau_{i-1} - \beta_i\theta_{i-1})/\alpha_i; W_i = W_{i-1} + \theta_i V_i;$
- $\bar{U}^{(i+1)} = \mathcal{M}(V_i) - \alpha_i U^{(i)};$
- $\beta_{i+1} = \|\bar{U}^{(i+1)}\|; U^{(i+1)} = \bar{U}^{(i+1)}/\beta_{i+1};$
- $\tau_i = -\tau_{i-1}\alpha_i/\beta_{i+1};$
- $\bar{V}_{i+1} = \mathcal{D}(U^{(i+1)}) - \beta_{i+1} V_i;$
- $\alpha_{i+1} = \|\bar{V}_{i+1}\|; V_{i+1} = \bar{V}_{i+1}/\alpha_{i+1};$
- $\gamma_i = \beta_{i+1}\xi_i/(\beta_{i+1}\theta_i - \tau_i);$
- $X_i = Z_i - \gamma_i W_i;$

3. If $\|\mathcal{G} - \mathcal{M}(X_i)\| \leq \epsilon$ then Stop; Otherwise go to 2.

3.2 Implementing the method

We employ the step-by-step method for solving (1). To do so, we choose $s \neq 0$, starting with $x_0 := t_0, Z_0 := P(x_0)$ and considering the points $x_i = x_0 + is, i = 1, 2, 3, \dots$. On each subinterval $[x_i, x_{i+1}]$ for $i = 0, 1, \dots, [\frac{t_f - t_0}{s}] - 1$, by solving the following equations

$$\begin{cases} Z'(t) = A(t)Z(t) + Z(t)B(t) + Q(t), & x_i \leq t < x_{i+1}, \\ Z(x_i) = Z_i, \end{cases}$$

with the framework described in Sect. 3.1, we consecutively approximate $P(t)$ by $Z(t)$. Then, to compute the approximate solution $Z(t)$ of $P(t)$ on the next subinterval, we set $Z_{i+1} = Z(x_{i+1})$.

3.3 Error estimation

We illustrate an error analysis based on the notion employed to the Volterra type integral equations [33, 34].

Definition 4 Let $F(x) = [f_{ij}(x)]$ be a matrix of order $p \times q$ defined on $[t_0, t_f]$ such that $f_{ij}(x) \in L^2_\omega[t_0, t_f]$. Then we define

$$\|F\|_\infty = \max_{i,j} \|f_{ij}(x)\|_{L^2_\omega}, \quad 1 \leq i \leq p, 1 \leq j \leq q.$$

Theorem 1 Consider the Sylvester matrix differential problem (1), where $p_{ij} \in H^k_\omega(x_i, x_{i+1}), h_l = x_{l+1} - x_l, A(t) = [a_{ij}(t)]_{p \times p}, B(t) = [b_{ij}(t)]_{q \times q}$, and $Q(t) = [q_{ij}(t)]_{p \times q}$ are given so that $a_{ij}(t), b_{ij}(t)$ and $q_{ij}(t)$ are sufficiently smooth. Also, assume that $P_m = \bar{A}(I_q \otimes \Phi)$ denotes the approximation of P . Furthermore, suppose that $M_1 = \max_{i,j} \max_{x_l \leq t \leq x_{l+1}} |a_{ij}(t)|$ and $M_2 =$

$\max_{i,j} \max_{x_l \leq t \leq x_{l+1}} |b_{ij}(t)|$. Then the following statement holds:

$$\|P - P_m\|_\infty \leq \tilde{C} h_l^{\min(k,m)} \left(h_l m^{-1-k} \max_{i,j} \left(M_1 \sum_{\nu=1}^p |p_{\nu j}|_{H_\omega^k(x_l, x_{l+1})} + M_2 \sum_{\nu=1}^q |p_{\nu i}|_{H_\omega^k(x_l, x_{l+1})} \right) + m^{-k} \max_{i,j} |p_{ij}|_{H_\omega^k(x_l, x_{l+1})} \right),$$

where \tilde{C} is a finite constant.

Proof Integrating (1) in $[x_l, x]$ results in

$$P(x) = \int_{x_l}^x (A(t)P(t) + P(t)B(t) + Q(t)) dt + P(x_l). \tag{11}$$

For $\xi_0 = x_l$ and the Chebyshev–Gauss nodes $\xi_n, 1 \leq n \leq m$, on the interval $[x_l, x_{l+1}]$, we have

$$P_m(\xi_n) = \int_{x_l}^{\xi_n} (A(t)P_m(t) + P_m(t)B(t) + Q(t)) dt + P(x_l), \quad n = 0, 1, \dots, m. \tag{12}$$

From (12), we obtain

$$P_m(\xi_n) = \int_{x_l}^{\xi_n} (A(t)E(t) + E(t)B(t)) dt + \int_{x_l}^{\xi_n} (A(t)P(t) + P(t)B(t) + Q(t)) dt + P(x_l), \tag{13}$$

where $E = [e_{ij}]_{p \times q} = P_m - P$. For L_n as the Lagrange interpolating polynomial, we have

$$\sum_{n=0}^m L_n(x) P_m(\xi_n) = \sum_{n=0}^m L_n(x) \left(\int_{x_l}^{\xi_n} (A(t)E(t) + E(t)B(t)) dt \right) + \sum_{n=0}^m L_n(x) \left(\int_{x_l}^{\xi_n} (A(t)P(t) + P(t)B(t) + Q(t)) dt + P(x_l) \right).$$

Subtracting this equation from (11) yields

$$\sum_{n=0}^m L_n(x) P_m(\xi_n) - P(x) = \int_{x_l}^x (A(t)E(t) + E(t)B(t)) dt + R_1(x) + R_2(x), \tag{14}$$

where

$$R_1(x) = \sum_{n=0}^m L_n(x) \int_{x_l}^{\xi_n} (A(t)E(t) + E(t)B(t)) dt - \int_{x_l}^x (A(t)E(t) + E(t)B(t)) dt$$

and

$$R_2(x) = \sum_{n=0}^m L_n(x) \left(\int_{x_l}^{\xi_n} (A(t)P(t) + P(t)B(t) + Q(t)) dt + P(x_l) \right) - \int_{x_l}^x (A(t)P(t) + P(t)B(t) + Q(t)) dt - P(x_l).$$

Since P_m is a polynomial of order m , we may rewrite (14) in the form

$$E(x) = \int_{x_l}^x (A(t)E(t) + E(t)B(t)) dt + R_1(x) + R_2(x). \tag{15}$$

By implying the Gronwall inequality on (15), we have

$$\|E\|_\infty \leq C \|R_1 + R_2\|_\infty. \tag{16}$$

Since $A(t)$, $B(t)$, and $Q(t)$ are sufficiently smooth, for $R_1(x)$ and $R_2(x)$ we obtain the following results. From Definition 4,

$$\|R_1(x)\|_\infty = \max_{ij} \|\mathcal{I}_m f - f\|_{L_\omega^2},$$

in which $f(x) = \int_{x_l}^x (\sum_{v=1}^p a_{iv}(t)e_{vj}(t) + \sum_{v=1}^q e_{iv}(t)b_{vj}(t)) dt$. Using (5) for $k = 1$ and (4), it can be deduced that

$$\begin{aligned} \|R_1(x)\|_\infty &\leq C_1 h_l m^{-1} \max_{ij} \left\| \sum_{v=1}^p a_{iv} e_{vj} + \sum_{v=1}^q e_{iv} b_{vj} \right\|_{L_\omega^2} \\ &\leq C_1 h_l^{\min(k,m)+1} m^{-1-k} \\ &\quad \times \max_{ij} \left(M_1 \sum_{v=1}^p |p_{vj}|_{H_\omega^k(x_l, x_{l+1})} + M_2 \sum_{v=1}^q |p_{iv}|_{H_\omega^k(x_l, x_{l+1})} \right). \end{aligned} \tag{17}$$

Also, for $R_2(x)$, we derive that

$$\begin{aligned} \|R_2(x)\|_\infty &= \max_{ij} \|\mathcal{I}_m p_{ij} - p_{ij}\|_{L_\omega^2} \\ &\leq C_2 h_l^{\min(k,m)} m^{-k} \max_{ij} |p_{ij}|_{H_\omega^k(x_l, x_{l+1})}. \end{aligned} \tag{18}$$

Now the assertion can be concluded from (16)–(18) immediately. □

4 Numerical simulations

In this section, we show the application of the Chebyshev collocation method to solve (1). We would like to point out that at each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \dots$, Algorithm 2 is applied. We use the notations

$$Err = \max_{ij} \|P_{ij} - Z_{ij}\|_\infty = \max_{ij} \max_{x_l \leq x \leq x_{l+1}} |P_{ij}(t) - Z_{ij}(t)|, \tag{19}$$

$$E(t) = \|P(t) - Z(t)\|_F, \tag{20}$$

Table 1 Err comparison of the spline method [8] and the proposed method for equations (21)–(22)

	Interval				
	[0, 0.1]	[0.1, 0.2]	[0.2, 0.3]	[0.3, 0.4]	[0.4, 0.5]
Our method	1.63806×10^{-12}	1.48828×10^{-12}	1.35303×10^{-12}	1.22071×10^{-12}	1.11135×10^{-12}
Spline algorithm [8]	2.6999×10^{-10}	5.1438×10^{-10}	7.36×10^{-10}	9.38797×10^{-10}	1.1268×10^{-9}
	Interval				
	[0.5, 0.6]	[0.6, 0.7]	[0.7, 0.8]	[0.8, 0.9]	[0.9, 1]
Our method	1.01325×10^{-12}	9.25324×10^{-13}	8.47024×10^{-13}	7.77505×10^{-13}	7.15577×10^{-13}
Spline algorithm [8]	1.30572×10^{-9}	1.48252×10^{-9}	1.66579×10^{-9}	1.86603×10^{-9}	2.09601×10^{-9}

where $Z(t) = [Z_{ij}(t)]_{p \times q}$ and $P(t) = [P_{ij}(t)]_{p \times q}$ denote the approximate and exact solutions, respectively.

Example 1 Consider the Sylvester equation [8]

$$\begin{cases} P'(t) = A(t)P(t) + P(t)B(t) + Q(t), \\ P(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0, 1], \end{cases} \tag{21}$$

where

$$\begin{aligned} A(t) &= \begin{pmatrix} 0 & te^{-t} \\ t & 0 \end{pmatrix}, & B(t) &= \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, \\ Q(t) &= \begin{pmatrix} -e^{-t}(1+t^2) & -2te^{-t} \\ 1-te^{-t} & -t^2 \end{pmatrix} \end{aligned} \tag{22}$$

with the exact solution

$$P(t) = \begin{pmatrix} e^{-t} & 0 \\ t & 1 \end{pmatrix}.$$

The obtained results of the spline method [8] and our method with $m = 5, h = 0.1$ are given in Table 1. The results show that our method has fewer errors in comparison with [8].

Example 2 Consider the periodic Lyapunov equation

$$\begin{cases} P'(t) = A(t)P(t) + P(t)A^T(t) + Q(t), \\ P(0) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad t \in [0, 30], \end{cases} \tag{23}$$

where

$$\begin{aligned} A(t) &= \begin{pmatrix} 0 & 1 \\ -10 \cos t - 1 & -24 - 10 \sin t \end{pmatrix}, \\ Q(t) &= \begin{pmatrix} -\sin t & 11 \cos t + 10 \cos^2 t - \sin t \\ 11 \cos t + 10 \cos^2 t - \sin t & 48 + \cos t + 68 \cos t + 20 \sin^2 t \end{pmatrix}. \end{aligned} \tag{24}$$

Table 2 Err of the proposed method for equations (23)–(24) with $h = 1$

Interval	Our method	
	$m = 4$	$m = 5$
[0, 5]	3.01372×10^{-4}	6.54493×10^{-6}
[5, 10]	1.29054×10^{-4}	6.23638×10^{-6}
[10, 15]	5.16822×10^{-4}	3.17433×10^{-6}
[15, 20]	3.76287×10^{-4}	2.97795×10^{-6}
[20, 25]	4.39640×10^{-4}	3.52549×10^{-6}
[25, 30]	4.87715×10^{-4}	5.88901×10^{-6}

Table 3 Err of the proposed method for equations (23)–(24) with $h = 0.1$

Interval	Our method	
	$m = 4$	$m = 5$
[0, 5]	3.73335×10^{-10}	1.65403×10^{-12}
[5, 10]	2.22099×10^{-9}	2.10642×10^{-12}
[10, 15]	3.11465×10^{-9}	2.77084×10^{-12}
[15, 20]	3.77280×10^{-9}	4.39272×10^{-12}
[20, 25]	5.53761×10^{-9}	5.19905×10^{-12}
[25, 30]	5.82448×10^{-9}	6.01671×10^{-12}

Table 4 CPU runtimes of Examples 1 and 2

	Interval	h	m	Runtime
Example 1	[0, 1]	0.1	5	6.350704534496123
Example 2	[0, 30]	1	4	0.7504827271177106
	[0, 30]	1	5	0.9672017
	[0, 30]	0.1	4	13.10867849194134
	[0, 30]	0.1	5	14.811903603279116

The period of the problem is 2π . The exact solution of this Lyapunov differential equation is

$$P(t) = \begin{pmatrix} 1 + \cos t & 0 \\ 0 & 1 + \sin t \end{pmatrix}.$$

The corresponding numerical results of Example 2 with $h = 1$ and $h = 0.1$ are reported in Tables 2 and 3, respectively.

The CPU runtimes of Examples 1 and 2 are reported in Table 4. For these computations, we applied the AbsoluteTiming[.] function in Wolfram Mathematica 12 on system with Pentium Dual Core T4500 2.3 GHz CPU and 4 GB RAM.

5 Conclusions

The Sylvester and Lyapunov equations have numerous notable applications in analysis and design of control systems. The properties of the Chebyshev basis have been employed to solve the Sylvester equations by a new framework. The Sylvester differential equations are useful in solving periodic eigenvalue assignment problems. Our approach converts the main problem to the coupled linear matrix equations. An iterative algorithm was proposed for solving the obtained coupled linear matrix equations. Also, an error estimation of the method was obtained. Numerical experiments have been explained to show the applicability of our scheme.

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