

Article

New Weighted Opial-Type Inequalities on Time Scales for Convex Functions

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Abstract: Our work is based on the multiple inequalities illustrated in 1967 by E. K. Godunova and V. I. Levin, in 1990 by Hwang and Yang and in 1993 by B. G. Pachpatte. With the help of the dynamic Jensen and Hölder inequality, we generalize a number of those inequalities to a general time scale. In addition to these generalizations, some integral and discrete inequalities will be obtained as special cases of our results.

Keywords: opial-type inequality; dynamic inequality; convexity; time scale

MSC: 26D10; 26D15; 26E70; 34A40

1. Introduction

The following inequality [1] is well-known in the literature as Opial's inequality.

Theorem 1. *If δ is an absolutely continuous function on $[0, h]_{\mathbb{R}}$ with $\delta(0) = \delta(h) = 0$, then*

$$\int_0^h |\delta(t)\delta'(t)| dt \leq \frac{h}{4} \int_0^h |\delta'(t)|^2 dt. \quad (1)$$

In 1967 E. K. Godunova and V. I. Levin [2] proved the following two theorems which are a generalization of Opial's inequality (1).

Theorem 2. *Let δ be real-valued absolutely continuous function on $[a, b]_{\mathbb{R}}$ with $\delta(a) = 0$. Let f be real-valued convex and increasing function on $[0, \infty)_{\mathbb{R}}$ with $f(0) = 0$. Then, the following inequality holds*

$$\int_a^b f'(|\delta(t)|) |\delta'(t)| dt \leq f\left(\int_a^b |\delta'(t)| dt\right).$$

Theorem 3. *Let δ be real-valued absolutely continuous function on $[a, b]_{\mathbb{R}}$ with $\delta(a) = \delta(b) = 0$. Assume f and g are real-valued convex and increasing functions on $[0, \infty)_{\mathbb{R}}$ with $f(0) = 0$. Further let $p \geq 0$ on $[a, b]_{\mathbb{R}}$ and $\int_a^b p(t) dt = 1$. Then, the following inequality holds*

$$\int_a^b f'(|\delta(t)|) |\delta'(t)| dt \leq 2f\left(g^{-1}\left(\int_a^b p(t)g\left(\frac{|\delta'(t)|}{2p(t)}\right) dt\right)\right). \quad (2)$$

In 1990, Hwang and Yang [3] established the following result:

Theorem 4. Assuming $f \geq 0$ and $g \geq 0$ are continuous functions on $[0, \infty)_{\mathbb{R}}$ with $f(0) = 0$ such that $f' \geq 0$ and $g' \geq 0$ exist and non-decreasing continuous functions on $[0, \infty)_{\mathbb{R}}$. Suppose x and y are absolutely continuous functions on $[a, \tau]_{\mathbb{R}}$, and $x(a) = y(a) = 0$. Then, for all $m \geq 1$, we get

$$\int_a^\tau [f(|\delta(t)|^m)g'(|\gamma(t)|^m)|\gamma'(t)|^m + g(|\gamma(t)|^m)f'(|\delta(t)|^m)|\delta'(t)|^m] dt \leq \frac{1}{\lambda}f\left(\lambda \int_a^\tau |\delta'(t)|^m dt\right)g\left(\lambda \int_a^\tau |\gamma'(t)|^m dt\right), \tag{3}$$

where $\lambda = (\tau - a)^{m-1}$.

S. Hilger [4], suggested time scales theory to unify discrete and continuous analysis. Continuous calculus, discrete calculus, and quantum calculus can be said as the three most common examples of calculus on time scales i.e., for continuous calculus $\mathbb{T} = \mathbb{R}$, for discrete calculus $\mathbb{T} = h\mathbb{Z}$ and for quantum calculus $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. The book due to Bohner and Peterson [5] on the subject of time scales briefs and organizes much of time scales calculus. For some Opial-type integral, dynamic inequalities and other types of inequalities on time scales, see the papers [6–32]. More results on inequalities see, [33–53].

The following essential relations on some time scales such as \mathbb{R} , \mathbb{Z} , $h\mathbb{Z}$ and $\overline{q^{\mathbb{Z}}}$ will be used in the following section. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(t) = t, \quad \mu(t) = 0, \quad \psi^\Delta(t) = \psi'(t), \quad \int_\alpha^\beta \psi(t)\Delta t = \int_\alpha^\beta \psi(t)dt. \tag{4}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(t) = t + 1, \quad \mu(t) = 1, \quad \psi^\Delta(t) = \psi(t + 1) - \psi(t), \quad \int_\alpha^\beta \psi(t)\Delta t = \sum_{t=\alpha}^{\beta-1} \psi(t). \tag{5}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$, then

$$\sigma(t) = t + h, \quad \mu(t) = h, \quad \psi^\Delta(t) = \frac{\psi(t + h) - \psi(t)}{h}, \quad \int_\alpha^\beta \psi(t)\Delta t = \sum_{t=\frac{\alpha}{h}}^{\frac{\beta}{h}-1} \psi(th)h. \tag{6}$$

(iv) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then

$$\sigma(t) = qt, \quad \mu(t) = (q - 1)t, \quad \psi^\Delta(t) = \frac{\psi(qt) - \psi(t)}{(q-1)t}, \quad \int_\alpha^\beta \psi(t)\Delta t = (q - 1) \sum_{t=(\log_q \alpha)}^{(\log_q \beta)-1} \psi(q^t)q^t. \tag{7}$$

Next is Hölder’s and Jensen’s inequality:

Lemma 1 (Hölder’s inequality [5]). Let $a, b \in \mathbb{T}$. For $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, we have

$$\int_a^b f(t)g(t)\Delta t \leq \left[\int_a^b f^p(t)\Delta t \right]^{\frac{1}{p}} \left[\int_a^b g^q(t)\Delta t \right]^{\frac{1}{q}},$$

where $p > 1$ and $1/p + 1/q = 1$.

Lemma 2 (Jensen's inequality [16]). Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Assume that $g \in C_{rd}([a, b]_{\mathbb{T}}, [c, d])$ and $r \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$ are nonnegative with $\int_a^b r(t) \Delta t > 0$. If $\check{\Phi} \in C_{rd}((c, d), \mathbb{R})$ is a convex function, then

$$\check{\Phi} \left(\frac{\int_a^b r(t) g(t) \Delta t}{\int_a^b r(t) \Delta t} \right) \leq \frac{\int_a^b r(t) \check{\Phi}(g(t)) \Delta t}{\int_a^b r(t) \Delta t}.$$

Lemma 3 (Chain rule [5]). Assume $g : \mathbb{R} \rightarrow \mathbb{R}$, is continuous, $g : \mathbb{T} \rightarrow \mathbb{R}$ is delta-differentiable on \mathbb{T}^{κ} , and $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with

$$(f \circ g)^{\Delta}(t) = f'(g(c)) g^{\Delta}(t). \quad (8)$$

In the proofs of our results, the following inequality will be used:

$$a^k + b^k \leq (a + b)^k \leq 2^{k-1} (a^k + b^k), \quad \text{if } a \geq 0, b \geq 0 \text{ and } k \geq 1. \quad (9)$$

In this article, we prove some dynamic Opial-type inequalities involving convex functions on time scales. Our results generalize some of the mentioned results of Pachpatte [54,55], E. K. Godunova and V. I. Levin [2] and Hwang and Yang [3], in time scales. Furthermore, our results extend some existing dynamic Opial-type inequalities in the literature, and give some integral and discrete inequalities as special cases.

2. Main Results

Theorem 5. Assuming \mathbb{T} is a time scale with $\Gamma, \alpha \in \mathbb{T}$, $g \geq 0$ and $f \geq 0$ are continuous functions on $[0, \infty)_{\mathbb{R}}$ with $f(0) = 0$ such that $g' \geq 0$ and $f' \geq 0$ exist and non-decreasing continuous functions on $[0, \infty)_{\mathbb{R}}$. Suppose x and y are rd-continuous functions on $[\alpha, \Gamma]_{\mathbb{T}}$, and $x(\alpha) = y(\alpha) = 0$. Then, for all $m \geq 1$, we get

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[f(|x(\theta)|^m) g'(|y(\theta)|^m) |y^{\Delta}(\theta)|^m + g(|y(\theta)|^m) f'(|x(\theta)|^m) |x^{\Delta}(\theta)|^m \right] \Delta \theta \\ & \leq \frac{1}{\lambda_1} f \left(\lambda_1 \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta \theta \right) g \left(\lambda_1 \int_{\alpha}^{\Gamma} |y^{\Delta}(\theta)|^m \Delta \theta \right), \end{aligned} \quad (10)$$

where $\lambda_1 = (\Gamma - \alpha)^{m-1}$.

Proof. For $t \in [\alpha, \Gamma]_{\mathbb{T}}$. Define $\varphi(t) := \int_{\alpha}^t |x^{\Delta}(\theta)|^m \Delta \theta$ and $\psi(t) := \int_{\alpha}^t |y^{\Delta}(\theta)|^m \Delta \theta$ so that $\varphi^{\Delta}(t) = |x^{\Delta}(t)|^m$ and $\psi^{\Delta}(t) = |y^{\Delta}(t)|^m$. Thus,

$$\begin{aligned} |x(t)|^m &= \left| \int_{\alpha}^t x^{\Delta}(\theta) \Delta \theta \right|^m \leq \left(\int_{\alpha}^t |x^{\Delta}(\theta)| \Delta \theta \right)^m, \\ |y(t)|^m &= \left| \int_{\alpha}^t y^{\Delta}(\theta) \Delta \theta \right|^m \leq \left(\int_{\alpha}^t |y^{\Delta}(\theta)| \Delta \theta \right)^m. \end{aligned} \quad (11)$$

Next, applying the dynamic Hölder's inequality (1) on (11) with indices m and $\frac{m}{m-1}$, we get

$$\begin{aligned} |x(t)|^m &\leq (t - \alpha)^{m-1} \left(\int_{\alpha}^t |x^{\Delta}(\theta)|^m \Delta \theta \right) \\ &\leq (t - \alpha)^{m-1} \left(\int_{\alpha}^t |x^{\Delta}(\theta)|^m \Delta \theta \right) \\ &\leq \lambda_1 \varphi(t). \end{aligned} \quad (12)$$

and similarly

$$|y(t)|^m \leq \lambda_1 \psi(t). \tag{13}$$

Since $f \geq 0, f' \geq 0, g \geq 0,$ and $g' \geq 0$ are non-decreasing continuous on $[0, \infty)_{\mathbb{R}}$. Since $\sigma(t) \geq t$, we get $g(\lambda_1 \psi) \leq g^\sigma(\lambda_1 \psi)$, and we find that

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[f(|x(\theta)|^m) g'(|y(\theta)|^m) |y^{\Delta}(\theta)|^m + g(|y(\theta)|^m) f'(|x(\theta)|^m) |x^{\Delta}(\theta)|^m \right] \Delta\theta \\ & \leq \int_{\alpha}^{\Gamma} \left[f(\lambda_1 \varphi(\theta)) g'(\lambda_1 \psi(\theta)) \psi^{\Delta}(\theta) + g(\lambda_1 \psi(\theta)) f'(\lambda_1 \varphi(\theta)) \varphi^{\Delta}(\theta) \right] \Delta\theta \\ & \leq \int_{\alpha}^{\Gamma} \left[f(\lambda_1 \varphi(\theta)) g'(\lambda_1 \psi(\theta)) \psi^{\Delta}(\theta) + g^{\sigma}(\lambda_1 \psi(\theta)) f'(\lambda_1 \varphi(\theta)) \varphi^{\Delta}(\theta) \right] \Delta\theta. \end{aligned} \tag{14}$$

From Lemma 3 for $c \in [t, \sigma(t)]_{\mathbb{R}}$, we get

$$\frac{1}{\lambda_1} f^{\Delta}(\lambda_1 \varphi)(t) = f'(\lambda_1 \varphi(c)) \varphi^{\Delta}(t) \geq f'(\lambda_1 \varphi(t)) \varphi^{\Delta}(t), \tag{15}$$

and similarly

$$\frac{1}{\lambda_1} g^{\Delta}(\lambda_1 \psi(t)) \geq g'(\lambda_1 \psi(t)) \psi^{\Delta}(t). \tag{16}$$

Then, by substituting (15) and (16) into (14), we get

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[f(|x(\theta)|^m) g'(|y(\theta)|^m) |y^{\Delta}(\theta)|^m + g(|y(\theta)|^m) f'(|x(\theta)|^m) |x^{\Delta}(\theta)|^m \right] \Delta\theta \\ & \leq \int_{\alpha}^{\Gamma} \left[f(\lambda_1 \varphi(\theta)) g'(\lambda_1 \psi(\theta)) \psi^{\Delta}(\theta) + g^{\sigma}(\lambda_1 \psi(\theta)) f'(\lambda_1 \varphi(\theta)) \varphi^{\Delta}(\theta) \right] \Delta\theta \\ & = \frac{1}{\lambda_1} \int_{\alpha}^{\Gamma} [f(\lambda_1 \varphi) g(\lambda_1 \psi)]^{\Delta}(\theta) \Delta\theta \\ & = \frac{1}{\lambda_1} f \left(\lambda_1 \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta\theta \right) g \left(\lambda_1 \int_{\alpha}^{\Gamma} |y^{\Delta}(\theta)|^m \Delta\theta \right), \end{aligned}$$

which is the desired inequality (10). This proves our claim. \square

Remark 1. When $\mathbb{T} = \mathbb{R}$ in Theorem 5, then, by the relation (4), we obtain Hwang and Yang inequality (3).

Corollary 1. If we take $\mathbb{T} = h\mathbb{Z}$ in Theorem 5, then, by the relation (6), inequality (10) becomes

$$\begin{aligned} & h \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} \left[f(|x(nh)|^m) g'(|y(nh)|^m) |\Delta y(nh)|^m + g(|y(nh)|^m) f'(|x(nh)|^m) |\Delta x(nh)|^m \right] \\ & \leq \frac{1}{\lambda_1} f \left(h \lambda_1 \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} |\Delta x(nh)|^m \right) g \left(h \lambda_1 \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} |\Delta y(nh)|^m \right). \end{aligned}$$

Remark 2. In Corollary 2, if we take $h = 1$, then inequality (10) becomes

$$\begin{aligned} & \sum_{n=\alpha}^{\Gamma-1} \left[f(|x(n)|^m) g'(|y(n)|^m) |\Delta y(n)|^m + g(|y(n)|^m) f'(|x(n)|^m) |\Delta x(n)|^m \right] \\ & \leq \frac{1}{\lambda_1} f \left(\lambda_1 \sum_{n=\alpha}^{\Gamma-1} |\Delta x(n)|^m \right) g \left(\lambda_1 \sum_{n=\alpha}^{\Gamma-1} |\Delta y(n)|^m \right). \end{aligned}$$

Corollary 2. If we take $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 5, then, by the relation (7), inequality (10) becomes

$$(q - 1) \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} \left[f(|x(q^n)|^m) g'(|y(q^n)|^m) |\Delta y(q^n)|^m + g(|y(q^n)|^m) f'(|x(q^n)|^m) |\Delta x(q^n)|^m \right] q^n \leq \frac{1}{\lambda_1} f \left((q - 1) \lambda_1 \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} |\Delta x(q^n)|^m q^n \right) g \left((q - 1) \lambda_1 \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} |\Delta y(q^n)|^m q^n \right).$$

Remark 3. If $g = m = 1$, and $x = y$ in inequality (10), then, we obtain

$$\int_{\alpha}^{\Gamma} f'(|x(\theta)|) |x^{\Delta}(\theta)| \Delta\theta \leq f \left(\int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)| \Delta\theta \right) \tag{17}$$

with weak conditions for f .

Remark 4. If $\mathbb{T} = \mathbb{R}$, then inequality (17) gives E. K. Godunova and V. I. Levin’s inequality in [2].

Corollary 3. Assuming $f(t) = t^{\frac{\ell+m}{m}}$, $g = 1$, $x = y$ and $\ell \geq 0$ in Theorem 5. Then, from (10), and Lemma 1 with indices $\frac{\ell+m}{\ell}$ and $\frac{\ell+m}{m}$, we obtain

$$\begin{aligned} \int_{\alpha}^{\Gamma} |x(\theta)|^{\ell} |x^{\Delta}(\theta)|^m \Delta\theta &\leq \frac{m}{\ell+m} \frac{1}{\lambda_1} \left(\lambda_1 \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta\theta \right)^{\frac{\ell+m}{m}} \\ &\leq \frac{m}{\ell+m} \lambda_1^{\frac{\ell}{m}} (\Gamma - \alpha)^{\frac{\ell}{m}} \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^{\ell+m} \Delta\theta \\ &= \frac{m}{\ell+m} (\Gamma - \alpha)^{\ell} \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^{\ell+m} \Delta\theta. \end{aligned} \tag{18}$$

Corollary 4. When $m = 1$ and $\ell = 1$ in (18), we obtain the following inequality

$$\int_{\alpha}^{\Gamma} |x(\theta)| |x^{\Delta}(\theta)| \Delta\theta \leq \frac{(\Gamma - \alpha)}{2} \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^2 \Delta\theta. \tag{19}$$

Corollary 5. When $\mathbb{T} = \mathbb{R}$, $m = 1$ and $\alpha = 0$ in (18), we obtain the inequality of Hua [56]

$$\int_0^{\Gamma} |x(\theta)|^{\ell} |x'(\theta)| d\theta \leq \frac{\Gamma^{\ell}}{\ell+1} \int_0^{\Gamma} |x'(\theta)|^{\ell+1} d\theta. \tag{20}$$

Moreover, in (20), equality holds if and only if $x(t) = ct$.

Corollary 6. If $\mathbb{T} = \mathbb{R}$, then (18), gives the inequality of Yang [57].

$$\int_{\alpha}^{\Gamma} |x(\theta)|^{\ell} |x'(\theta)|^m d\theta \leq \frac{m}{\ell+m} (\Gamma - \alpha)^{\ell} \int_{\alpha}^{\Gamma} |x'(\theta)|^{\ell+m} d\theta. \tag{21}$$

Corollary 7. Assuming $p \geq 0$ is rd-continuous on $[\alpha, \Gamma]_{\mathbb{T}}$ with $\int_{\alpha}^{\Gamma} \frac{\Delta\theta}{p(\theta)} < \infty$, and let $q > 0$ is non-increasing bounded on $[\alpha, \Gamma]_{\mathbb{T}}$. By using $f(t) = t^2$, $g(t) = 1$, $m = 1$ and $x(t) = y(t) = \int_{\alpha}^t \sqrt{q(\theta)} |z^{\Delta}(\theta)| \Delta\theta$ and $x^{\Delta}(t) = \sqrt{q(t)} |z^{\Delta}(t)|$. Then, we get from Theorem 5, the following inequality

$$2 \int_{\alpha}^{\Gamma} \left(\int_{\alpha}^{\theta} \sqrt{q(\xi)} |x^{\Delta}(\xi)| \Delta\xi \right) \sqrt{q(\theta)} |z^{\Delta}(\theta)| \Delta\theta \leq \left(\int_{\alpha}^{\Gamma} \sqrt{q(\theta)} |z^{\Delta}(\theta)| \Delta\theta \right)^2.$$

However, since

$$\int_{\alpha}^t \sqrt{q(\theta)} |z^{\Delta}(\theta)| \Delta\theta \geq \sqrt{q(t)} \int_{\alpha}^t |z^{\Delta}(\theta)| \Delta\theta \geq \sqrt{q(t)} z(t),$$

it follows that

$$2 \int_{\alpha}^{\Gamma} q(\theta) |x(\theta)| |x^{\Delta}(\theta)| \Delta\theta \leq \left(\int_{\alpha}^{\Gamma} \frac{1}{\sqrt{p(\theta)}} \sqrt{p(\theta)q(\theta)} |x^{\Delta}(\theta)| \Delta\theta \right)^2.$$

By applying the Cauchy–Schwarz inequality, we get that

$$\int_{\alpha}^{\Gamma} q(\theta) |x(\theta)| |x^{\Delta}(\theta)| \Delta\theta \leq \frac{1}{2} \int_{\alpha}^{\Gamma} \frac{\Delta\theta}{p(\theta)} \int_{\alpha}^{\Gamma} p(\theta) q(\theta) |x^{\Delta}(\theta)|^2 \Delta\theta. \tag{22}$$

Remark 5. When $\mathbb{T} = \mathbb{R}$, the inequality (22), reduces to Yang inequality [57]

$$\int_{\alpha}^{\Gamma} q(\theta) |x(\theta)| |x'(\theta)| d\theta \leq \frac{1}{2} \int_{\alpha}^{\Gamma} \frac{d\theta}{p(\theta)} \int_{\alpha}^{\Gamma} p(\theta) q(\theta) |x'(\theta)|^2 d\theta.$$

Corollary 8. Take $f(t) = t^2$, $g(t) = 1$, $m = 1$ and $x = y$ in Theorem 5, the inequality (10) becomes

$$\begin{aligned} \int_{\alpha}^{\Gamma} |x(\theta)| |x^{\Delta}(\theta)| \Delta\theta &\leq \frac{1}{2} \left(\int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)| \Delta\theta \right)^2 \\ &= \frac{1}{2} \left(\int_{\alpha}^{\Gamma} p^{\frac{1}{\hat{\nu}}}(\theta) p^{-\frac{1}{\hat{\nu}}}(\theta) |x^{\Delta}(\theta)| \Delta\theta \right)^2. \end{aligned}$$

From Lemma 1 with indices $\hat{\nu}$, $\hat{\mu}$ such that $\frac{1}{\hat{\nu}} + \frac{1}{\hat{\mu}} = 1$, we obtain

$$\int_{\alpha}^{\Gamma} |x(\theta)| |x^{\Delta}(\theta)| \Delta\theta \leq \frac{1}{2} \left(\int_{\alpha}^{\Gamma} p^{1-\hat{\mu}}(\theta) \Delta\theta \right)^{\frac{2}{\hat{\mu}}} \left(\int_{\alpha}^{\Gamma} p(\theta) |x^{\Delta}(\theta)|^{\hat{\nu}} \Delta\theta \right)^{\frac{2}{\hat{\nu}}}. \tag{23}$$

Remark 6. If $\mathbb{T} = \mathbb{R}$, then (23), gives the inequality of Maroni [58].

$$\int_{\alpha}^{\Gamma} |x(\theta)| |x^{\Delta}(\theta)| d\theta \leq \frac{1}{2} \left(\int_{\alpha}^{\Gamma} p^{1-\hat{\mu}}(\theta) d\theta \right)^{\frac{2}{\hat{\mu}}} \left(\int_{\alpha}^{\Gamma} p(\theta) |x'(\theta)|^{\hat{\nu}} d\theta \right)^{\frac{2}{\hat{\nu}}}. \tag{24}$$

Theorem 6. Under the hypotheses of Theorem 5. Let $p^* > 0$ and $p > 0$ be defined on $[\alpha, \Gamma]_{\mathbb{T}}$ with $\int_{\alpha}^{\Gamma} p(\theta) \Delta\theta = 1$ and $\int_{\alpha}^{\Gamma} p^*(\theta) \Delta\theta = 1$. Further, let $\eta > 0$ be convex and increasing on $[0, \infty)_{\mathbb{R}}$. Then, for all $m \geq 1$, the following inequality holds

$$\begin{aligned} &\int_{\alpha}^{\Gamma} [f(|x(\theta)|^m) g'(|y(\theta)|^m) |y^{\Delta}(\theta)|^m + g(|y(\theta)|^m) f'(|x(\theta)|^m) |x^{\Delta}(\theta)|^m] \Delta\theta \\ &\leq \frac{1}{\lambda_1} f \left(2\lambda_1 \eta^{-1} \left(\int_{\alpha}^{\Gamma} p(\theta) \eta \left(\frac{|x^{\Delta}(\theta)|^m}{2p(\theta)} \right) \Delta\theta \right) \right) \\ &\quad \times g \left(2\lambda_1 \eta^{-1} \left(\int_{\alpha}^{\Gamma} p^*(\theta) \eta \left(\frac{|y^{\Delta}(\theta)|^m}{2p^*(\theta)} \right) \Delta\theta \right) \right), \end{aligned} \tag{25}$$

where λ_1 defined as in Theorem 5.

Proof. Dynamic Jensen inequality (2) provides

$$\eta \left(\frac{1}{2} \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta\theta \right) \leq \int_{\alpha}^{\Gamma} p(\theta) \eta \left(\frac{|x^{\Delta}(\theta)|^m}{2p(\theta)} \right) \Delta\theta.$$

Further, since η is increasing, we have

$$\int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta\theta \leq 2\eta^{-1} \left(\int_{\alpha}^{\Gamma} p(\theta) \eta \left(\frac{|x^{\Delta}(\theta)|^m}{2p(\theta)} \right) \Delta\theta \right). \quad (26)$$

Similarly, we have

$$\int_{\alpha}^{\Gamma} |y^{\Delta}(\theta)|^m \Delta\theta \leq 2\eta^{-1} \left(\int_{\alpha}^{\Gamma} p^*(\theta) \eta \left(\frac{|y^{\Delta}(\theta)|^m}{2p^*(\theta)} \right) \Delta\theta \right). \quad (27)$$

From (10), (26) and (27) immediately gives

$$\begin{aligned} & \int_{\alpha}^{\Gamma} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ & \leq \frac{1}{\lambda_1} f \left(2\lambda_1 \eta^{-1} \left(\int_{\alpha}^{\Gamma} p(\theta) \eta \left(\frac{|x^{\Delta}(\theta)|^m}{2p(\theta)} \right) \Delta\theta \right) \right) \\ & \quad \times g \left(2\lambda_1 \eta^{-1} \left(\int_{\alpha}^{\Gamma} p^*(\theta) \eta \left(\frac{|y^{\Delta}(\theta)|^m}{2p^*(\theta)} \right) \Delta\theta \right) \right). \end{aligned}$$

This gives our claim. \square

Remark 7. When $\mathbb{T} = \mathbb{R}$ in Theorem 6, then, by the relation (4), we get the inequality of Hwang and Yang [3].

Corollary 9. If we take $\mathbb{T} = h\mathbb{Z}$ in Theorem 6, then, by the relation (6), inequality (25) becomes

$$\begin{aligned} & h \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} [f(|x(nh)|^m)g'(|y(nh)|^m)|\Delta y(nh)|^m + g(|y(nh)|^m)f'(|x(nh)|^m)|\Delta x(nh)|^m] \\ & \leq \frac{1}{\lambda_1} f \left(2h\lambda_1 \eta^{-1} \left(\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} p(nh) \eta \left(\frac{|\Delta x(nh)|^m}{2p(nh)} \right) \right) \right) \\ & \quad \times g \left(2h\lambda_1 \eta^{-1} \left(\sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} p^*(nh) \eta \left(\frac{|\Delta y(nh)|^m}{2p^*(nh)} \right) \right) \right). \end{aligned}$$

Remark 8. In Corollary 10, if we take $h = 1$, then inequality (25) becomes

$$\begin{aligned} & \sum_{n=\alpha}^{\Gamma-1} [f(|x(n)|^m)g'(|y(n)|^m)|\Delta y(n)|^m + g(|y(n)|^m)f'(|x(n)|^m)|\Delta x(n)|^m] \\ & \leq \frac{1}{\lambda_1} f \left(2\lambda_1 \eta^{-1} \left(\sum_{n=\alpha}^{\Gamma-1} p(n) \eta \left(\frac{|\Delta x(n)|^m}{2p(n)} \right) \right) \right) \\ & \quad \times g \left(2\lambda_1 \eta^{-1} \left(\sum_{n=\alpha}^{\Gamma-1} p^*(n) \eta \left(\frac{|\Delta y(n)|^m}{2p^*(n)} \right) \right) \right). \end{aligned}$$

Corollary 10. If we take $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Theorem 6, then, by the relation (7), inequality (25) becomes

$$\begin{aligned} & (q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} [f(|x(q^n)|^m)g'(|y(q^n)|^m)|\Delta y(q^n)|^m + g(|y(q^n)|^m)f'(|x(q^n)|^m)|\Delta x(q^n)|^m]q^n \\ & \leq \frac{1}{\lambda_1} f \left(2(q-1)\lambda_1 \eta^{-1} \left(\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} q^n p(q^n) \eta \left(\frac{|\Delta x(q^n)|^m}{2p(q^n)} \right) \right) \right) \\ & \quad \times g \left(2(q-1)\lambda_1 \eta^{-1} \left(\sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} q^n p^*(q^n) \eta \left(\frac{|\Delta y(q^n)|^m}{2p^*(q^n)} \right) \right) \right). \end{aligned}$$

Remark 9. When $\mathbb{T} = \mathbb{R}$, if we take $m = 1$ and $p(t) = p^*(t)$, in the inequality (25), we get Pachpatte inequality [54]

$$\begin{aligned} & \int_{\alpha}^{\Gamma} [f(|x(\theta)|)g'(|y(\theta)|)|y'(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x'(\theta)|]d\theta \\ & \leq f \left(\eta^{-1} \left(\int_{\alpha}^{\Gamma} p^*(\theta) \eta \left(\frac{|x'(\theta)|}{p^*(\theta)} \right) d\theta \right) \right) \\ & \quad \times g \left(\eta^{-1} \left(\int_{\alpha}^{\Gamma} p^*(\theta) \eta \left(\frac{|y'(\theta)|}{p^*(\theta)} \right) d\theta \right) \right). \end{aligned} \tag{28}$$

Remark 10. Taking $f(\theta) = g(\theta)$ and $x(\theta) = y(\theta)$ in inequality (28), we get the Pachpatte inequality [54]

$$\int_{\alpha}^{\Gamma} f(|x(\theta)|)f'(|x(\theta)|)|x'(\theta)|d\theta \leq \frac{1}{2} \left[f \left(\eta^{-1} \left(\int_{\alpha}^{\Gamma} p(\theta) \eta \left(\frac{|x'(\theta)|}{p(\theta)} \right) d\theta \right) \right) \right]^2.$$

Corollary 11. Suppose \mathbb{T} is a time scale, $\alpha, \beta, (\alpha + \beta)/2 \in \mathbb{T}$, and f, g are defined as in Theorem 5. Furthermore, Assume x and y are rd-continuous functions on $[\alpha, \beta]_{\mathbb{T}}$ such that $x(\alpha) = y(\alpha) = 0$ and $x(\beta) = y(\beta) = 0$. Then, we get

$$\begin{aligned} & \int_{\alpha}^{\beta} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ & \leq \frac{2^{m-1}}{\lambda_2} g \left(2^{1-m} \lambda_2 \int_{\alpha}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta \right) \\ & \quad \times \left[f \left(2^{1-m} \lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta \right) + f \left(2^{1-m} \lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta \right) \right], \end{aligned} \tag{29}$$

where $\lambda_2 = (\beta - \alpha)^{m-1}$.

Proof. Let $\Gamma \in [\alpha, \beta]_{\mathbb{T}}$, the functions x and y satisfy the conditions of Theorem 5 on $[\alpha, \Gamma]_{\mathbb{T}}$. Thus, inequality (10) holds. Next, in the interval $[\Gamma, \beta]_{\mathbb{T}}$, the functions x and y are rd-continuous, and $x(\beta) = y(\beta) = 0$. Thus, by defining $\varphi(t) = \int_t^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta$, $\phi(t) = \int_t^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta$ $t \in [\Gamma, \beta]$, and following an argument similar to Theorem 5, we obtain

$$\begin{aligned} & \int_{\Gamma}^{\beta} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ & \leq \frac{1}{\lambda_3} f \left(\lambda_3 \int_{\Gamma}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta \right) g \left(\lambda_3 \int_{\Gamma}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta \right), \end{aligned} \tag{30}$$

where $\lambda_3 = (\beta - \Gamma)^{m-1}$. A combination of the inequalities (10) and (30), we get

$$\begin{aligned} & \int_{\alpha}^{\beta} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ &= \int_{\alpha}^{\Gamma} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ &\quad + \int_{\Gamma}^{\beta} [f(|x(\theta)|^m)g'(|y(t)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ &\leq \frac{1}{\lambda_1} f\left(\lambda_1 \int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(\lambda_1 \int_{\alpha}^{\Gamma} |y^{\Delta}(\theta)|^m \Delta\theta\right) \\ &\quad + \frac{1}{\lambda_3} f\left(\lambda_3 \int_{\Gamma}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(\lambda_3 \int_{\Gamma}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta\right), \end{aligned} \tag{31}$$

for $\Gamma = \frac{\alpha+\beta}{2}$, we find that $\lambda_1 = \lambda_3 = (\frac{\beta-\alpha}{2})^{m-1}$, where $\lambda_2 = (\beta - \alpha)^{m-1}$, then $\lambda_1 = \lambda_3 = \frac{\lambda_2}{2^{m-1}}$. Then, by substituting in (31), we get

$$\begin{aligned} & \int_{\alpha}^{\beta} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ &\leq \frac{2^{m-1}}{\lambda_2} f\left(2^{1-m}\lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(2^{1-m}\lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |y^{\Delta}(\theta)|^m \Delta\theta\right) \\ &\quad + \frac{2^{m-1}}{\lambda_2} f\left(2^{1-m}\lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) g\left(2^{1-m}\lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta\right). \end{aligned} \tag{32}$$

Since g is non-decreasing function, we have

$$\begin{aligned} & \int_{\alpha}^{\beta} [f(|x(\theta)|^m)g'(|y(\theta)|^m)|y^{\Delta}(\theta)|^m + g(|y(\theta)|^m)f'(|x(\theta)|^m)|x^{\Delta}(\theta)|^m] \Delta\theta \\ &\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2 \int_{\alpha}^{\beta} |y^{\Delta}(\theta)|^m \Delta\theta\right) \left[f\left(2^{1-m}\lambda_2 \int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right) \right. \\ &\quad \left. + f\left(2^{1-m}\lambda_2 \int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) \right]. \end{aligned}$$

This gives our claim (29). \square

Corollary 12. If we take $\mathbb{T} = h\mathbb{Z}$ in Corollary 11, then, by the relation (6), inequality (29) becomes

$$\begin{aligned} & h \sum_{n=\frac{\alpha}{h}}^{\frac{\beta}{h}-1} [f(|x(nh)|^m)g'(|y(nh)|^m)|\Delta y(nh)|^m + g(|y(nh)|^m)f'(|x(nh)|^m)|\Delta x(nh)|^m] \\ &\leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2 h \sum_{n=\frac{\alpha}{h}}^{\frac{\beta}{h}-1} |\Delta y(nh)|^m\right) \\ &\quad \times \left[f\left(2^{1-m}\lambda_2 h \sum_{n=\frac{\alpha}{h}}^{\frac{\alpha+\beta}{2h}-1} |\Delta x(nh)|^m\right) + f\left(2^{1-m}\lambda_2 h \sum_{n=\frac{\alpha+\beta}{2h}}^{\frac{\beta}{h}-1} |\Delta x(nh)|^m\right) \right]. \end{aligned}$$

Remark 11. In Corollary 12, if we take $h = 1$, then inequality (29) becomes

$$\begin{aligned} & \sum_{n=\alpha}^{\beta-1} [f(|x(n)|^m)g'(|y(n)|^m)|\Delta y(n)|^m + g(|y(n)|^m)f'(|x(n)|^m)|\Delta x(n)|^m] \\ & \leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2 \sum_{n=\alpha}^{\beta-1} |\Delta y(n)|^m\right) \\ & \quad \times \left[f\left(2^{1-m}\lambda_2 \sum_{n=\alpha}^{\frac{\alpha+\beta}{2}-1} |\Delta x(n)|^m\right) + f\left(2^{1-m}\lambda_2 \sum_{n=\frac{\alpha+\beta}{2}}^{\beta-1} |\Delta x(n)|^m\right) \right]. \end{aligned}$$

Corollary 13. If we take $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ in Corollary 11, then, by the relation (7), inequality (29) becomes

$$\begin{aligned} (q-1) & \sum_{n=(\log_q \alpha)}^{(\log_q \beta)-1} [f(|x(q^n)|^m)g'(|y(q^n)|^m)|\Delta y(q^n)|^m + g(|y(q^n)|^m)f'(|x(q^n)|^m)|\Delta x(q^n)|^m] q^n \\ & \leq \frac{2^{m-1}}{\lambda_2} g\left(2^{1-m}\lambda_2(q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \beta)-1} |\Delta y(q^n)|^m q^n\right) \\ & \quad \times \left[f\left(2^{1-m}\lambda_2(q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \frac{\alpha+\beta}{2})-1} |\Delta x(q^n)|^m q^n\right) \right. \\ & \quad \left. + f\left(2^{1-m}\lambda_2(q-1) \sum_{n=(\log_q \frac{\alpha+\beta}{2})}^{(\log_q \beta)-1} |\Delta x(q^n)|^m q^n\right) \right]. \end{aligned}$$

Corollary 14. In Corollary 11. For $m = 1$, we can get the following inequality

$$\begin{aligned} & \int_{\alpha}^{\beta} [f(|x(\theta)|)g'(|y(\theta)|)|y^{\Delta}(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x^{\Delta}(\theta)|] \Delta\theta \\ & \leq 2f\left(\frac{1}{2} \int_{\alpha}^{\beta} |x^{\Delta}(\theta)| \Delta\theta\right) g\left(\frac{1}{2} \int_{\alpha}^{\beta} |y^{\Delta}(\theta)| \Delta\theta\right). \end{aligned} \tag{33}$$

Proof. By substituting in (32) by $m = 1$, we get

$$\begin{aligned} & \int_{\alpha}^{\beta} [f(|x(\theta)|)g'(|y(\theta)|)|y^{\Delta}(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x^{\Delta}(\theta)|] \Delta\theta \\ & \leq f\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)| \Delta\theta\right) g\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} |y^{\Delta}(\theta)| \Delta\theta\right) \\ & \quad + f\left(\int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)| \Delta\theta\right) g\left(\int_{\frac{\alpha+\beta}{2}}^{\beta} |y^{\Delta}(\theta)| \Delta\theta\right). \end{aligned} \tag{34}$$

If we choose Γ such that

$$\int_{\alpha}^{\Gamma} |x^{\Delta}(\theta)| \Delta\theta = \int_{\Gamma}^{\beta} |x^{\Delta}(\theta)| \Delta\theta = \frac{1}{2} \int_{\alpha}^{\beta} |x^{\Delta}(\theta)| \Delta\theta. \tag{35}$$

Then, by substituting from (35) in (34), we can get

$$\int_{\alpha}^{\beta} [f(|x(\theta)|)g'(|y(\theta)|)|y^{\Delta}(\theta)| + g(|y(\theta)|)f'(|x(\theta)|)|x^{\Delta}(\theta)|] \Delta\theta \leq 2f\left(\frac{1}{2} \int_{\alpha}^{\beta} |x^{\Delta}(\theta)| \Delta\theta\right) g\left(\frac{1}{2} \int_{\alpha}^{\beta} |y^{\Delta}(\theta)| \Delta\theta\right).$$

This gives our claim. \square

Corollary 15. Let $f(t) = t^{\frac{\ell+m}{m}}$, $\ell \geq 0$, $g(t) = 1$ and $x(t) = y(t)$ in (29), to obtain

$$\int_{\alpha}^{\beta} \frac{\ell+m}{m} |x(\theta)|^{\ell} |x^{\Delta}(\theta)|^m \Delta\theta \leq \left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}} \left[\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right)^{\frac{\ell+m}{m}} + \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right)^{\frac{\ell+m}{m}} \right]. \tag{36}$$

By using the inequality (9) in inequality (36), we get

$$\begin{aligned} \int_{\alpha}^{\beta} |x(\theta)|^{\ell} |x^{\Delta}(\theta)|^m \Delta\theta &\leq \frac{m}{\ell+m} \left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}} \\ &\times \left[\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right)^{\frac{\ell+m}{m}} + \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right)^{\frac{\ell+m}{m}} \right] \\ &\leq \frac{m}{\ell+m} \left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}} \left[\left(\int_{\alpha}^{\frac{\alpha+\beta}{2}} |x^{\Delta}(\theta)|^m \Delta\theta\right) + \left(\int_{\frac{\alpha+\beta}{2}}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right) \right]^{\frac{\ell+m}{m}} \\ &= \frac{m}{\ell+m} \left(\frac{\beta-\alpha}{2}\right)^{\frac{\ell(m-1)}{m}} \left(\int_{\alpha}^{\beta} |x^{\Delta}(\theta)|^m \Delta\theta\right)^{\frac{\ell+m}{m}}. \end{aligned} \tag{37}$$

From Lemma 1 with indices $\frac{\ell+m}{\ell}$ and $\frac{\ell+m}{m}$ on (37), we have

$$\int_{\alpha}^{\beta} |x(\theta)|^{\ell} |x^{\Delta}(\theta)|^m \Delta\theta \leq \frac{m}{\ell+m} \left(\frac{\beta-\alpha}{2}\right)^{\ell} \int_{\alpha}^{\beta} |x^{\Delta}(\theta)|^{\ell+m} \Delta\theta. \tag{38}$$

Remark 12. Taking $\mathbb{T} = \mathbb{R}$, then (38) gives Yang inequality [59]

$$\int_{\alpha}^{\beta} |x(\theta)|^{\ell} |x'(\theta)|^m d\theta \leq \frac{m}{\ell+m} \left(\frac{\beta-\alpha}{2}\right)^{\ell} \int_{\alpha}^{\beta} |x'(\theta)|^{\ell+m} d\theta.$$

Theorem 7. Under the hypotheses of Theorem 5. Assuming that $s \geq 0$, $r \geq 0$ and $s^{\Delta} \geq 0$, $r^{\Delta} \geq 0$, $t \in [\alpha, \Gamma]_{\mathbb{T}}$ and $r(\alpha) = 0$, $s(\alpha) = 0$ and $\chi \geq 0$ and $\pi \geq 0$ are convex and increasing functions on $(0, \infty)_{\mathbb{R}}$. Then, we get

$$\begin{aligned} &\int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi \left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)} \right) f \left(r(\theta) \chi \left(\frac{|x(\theta)|}{r(\theta)} \right) \right) g' \left(s(\theta) \pi \left(\frac{|y(\theta)|}{s(\theta)} \right) \right) \right. \\ &\quad \left. + r^{\Delta}(\theta) \chi \left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)} \right) f' \left(r(\theta) \chi \left(\frac{|x(\theta)|}{r(\theta)} \right) \right) g \left(s(\theta) \pi \left(\frac{|y(\theta)|}{s(\theta)} \right) \right) \right] \Delta\theta \\ &\leq f \left(\int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi \left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)} \right) \Delta\theta \right) g \left(\int_{\alpha}^{\Gamma} s^{\Delta}(\theta) \pi \left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)} \right) \Delta\theta \right). \end{aligned} \tag{39}$$

Proof. For $t \in [\alpha, \Gamma]_{\mathbb{T}}$, we define $\varphi(t) := \int_{\alpha}^t |x^{\Delta}(\theta)| \Delta\theta$ and $\psi(t) := \int_{\alpha}^t |y^{\Delta}(\theta)| \Delta\theta$ so that $\varphi^{\Delta}(t) = |x^{\Delta}(t)|$ and $\psi^{\Delta}(t) = |y^{\Delta}(t)|$.

Thus,

$$|x(t)| = \left| \int_{\alpha}^t x^{\Delta}(\theta) \Delta\theta \right| \leq \int_{\alpha}^t |x^{\Delta}(\theta)| \Delta\theta = \varphi(t),$$

$$|y(t)| = \left| \int_{\alpha}^t y^{\Delta}(\theta) \Delta\theta \right| \leq \int_{\alpha}^t |y^{\Delta}(\theta)| \Delta\theta = \psi(t).$$

Then, we obtain

$$\frac{|x(t)|}{|r(t)|} \leq \frac{\int_{\alpha}^t r^{\Delta}(\theta) \varphi^{\Delta}(\theta) \Delta\theta}{\int_{\alpha}^t r^{\Delta}(\theta) \Delta\theta},$$

$$\frac{|y(t)|}{|s(t)|} \leq \frac{\int_{\alpha}^t s^{\Delta}(\theta) \psi^{\Delta}(\theta) \Delta\theta}{\int_{\alpha}^t s^{\Delta}(\theta) \Delta\theta}.$$

Thus, from Lemma 2, we get

$$\chi\left(\frac{|x(t)|}{|r(t)|}\right) \leq \frac{1}{r(t)} \int_{\alpha}^t r^{\Delta}(\theta) \chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right) \Delta\theta.$$

$$\pi\left(\frac{|y(t)|}{|s(t)|}\right) \leq \frac{1}{s(t)} \int_{\alpha}^t s^{\Delta}(\theta) \pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right) \Delta\theta.$$

Using the above inequalities, we get

$$\int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right) f\left(r(\theta) \chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) g'\left(s(\theta) \pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right) \right. \\ \left. + r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right) f'\left(r(\theta) \chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) g\left(s(\theta) \pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right) \right] \Delta\theta \tag{40}$$

$$\leq \int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right) f\left(\int_{\alpha}^t r^{\Delta}(\theta) \chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right) \Delta\theta\right) g'\left(\int_{\alpha}^t s^{\Delta}(\theta) \pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right) \Delta\theta\right) \right. \\ \left. + r^{\Delta}(\theta) \chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right) f'\left(\int_{\alpha}^t r^{\Delta}(\theta) \chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right) \Delta\theta\right) g\left(\int_{\alpha}^t s^{\Delta}(\theta) \pi\left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)}\right) \Delta\theta\right) \right] \Delta\theta.$$

Since f , r , χ , and φ are increasing and $t \leq \sigma(t)$, we get

$$f\left(\int_{\alpha}^t r^{\Delta}(\theta) \chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right) \Delta\theta\right) \leq f^{\sigma}\left(\int_{\alpha}^t r^{\Delta}(\theta) \chi\left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)}\right) \Delta\theta\right).$$

Then by substituting in (40), we get

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi \left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)} \right) f \left(r(\theta) \chi \left(\frac{|x(\theta)|}{r(\theta)} \right) \right) g' \left(s(\theta) \pi \left(\frac{|y(\theta)|}{s(\theta)} \right) \right) \right. \\ & \quad \left. + r^{\Delta}(\theta) \chi \left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)} \right) f' \left(r(\theta) \chi \left(\frac{|x(\theta)|}{r(\theta)} \right) \right) g \left(s(\theta) \pi \left(\frac{|y(\theta)|}{s(\theta)} \right) \right) \right] \Delta\theta \\ & \leq \int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) f^{\sigma} \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) g' \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) \right. \\ & \quad \left. + r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) f' \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) g \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) \right] \Delta\theta. \end{aligned} \tag{41}$$

From Lemma 3 for $c \in [t, \sigma(t)]$, we get

$$\begin{aligned} f^{\Delta} \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) &= r^{\Delta}(t) \chi \left(\frac{\varphi^{\Delta}(t)}{r^{\Delta}(t)} \right) f' \left(\int_{\alpha}^c r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) \\ &\geq r^{\Delta}(t) \chi \left(\frac{\varphi^{\Delta}(t)}{r^{\Delta}(t)} \right) f' \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right). \end{aligned} \tag{42}$$

$$\begin{aligned} g^{\Delta} \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) &= s^{\Delta}(t) \chi \left(\frac{\psi^{\Delta}(t)}{s^{\Delta}(t)} \right) g' \left(\int_{\alpha}^c s^{\Delta}(\theta) \chi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) \\ &\geq s^{\Delta}(t) \chi \left(\frac{\psi^{\Delta}(t)}{s^{\Delta}(t)} \right) g' \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right). \end{aligned} \tag{43}$$

Then, by substituting from (42) and (43) in (41), we get

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi \left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)} \right) f \left(r(\theta) \chi \left(\frac{|x(\theta)|}{r(\theta)} \right) \right) g' \left(s(\theta) \pi \left(\frac{|y(\theta)|}{s(\theta)} \right) \right) \right. \\ & \quad \left. + r^{\Delta}(\theta) \chi \left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)} \right) f' \left(r(\theta) \chi \left(\frac{|x(\theta)|}{r(\theta)} \right) \right) g \left(s(\theta) \pi \left(\frac{|y(\theta)|}{s(\theta)} \right) \right) \right] \Delta\theta \\ & \leq \int_{\alpha}^{\Gamma} \left[g^{\Delta} \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) f^{\sigma} \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) \right. \\ & \quad \left. + f^{\Delta} \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) g \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) \right] \Delta\theta \\ & = \int_{\alpha}^{\Gamma} \left[f \left(\int_{\alpha}^t r^{\Delta}(\theta) \chi \left(\frac{\varphi^{\Delta}(\theta)}{r^{\Delta}(\theta)} \right) \Delta\theta \right) g \left(\int_{\alpha}^t s^{\Delta}(\theta) \pi \left(\frac{\psi^{\Delta}(\theta)}{s^{\Delta}(\theta)} \right) \Delta\theta \right) \right]^{\Delta} \\ & = f \left(\int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi \left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)} \right) \Delta\theta \right) g \left(\int_{\alpha}^{\Gamma} s^{\Delta}(\theta) \pi \left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)} \right) \Delta\theta \right). \end{aligned}$$

This gives our claim. \square

Remark 13. Taking $\mathbb{T} = \mathbb{R}$, then, by the relation (6), inequality (39) gives Pachpatte inequality in [55]

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[s'(\theta)\pi\left(\frac{|y'(\theta)|}{s'(\theta)}\right) f\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) g'\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right) \right. \\ & \quad \left. + r'(\theta)\chi\left(\frac{|x'(\theta)|}{r'(\theta)}\right) f'\left(r(\theta)\chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) g\left(s(\theta)\pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right) \right] d\theta \\ & \leq f\left(\int_{\alpha}^{\Gamma} r'(\theta)\chi\left(\frac{|x'(\theta)|}{r'(\theta)}\right) d\theta\right) g\left(\int_{\alpha}^{\Gamma} s'(\theta)\pi\left(\frac{|y'(\theta)|}{s'(\theta)}\right) d\theta\right). \end{aligned}$$

Corollary 16. If we take $\mathbb{T} = h\mathbb{Z}$ in Theorem 7, then, by the relation (6), inequality (39) becomes

$$\begin{aligned} & h \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} \left[\Delta s(nh)\pi\left(\frac{|\Delta y(nh)|}{\Delta s(nh)}\right) f\left(r(nh)\chi\left(\frac{|x(nh)|}{r(nh)}\right)\right) g'\left(s(nh)\pi\left(\frac{|y(nh)|}{s(nh)}\right)\right) \right. \\ & \quad \left. + \Delta r(nh)\chi\left(\frac{|\Delta x(nh)|}{\Delta r(nh)}\right) f'\left(r(nh)\chi\left(\frac{|x(nh)|}{r(nh)}\right)\right) g\left(s(nh)\pi\left(\frac{|y(nh)|}{s(nh)}\right)\right) \right] \\ & \leq f\left(h \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} \Delta r(nh)\chi\left(\frac{|\Delta x(nh)|}{\Delta r(nh)}\right)\right) g\left(h \sum_{n=\frac{\alpha}{h}}^{\frac{\Gamma}{h}-1} \Delta s(nh)\pi\left(\frac{|\Delta y(nh)|}{\Delta s(nh)}\right)\right). \end{aligned}$$

Remark 14. In Corollary 16, if we take $h = 1$, then inequality (39) becomes

$$\begin{aligned} & \sum_{n=\alpha}^{\Gamma-1} \left[\Delta s(n)\pi\left(\frac{|\Delta y(n)|}{\Delta s(n)}\right) f\left(r(n)\chi\left(\frac{|x(n)|}{r(n)}\right)\right) g'\left(s(n)\pi\left(\frac{|y(n)|}{s(n)}\right)\right) \right. \\ & \quad \left. + \Delta r(n)\chi\left(\frac{|\Delta x(n)|}{\Delta r(n)}\right) f'\left(r(n)\chi\left(\frac{|x(n)|}{r(n)}\right)\right) g\left(s(n)\pi\left(\frac{|y(n)|}{s(n)}\right)\right) \right] \\ & \leq f\left(\sum_{n=\alpha}^{\Gamma-1} \Delta r(n)\chi\left(\frac{|\Delta x(n)|}{\Delta r(n)}\right)\right) g\left(\sum_{n=\alpha}^{\Gamma-1} \Delta s(n)\pi\left(\frac{|\Delta y(n)|}{\Delta s(n)}\right)\right). \end{aligned}$$

Corollary 17. If we take $\mathbb{T} = q^{\mathbb{Z}}$ in Theorem 7, then, by the relation (7), inequality (39) becomes

$$\begin{aligned} & (q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} \left[\Delta s(q^n)\pi\left(\frac{|\Delta y(q^n)|}{\Delta s(q^n)}\right) f\left(r(q^n)\chi\left(\frac{|x(q^n)|}{r(q^n)}\right)\right) g'\left(s(q^n)\pi\left(\frac{|y(q^n)|}{s(q^n)}\right)\right) \right. \\ & \quad \left. + \Delta r(q^n)\chi\left(\frac{|\Delta x(q^n)|}{\Delta r(q^n)}\right) f'\left(r(q^n)\chi\left(\frac{|x(q^n)|}{r(q^n)}\right)\right) g\left(s(q^n)\pi\left(\frac{|y(q^n)|}{s(q^n)}\right)\right) \right] q^n \\ & \leq f\left((q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} q^n \Delta r(q^n)\chi\left(\frac{|\Delta x(q^n)|}{\Delta r(q^n)}\right)\right) \\ & \quad \times g\left((q-1) \sum_{n=(\log_q \alpha)}^{(\log_q \Gamma)-1} q^n \Delta s(q^n)\pi\left(\frac{|\Delta y(q^n)|}{\Delta s(q^n)}\right)\right). \end{aligned}$$

Remark 15. For $s(t) = r(t)$, $\chi(t) = \pi(t)$, $y(t) = x(t)$ and $g(t) = f(t)$, the inequality (39) becomes

$$\begin{aligned} & \int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right) f\left(r(\theta) \chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) f'\left(r(\theta) \chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) \Delta\theta \\ & \leq \frac{1}{2} \left[f\left(\int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right) \Delta\theta\right) \right]^2. \end{aligned} \tag{44}$$

Corollary 18. With the assumptions of Theorem 7. Suppose $\omega \geq 0$, $t \in [\alpha, \Gamma]_{\mathbb{T}}$ and $\int_{\alpha}^{\Gamma} \omega(\theta) \Delta\theta = 1$. Assuming $\Psi \geq 0$ is increasing and convex on $[0, \infty)_{\mathbb{R}}$. Then, we get

$$\begin{aligned} & \int_{\alpha}^{\Gamma} \left[s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right) f\left(r(\theta) \chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) g'\left(s(\theta) \pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right) \right. \\ & \quad \left. + r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right) f'\left(r(\theta) \chi\left(\frac{|x(\theta)|}{r(\theta)}\right)\right) g\left(s(\theta) \pi\left(\frac{|y(\theta)|}{s(\theta)}\right)\right) \right] \Delta\theta \\ & \leq f\left(\Psi^{-1}\left(\int_{\alpha}^{\Gamma} \omega(\theta) \Psi\left(\frac{r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)}{\omega(\theta)}\right) \Delta\theta\right)\right) g\left(\Psi^{-1}\left(\int_{\alpha}^{\Gamma} \omega(\theta) \Psi\left(\frac{s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)}{\omega(\theta)}\right) \Delta\theta\right)\right). \end{aligned} \tag{45}$$

Proof. Since

$$\int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right) \leq \frac{\int_{\alpha}^{\Gamma} \frac{r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)}{\omega(\theta)} \omega(\theta) \Delta\theta}{\int_{\alpha}^{\Gamma} \omega(\theta) \Delta\theta}$$

and

$$\int_{\alpha}^{\Gamma} s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right) \leq \frac{\int_{\alpha}^{\Gamma} \frac{s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)}{\omega(\theta)} \omega(\theta) \Delta\theta}{\int_{\alpha}^{\Gamma} \omega(\theta) \Delta\theta}.$$

From Lemma 2, we have

$$\Psi\left(\int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)\right) \leq \int_{\alpha}^{\Gamma} \omega(\theta) \Psi\left(\frac{r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)}{\omega(\theta)}\right) \Delta\theta$$

and

$$\Psi\left(\int_{\alpha}^{\Gamma} s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)\right) \leq \int_{\alpha}^{\Gamma} \omega(\theta) \Psi\left(\frac{s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)}{\omega(\theta)}\right) \Delta\theta$$

and hence

$$\int_{\alpha}^{\Gamma} r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right) \leq \Psi^{-1}\left(\int_{\alpha}^{\Gamma} \omega(\theta) \Psi\left(\frac{r^{\Delta}(\theta) \chi\left(\frac{|x^{\Delta}(\theta)|}{r^{\Delta}(\theta)}\right)}{\omega(\theta)}\right) \Delta\theta\right) \tag{46}$$

and

$$\int_{\alpha}^{\Gamma} s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right) \leq \Psi^{-1}\left(\int_{\alpha}^{\Gamma} \omega(\theta) \Psi\left(\frac{s^{\Delta}(\theta) \pi\left(\frac{|y^{\Delta}(\theta)|}{s^{\Delta}(\theta)}\right)}{\omega(\theta)}\right) \Delta\theta\right) \tag{47}$$

Using (46) and (47) in (39), we get (45). This completes the proof. \square

3. Conclusions

In this paper, with the help of the dynamic Jensen inequality, dynamic Hölder inequality and a simple consequence of Keller’s chain rule on time scales, we generalized a number of Opial-type

inequalities to a general time scale. Besides that, in order to illustrate the theorems for each type of inequality applied to various time scales such as \mathbb{R} , $h\mathbb{Z}$, $q^{\mathbb{Z}}$ and \mathbb{Z} as a sub case of $h\mathbb{Z}$. For future studies, researchers may obtain some different generalizations for dynamic Opial inequality and its companion inequalities by using the results presented in this paper.

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