Residual power series algorithm for fractional cancer tumor models

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Abstract   In this paper, the new series solutions of some fractional cancer tumor models are investigated by using residual power series method (RPSM). The RPSM is explained with Maclaurin expansion for the solution. One of the advantages of this method is quick and easy calculation to find series solutions by using mathematica software package. Graphical presentations for series solutions are given to explanation of the method. The obtained outcomes explain that process is applicable and reliable method to obtain numerical solutions of fractional equations.

1. Introduction

Tumours are dynamic systems in which cancer cells grow and spread in the end killing good cells by deficiency of oxygen and nutrients from the blood. The tumour cells spread around the area where it is located and they usually die because oxygen and nutrients in the blood are low. This movement can be compared to a fire [1]. If it is desired to destroy the tumor in an effective treatment, the treatment should move faster than the spread of the tumor. However, it should be remembered that tumors grow rapidly. In [2], the tumour growth is sup-

posed to be monoton and is claimed to be an treatment to the assumption of monotony can be applied. Because many issues need to be considered in such problems. So it is only necessary to focus on the treatment aspect of the problem.

The importance of mathematical modeling in cancer treatment is that it provides an analytical outline of which components of the immune system are important in cancer treatment. Gompertz first used mathematical modeling in cancer treatment, in 1825. He modeled tumor growth taking into account cell proliferation and death. According to his model, the more cells, the faster the growth. Then, many mathematical models have been developed to investigate the effects of different components in tumor microenvironment studies [3–8]. The tumor microenvironment contains growth elements (having hormones and cytokines), the extracellular matrix, immune cells, fibroblasts, signaling molecules (cytokines and chemokines), and other connective tissue cells. These interactions are

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important to investigate effective cancer immunotherapies. Here, mathematical modeling is important because it is necessary to model real data in the most accurate way. From this point, fractional calculus is more effective in modeling real data [9]. Fractional derivatives are used effectively in almost all biological systems contrary to the integer order derivatives.

Nonlinear partial differential equations and nonlinear evolution equations form the basis of many researches in mathematical physics [10–13]. In addition fractional calculus gained considerable interests and significant theoretical developments in many fields and many studies have been done in this field, recently [14–21]. Many studies have been made on fractional differential equations and its applications. In [22]; researchers investigated the numerical solutions of time fractional Jaulent-Miodek equations by using coupled fractional reduced differential transform method and q-homotopy analysis transform method, in [23]; local fractional wave equation is analysed in fractal strings by using local fractional homotopy perturbation Laplace transform scheme, in [24]; are studied nonlinear Fisher's equation of fractional order with q-homotopy analysis transform method, in [25]; are investigated the uniqueness of solutions for a fractional differential equation with dependence on the first order derivative, in [26]; authors analyzed the system of fractional Burger differential equations gived as a new fractional form for Atangana-Baleanu fractional derivative in the case of Mittag leffler kernel, in [27]; is investigated theory and application for the time fractional Gardner equation with the aid of Mittag-Leffler kernel and in [28]; researchers obtained new soliton solutions of the fractional Regularized Long Wave Burger equation with the aid of conformable derivative. For further some articles, Ref. [29–31] can be viewed.

In this article, we investigate RPSM to find influential series solution for several nonlinear problems. The applied algorithm gives the solutions in the form of a convergence series. An iterated transactions are created for obtain the infinite series solutions. The RPSM was expressed as an effectual algorithm for Fuzzy differential equations [32]. Emad Az-Zo'bi generalized the recently devised technique, known as the residual power series method, for analytic treatment of higher-order nonlinear partial differential equations in [33], in [34] are studied non-compound fractional differential equations, in [35]; the residual power series scheme is developed for mixed-type systems of conservation laws, in [36], this algorithm is tested on Fitzhugh–Nagumo and generalized Fisher equations with nonlinearity ranging, the comparative solution of the nonlinear fractional KdV Burgers equation [37], in [38] is investigated construction of fractional power series solutions to fractional stiff system using residual functions algorithm and are obtained analytic-approximate solution of time-fractional Zakharov-Kuznetsov equation by using this method in [39].

RPSM is quick and easy calculation to find series solutions by using mathematica software package. Also, unlike Taylor series method, RPSM requires easy computation state with high reliability and less time.

In this paper we investigate on a fractional diffusion model by using estimate time and spatial dependency of concentration of tumor cells as well as that of the killing ratio. In [40], Burgess et al. was presented a diffusion model. In this model, is considered a globular tumor in which having the reproduction ratio p and therapy dependent killing ratio k.

$$\frac{\partial P(x,\tau)}{\partial \tau} = D \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial P(x,\tau)}{\partial r}\right) + pP(x,\tau) - kP(x,\tau), \quad (1.1)$$

where $P(x,\tau)$ refer to the concentration of tumor cells at location r and time t, D gives the diffusivity factor.

In Ref. [41] are researched the one dimensional form for above model by using variable killing ratio with the aid of Lie symmetry method,

$$\frac{\partial^2 P(x,\tau)}{\partial x^2} - K(x,\tau)P(x,\tau) - \frac{\partial P(x,\tau)}{\partial \tau} = 0, \quad (1.2)$$

where $K(x,\tau)$ refers to the temporary view of the treatment. It is the clear ratio of remove of the tumor cells. The therapy dependent kill ratio K can be expressed in three cases: (1) may be constant, (2) may be a function of time, (3) may not be dependent solely on time.

Additionally, In [42–44] are investigated Eq. (1.2) for causes where the killing ratio of cancer cells K depends on the concentration of cells. Therefore this model was converted to nonlinear partial differential equation to study many different situations.

The majority of nonlinear phenomena are modelled by the aid of differential and integral equations of fractional order. Fractional calculus is more effective in modeling real data. Fractional derivatives are used effectively in almost all biological systems contrary to the integer order derivatives. Many physical phenomena modeled with fractional derivatives have been investigated. One of them is Caputo's modeling of the term memory from a different angle [45]. In [44], also is studied the super diffusion of cancer on the comb construction. In this study, tumor improving was shown to correspond to fractional transport of cells and obtained several analytical solutions of discussed problem.

In this paper, we investigated some fractional order cancer tumor models [42–44]. We studied in three cases of the therapy dependent kill ratio K, because of, it helps explain the growth or deterioration of the tumor. It can also help a person prefer a specific treatment type. We obtained numerical solutions for this problem by using the Residual power series method (RPSM) and we presented the convergence anaysis of the method.

The main aim of our article is to analyse processes of RPSM with the aid of the Caputo's fractional differential operators to obtain approximate solutions of several test problems [42–44]

$$\frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{\partial^2 P(x,\tau)}{\partial x^2} - \tau^2 P(x,\tau), \quad (1.3)$$

$$\frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{\partial^2 P(x,\tau)}{\partial x^2} - \frac{2}{x^2} P(x,\tau), \quad (1.4)$$

$$\frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{\partial^2 P(x,\tau)}{\partial x^2} - \frac{2}{x} \frac{\partial P(x,\tau)}{\partial x} - P(x,\tau)^2. \quad (1.5)$$

Our aim in this work is to obtain new series solutions of the (1.1), (1.2) and (1.3) equations with some initial conditions by using Caputo's fractional derivatives. We use the RPSM to produce series solutions. Several graphical expressions are presented to show the reliableness and efficiency of the method. Furthermore, results are presented in last section.

This work is prepared as follows. Formulation of fractional order cancer tumor models are given in section introduction.

Section 2 describes the some important definitions and several statements for the fractional calculus. Convergence of RPSM algorithm are given in Section 3. In Section 4, applications of RPSM algorithm are investigated for three examples. Physical Reviews and graphics are given in the last section.

2. Some necessary definitions and results from fractional calculus theory

In this section, we first give the important definitions and several statements for the fractional calculus [11].

Definition 2.1. The Riemann-Liouville fractional integral operator for order $\alpha(\alpha \geqslant 0)$ is given by [34,37],

$$J^\alpha p(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} p(\tau) d\tau, \quad \alpha > 0, \ x > 0, \quad (2.1)$$

$$J^0 p(x) = p(x).$$

Definition 2.2. The Caputo fractional derivatives for order α is below,

$$D^\alpha p(x) = J^{n-\alpha} D^n p(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} p(\tau) d\tau, \quad (2.2)$$

$$x > 0, \quad n-1 < \alpha \leqslant n,$$

where D^n is the classic differential operator for order n [34,37].

One of the features of the Caputo derivative is written is below,

$$D^\alpha x^\eta = 0, \eta < \alpha,$$
$$D^\alpha x^\eta = \frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\alpha)} x^{\eta-\alpha}, \eta \geqslant \alpha.$$

Definition 2.3. When $n < \alpha$, the Caputo time-fractional differential operator with order α for $P(x,\tau)$ is defined as follows [34,37],

$$D^\alpha_\tau P(x,\tau) = \frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^\tau (\tau-\tau)^{n-\alpha-1} \frac{\partial^n P(x,\tau)}{\partial \tau^n} d\tau, \quad (2.3)$$

$$n-1 < \alpha < n,$$

$$D^n_\tau P(x,\tau) = \frac{\partial^n P(x,\tau)}{\partial \tau^n}, \quad n \in N,$$

and the space–time fractional differential for order η of $P(x,\tau)$ is defined as follows,

$$D^\eta_x P(x,\tau) = \frac{\partial^\eta P(x,\tau)}{\partial x^\eta} = \frac{1}{\Gamma(n-\eta)} \int_0^x (x-\tau)^{n-\eta-1} \frac{\partial^n P(\tau,\tau)}{\partial \tau^n} d\tau, \quad (2.4)$$

$$n-1 < \eta < n,$$

$$D^n_x P(x,\tau) = \frac{\partial^n P(x,\tau)}{\partial x^n}, \quad n \in N,$$

Definition 2.4. A power series (PS) are expressed as follows,

$$\sum_{n=0}^\infty c_n(\tau-\tau_0)^{n\alpha} = c_0 + c_1(\tau-\tau_0)^\alpha + c_2(\tau-\tau_0)^{2\alpha} + \dots,$$

$$0 \leqslant n-1 < \alpha \leqslant n, \quad \tau \geqslant \tau_0,$$

This is called fractional PS at $\tau = \tau_0$ [30].

Definition 2.5. A PS can express as follows,

$$\sum_{n=0}^\infty p_n(x)(\tau-\tau_0)^{n\alpha} = p_0(x) + p_1(x)(\tau-\tau_0)^\alpha$$

$$+ p_2(x)(\tau-\tau_0)^{2\alpha} + \dots,$$

$$0 \leqslant n-1 < \alpha \leqslant n, \quad \tau \geqslant \tau_0, \quad (2.5)$$

This is called fractional PS at $\tau = \tau_0$ [30].

Theorem 2.1. Only if $P(x,\tau)$ is a polynomial fractional PS at point $\tau = \tau_0$ of the form

$$P(x,\tau) = \sum_{n=0}^\infty p_n(x)(\tau-\tau_0)^{n\alpha}, \quad (2.6)$$

$$0 \leqslant n-1 < \alpha \leqslant n, \quad x \in I, \quad \tau_0 \leqslant \tau < \tau_0 + R.$$

If $D^{n\alpha}_\tau P(x,\tau)$ are continuous on $I \times (\tau_0, \tau_0 + R)$, coefficients $p_n(x)$ are expressed as follows

$$p_n(x) = \frac{D^{n\alpha}_\tau P(x,\tau_0)}{\Gamma(n\alpha+1)}, n = \overline{0, \infty}.$$

where $D^{n\alpha}_\tau = \frac{\partial^{n\alpha}}{\partial \tau^{n\alpha}} = \frac{\partial^\alpha}{\partial \tau^\alpha} \cdot \frac{\partial^\alpha}{\partial \tau^\alpha} \dots \frac{\partial^\alpha}{\partial \tau^\alpha}$ (n-times) and $R = \min_{c \in I} R_c$. Where R_c is domain of convergency for the fractional PS $\sum_{n=0}^\infty p_n(c)(\tau-\tau_0)^{n\alpha}$. The function $p(x)$ is analytic on $x > 0$. (see [37] for proof.)

Result 2.1. The fractional PS expanded of $P(x,\tau)$ ($\tau = \tau_0$) is given by,

$$P(x,\tau) = \sum_{n=0}^\infty \frac{D^{n\alpha}_\tau P(x,\tau_0)}{\Gamma(n\alpha+1)} (\tau-\tau_0)^{n\alpha}, \quad (2.7)$$

$$0 \leqslant n-1 < \alpha \leqslant n, \quad x \in I, \quad \tau_0 \leqslant \tau < \tau_0 + R,$$

This is a Generalized Taylor's series expansion. To particularise, if $\alpha = 1$ in Eq. (2.7), this equiation is the classical Taylor's series expansion as follows [37],

$$P(x,\tau) = \sum_{n=0}^\infty \frac{\partial^n P(x,\tau_0)}{\partial \tau^n} \frac{(\tau-\tau_0)}{n!}, x \in I, \quad \tau_0 \leqslant \tau < \tau_0 + R,$$

3. Convergence of RPSM algorithm

Theorem. When $0 < K < 1, \|P_{m+1}(x,\tau)\| \leqslant K\|P_m(x,\tau)\|$ gives $\forall m \in N$ and $0 < \tau < T < 1$, then the series of numerical solutions converges to an exact solution.

Proof. We can consider

$$\|P(x,\tau) - P_m(x,\tau)\| = \left\| \sum_{n=m+1}^\infty P_n(x,\tau) \right\|$$

$$\leqslant \sum_{n=m+1}^\infty \|P_n(x,\tau)\|, \quad \forall \ 0 < \tau < T < 1.$$

$$\leqslant \|g(y)\| \left\| \sum_{n=m+1}^\infty K^n \right\|$$

$$= \frac{K^{m+1}}{1-K} \|g(y)\| \to 0 \text{ as } m \to \infty.$$

Lemma. When $-\infty < x < \infty$, the classical power series expansion $\sum_{n=0}^\infty P_n(x,\tau)x^n$ has a range of convergence T, if the fractional power series $\sum_{n=0}^\infty P_n(x,\tau)x^n, x \geqslant 0$ has a range of convergence $T^{\frac{1}{\alpha}}$. (See [34] for proof.).

4. Applications of RPSM algorithm

In this section, our aim is to apply the proposed algorithm for (1.1) cancer tumor models

$$D^\alpha_\tau P(x,\tau) = \frac{\partial^2 P(x,\tau)}{\partial x^2} - K(x,\tau)P(x,\tau),$$

$$\tau > 0, \quad 0 < \alpha \leqslant 1.$$

with the initial conditions

$$P(x,0) = f(x) \text{ and } P_\tau(x,0) = g(x).$$

4.1. Example 1.

The clear killing ratio of the cancer cells is just time dependent

$$\frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{\partial^2 P(x,\tau)}{\partial x^2} - \tau^2 P(x,\tau), \qquad (4.1)$$
$$\tau \geqslant 1, 0 < \alpha \leqslant 1,$$

subjected to the initial conditions

$$P(x,0) = e^{kx}. \qquad (4.2)$$

The RPSM gives the series solutions for Eqs. (4.1) and (4.2). This solutions is form a fractional PS at point $\tau = 0$ [32]. Suppose that the solution is expansion form as follows,

$$P(x,\tau) = \sum_{n=0}^{\infty} p_n(x) \frac{\tau^{n\alpha}}{\Gamma(1+n\alpha)} \qquad 0 < \alpha \leqslant 1, \quad x \in I, \quad 0$$
$$\leqslant \tau < R. \qquad (4.3)$$

We consider that $P_j(x,\tau)$ is j. truncated series of $P(x,\tau)$,

$$P_j(x,\tau) = \sum_{n=0}^{j} p_n(x) \frac{\tau^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \leqslant 1, x \in I, 0$$
$$\leqslant \tau < R. \qquad (4.4)$$

where $P_0(x,\tau) = p_0(x) = P(x,0) = p(x)$.

Then, Eq. (4.4) can express as follows

$$P_j(x,\tau) = p(x) + \sum_{n=1}^{j} p_n(x) \frac{\tau^{n\alpha}}{\Gamma(1+n\alpha)}, \qquad (4.5)$$
$$0 < \alpha \leqslant 1, \quad 0 \leqslant \tau < R, \quad x \in I, j = \overline{1,\infty}.$$

To obtain the value of coefficients $p_n(x), n = 1, 2, 3, \ldots, j$ in series expanded of Eq. (4.5), Residual function Res is given by

$$Res(x,\tau) = \frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P(x,\tau)}{\partial x^2} + \tau^2 P(x,\tau),$$

and the j-th residual function, Res_j is given by:

$$Res_j(x,\tau) = \frac{\partial^\alpha P_j(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_j(x,\tau)}{\partial x^2} + \tau^2 P_j(x,\tau), \quad j = 1, 2, 3, \ldots \qquad (4.6)$$

$$\lim_{j \to \infty} Res_j(x,\tau) = Res(x,\tau) \quad \text{for} \quad \forall x \in I \quad \text{and} \quad \tau \geqslant 0 \quad \text{and}$$
$Res(x,\tau) = 0$ [32–35].

Then, $D_\tau^{r\alpha} Res(x,\tau) = 0$ and $Res_j(x,\tau)$ are at $\tau = 0$ with $\forall r = \overline{0,j}$. To present RPS process: At first, we write the j-th residual series expansion of $P(x,\tau)$ in Eq. (4.1). Then, we obtain the fractional derivative $D_\tau^{(j-1)\alpha}$ of both $Res_{P_j}(x,\tau)$, $j = \overline{1,\infty}$ and finally, we can solve obtained system

$$D_\tau^{(j-1)\alpha} Res_{P_j}(x,0) = 0, 0 < \alpha \leqslant 1, x \in I, j = \overline{1,\infty}. \qquad (4.7)$$

to find the needed coefficients $p_n(x)$ for $n = \overline{1,j}$. in Eq. (4.5).

To obtain $p_1(x)$, we consider $j = 1$ in Eq. (4.6),

$$Res_1(x,\tau) = \frac{\partial^\alpha P_1(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_1(x,\tau)}{\partial x^2} + \tau^2 P_1(x,\tau), \qquad (4.8)$$

where

$$P_1(x,\tau) = \frac{\tau^\alpha}{\Gamma(1+\alpha)} p_1(x) + p(x),$$

for

$$P(x,0) = p_0(x) = p(x) = P(x,0) = e^{kx}.$$

where, we know that $Res_1(x,0) = 0$ and thus,

$$p_1(x) = e^{kx} k^2, \qquad (4.9)$$

and

$$P_1(x,\tau) = e^{kx} k^2 \frac{\tau^\alpha}{\Gamma(1+\alpha)}, \qquad (4.10)$$

Likewise, to obtain the form of the second unknown coefficient $p_2(x)$, we write $j = 2$ in Eq. (4.6)

$$Res_2(x,\tau) = \frac{\partial^\alpha P_2(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_2(x,\tau)}{\partial x^2} + \tau^2 P_2(x,\tau),$$

where

$$P_2(x,\tau) = e^{kx} + e^{kx} k^2 \frac{\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} p_2(x),$$

we know that $D_\tau^\alpha Res_2(x,0) = 0$ and thus,

$$p_2(x) = e^{kx} k^4, \qquad (4.11)$$

and

$$P_2(x,\tau) = e^{kx} + e^{kx} k^2 \frac{\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} e^{kx} k^4, \qquad (4.12)$$

Similarly to obtain $p_3(x)$, we consider $j = 3$ in Eq. (4.6),

$$Res_3(x,\tau) = \frac{\partial^\alpha P_3(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_3(x,\tau)}{\partial x^2} + \tau^2 P_3(x,\tau),$$

where

$$P_3(x,\tau) = e^{kx} + e^{kx} k^2 \frac{\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} e^{kx} k^4$$
$$+ \frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)} p_3(x),$$

$D_\tau^{2\alpha} Res_3(x,0) = 0$ and thus,

$$p_3(x) = \frac{1}{2} e^{kx} (-2 + k^6), \qquad (4.13)$$

and

$$P_3(x,\tau) = e^{kx} + e^{kx} k^2 \frac{\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} e^{kx} k^4$$
$$+ \frac{\tau^{3\alpha}}{2\Gamma(1+3\alpha)} e^{kx} (-2 + k^6), \qquad (4.14)$$

Repeating the above operation for $j = 4$ we obtain $p_4(x)$,

$$p_4(x) = \frac{1}{6} e^{kx} k^2 (-8 + k^6),$$
$$p_5(x) = \frac{1}{24} e^{kx} k^4 (-20 + k^6), \qquad (4.15)$$

and

$$P_5(x,\tau) = e^{kx} + e^{kx} k^2 \frac{\tau^\alpha}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} e^{kx} k^4$$
$$+ \frac{\tau^{3\alpha}}{2\Gamma(1+3\alpha)} e^{kx} (-2 + k^6)$$
$$+ \frac{\tau^{4\alpha}}{6\Gamma(1+4\alpha)} e^{kx} k^2 (-8 + k^6)$$
$$+ \frac{\tau^{4\alpha}}{24\Gamma(1+4\alpha)} e^{kx} k^4 (-20 + k^6). \qquad (4.16)$$

4.2. Example 2.

Consider the following initial value problem

$$\frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{\partial^2 P(x,\tau)}{\partial x^2} - \frac{2}{x^2}P(x,\tau), \tag{4.17}$$
$$\tau > 0, 0 \leqslant x \leqslant 1, 0 < \alpha \leqslant 1,$$

by the initial condition

$$P_0(x,\tau) = p_0(x) = P(x,0) = p(x) = \frac{a}{x} + bx^2,$$

To obtain $p_1(x)$, we consider $j = 1$ in Eq. (4.6),

$$Res_1(x,\tau) = \frac{\partial^\alpha P_1(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_1(x,\tau)}{\partial x^2} + \frac{2}{x^2}P_1(x,\tau), \tag{4.18}$$

where

$$P_1(x,\tau) = \frac{\tau^\alpha}{\Gamma(1+\alpha)}p_1(x) + p(x),$$

We know that $Res_1(x,0) = 0$ and we obtain,

$$p_1(x) = 0,$$

and

$$P_1(x,\tau) = \frac{a}{x} + bx^2, \tag{4.19}$$

Similarly, to find the second unknown coefficient $p_2(x)$, we consider $j = 2$ in Eq. (4.6)

$$Res_2(x,\tau) = \frac{\partial^\alpha P_2(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_2(x,\tau)}{\partial x^2} + \frac{2}{x^2}P_2(x,\tau), \tag{4.20}$$

where

$$P_2(x,\tau) = \frac{a}{x} + bx^2 + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}p_2(x),$$

We know that $D_\tau^\alpha Res_2(x,0) = 0$ and we obtain,

$$p_2(x) = 0,$$

and

$$P_2(x,\tau) = \frac{a}{x} + bx^2.$$

Similarly to find $p_3(x)$, we consider $j = 3$ in Eq. (4.6),

$$Res_3(x,\tau) = \frac{\partial^\alpha P_3(x,\tau)}{\partial \tau^\alpha} - \frac{\partial^2 P_3(x,\tau)}{\partial x^2} + \frac{2}{x^2}P_3(x,\tau),$$

where

$$P_3(x,\tau) = \frac{a}{x} + bx^2 + \frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}p_3(x),$$

$D_\tau^{2\alpha} Res_3(x,0) = 0$ and thus,

$$p_3(x) = 0,$$

and

$$P_3(x,\tau) = \frac{a}{x} + bx^2.$$

Repeating the above operation for $j = 4$ we obtain $p_4(x)$,

$$p_4(x) = 0,$$
$$p_5(x) = 0,$$

and

$$P_5(x,\tau) = \frac{a}{x} + bx^2. \tag{4.21}$$

4.3. Example 3.

Consider the following initial value problem

$$\frac{\partial^\alpha P(x,\tau)}{\partial \tau^\alpha} = \frac{\partial^2 P(x,\tau)}{\partial x^2} - \frac{2}{x}\frac{\partial P(x,\tau)}{\partial x} - P(x,\tau)^2, \tag{4.22}$$
$$\tau > 0, 0 \leqslant x \leqslant 1, 0 < \alpha \leqslant 2,$$

by the initial condition

$$P_0(x,\tau) = p_0(x) = P(x,0) = p(x) = x^p.$$

If operations are performed as in the examples above, we obtain that
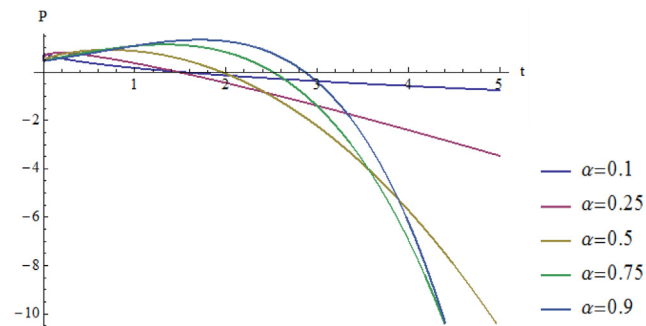
Fig. 2 The 2D graphics of the $P_5(x,\tau)$ for different value of α in Example 1 $(k = -1, x = 0.8)$.
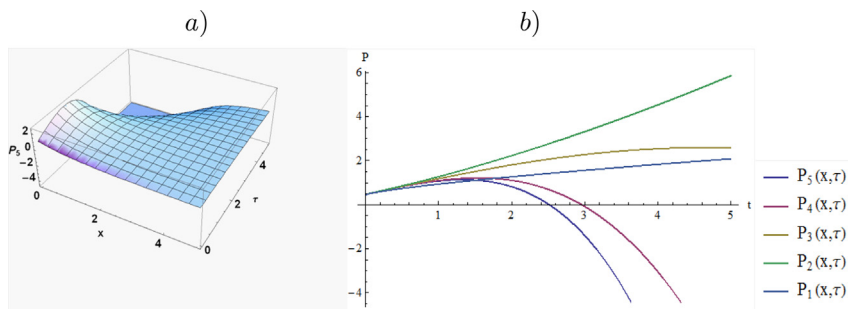
Fig. 1 **(a)** The 3D graphic for the $P_5(x,\tau)$ in Example 1 $(k = -1, \alpha = 0.75)$, **(b)**2D graphics of $P_n(x,\tau)$ for different value of n in Example 1$(k = -1, \alpha = 0.75, x = 0.8)$.

$$p_1(x) = x^{-2+p}(-3p + p^2 - x^{2+p}),$$

$$p_2(x) = x^{-4+p}(-10p^3 + p^4 + 2x^{4+2p} + p^2(31 - 6x^{2+p})$$
$$+6p(-5 + 2x^{2+p})),$$

$$p_3(x) = -\tfrac{1}{2}x^{-6+p}(840p - 1198p^2 + 651p^3 - 169p^4 + 21p^5$$
$$-p^6 - 180px^{2+p} + 308p^2x^{2+p} - 164p^3x^{2+p} + 28p^4x^{2+p}$$
$$+54px^{4+2p} - 34p^2x^{4+2p} + 6x^{6+3p}),$$

$$p_4(x) = \tfrac{1}{6}x^{-8+p}(-45360p + 77292p^2 - 53964p^3 + 20089p^4$$
$$-4320p^5 + 538p^6 - 36p^7 + p^8 + 6720px^{2+p}$$
$$-15520p^2x^{2+p} + 14128p^3x^{2+p} - 6328p^4x^{2+p}$$
$$+1392p^5x^{2+p} - 120p^6x^{2+p} - 1080px^{4+2p} + 2492p^2x^{4+2p}$$
$$+404p^4x^{4+2p} + 288px^{6+3p} - 212p^2x^{6+3p} + 24x^{8+4p}),$$

$$p_5(x) = -\tfrac{1}{24}x^{-10+p}(3991680p - 7663536p^2 + 6262740p^3$$
$$-2870440p^4 + 815815p^5 - 149513p^6 + 17710p^7$$
$$-1310p^8 + 55p^9 - p^{10} - 453600px^{2+p} + 1219824p^2x^{2+p}$$
$$-1409952p^3x^{2+p} + 902384p^4x^{2+p} - 343504p^5x^{2+p}$$
$$+77456p^6x^{2+p} - 9552p^7x^{2+p} + 496p^8x^{2+p}$$
$$+50400px^{4+2p} + \ldots).$$

and the following solution is obtained;

$$P_5(x,\tau) = x^p + x^{-2+p}(-3p + p^2 - x^{2+p})\frac{\tau^\alpha}{\Gamma(1+\alpha)}$$

$$+\frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)}x^{-4+p}(-10p^3 + p^4 + 2x^{4+2p}$$
$$+p^2(31 - 6x^{2+p}) + 6p(-5 + 2x^{2+p}))$$

$$+\frac{\tau^{3\alpha}}{\Gamma(1+3\alpha)}\left(-\tfrac{1}{2}x^{-6+p}(840p - 1198p^2 + 651p^3\right.$$
$$-169p^4 + 21p^5 - p^6 - 180px^{2+p} + 308p^2x^{2+p}$$
$$-164p^3x^{2+p} + 28p^4x^{2+p} + 54px^{4+2p}$$
$$\left.-34p^2x^{4+2p} + 6x^{6+3p}\right)$$

$$+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}\left(\tfrac{1}{6}x^{-8+p}(-45360p + 77292p^2 - 53964p^3\right.$$
$$+20089p^4 - 4320p^5 + 538p^6 - 36p^7 + p^8 + 6720px^{2+p}$$
$$-15520p^2x^{2+p} + 14128p^3x^{2+p} - 6328p^4x^{2+p}$$
$$+1392p^5x^{2+p} - 120p^6x^{2+p} - 1080px^{4+2p} + 2492p^2x^{4+2p}$$
$$\left.+404p^4x^{4+2p} + 288px^{6+3p} - 212p^2x^{6+3p} + 24x^{8+4p}\right)$$

$$+\frac{\tau^{4\alpha}}{\Gamma(1+4\alpha)}\left(-\tfrac{1}{24}x^{-10+p}(3991680p - 7663536p^2\right.$$
$$+6262740p^3 - 2870440p^4 + 815815p^5 - 149513p^6$$
$$+17710p^7 - 1310p^8 + 55p^9 - p^{10} - 453600px^{2+p}$$
$$+1219824p^2x^{2+p} - 1409952p^3x^{2+p} + 902384p^4x^{2+p}$$
$$-343504p^5x^{2+p} + 77456p^6x^{2+p} - 9552p^7x^{2+p}$$
$$\left.+496p^8x^{2+p} + 50400px^{4+2p} + \ldots\right). \qquad (4.23)$$

5. Physical reviews

In this section, we drawn some pictures to investigate the behaviour of the obtained solutions of Examples 1–3. We researched how fractional derivative affects in time on the concentration of cancer cells.

In Fig. 1, we draw 2D and 3D graphics with 5-term of the series solution for Example 1. We can see that good results are obtained as the number of terms increases in serial solution. The concentration of cancer cells decreases and finally arrives zero over time.

In Fig. 2, we draw 2D graphics with 5-term of the series solution for example 1. We can see that good results are obtained as αapproaches 1 in serial solution. The concentration of cancer cells decreases and finally arrives zero over time.

In Fig. 3, we draw 3D graphic with 5-term of the series solution for example 2. We can see the effect of αfractional order is not effective here but it is not right to say it is insignificant. In Example 2, appropriate choices should be made in specified factors and the initial case.

In Fig. 4, we draw 2D and 3D graphics with 5-term of the series solution for Example 3. We can see that good results are obtained as the number of terms increases in serial solution. The concentration of cancer cells decreases and finally arrives zero over time.

In Fig. 5, we draw 2D graphics with 5-term of the series solution for example 3. We can see that good results are obtained as αapproaches 2 in serial solution. In a very short time, the concentration of cancer cells reduces for every $0 < \alpha < 2$ for the specified factors and the initial case. As time goes by, the concentration of cancer cells increases for some α. We can say that at $\alpha = 0.9$, the concentration of cancer cells increase, as time goes by. However, at $\alpha = 1.8$, we cannot say this. Therefore, $\alpha = 1.8$ can be recommended as the most suitable case in time.

6. Final remarks

In this study, we have studied how fractional derivative will affect as time goes by on the concentration of cancer cells. The clear killing ratio of the cancer cells could also be based on the concentration of the cells. We applied RPSM to obtain numerical solutions for the different three cases: (1) K may be constant, (2) K may be a function of time, (3) K may not be dependent solely on time. We investigated these cases and we obtained analytical solutions for these cases. The results indicate that the killing rate K indicates that the appropriate selected parameter and the starting condition are effective for the concentration of cancer cells decreases and disappears over time. RPSM provides almost accurate estimation of solutions and is directly applicable without considering lineariza-
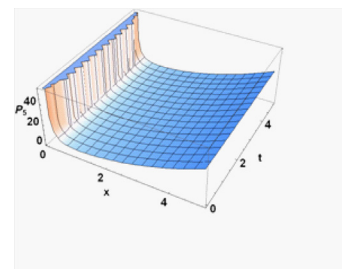

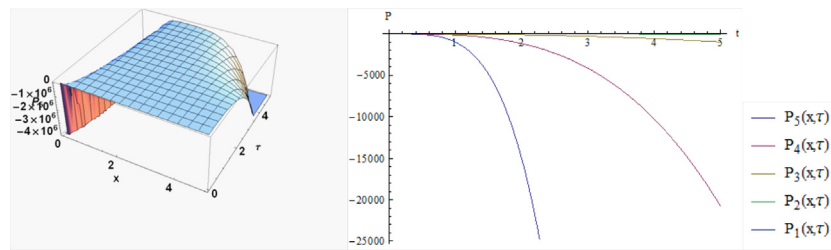**Fig. 3** The 3D graphic for the $P_5(x,\tau)$ in Example 2 ($a = 2.5, b = 1.3$).

Fig. 4 (a) The 3D graphic for the $P_5(x, \tau)$ in Example 3 ($\alpha = 0.75, p = 1.2$), **(b)** 2D graphics of $P_n(x, \tau)$ for different value of n in Example 3 ($\alpha = 0.75, p = 1.2, x = 0.8$).
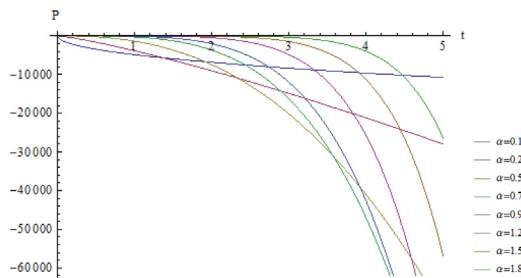

Fig. 5 The 2D graphics of the $P_5(x, \tau)$ for different value of α in Example 3 ($p = 1.2, x = 0.8$).

tion, discretization or any other restrictive assumptions. We say that this study is important, due to this method can be used as an alternating to obtain analytic solutions of different types of cancer tumor problems.

Declaration of Competing Interest

None.

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