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The Analytical Analysis of Time-Fractional Fornberg–Whitham Equations

A. A. Alderremy [1], Hassan Khan [2,*], Rasool Shah [2], Shaban Aly [1,3] and Dumitru Baleanu [4,5]

[1] Department of Mathematics, Faculty of Science, King Khalid University, Abha 61413, Saudi Arabia; aaldramy@kku.edu.sa (A.A.A.); shhaly70@yahoo.com (S.A.)

[2] Department of Mathematics, Abdul Wali khan University, Mardan 23200, Pakistan; rasoolshah@awkum.edu.pk

[3] Department of Mathematics, Faculty of Science, AL-Azhar University, Assiut 71516, Egypt

[4] Department of Mathematics, Faculty of Arts and Sciences, Cankaya University, 06530 Ankara, Turkey; dumitru@cankaya.edu.tr

[5] Institute of Space Sciences, 077125 Magurele, Romania

[*] Correspondence: hassanmath@awkum.edu.pk

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Abstract: This article is dealing with the analytical solution of Fornberg–Whitham equations in fractional view of Caputo operator. The effective method among the analytical techniques, natural transform decomposition method, is implemented to handle the solutions of the proposed problems. The approximate analytical solutions of nonlinear numerical problems are determined to confirm the validity of the suggested technique. The solution of the fractional-order problems are investigated for the suggested mathematical models. The solutions-graphs are then plotted to understand the effectiveness of fractional-order mathematical modeling over integer-order modeling. It is observed that the derived solutions have a closed resemblance with the actual solutions. Moreover, using fractional-order modeling various dynamics can be analyzed which can provide sophisticated information about physical phenomena. The simple and straight-forward procedure of the suggested technique is the preferable point and thus can be used to solve other nonlinear fractional problems.

Keywords: Adomian decomposition method; Caputo operator; Natural transform; Fornberg–Whitham equations

1. Introduction

It is well known that in many fields of physics, the studies of non-linear wave problems and their effects are of wide significance. Traveling wave solutions are a significant kind of result for the non-linear partial differential condition and numerous non-linear fractional differential equations (FDEs) have been shown to an assortment of traveling wave results. Although water wave are among the extremely important of all-natural phenomena, they have an extraordinarily rich mathematical structure. Water waves are one of the most complicated fields in wave dynamics, including the study in non-linear, electromagnetic waves in 1 space and 3-time dimensions [1–5]. For illustration, the well-known Korteweg–de Vries equation

$$D_t\mu - 6\mu D_x\mu + \mu D_{xxx}\mu = 0,$$

has a simple solitary-wave solution [6]. Camassa-Holm equation

$$D_t\mu - D_{xxt}\mu + 3\mu D_x\mu = 2D_x\mu D_{xx}\mu + \mu D_{xxx}\mu,$$

a model approximation for symmetric non-linear dispersive waves in shallow water, was suggested by Camassa and Holm [7]. Due to its useful mathematical proprieties, this scenario has attracted much attention during the past decade. It has been found that the Camassa Holm equation includes poles, composite wave, stumpons, and cuspons solutions [8]. The specific Camassa–Holm equation solutions were studied by Vakhnenko and Parkes [9]. In mathematical physics the Fornberg-Whitham (FW) model is a significant mathematical equation. The FWE [10,11] is expressed as

$$D_t\mu - D_{xxt}\mu + D_x\mu = \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu,$$

where $\mu(x,t)$ is the fluid velocity, x is the spatial co-ordinate and t is the time. In 1978 Fornberg and Whitham derived a $\mu(x,t) = Ce^{\frac{x}{2}-\frac{4t}{3}}$ peaked solution with an arbitrary constant of C [12]. This algorithm was developed to analyze the breakup of dispersive nonlinear water waves. The FWE has been found to require peakon results as a simulation for limiting wave heights as well as the frequency of wave breaks. In fractional calculus (FC) has gained considerable significance and popularity, primarily because of its well-shown applications in a wide range of apparently disparate areas of engineering and science [13]. Many scholars, such as Singh et al. [14], Merdan et al. [15], Saker et al. [16], Gupta and Singh [17] etc., have therefore researched the fractional extensions of the FW model for the Caputo fractional-order derivative [18].

The existence, uniqueness and stability are the important ingredient to show for any mathematical problems in science and engineering. In this connection Li et al. have determine the existance and unique of the solutions for some nonlinear fractional differential equations [19]. Becani et al. have discussed the theory of existence and uniqueness for some singular PDEs [20]. The generalized theorem of existence and uniqueness for nth order fractional DEs was analyzed by Dannan et al. in [21]. Similarly the stability of solutions for the Fornberg-Whitham equation was investigated by Xiujuan Gao et al. in [22]. Shan et al. have discussed the optimal control of the Fornberg-Whitham equation [23].

Recently, the researchers have taken greater interest in FC, i.e., the study of integrals and derivatives of fractional-order non-integer. Major importance have been demonstrated in the analysis of the FC and its various implementations in the field engineering [24–27]. FDEs are widely utilized to model in a variety of fields of study, including an analysis of fractional random walking, kinetic control schemes theory, signal processing, electrical networks, reaction and diffusion procedure [28,29]. FD provides a splendid method for characterizing the memories and genetic properties of different procedures [30,31].

Over the last few years, FDEs have become the subject of several studies owing to their frequent use in numerous implementations in viscoelasticity, biology, fluid mechanics, physics, dynamical schemes, electrical network, physics, signal and optics process, as they can be modelled by linear and nonlinear FDEs [32–36]. FD offer an outstanding method for explaining the memories and inherited properties of specific materials and processes. Fractional-order integrals and derivatives have proven more effective in formulating such electrical and chemical problems than the standard models. Non-linear FPDEs have many applications in various areas of engineering such as heat and mass transfer, thermodynamics and micro-electro mechanics scheme [37–39].

The technique of natural decomposition (NDM) was initially developed by Rawashdeh and Maitama in 2014 [40–42], to solve ODEs and PDEs that appear in different fields of mathematics. The suggested technique is mixing of the Adomian technique (ADM) and natural transformation. The key benefit of this suggested technique is the potential to integrate two important methods of achieving fast convergent series for PDEs. Many scholars have recently solved different types of fractional-order PDEs, for example heat and wave equations [43], coupled Burger equations [44], hyperbolic telegraph equation [45], Harry Dym equation [46] and diffusion equations [47].

2. Preliminaries

Definition 1. *Let $g \in C_\beta$ and $\beta \geq -1$, then the Riemann–Liouville integral of order γ, $\gamma > 0 >$ is given by [48–50]:*

$$J_t^\gamma g(x,t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-\theta)^{\gamma-1} g(x,\theta) d\theta, \qquad t > 0. \tag{1}$$

Definition 2. *Let $g \in \mathbb{C}_t$ and $t \geq -1$, then Caputo definition of fractional derivative of order γ if $m-1 < \gamma \leq m$ with $m \in \mathbb{N}$ is describe as [48–50]*

$$D_t^\gamma g(t) = \begin{cases} \frac{d^m g(t)}{dt^m}, & \gamma = m \in N, \\ \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-\theta)^{m-\gamma-1} g^{(m)}(\theta) d\theta, & m-1 < \gamma < m, \quad m \in N, \end{cases} \tag{2}$$

Remark 1. *Some basic properties are below [48–50]*

$$D_x^\gamma I_x^\gamma g(x) = g(x),$$

$$I^\gamma x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\gamma+\lambda+1)} x^{\gamma+\lambda}, \qquad \gamma > 0, \lambda > -1, \quad x > 0,$$

$$D_x^\gamma I_x^\gamma g(x) = g(x) - \sum_{k=0}^m g^{(k)}(0^+) \frac{x^k}{k!}, \qquad for \quad x > 0.$$

Definition 3. *The natural transform of the function $g(t)$ is expressed by $N[g(t)]$ for $t \in R$ and is given by [40–42,51]*

$$N[g(t)] = \mathcal{G}(s,\omega) = \int_{-\infty}^\infty e^{-st} g(\omega t) dt; \quad s,\omega \in (-\infty,\infty), aa$$

where s and ω are the NT variables. If $g(t)H(t)$ is defined for positive real numbers, then NT can be presented as [40–42,51]

$$N[g(t)Q(t)] = N^+[g(t)] = \mathcal{G}^+(s,\omega) = \int_0^\infty e^{-st} g(\omega t) dt; \quad s,\omega \in (0,\infty), \quad and \quad t \in R, \tag{3}$$

where $Q(t)$ denotes the Heaviside function.

Theorem 1. *The NT of the Caputo derivative of fractional order of any function $g(t)$ is defined as [40–42,51]*

$$N^+[{}^c D^\gamma g(t)] = \frac{s^\gamma}{\omega^\gamma} \mathcal{G}(s,\omega) - \sum_{k=0}^{m-1} \frac{s^{\gamma-(k+1)}}{\omega^{\gamma-k}} [D^k g(t)]_{t=0}, \quad m-1 \leq \gamma < m. \tag{4}$$

where m is the natural number and γ represent the order of the derivative with fractional order.

Remark 2. *Some basic NT properties are listed below [40–42,51]*

$$N^+[1] = \frac{1}{s},$$

$$N^+[t^\gamma] = \frac{\Gamma(\gamma+1)\omega^\gamma}{s^{\gamma+1}},$$

$$N^+[g^{(m)}(t)] = \frac{s^m}{\omega^m} \mathcal{R}(s,\omega) - \sum_{k=0}^{m-1} \frac{s^{m-(k+1)}}{\omega^{m-k}} \frac{\Gamma(\gamma+1)\omega^\gamma}{s^{\gamma+1}}.$$

3. NDM Procedure

In this section, NDM procedure is introduced to solve general FPDEs of the form [41,42]

$$D^\gamma \mu(x,t) + L\mu(x,t) + N\mu(x,t) = P(x,t), \quad x,t \geq 0, \quad \ell - 1 < \gamma < \ell, \tag{5}$$

The fractional derivative in Equation (5) is represented by Caputo operator. The linear and nonlinear terms are denoted by L and N respectively and the source term is $P(x,t)$.

The solution at $t = 0$ is

$$\mu(x,0) = h(x). \tag{6}$$

Using NT to Equation (5), we get [41,42]

$$N^+ \left[D^\gamma \mu(x,t) \right] + N^+ \left[L\mu(x,t) + N\mu(x,t) \right] = N^+ \left[P(x,t) \right], \tag{7}$$

Applying the differential property of NT [41,42]

$$\frac{s^\gamma}{\omega^\gamma} N^+ \left[\mu(x,t) \right] - \frac{s^{\gamma-1}}{\omega^\gamma} \mu(x,0) = N^+ \left[P(x,t) \right] - N^+ \left[L\mu(x,t) + N\mu(x,t) \right],$$

$$N^+ \left[\mu(x,t) \right] = \frac{1}{s} \mu(x,0) + \frac{\omega^\gamma}{s^\gamma} N^+ \left[P(x,t) \right] - \frac{\omega^\gamma}{s^\gamma} N^+ \left[L\mu(x,t) + N\mu(x,t) \right].$$

Now $\mu(x,0) = k(x)$,

$$N^+ \left[\mu(x,t) \right] = \frac{h(x)}{s} + \frac{\omega^\gamma}{s^\gamma} N^+ \left[P(x,t) \right] - \frac{\omega^\gamma}{s^\gamma} N^+ \left[L\mu(x,t) + N\mu(x,t) \right]. \tag{8}$$

The infinite series of NDM $\mu(x,t)$ is shown by

$$\mu(x,t) = \sum_{\ell=0}^{\infty} \mu_\ell(x,t). \tag{9}$$

Adomian polynomial for nonlinear terms is

$$N\mu(x,t) = \sum_{\ell=0}^{\infty} A_\ell, \tag{10}$$

$$A_\ell = \frac{1}{\ell!} \left[\frac{d^\ell}{d\lambda^\ell} \left[N \sum_{\ell=0}^{\infty} (\lambda^\ell \mu_\ell) \right] \right]_{\lambda=0}, \qquad \ell = 0,1,2\cdots \tag{11}$$

putting Equations (9) and (11) into Equation (8), we have

$$N^+ \left[\sum_{\ell=0}^{\infty} \mu_\ell(x,t) \right] = \frac{h(x)}{s} + \frac{\omega^\gamma}{s^\gamma} N^+ \left[P(x,t) \right] - \frac{\omega^\gamma}{s^\gamma} N^+ \left[L \sum_{\ell=0}^{\infty} \mu_\ell(x,t) + \sum_{\ell=0}^{\infty} A_\ell \right]. \tag{12}$$

$$N^+ \left[\mu_0(x,t) \right] = \frac{h(x)}{s} + \frac{\omega^\gamma}{s^\gamma} N^+ \left[P(x,t) \right], \tag{13}$$

$$N^+ \left[\mu_1(x,t) \right] = -\frac{\omega^\gamma}{s^\gamma} N^+ \left[L\mu_0(x,t) + A_0 \right]. $$

We will usually compose

$$N^+ \left[\mu_{\ell+1}(x,t) \right] = -\frac{\omega^\gamma}{s^\gamma} N^+ \left[L\mu_\ell(x,t) + A_\ell \right], \quad \ell \geq 1. \tag{14}$$

Using the inverse NT to Equations (13) and (14) [41,42].

$$\mu_0(x,t) = h(x) + N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[P(x,t) \right] \right],$$

$$v_{\ell+1}(x,t) = -N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[L\mu_\ell(x,t) + A_\ell \right] \right]. \tag{15}$$

4. NDM Implementation

Example 1. *The following nonlinear Fornberg-Whitham with fractional derivative is considered [14]*

$$D_t^\gamma \mu - D_{xxt}\mu + D_x\mu = \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu, \quad t > 0, \quad 0 < \gamma \leq 1, \tag{16}$$

having initial solution as

$$\mu(x,0) = \exp\left(\frac{x}{2}\right). \tag{17}$$

Applying NT to Equation (16), we have

$$\frac{s^\gamma}{\omega^\gamma} N^+ \left[\mu(x,t) \right] - \frac{s^{\gamma-1}}{\omega^\gamma} \mu(x,0) = N^+ \left[D_{xxt}\mu - D_x\mu + \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu \right].$$

$$N^+ \left[\mu(x,t) \right] - \frac{1}{s} \mu(x,0) = \frac{\omega^\gamma}{s^\gamma} N^+ \left[D_{xxt}\mu - D_x\mu + \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu \right].$$

Using inverse natural transformation

$$\mu(x,t) = N^- \left[\frac{\mu(x,0)}{s} - \frac{\omega^\gamma}{s^\gamma} N^+ \left[D_{xxt}\mu - D_x\mu + \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu \right] \right].$$

Applying the ADM process, we have

$$\mu_0(x,t) = N^- \left[\frac{\mu(x,0)}{s} \right] = N^- \left[\frac{\exp\left(\frac{x}{2}\right)}{s} \right],$$

$$\mu_0(x,t) = \exp\left(\frac{x}{2}\right), \tag{18}$$

$$\sum_{\ell=0}^\infty \mu_{\ell+1}(x,t) = N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[\sum_{\ell=0}^\infty (D_{xxt}\mu)_\ell - \sum_{\ell=0}^\infty (D_x\mu)_\ell + \sum_{\ell=0}^\infty A_\ell - \sum_{\ell=0}^\infty B_\ell + 3\sum_{\ell=0}^\infty C_\ell \right] \right], \quad \ell = 0,1,2,\cdots$$

$$A_0(\mu D_{xxx}\mu) = \mu_0 D_{xxx}\mu_0,$$
$$A_1(\mu D_{xxx}\mu) = \mu_0 D_{xxx}\mu_1 + \mu_1 D_{xxx}\mu_0,$$
$$A_2(\mu D_{xxx}\mu) = \mu_1 D_{xxx}\mu_2 + \mu_1 D_{xxx}\mu_1 + \mu_2 D_{xxx}\mu_0,$$

$$B_0(\mu D_x\mu) = \mu_0 D_x\mu_0,$$
$$B_1(\mu D_x\mu) = \mu_0 D_x\mu_1 + \mu_1 D_x\mu_0,$$
$$B_2(\mu D_x\mu) = \mu_1 D_x\mu_2 + \mu_1 D_x\mu_1 + \mu_2 D_x\mu_0,$$

$$C_0(D_x\mu D_{xx}\mu) = D_x\mu_0 D_{xx}\mu_0,$$
$$C_1(D_x\mu D_{xx}\mu) = D_x\mu_0 D_{xx}\mu_1 + D_x\mu_1 D_{xx}\mu_0,$$
$$C_2(D_x\mu D_{xx}\mu) = D_x\mu_1 D_{xx}\mu_2 + D_x\mu_1 D_{xx}\mu_1 + D_x\mu_2 D_{xx}\mu_0,$$

for $\ell = 1$

$$\mu_1(x,t) = N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[(D_{xxt}\mu)_0 - (D_x\mu)_0 + A_0 - B_0 + 3C_0 \right] \right],$$

$$\mu_1(x,t) = -\frac{1}{2} N^- \left[\frac{\omega^\gamma \exp(\frac{x}{2})}{s^{\gamma+1}} \right] = -\frac{1}{2} \exp\left(\frac{x}{2}\right) \frac{t^\gamma}{\Gamma(\gamma+1)}. \tag{19}$$

for $\ell = 2$

$$\mu_2(x,t) = N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[(D_{xxt}\mu)_1 - (D_x\mu)_1 + A_1 - B_1 + 3C_1 \right] \right],$$

$$\mu_2(x,t) = -\frac{1}{8} \exp\left(\frac{x}{2}\right) \frac{t^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{1}{4} \exp\left(\frac{x}{2}\right) \frac{t^{2\gamma}}{\Gamma(2\gamma+1)}, \tag{20}$$

for $\ell = 3$

$$\mu_3(x,t) = N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[(D_{xxt}\mu)_2 - (D_x\mu)_2 + A_2 - B_2 + 3C_2 \right] \right],$$

$$\mu_3(x,t) = -\frac{1}{32} \exp\left(\frac{x}{2}\right) \frac{t^{3\gamma-2}}{\Gamma(3\gamma-1)} + \frac{1}{8} \exp\left(\frac{x}{2}\right) \frac{\gamma^{3\gamma-1}}{\Gamma(3\gamma)} - \frac{1}{8} \exp\left(\frac{x}{2}\right) \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}, \tag{21}$$

The NDM solution for problem (16) is

$$\mu(x,t) = \mu_0(x,t) + \mu_1(x,t) + \mu_2(x,t) + \mu_3(x,t) + \mu_4(x,t) \cdots .$$

$$\mu(x,t) = \exp\left(\frac{x}{2}\right) - \frac{1}{2} \exp\left(\frac{x}{2}\right) \frac{t^\gamma}{\Gamma(\gamma+1)}$$

$$- \frac{1}{8} \exp\left(\frac{x}{2}\right) \frac{t^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{1}{4} \exp\left(\frac{x}{2}\right) \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{1}{32} \exp\left(\frac{x}{2}\right)$$

$$\frac{t^{3\gamma-2}}{\Gamma(3\gamma-1)} + \frac{1}{8} \exp\left(\frac{x}{2}\right) \frac{\gamma^{3\gamma-1}}{\Gamma(3\gamma)} - \frac{1}{8} \exp\left(\frac{x}{2}\right) \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} - \cdots . \tag{22}$$

The simplification of Equation (22);

$$\mu(x,t) = \exp\left(\frac{x}{2}\right) \left[1 - \frac{t^\gamma}{2\Gamma(\gamma+1)} - \frac{1}{8}\frac{t^{2\gamma-1}}{\Gamma(2\gamma)} + \frac{1}{4}\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - \frac{1}{32}\frac{t^{3\gamma-2}}{\Gamma(3\gamma-1)} + \frac{1}{8}\frac{t^{3\gamma-1}}{\Gamma(3\gamma)} - \frac{1}{8}\frac{t^{3\gamma}}{\Gamma(3\gamma+1)} + \cdots \right]. \tag{23}$$

The exact result of Example 1

$$\mu(x,t) = \exp\left(\frac{x}{2} - \frac{2t}{3}\right), \tag{24}$$

In Table 1, the NDM-solutions at different fractional-order derivatives, $\gamma = 0.5, 0.7$ and 1 are shown. The NDM-solutions at various time level, $t = 0.2, 0.4$ and $t = 1$ are determined. The absolute error of the proposed method at $\gamma = 1$ is also displayed. From Table 1, it is investigated that suggested method has the desire rate of convergence and considered to be the best tool for the analytical solution of FPDEs. In Table 2, the NDM and LADM solutions are compared at various fractional-order of the derivatives. It is observed that the NDM has the higher degree of accuracy as compared to LDM. The comparison has been done at $\gamma = 0.5, 0.7$ and 0.9. It is also investigated that the fractional-order solutions of NDM have the higher accuracy as compared LDM.

Table 1. The NDM solutions and absolute error of Example 1 at $\gamma = 0.5$, 0.7 and 1.

t	x	$\gamma = 0.50$	$\gamma = 0.7$	NDM ($\gamma = 1$)	Exact	NDM (AE) ($\gamma = 1$)
	0.5	1.168497921	1.229840967	1.266952492	1.267018708	6.620×10^{-5}
0.2	1.0	1.500381030	1.579147061	1.626799201	1.626884224	8.500×10^{-5}
	1.5	1.926527377	2.027664962	2.088851523	2.088960694	1.090×10^{-4}
	2.0	2.473710118	2.603573348	2.682138447	2.682278626	1.400×10^{-4}
	0.5	1.123744786	1.197168588	1.250129766	1.25023725	1.070×10^{-4}
0.4	1.0	1.442916867	1.537194895	1.605198393	1.60533640	1.380×10^{-4}
	1.5	1.852741931	1.973797315	2.061115536	2.06129274	1.770×10^{-4}
	2.0	2.378967730	2.534405920	2.646524735	2.64675227	2.270×10^{-4}
	0.5	1.039959208	1.124099772	1.201030155	1.20114746	1.84×10^{-4}
1	1.0	1.335334055	1.443372678	1.542153245	1.542390265	2.370×10^{-4}
	1.5	1.714602867	1.853327204	1.980163963	1.980468303	3.004×10^{-4}
	2.0	2.201593661	2.379719235	2.542580858	2.542971638	3.900×10^{-4}

Table 2. Two terms comparison of NDM and LDM [16] of different fractional-order at $\gamma = 0.5$, 0.7 and 0.9 of Example 1.

x	t	NDM	LDM	NDM	LDM	NDM	LDM
		$\gamma = 0.5$	$\gamma = 0.5$	$\gamma = 0.7$	$\gamma = 0.7$	$\gamma = 0.9$	$\gamma = 0.9$
	0.1	1.259×10^{-1}	4.021×10^{-1}	5.006×10^{-2}	2.158×10^{-1}	2.833×10^{-3}	4.973×10^{-2}
	0.2	1.408×10^{-1}	4.171×10^{-1}	5.916×10^{-2}	2.249×10^{-1}	2.978×10^{-3}	5.227×10^{-2}
	0.3	1.411×10^{-1}	4.174×10^{-1}	6.148×10^{-2}	2.272×10^{-1}	3.131×10^{-3}	5.496×10^{-2}
	0.4	1.356×10^{-1}	4.119×10^{-1}	6.153×10^{-2}	2.273×10^{-1}	3.291×10^{-3}	5.777×10^{-2}
0.2	0.5	1.276×10^{-1}	4.039×10^{-1}	6.107×10^{-2}	2.268×10^{-1}	3.460×10^{-3}	6.074×10^{-2}
	0.6	1.186×10^{-1}	3.949×10^{-1}	6.096×10^{-2}	2.267×10^{-1}	3.638×10^{-3}	6.385×10^{-2}
	0.7	1.095×10^{-1}	3.858×10^{-1}	6.164×10^{-2}	2.274×10^{-1}	3.824×10^{-3}	6.712×10^{-2}
	0.8	1.007×10^{-1}	3.771×10^{-1}	6.337×10^{-2}	2.291×10^{-1}	4.020×10^{-3}	7.057×10^{-2}
	0.9	9.288×10^{-2}	3.691×10^{-1}	6.626×10^{-2}	2.320×10^{-1}	4.226×10^{-3}	7.418×10^{-2}
	1.0	8.576×10^{-2}	3.622×10^{-1}	7.038×10^{-2}	2.361×10^{-1}	4.443×10^{-3}	7.799×10^{-2}

In Figures 1 and 2, the NDM and actual solution of Example 1 are plotted. It is observed that NDM solutions are in closed contact with the exact solutions of Example 1. In Figures 3 and 4, the solutions of Example 1 at various fractional-order of the derivatives are plotted. The graphical representation has shown the convergence phenomena of fractional-order solution towards the solution at integer order of Example 1.
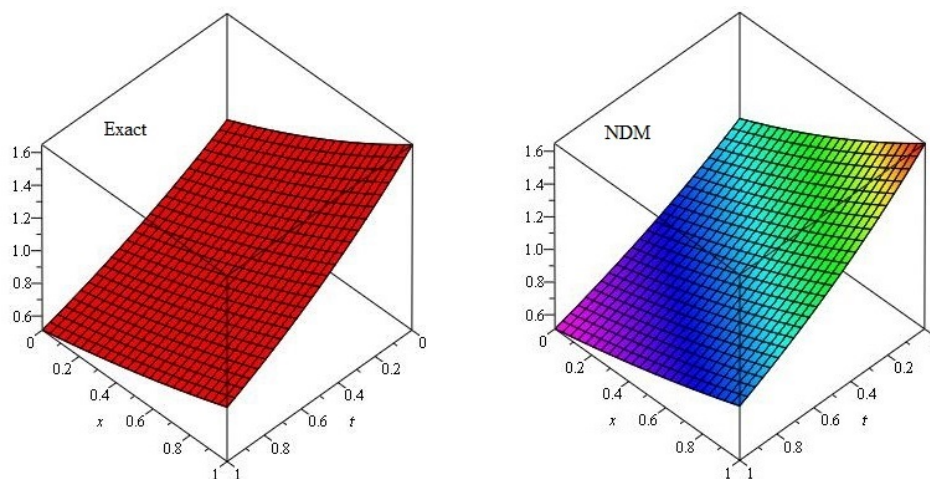

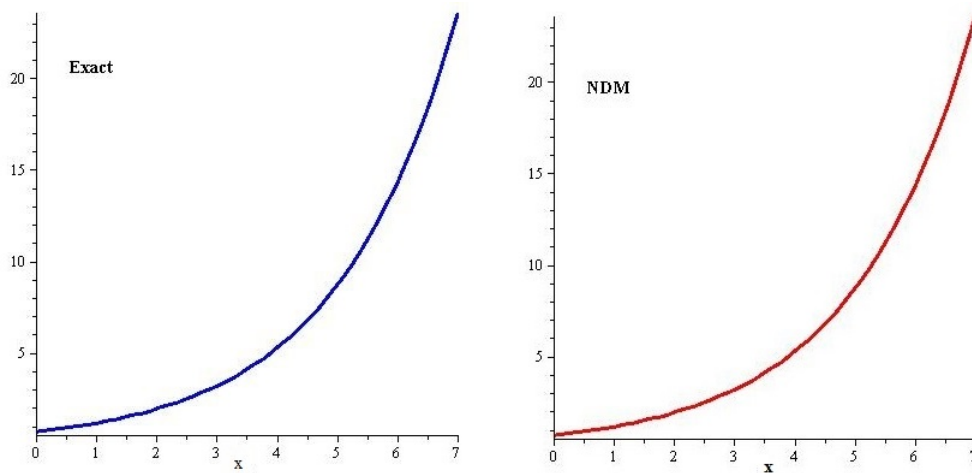**Figure 1.** Exact and NDM solutions at $\gamma = 1$ of Example 1.

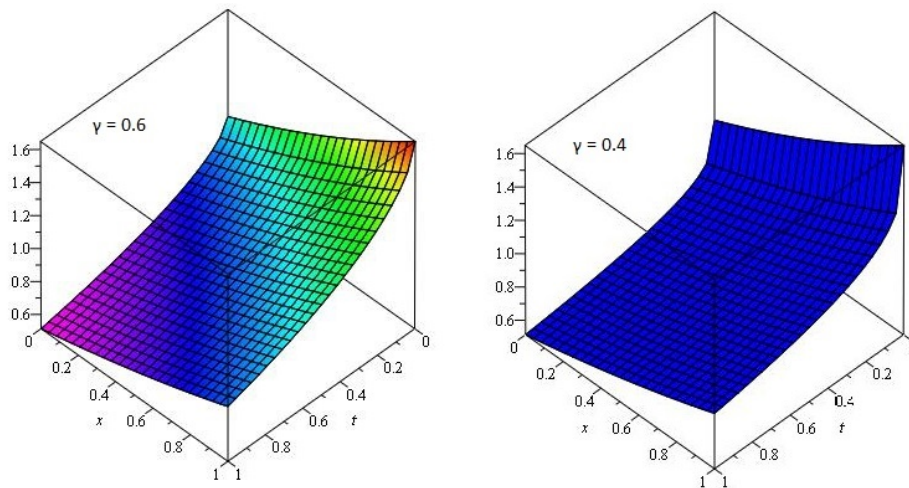**Figure 2.** Exact and NDM solutions at $\gamma = 1$ of Example 1.

Figure 3. The NDM solutions of different valve of γ of Example 1.
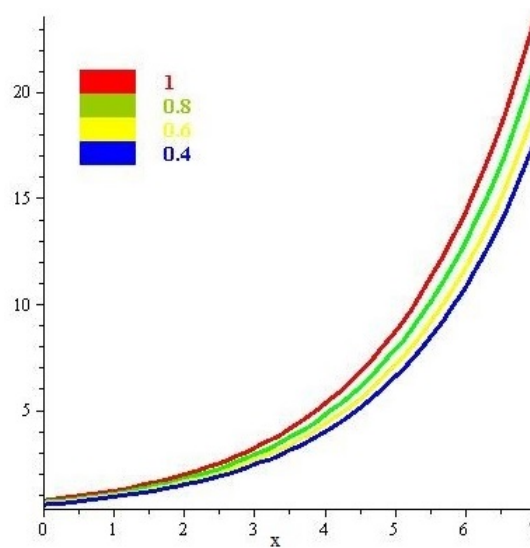

Figure 4. Solution graph of Example 1, at various value of γ.

Example 2. *Consider the following nonlinear time-fractional Fornberg–Whitham equation* [18]

$$D_t^\gamma \mu - D_{xxt}\mu + D_x\mu = \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu, \quad t > 0, \quad 0 < \gamma \le 1, \tag{25}$$

with initial condition

$$\mu(x,0) = \cosh^2\left(\frac{x}{4}\right), \tag{26}$$

Applying natural transformation of Equation (25),

$$\frac{s^\gamma}{\omega^\gamma} N^+\left[\mu(x,t)\right] - \frac{s^{\gamma-1}}{\omega^\gamma}\mu(x,0) = N^+\left[D_{xxt}\mu - D_x\mu + \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu\right].$$

$$N^+\left[\mu(x,t)\right] - \frac{1}{s}\mu(x,0) = \frac{\omega^\gamma}{s^\gamma} N^+\left[D_{xxt}\mu - D_x\mu + \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu\right].$$

Using inverse natural transformation

$$\mu(x,t) = N^-\left[\frac{\mu(x,0)}{s} - \frac{\omega^\gamma}{s^\gamma} N^+\left[D_{xxt}\mu - D_x\mu + \mu D_{xxx}\mu - \mu D_x\mu + 3D_x\mu D_{xx}\mu\right]\right].$$

Applying the ADM process, we have

$$\mu_0(x,t) = N^-\left[\frac{\mu(x,0)}{s}\right] = N^-\left[\frac{\cosh^2\left(\frac{x}{4}\right)}{s}\right],$$

$$\mu_0(x,t) = \cosh^2\left(\frac{x}{4}\right), \tag{27}$$

$$\sum_{\ell=0}^\infty \mu_{\ell+1}(x,t) = N^-\left[\frac{\omega^\gamma}{s^\gamma} N^+\left[\sum_{\ell=0}^\infty (D_{xxt}\mu)_\ell - \sum_{\ell=0}^\infty (D_x\mu)_\ell + \sum_{\ell=0}^\infty A_\ell - \sum_{\ell=0}^\infty B_\ell + 3\sum_{\ell=0}^\infty C_\ell\right]\right], \quad \ell = 0,1,2,\cdots$$

$$A_0(\mu D_{xxx}\mu) = \mu_0 D_{xxx}\mu_0,$$
$$A_1(\mu D_{xxx}\mu) = \mu_0 D_{xxx}\mu_1 + \mu_1 D_{xxx}\mu_0,$$
$$A_2(\mu D_{xxx}\mu) = \mu_1 D_{xxx}\mu_2 + \mu_1 D_{xxx}\mu_1 + \mu_2 D_{xxx}\mu_0,$$

$$B_0(\mu D_x\mu) = \mu_0 D_x\mu_0,$$
$$B_1(\mu D_x\mu) = \mu_0 D_x\mu_1 + \mu_1 D_x\mu_0,$$
$$B_2(\mu D_x\mu) = \mu_1 D_x\mu_2 + \mu_1 D_x\mu_1 + \mu_2 D_x\mu_0.$$

$$C_0(D_x\mu D_{xx}\mu) = D_x\mu_0 D_{xx}\mu_0,$$
$$C_1(D_x\mu D_{xx}\mu) = D_x\mu_0 D_{xx}\mu_1 + D_x\mu_1 D_{xx}\mu_0,$$
$$C_2(D_x\mu D_{xx}\mu) = D_x\mu_1 D_{xx}\mu_2 + D_x\mu_1 D_{xx}\mu_1 + D_x\mu_2 D_{xx}\mu_0,$$

for $\ell = 1$

$$\mu_1(x,t) = N^-\left[\frac{\omega^\gamma}{s^\gamma} N^+\left[(D_{xxt}\mu)_0 - (D_x\mu)_0 + A_0 - B_0 + 3C_0\right]\right],$$

$$\mu_1(x,t) = -\frac{11}{32} N^-\left[\frac{\omega^\gamma \sinh\left(\frac{x}{2}\right)}{s^{\gamma+1}}\right] = -0.3437 \sinh\left(\frac{x}{4}\right)\frac{t^\gamma}{\Gamma(\gamma+1)}, \tag{28}$$

for $\ell = 2$

$$\mu_2(x,t) = N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[(D_{xxt}\mu)_1 - (D_x\mu)_1 + A_1 - B_1 + 3C_1 \right] \right],$$

$$\mu_2(x,t) = -0.08593 \sinh\left(\frac{x}{4}\right) \frac{t^\gamma}{\Gamma(\gamma+1)} + 0.11816 \cosh\left(\frac{x}{4}\right) \frac{t^{2\gamma}}{\Gamma(2\gamma+1)},$$

(29)

for $\ell = 3$

$$\mu_3(x,t) = N^- \left[\frac{\omega^\gamma}{s^\gamma} N^+ \left[(D_{xxt}\mu)_2 - (D_x\mu)_2 + A_2 - B_2 + 3C_2 \right] \right],$$

$$\mu_3(x,t) = -0.08593 \sinh\left(\frac{x}{4}\right) \frac{t^\gamma}{\Gamma(\gamma+1)} + 0.11816 \cosh\left(\frac{x}{4}\right) \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - 0.02707 \sinh\left(\frac{x}{4}\right) \frac{t^{3\gamma}}{\Gamma(3\gamma+1)}.$$

(30)

The NDM result for problem 2 is

$$\mu(x,t) = \mu_0(x,t) + \mu_1(x,t) + \mu_2(x,t) + \mu_3(x,t) + \mu_4(x,t) \cdots.$$

$$\mu(x,t) = \cosh^2\left(\frac{x}{4}\right) - 0.3437 \sinh\left(\frac{x}{4}\right) \frac{t^\gamma}{\Gamma(\gamma+1)} - 0.08593 \sinh\left(\frac{x}{4}\right) \frac{t^\gamma}{\Gamma(\gamma+1)}$$

$$+ 0.11816 \cosh\left(\frac{x}{4}\right) \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - 0.08593 \sinh\left(\frac{x}{4}\right) \frac{t^\gamma}{\Gamma(\gamma+1)}$$

$$+ 0.11816 \cosh\left(\frac{x}{4}\right) \frac{t^{2\gamma}}{\Gamma(2\gamma+1)} - 0.02707 \sinh\left(\frac{x}{4}\right) \frac{t^{3\gamma}}{\Gamma(3\gamma+1)} - \cdots.$$

The exact result is;

$$\mu(x,t) = \cosh^2\left(\frac{x}{4} - \frac{11t}{24}\right).$$

In Figures 5 and 6, the solution graph of exact and NDM of Example 2 at integer-order are plotted. The closed relation is observed between NDM and exact solution of Example 2. In Figures 7 and 8, the fractional-order solutions of Example 2 are presented. The graphical representation have confirmed the different dynamics of Example 2, which are correlated with each other.
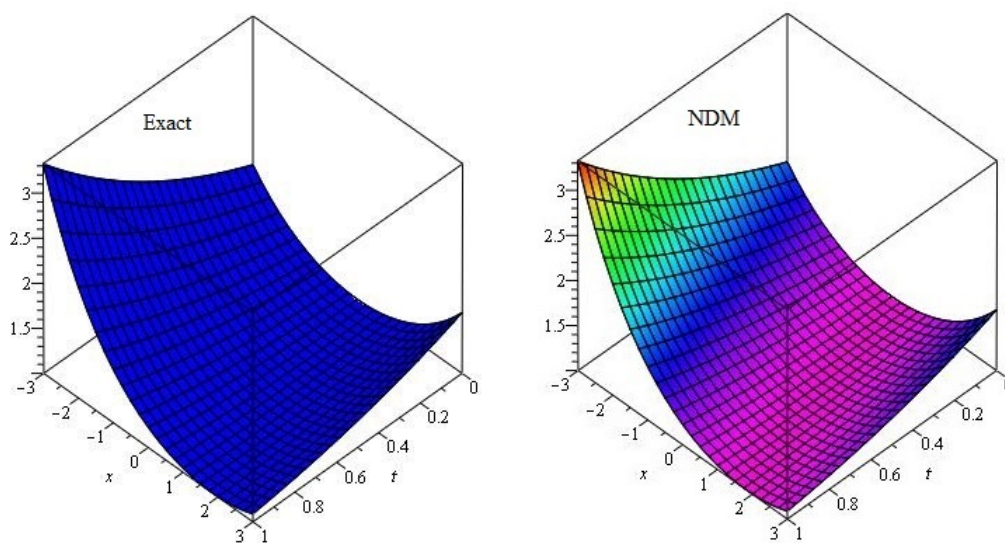

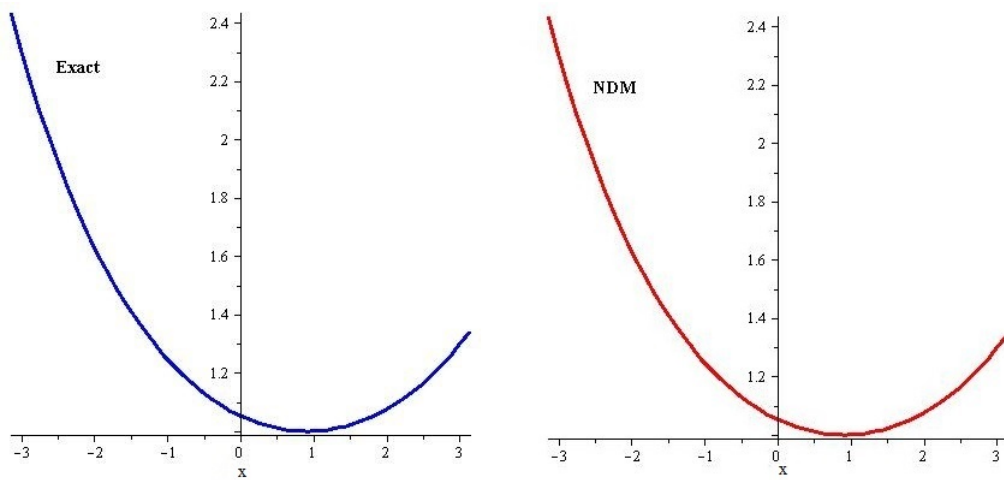**Figure 5.** The graph of exact and approximate solution of Example 2.

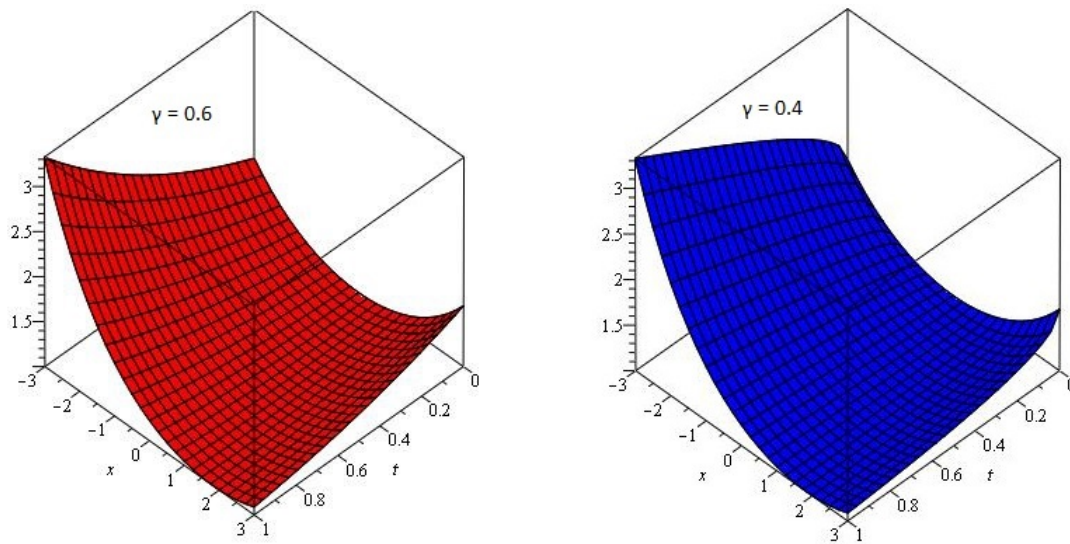**Figure 6.** The graph of exact and approximate solution of Example 2.

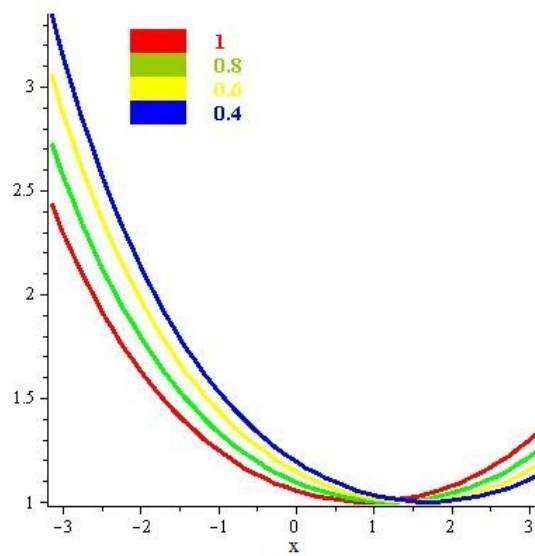**Figure 7.** The NDM solutions of different valve of γ of Example 2.

Figure 8. The graph of Example 2, for different value of γ.

5. Conclusions

In the current work. an innovative technique is used to find the solution of fractional Fornberg-Whitham equations. The fractional-derivatives are discussed within Caputo operator. The solutions are determined for fractional-order problems and an aesthetically a strong relation is found. The fractional models have shown convergence to the ordinary model as the order of the derivative tends towards to an integer. The graphical representation has provided similar behavior of actual and derived results. It is also noted the current method needs small calculation and higher convergence to achieve the solution of the targeted problems.

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