Research Article

A Novel 2-Stage Fractional Runge–Kutta Method for a Time-Fractional Logistic Growth Model

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In this paper, the fractional Euler method has been studied, and the derivation of the novel 2-stage fractional Runge–Kutta (FRK) method has been presented. The proposed fractional numerical method has been implemented to find the solution of fractional differential equations. The proposed novel method will be helpful to derive the higher-order family of fractional Runge–Kutta methods. The nonlinear fractional Logistic Growth Model is solved and analyzed. The numerical results and graphs of the examples demonstrate the effectiveness of the method.

1. Introduction

In the 20th century, important research in fractional calculus was published in the engineering and science literature. Progress of fractional calculus is reported in various applications in the field of integral equations, fluid mechanics, viscoelastic models, biological models, and electrochemistry [1–3]. Undoubtedly, fractional calculus is an efficient mathematical tool to solve various problems in mathematics, engineering, and sciences. To get more attention in this field and to validate its effectiveness, this paper contributes the solution of new and recent applications of fractional calculus in biological and engineering sciences [4, 5]. Recently, the tool of fractional calculus has been used to analyze the nonlinear dynamics of different problems [6–8].

Mostly, the analytical solutions cannot be obtained for fractional differential equations, so that there is a need of semianalytical and numerical methods to understand the effects of the solutions to the nonlinear problems [9]. In the recent decades, different methods have been implemented to solve the linear as well as the nonlinear dynamical systems, such as the

Adomian decomposition method (ADM) [10], variational iteration method (VIM) [11], Homotopy perturbation method (HPM) [12], Homotopy perturbation method in association with the Laplace transform method [6], Homotopy analysis method (HAM) [13], and Homotopy analysis transform method (HATM) [7]. In the recent years, the novel numerical techniques have also been applied on a two-dimensional telegraph equation on arbitrary domains and modified diffusion equations with nonlinear source terms [14–16].

In the recent past, many numerical methods have been used just for linear equations or often more smaller classes. The generalization of the classical Adams–Bashforth–Moulton method has been introduced for the numerical solutions of nonlinear fractional differential equations [17]. Odibat and Momani also develop the new method with the connection of fractional Euler method and modified Trapezoidal rule by using the generalized Taylor series expansion [18].

Moreover, scientists have been actively worked on logistic growth that is typically the common model of population growth. A biological population with a lot of food, space to grow and no threats from predators, and trends to

grow at a rate that is proportional to the population in each unit of time is a certain percentage of the individuals who produce new individuals [19–21].

In this paper, we derived the 2-stage fractional Runge–Kutta method by using the generalized Taylor series expansion in Section 2. Afterwards, we applied the proposed numerical method on different nonlinear fractional differential equations and present the numerical results in Section 3. More specifically, we have used the fractional Runge–Kutta Method to solve the fractional logistic growth model. The conclusion is drawn in Section 4.

2. Method Description

In order to study the fractional differential equation, we will consider Caputo's fractional order derivative. Caputo's fractional order derivative is the modified form of the Riemann–Liouville definition and beneficial in dealing with the initial value problem more efficiently. Generalized Taylor's formula is defined as follows.

2.1. Generalized Taylor's Formula. Here, we are defining generalized Taylor's formula as given in [18], i.e, suppose that $D^{k\alpha}\phi(x) \in \mathbf{C}(0, a)$ for $k = 0, 1, 2, \ldots, n + 1$, where $0 < \alpha \leq 1$. We have

$$u(x) = \sum_{i=0}^{n} \frac{x^{i\alpha}}{\Gamma(i\alpha + 1)}\left(D^{i\alpha}\phi\right)(0) + \frac{\left(D^{(n+1)\alpha}\phi\right)(\varsigma)}{\Gamma((n + 1)\alpha + 1)}x^{(n+1)\alpha}, \tag{1}$$

with $0 \leq \varsigma \leq x, \forall x \in (0, a]$.

2.2. Fractional Euler Method. In order to derive the fractional Euler's method to find the numerical solution of initial value problem with time-fractional derivative in Caputo's sense, we consider the initial value problem of the form

$$\frac{d^{\alpha}}{dt^{\alpha}}u(t) = \phi(t, u(t));$$

$$u(0) = u_0, \tag{2}$$

$$\alpha \in (0, 1],$$

where D^{α} represents the Caputo fractional differential operator [22]. Consider the initial value problem. Let $[0, a]$ be an interval for which we are finding the solution of the problem in equation (2). The collection of points $(t_j, u(t_j))$ are used to find the approximation. The interval $[0, a]$ is subdivided into r subintervals $[t_j, t_{j+1}]$ of equal step size $h = (a/r)$ using the nodal points $t_j = jh$ for $j = 0, 1, 2, \ldots, r$. Suppose that $u(t)$, $D^{\alpha}u(t)$, and $D^{2\alpha}u(t)$ are continuous functions on the interval $[0, a]$, and applying Taylor's formula involving fractional derivatives, we have

$$u(t + h) = u(t) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)}D^{\alpha}u(t) + \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)}D^{2\alpha}u(t) + \cdots. \tag{3}$$

For the very small step size, we neglect the higher terms involving $h^{2\alpha}$ or higher, and substituting the value of $D^{\alpha}u(t)$ from equation (2), we obtain

$$u(t + h) = u(t) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)}\phi(t, u(t)). \tag{4}$$

By using the abovementioned equation, we can obtain the following iterative formula.

$$u_{n+1} = u_n + \frac{h^{\alpha}}{\Gamma(\alpha + 1)}\phi(t_n, u_n). \tag{5}$$

It is worth mentioning here that if $\alpha = 1$, then fractional Euler's method 2.3 reduced to classical Euler's method. This is the generalization of classical Euler's method.

2.3. Fractional Runge–Kutta Method. This method is the generalization of the Runge–Kutta (RK) method of order 2. Consider fractional order initial value problem (2). The generalized Taylor expansion is

$$u(t + h) = u(t) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)}D^{\alpha}u(t) + \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)}D^{2\alpha}u(t) + \cdots, \tag{6}$$

and using the formula $D_t^{2\alpha}u = D_t^{\alpha}\phi(t, u) + \phi(t, u)D_u^{\alpha}\phi(t, u)$ in equation (6) gives

$$u(t + h) = u(t) + \frac{h^{\alpha}}{\Gamma(\alpha + 1)}\phi(t, u(t)) + \frac{h^{2\alpha}}{\Gamma(2\alpha + 1)} \cdot \left\{D_t^{\alpha}\phi(t, u) + \phi(t, u)D_u^{\alpha}\phi(t, u)\right\} + \cdots. \tag{7}$$

Rearranging the abovementioned equation, we have

$$u(t + h) = u(t) + \frac{h^{\alpha}}{2\Gamma(\alpha + 1)}\phi(t, u(t)) + \frac{h^{\alpha}}{2\Gamma(\alpha + 1)}$$

$$\cdot \left\{\phi(t, u(t)) + \frac{2h^{\alpha}\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}D_t^{\alpha}\phi(t, u) + \frac{2h^{\alpha}\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}\right.$$

$$\left. \cdot \phi(t, u)D_u^{\alpha}\phi(t, u)\right\} + \cdots. \tag{8}$$

It can also be written as

$$u(t + h) = u(t) + \frac{h^{\alpha}}{2\Gamma(\alpha + 1)}\phi(t, u(t)) + \frac{h^{\alpha}}{2\Gamma(\alpha + 1)}$$

$$\cdot \phi\left(t + \frac{2h^{\alpha}\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}, u(t) + \frac{2h^{\alpha}\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)}\phi(t, u)\right). \tag{9}$$

In view of the abovementioned expression, the following formula is the 2-stage fractional Runge–Kutta method.

$$u_{n+1} = u_n + \frac{h^{\alpha}}{2\Gamma(\alpha + 1)}\{K_1 + K_2\}, \tag{10}$$

where

$$K_1 = \phi(t_n, u_n),$$

$$K_2 = \phi\left(t_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}, u_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\phi(t_n, u_n)\right).$$

(11)

One can easily verify that if $\alpha = 1$, then the fractional order Runge–Kutta method 2.5 reduced to the classical Runge–Kutta method of order 2.

3. Numerical Examples

To understand the methodology to apply the fractional Runge–Kutta method, we have solved three examples and also made a comparison with the exact solution.

Example 1. In the first example, we consider the inhomogeneous linear fractional differential equation

$$D^\alpha u(t) = \frac{2}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}t^{1-\alpha} - u(t) + t^2 - t,$$

(12)

subject to the conditions

$$u(0) = 0;$$
$$0 < \alpha \le 1;$$
$$t > 0;$$

(13)

with the exact solution

$$u(t) = t^2 - t.$$

(14)

By using the fractional R–K Method, we obtain the iterative relation for equation (12).

$$u_{n+1} = u_n + \frac{h^\alpha}{2\Gamma(\alpha+1)}(K_1 + K_2),$$

(15)

where

$$K_1 = \frac{2}{\Gamma(3-\alpha)}(t_n)^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}(t_n)^{1-\alpha} - u_n + t_n^2 - t_n,$$

$$K_2 = \frac{2}{\Gamma(3-\alpha)}\left(t_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)}\left(t_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)^{1-\alpha}$$

(16)

$$- \left(u_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}K_1\right) + \left(t_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)^2 - \left(t_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right).$$

Figure 1 expresses the numerical solutions of equation (12) for different values of α using the fractional Runge–Kutta method. Here, we can easily visualize in Table 1 that when we put $\alpha = 1$ the approximate solution coincide with the exact solution $u(t) = t^2 - t$. In Table 2, we can further analyze the solutions of the problem for $\alpha = 0.96$. Moreover, in Figure 2, hidden effects are visible by changing the values of α which cannot be obtained by using integer order derivative. Accuracy will be improved by using the small mesh size.

Example 2. Consider the nonlinear fractional differential equation

$$D^\alpha u(t) = (u(t))^2 - \frac{2}{(t+1)^2},$$

(17)

along with the conditions

$$u(0) = -2;$$
$$0 < \alpha \le 1;$$
$$t > 0.$$

(18)

The exact solution of equation (17) for $\alpha = 1$ is given by

$$u(t) = -\frac{2}{t+1}.$$

(19)

By using the fractional R–K method, we get the iterative relation for equation (12).

$$u_{n+1} = u_n + \frac{h^\alpha}{2\Gamma(\alpha+1)}(K_1 + K_2),$$

(20)

where

$$K_1 = u_n^2 - \frac{2}{(t_n+1)^2},$$

$$K_2 = \left(u_n + \frac{2h^\alpha\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}K_1\right)^2$$

(21)

$$- \frac{2}{(t_n + (2h^\alpha\Gamma(\alpha+1)/\Gamma(2\alpha+1)) + 1)^2}.$$

Figures 3 and 4 show numerical solutions of equation (17) for different values of α using the fractional Runge–Kutta method. We can see in Table 3 that when $\alpha = 1$, the approximate solution has excellent agreement with the exact solution $u(t) = -(2/t + 1)$. In Table 4, we can further analyze the solutions of the problem for $\alpha = 0.96$. Moreover, the
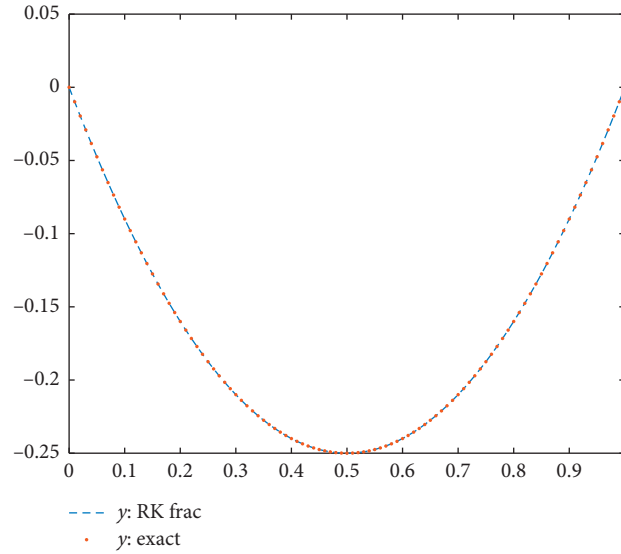
FIGURE 1: Numerical results of Example 1 for $\alpha = 1$ having discretization $h = 0.01$, respectively.

TABLE 1: Numerical results of Example 1 for $\alpha = 1$, with discretization $h = 0.01$.

T	y_{exact}	y_{approx}	AbsError ($\alpha = 1$)
0.0	0	0	0
0.1	−0.0900	−0.0900	$4.7820e - 06$
0.2	−0.1600	−0.1600	$9.1089e - 06$
0.3	−0.2100	−0.2100	$1.3024e - 05$
0.4	−0.2400	−0.2400	$1.6567 - e05$
0.5	−0.2500	−0.2500	$1.9772 - e05$
0.6	−0.2400	−0.2400	$2.2673e - 05$
0.7	−0.2100	−0.2100	$2.5297e - 05$
0.8	−0.1600	−0.1600	$2.7672e - 05$
0.9	−0.0900	−0.0900	$2.9820e - 05$
1.0	0	$3.1765e - 05$	$3.1765e - 05$

TABLE 2: Numerical results of Example 1 for $\alpha = 0.96$, with discretization $h = 0.01$.

T	y_{exact}	y_{approx}	AbsError ($\alpha = 0.96$)
0.0	0	0	0
0.1	−0.0900	−0.0912	0.0012
0.2	−0.1600	−0.1700	0.0100
0.3	−0.2100	−0.2274	0.0174
0.4	−0.2400	−0.2626	0.0226
0.5	−0.2500	−0.2752	0.0252
0.6	−0.2400	−0.2650	0.0250
0.7	−0.2100	−0.2322	0.0222
0.8	−0.1600	−0.1766	0.0166
0.9	−0.0900	−0.0985	0.0085
1.0	0	0.0021	0.0021

hidden nonlinearity effects are also visible in Table 2 by changing the value of α. Accuracy will be improved by using the small mesh size.

Example 3. Time-Fractional Logistic Growth Model

We consider the time-fractional logistic growth model represented by the equation

$$\frac{d^\alpha P}{dt^\alpha} = rP\left(1 - \frac{P}{M}\right);$$

$$P(t_0) = P_0,$$

(22)

where P_0 is the initial density of the population, r is intrinsic growth rate of the population, and M is the
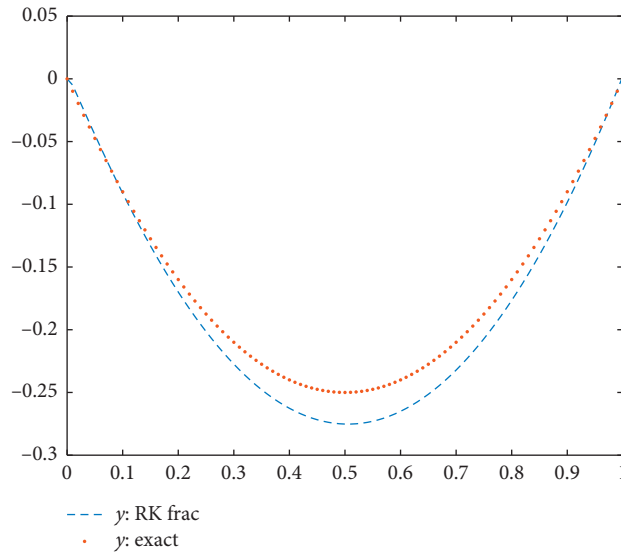
FIGURE 2: Numerical results of Example 1 for $\alpha = 0.96$ having discretization $h = 0.01$, respectively.
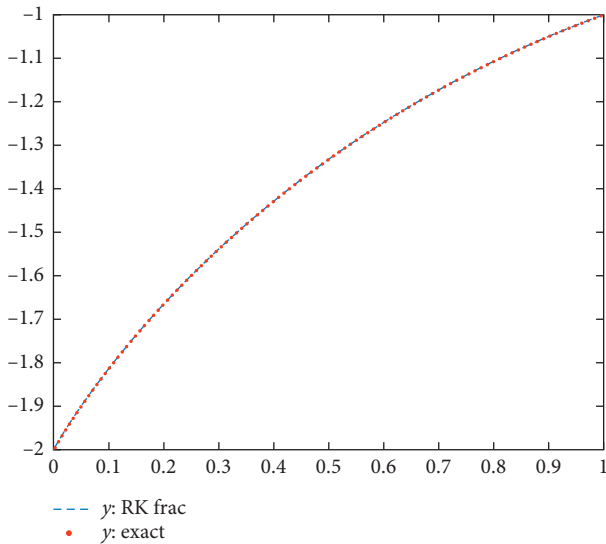

FIGURE 3: Numerical results of Example 2 for $\alpha = 1$, with mesh size $h = 0.01$.
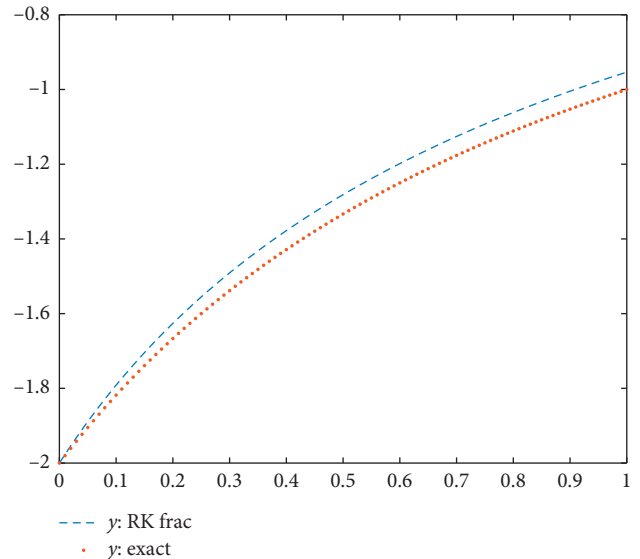

FIGURE 4: Numerical results of Example 2 for $\alpha = 0.96$, with mesh size $h = 0.01$.

carrying capacity. The analytical solution of equation (22) is given by

$$P = \frac{MP_0}{P_0 + (M - P_0)e^{-rt}}. \tag{23}$$

In the review of the fractional Runge–Kutta method, we have

$$P_{n+1} = P_n + \frac{h^\alpha}{2\Gamma(\alpha + 1)}\{K_1 + K_2 d\}, \tag{24}$$

where

TABLE 3: Numerical results of Example 2 for $\alpha = 1$, with discretization $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError $(\alpha = 1)$
0.0	-2	-2	0
0.1	-1.8182	-1.8182	$2.0884e - 05$
0.2	-1.6667	-1.6667	$2.9468e - 05$
0.3	-1.5385	-1.5385	$3.2069e - 05$
0.4	-1.4286	-1.4286	$3.1769 - e05$
0.5	-1.3333	-1.3334	$3.0117 - e05$
0.6	-1.2500	-1.2500	$2.7902e - 05$
0.7	-1.1765	-1.1765	$2.5530e - 05$
0.8	-1.1111	-1.1111	$2.3203e - 05$
0.9	-1.0526	-1.0527	$2.1018e - 05$
1.0	-1	-1	$1.9014e - 05$

TABLE 4: Numerical results of Example 2 for $\alpha = 0.96$, with discretization $h = 0.01$.

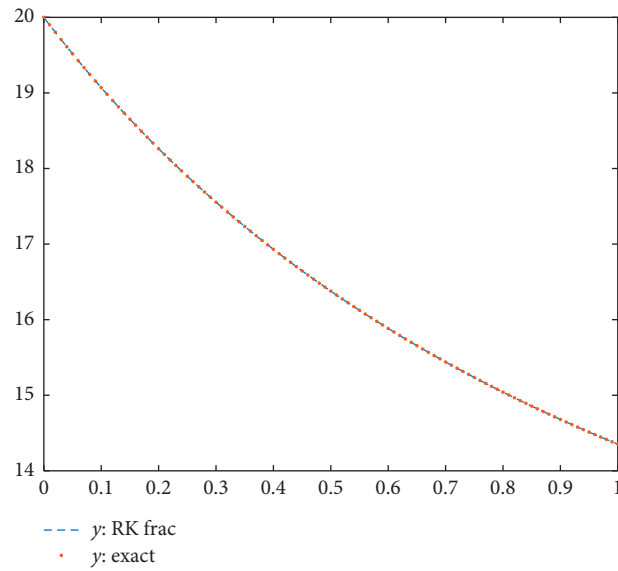| t	y_{exact}	y_{approx}	AbsError ($\alpha = 0.96$)
0.0	−2	−2	0
0.1	−1.8182	−1.7914	0.0268
0.2	−1.6667	−1.6260	0.0406
0.3	−1.5385	−1.4909	0.0476
0.4	−1.4286	−1.3778	0.0507
0.5	−1.3333	−1.2816	0.0518
0.6	−1.2500	−1.1984	0.0516
0.7	−1.1765	−1.1258	0.0507
0.8	−1.1111	−1.0617	0.0494
0.9	−1.0526	−1.0046	0.0480
1.0	−1	−0.9535	0.0465

FIGURE 5: Numerical results of fractional logistic growth model for $\alpha = 1$; $r = 0.5$; and $M = 10$ with mesh size $h = 0.01$.
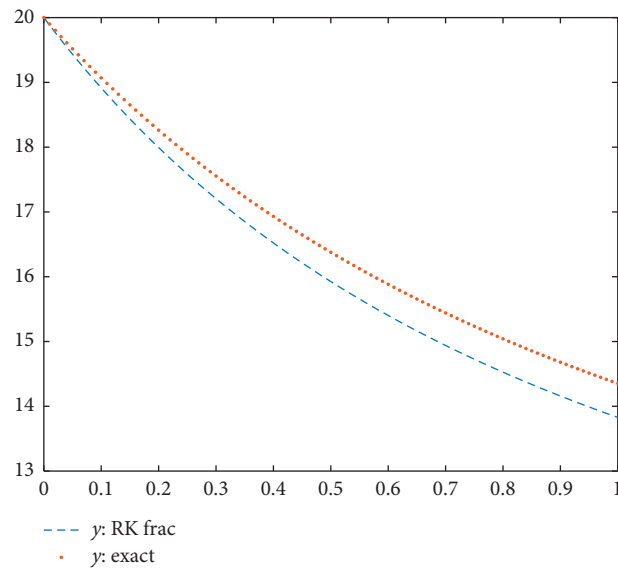

FIGURE 6: Numerical results of fractional logistic growth model for $\alpha = 0.96$; $r = 0.5$; and $M = 10$ with mesh size $h = 0.01$.

Table 5: Numerical results of Example 3 for $\alpha = 1$, with mesh size $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError $(\alpha = 1)$
0.0	20	20	0
0.1	19.0699	19.0700	$2.3897e - 05$
0.2	18.2621	18.2622	$3.9362e - 05$
0.3	17.5548	17.5548	$4.9190e - 05$
0.4	16.9309	16.9310	$5.5198 - e05$
0.5	16.3773	16.3774	$5.8593 - e05$
0.6	15.8833	15.8834	$6.0190e - 05$
0.7	15.4403	15.4404	$6.0546e - 05$
0.8	15.0412	15.0413	$6.0048e - 05$
0.9	14.6803	14.6803	$5.8968e - 05$
1.0	14.3527	14.3527	$5.7498e - 05$

Table 6: Numerical results of Example 3 for $\alpha = 0.96$, with mesh size $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError $(\alpha = 0.96)$
0.0	20	20	0
0.1	19.0699	18.9146	0.1553
0.2	18.2621	17.9941	0.2680
0.3	17.5548	17.2048	0.3500
0.4	16.9309	16.5217	0.4093
0.5	16.3773	15.9256	0.4518
0.6	15.8833	15.4018	0.4816
0.7	15.4403	14.9386	0.5017
0.8	15.0412	14.5269	0.5144
0.9	14.6803	14.1590	0.5213
1.0	14.3527	13.8288	0.5238

$$K_1 = rP_n\left(1 - \frac{P_n}{M}\right),$$

$$K_2 = \frac{rP_n}{M}\left\{1 + \frac{2rh^{\alpha}\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} - \frac{2rP_n h^{\alpha}\Gamma(\alpha + 1)}{M\Gamma(2\alpha + 1)}\right\}$$

$$\cdot\left\{M - P_n\left\{1 + \frac{2rh^{\alpha}\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} - \frac{2rP_n h^{\alpha}\Gamma(\alpha + 1)}{M\Gamma(2\alpha + 1)}\right\}\right\}.$$

$$(25)$$

Figures 5 and 6 demonstrate the approximate solutions of fractional Logistic Growth Model represented by equation (22) for different values of α using the fractional Runge–Kutta method.

Table 5 shows that when we put $\alpha = 1$, the approximate solution has excellent agreement with the exact solution given in equation (23). In Table 6, we can further analyze the solutions of the problem for $\alpha = 0.96$. Moreover, we can get better accuracy by using the small mesh size.

4. Conclusions

The fundamental objective of this research is to construct the numerical scheme to solve fractional differential equations. The objective has been achieved by implementing the fractional numerical method (fractional Runge–Kutta method). The derivation of the method is also presented. The method is a new contribution and is reliable to find the solutions of problems which arise in applied sciences. The comparison of numerical results has been made with exact solutions. The proposed method is useful to derive the higher order family of fractional Runge Kutta Methods. Finally, the recent development in the field of fractional differential equations in applied mathematics makes it needed to implement on such equations to get the numerical solutions. We are hoping that this work is the active contribution in this direction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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