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# Study on Krasnoselskii's fixed point theorem for Caputo–Fabrizio fractional differential equations

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**Abstract**

This note is concerned with establishing existence theory of solutions to a class of implicit fractional differential equations (FODEs) involving nonsingular derivative. By using usual classical fixed point theorems of Banach and Krasnoselskii, we develop sufficient conditions for the existence of at least one solution and its uniqueness. Further, some results about Ulam–Hyers stability and its generalization are also discussed. Two suitable examples are given to demonstrate the results.

**Keywords:** Krasnoselskii's fixed point theorem; Caputo–Fabrizio fractional differential equations; Hyers–Ulam stability

## 1 Introduction

FODEs have many applications in real world problems; see [1–3]. The concerned area has been investigated from different aspects in the last several years. These investigations include the existence theory of solutions by the fixed point theory, numerical analysis and stability theory by taking Hadamard, Riemann–Liouville, Caputo, etc., type fractional derivatives (for details, see [4–7]). But recently another form of derivative, called nonsingular type, has attracted much attention from the researchers. The existence theory, together with stability results, has been very well investigated for other FODEs; for details, see [8–10]. The considered differential operator has been introduced in 2015 by Caputo and Fabrizio [11] (in short, we write it as (CFFD)), which replaces the singular kernel by a nonsingular kernel of exponential type. In this research work, we establish the existence theory for the following class of fractional differential equations involving the CFFD:

$$\begin{cases} {}_0^{\mathrm{CF}}\mathbf{D}_x^{\theta}u(x) = f(x, u(x), {}_0^{\mathrm{CF}}\mathbf{D}_x^{\theta}u(x)), & x \in [0, T] = \mathcal{J}, \\ u(0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $\theta \in (0, 1]$, $f : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. The considered differential operator replaces the singular kernel by a nonsingular kernel of exponential type in (1). The mentioned operator has been observed to be more practical than the usual Caputo operator in some problems; for details, see [12–15].

So in this paper, we are using the fixed point theory to obtain some results for the existence and uniqueness of a solution to the considered problem (1). Also the stability theory of Ulam–Hyers type has been properly investigated for ordinary FODEs. Some results in this regards can be traced back in [16–19]. In recent years some remarkable work has been carried out about the mentioend FODEs; see [20–24] Therefore in this article, we also developed some results about the stability for the proposed problem. Two proper examples are also given in the end.

## 2 Background materials

Some basic notions and results are provided bellow.

**Definition 1** ([25, 26]) Letting $u \in H^1(\mathcal{J})$, where $H^1(0,T)$ is a Hilbert space, we define the nonsingular derivatives for $\theta \in (0,1]$ as

$$ {}_{0}^{\mathrm{CF}}\mathbf{D}_{x}^{\theta}u(x) = \frac{\mathbb{M}(\theta)}{1-\theta}\int_0^x u'(\eta)\exp\left(\frac{-\theta(x-\eta)}{1-\theta}\right)d\eta, \tag{2} $$

provided the integral on the right-hand side of (2) converges on $(0,\infty)$, where $\mathbb{M}(\theta)$ is a normalization function with $\mathbb{M}(0) = \mathbb{M}(1) = 1$. Further, if $u$ does not exist in $H^1(\mathcal{J})$, then the listed derivative of fractional order is defined as

$$ {}_{0}^{\mathrm{CF}}\mathbf{D}_{x}^{\theta}u(x) = \frac{\mathbb{M}(\theta)}{1-\theta}\int_0^x \bigl(u(x)-u(\eta)\bigr)\exp\left(\frac{-\theta(x-\eta)}{1-\theta}\right)d\eta, \tag{3} $$

provided that the integral on the right-hand side of (3) converges on $(0,\infty)$. Further, let $\lambda = \frac{1-\theta}{\theta}$, $\theta \in [0,1]$, $\lambda \in [0,\infty]$, and then

$$ \lim_{\lambda\to 0}\frac{1}{\lambda}\exp\left(-\frac{x-\eta}{\lambda}\right) = \delta(x-\eta). $$

Further,

$$ \begin{aligned} \lim_{\theta\to 1}\bigl[{}_{0}^{\mathrm{CF}}\mathbf{D}_{x}^{\theta}u(x)\bigr] &= \lim_{\theta\to 1}\frac{\mathbb{M}(\theta)}{1-\theta}\int_0^x u'(\eta)\exp\left(\frac{-\theta(x-\eta)}{1-\theta}\right)d\eta \\ &= \lim_{\lambda\to 0}\frac{\theta\mathbb{N}(\theta)}{\lambda}\int_0^x u'(\eta)\exp\left(-\frac{x-\eta}{\lambda}\right)d\eta \\ &= u(x), \end{aligned} $$

where $\mathbb{N}(\theta)$ is the corresponding normalization term of $\mathbb{M}(\theta)$ with the property $\mathbb{N}(0) = \mathbb{N}(\infty) = 1$.

**Definition 2** ([25, 26]) The nonsingular kernel type fractional integral is given by

$$ {}_{0}^{\mathrm{CF}}\mathbf{I}_{x}^{\theta}u(x) = \frac{(1-\theta)}{M(\theta)}u(x) + \frac{\theta}{\mathbb{M}(\theta)}\int_0^x u(\eta)\,d\eta, \tag{4} $$

provided that the integral on right-hand side converges on $(0,\infty)$. Further, if we set $\theta = 1$, then $\mathbb{M}(\theta) = 1$ in (4), and we get the following classical integral:

$$ \lim_{\theta\to 1}\bigl[{}_{0}^{\mathrm{CF}}\mathbf{I}_{x}^{\theta}u(x)\bigr] = \lim_{\theta\to 1}\left[\frac{(1-\theta)}{M(\theta)}u(x) + \frac{\theta}{\mathbb{M}(\theta)}\int_0^x u(\eta)\,d\eta\right] = \int_0^x u(\eta)\,d\eta. $$

**Lemma 1** ([11]) *Let $y \in C[0,T]$, then the solution of FODE (5)*

$$\begin{aligned} &{}_0^{\mathrm{CF}}\mathbf{D}_x^{\theta} u(x) = y(x), \quad x \in [0,T], 0 < \theta \le 1, \\ &u(0) = u_0, \quad u_0 \in \mathbb{R} \end{aligned} \tag{5}$$

*is given as*

$$u(x) = u_0 + D_\theta\big[y(x) - y_0\big] + \bar{D}_\theta \int_0^t y(\eta)\, d\eta, \tag{6}$$

*where* $D_\theta = \frac{(1-\theta)}{\mathbb{M}(\theta)}$, $\bar{D}_\theta = \frac{\theta}{\mathbb{M}(\theta)}$.

*Proof* Using the definition of ${}_0^{\mathrm{CF}}\mathbf{I}_x^{\theta}$, (5) implies that

$$u(x) = c + D_\theta y(x) + \bar{D}_\theta \int_0^x y(\eta)\, d\eta, \quad c \in \mathbb{R}. \tag{7}$$

Using the initial condition $u(0) = u_0$ and $y(0) = y_0 \in \mathbb{R}$, from (7), we get $c = u_0 - D_\theta y_0$. Hence by plugging the value of $c$ in (7), we get (6). □

*Remark* 1 Henceforth, for simplicity, we use ${}_0^{\mathrm{CF}}\mathbf{D}_x^{\theta} u(x) = h_u(x)$ for the implicit term in our analysis. Further, for simplicity, we use $f(0, u(0), h_{\bar{u}}(0)) = f_0$.

## 3 Main work

**Lemma 2** *Under the conditions of Lemma 1, the solution of (1) is given by*

$$u(x) = u_0 + D_\theta\big[f\big(x, u(x), h_u(x)\big) - f_0\big] + \bar{D}_\theta \int_0^t f\big(\eta, u(\eta), h_u(\eta)\big)\, d\eta. \tag{8}$$

To proceed further, we assume that

($C_1$) There exist $L_f > 0$ and $0 < M_f < 1$ such that

$$\big|f(x,u,h_u) - f(x,\bar{u},h_{\bar{u}})\big| \le L_f|u - \bar{u}| + M_f|h_u - h_{\bar{u}}|$$

for all $u, \bar{u}, h_u, h_{\bar{u}} \in \mathbb{R}$.

Let $X = C(\mathcal{J})$ be a Banach space with norm $\|x\| = \max_{x\in\mathcal{J}} |u(x)|$.

**Theorem 1** *Under the assumption ($C_1$), if the condition* $(D_\theta + \bar{D}_\theta T)\frac{L_f}{1-M_f} < 1$ *holds, then the considered problem (1) has a unique solution.*

*Proof* Define an operator $S : X \to X$ by using (8) as

$$Su(x) = u_0 + D_\theta\big[f\big(x, u(x), h_u(x)\big) - f_0\big] + \bar{D}_\theta \int_0^x f\big(\eta, u(\eta), h_u(\eta)\big)\, d\eta. \tag{9}$$

Then for any $u, \bar{u} \in X$, from (9), we have

$$\begin{aligned} \|Su - S\bar{u}\| &= \max_{x\in\mathcal{J}}\big|Su(x) - S\bar{u}(x)\big| \\ &= \max_{x\in\mathcal{J}}\Big|D_\theta\big[f\big(x, u(x), h_u(x)\big) - f\big(x, \bar{u}(x), h_{\bar{u}}(x)\big)\big] \end{aligned}$$

$$
\begin{aligned}
&+ D_\theta \int_0^x \bigl[f\bigl(\eta, u(\eta), h_u(\eta)\bigr) - f\bigl(\eta, \bar{u}(\eta), h_{\bar{u}}(\eta)\bigr)\bigr]\, d\eta \Bigr| \\
&\leq D_\theta \left(\frac{L_f}{1-M_f}\right)\|u-\bar{u}\| + \bar{D}_\theta \left(\frac{TL_f}{1-M_f}\right)\|u-\bar{u}\| \\
&= [D_\theta + \bar{D}_\theta T]\frac{L_f}{1-M_f}\|u-\bar{u}\|.
\end{aligned}
$$

Hence $S$ is a contraction, therefore $S$ has a unique fixed point. Hence the corresponding problem (1) has a unique solution. □

**Theorem 2** ([27]) *Let $E \subset X$ be a closed, convex, and nonempty subset of $X$, and suppose there exist two operators $S_1, S_2$ such that*

1. $S_1u_1 + S_2u_2 \in E$ *for all* $u_1, u_2 \in E$;
2. $S_1$ *is a contraction and* $S_2$ *is compact and continuous.*

*Then there exists at least one solution $u \in E$ to the operator equation $S_1u + S_2u = u$.*

For further analysis, let the given assumption hold:

$(C_2)$ There exist constants $a_f, b_f, c_f > 0$ with $0 < c_f < 1$ such that

$$
\bigl|f(x,u,v)\bigr| \leq a_f + b_f|u| + c_f|v|.
$$

**Theorem 3** *Under the assumption $(C_2)$, if $0 < D_\theta \frac{L_f}{1-M_f} < 1$ holds, then the considered problem (1) has at least one solution.*

*Proof* Let us define two operators from (8) as

$$
S_1u(x) = u_0 + D_\theta\bigl[f\bigl(x, u(x), h_u(x)\bigr) - f_0\bigr] \tag{10}
$$

and

$$
S_2u(x) = \bar{D}_\theta \int_0^x f\bigl(\eta, u(\eta), h_{\bar{u}}(\eta)\bigr). \tag{11}
$$

Let us define a set $E = \{u \in X : \|u\| \leq r\}$. Since $f$ is continuous, so is $S_1$, and letting $u, \bar{u} \in E$, from (10), we have

$$
\begin{aligned}
\|S_1u - S_1\bar{u}\| &= \max_{x\in\mathcal{J}}\bigl|D_\theta\bigl(f\bigl(x, u(x), h_u(x)\bigr) - f\bigl(x, \bar{u}(x), h_{\bar{u}}(x)\bigr)\bigr)\bigr| \\
&\leq \frac{D_\theta L_f}{1-M_f}\|u-\bar{u}\|.
\end{aligned}
$$

Hence $S_1$ is a contraction. Next to prove that $S_2$ is compact and continuous, for any $u \in E$, we have from (11)

$$
\begin{aligned}
\|S_2u\| &= \max_{x\in\mathcal{J}}\bigl|S_2u(x)\bigr| = \max_{x\in\mathcal{J}}\left|\bar{D}_\theta \int_0^x f\bigl(\eta, u(\eta), h_u(\eta)\bigr)\, d\eta\right| \\
&\leq \frac{\bar{D}_\theta(a_f + b_f r)}{1-c_f} = A,
\end{aligned}
$$

which implies that $\|S_2u\| \leq A$. Thus $S_2$ is bounded. Next, letting $x_1 < x_2$ in $\mathcal{J}$, we have

$$\begin{aligned}\left|S_2u(x_2) - S_2u(x_1)\right| &= \left|\bar{D}_\theta \int_0^{x_2} f\bigl(\eta, u(\eta), h_u(\eta)\bigr)\, d\eta - \bar{D}_\theta \int_0^{x_1} f\bigl(\eta, u(\eta), h_u(\eta)\bigr)\, d\eta\right| \\ &\leq \bar{D}_\theta \int_0^{x_2} \bigl|f\bigl(\eta, u(\eta), h_u(\eta)\bigr)\bigr|\, d\eta + \bar{D}_\theta \int_0^{x_1} \bigl|f\bigl(\eta, u(\eta), h_u(\eta)\bigr)\bigr|\, d\eta \\ &\leq \bar{D}_\theta \int_0^{x_2} \frac{(a_f + b_f r)}{1 - c_f}\, d\eta + \bar{D}_\theta \int_0^{x_1} \frac{(a_f + b_f r)}{1 - c_f}\, d\eta,\end{aligned}$$

which implies that

$$\left|S_2u(x_2) - S_2u(x_1)\right| \leq \bar{D}_\theta \left(\frac{a_f + b_f r}{1 - c_f}\right)(x_2 - x_1). \tag{12}$$

From (12), we see that if $x_1 \to x_2$, then the right-hand side of (12) goes to zero, so $|S_2u(x_2) - S_2u(x_1)| \to 0$ as $x_1 \to x_2$. Thus the operator defined in (11), $S_2$, is continuous. Also $S_2(E) \subset E$, therefore $S_2$ is compact and, due to Arzelá–Ascoli theorem, $S$ has at least one fixed point. Hence the corresponding problem has at least one solution. □

## 4 Stability theory

In this section, we establish some results regarding stability of Ulam type. Before proceeding further, we give some notion and a definition:

**Definition 3** The considered problem (1) is Ulam–Hyers stable if for any $\varepsilon > 0$ such that the inequality

$$\left|{}_0^{\mathrm{CF}}\mathbf{D}_x^\theta u(x) - f\bigl(x, u(x), {}_0^{\mathrm{CF}}\mathbf{D}_x^\theta u(x)\bigr)\right| < \varepsilon, \quad \forall x \in \mathcal{J},$$

holds, there exists a unique solution $\bar{u}$ with a constant $\mathcal{C}_f$ such that

$$\left|u(x) - \bar{u}(x)\right| \leq \mathcal{C}_f \varepsilon, \quad \forall x \in \mathcal{J}.$$

Further the mentioned problem will be generalized Ulam–Hyers stable if there exists a nondecreasing function $\vartheta : (0,1) \to (0,\infty)$ such that

$$\left|u(x) - \bar{u}(x)\right| \leq \mathcal{C}_f \vartheta(\varepsilon), \quad \forall x \in \mathcal{J}$$

with $\vartheta(0) = 0$.

Also we state an important remark.

*Remark* 2 There exists a function $\ell(x)$ depending on $u \in X$ with $\ell(0) = 0$ and such that

1. $|\ell(x)| \leq \varepsilon$, $\forall x \in \mathcal{J}$;
2. ${}_0^{\mathrm{CF}}\mathbf{D}_x^\theta u(x) = f(x, u(x), h_u(x)) + \ell(x)$, $\forall x \in \mathcal{J}$.

**Lemma 3** *The solution of the given perturbed problem*

$$\begin{aligned}&{}_0^{\mathrm{CF}}\mathbf{D}_x^\theta u(t) = f\bigl(x, u(x), h_u(x)\bigr) + \ell(x), \quad \forall x \in \mathcal{J}, \\ &u(0) = u_0\end{aligned}$$

*is given as*

$$u(x) = u_0 + D_\theta\big[f\big(x, u(x), h_u(x)\big) - f_0\big] + \bar{D}_\theta \int_0^x f\big(\eta, u(\eta), h_u(\eta)\big)\, d\eta$$
$$+ D_\theta \ell(x) + \bar{D}_\theta \int_0^x \ell(\eta)\, d\eta, \quad \forall x \in \mathcal{J}. \tag{13}$$

*Moreover, the solution satisfies the following inequality*:

$$\left|u(x) - \left[u_0 + D_\theta\big[f\big(x, u(x), h_u(x)\big) - f_0\big] + \bar{D}_\theta \int_0^x f\big(\eta, u(\eta), h_u(\eta)\big)\, d\eta\right]\right|$$
$$\leq \Omega\varepsilon, \quad \forall x \in \mathcal{J}, \tag{14}$$

*where* $\Omega = D_\theta + \bar{D}_\theta T$.

*Proof* The solution (13) can be obtained easily by using Lemma 2. From it, it is obvious how to get result (14) using Remark 2. □

**Theorem 4** *Under the assumptions of Lemma 3, the solution of the considered problem (1) is Ulam–Hyers stable and also generalized Ulam–Hyers stable if* $\frac{L_f \Omega}{1-M_f} < 1$.

*Proof* Let $u \in X$ be any solution of problem (1) and $\bar{u} \in X$ be the unique solution of the considered problem. Then take

$$\begin{aligned}\|u - \bar{u}\| &= \max_{x\in\mathcal{J}}\left|u - \left[u_0 + D_\theta\big[f\big(x, \bar{u}(x), h_{\bar{u}}(x)\big) - f_0\big] + \bar{D}_\theta \int_0^x f\big(\eta, \bar{u}(\eta), h_{\bar{u}}(\eta)\big)\, d\eta\right]\right| \\ &\leq \max_{x\in\mathcal{J}}\left|u - \left[u_0 + D_\theta\big[f\big(x, u(x), h_u(x)\big) - f_0\big] + \bar{D}_\theta \int_0^x f\big(\eta, u(\eta), h_u(\eta)\big)\, d\eta\right]\right| \\ &\quad + \max_{x\in\mathcal{J}}\left|D_\theta\big[f\big(x, u(x), h_u(x)\big) - f\big(x, \bar{u}(x), h_{\bar{u}}(x)\big)\big]\right| \\ &\quad + \max_{x\in\mathcal{J}} \bar{D}_\theta \int_0^x \left|f\big(\eta, u(\eta), h_u(\eta)\big) - f\big(\eta, \bar{u}(\eta), h_{\bar{u}}(\eta)\big)\right| d\eta \\ &\leq \Omega\varepsilon + \frac{\Omega L_f}{1 - M_f}\|u - \bar{u}\|.\end{aligned} \tag{15}$$

Hence from (15), we have

$$\|u - \bar{u}\| \leq \frac{\Omega}{1 - \frac{L_f \Omega}{1-M_f}}\varepsilon. \tag{16}$$

Hence (16) yields that the solution is Ulam–Hyers stable. Further let $\mathcal{C}_f = \frac{\Omega}{1-\frac{L_f\Omega}{1-M_f}}$ and suppose there exists a nondecreasing function $\vartheta \in C((0,1),(0,\infty))$. Then from (16) we can write

$$\|u - \bar{u}\| \leq \mathcal{C}_f \vartheta(\varepsilon), \quad \text{with } \vartheta(0) = 0. \tag{17}$$

Therefore (17) implies that the solution is also generalized Ulam–Hyers stable. □

## 5 Application of our analysis

In this part of the paper, we test our obtained results on some problems given bellow.

*Example* 1 Take an implicit-type problem

$$\begin{cases} {}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{1}{2}} u(x) = \frac{x^2}{10} + \frac{\sin|u(x)|+\sin|{}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{1}{2}} u(x)|}{50+x^2}, \quad x \in [0,1], \\ u(0) = 0. \end{cases} \tag{18}$$

Clearly, from (18), $T = 1$ and

$$f(x,u,h_u) = \frac{x^2}{10} + \frac{\sin|u(x)| + \sin|{}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{1}{2}} u(x)|}{50 + x^2}$$

is continuous for all $x \in [0,1]$. Further, let $u, \tilde{u}, h_u, h_{\tilde{u}} \in \mathbb{R}$, then one has

$$\left|f(x,u,h_u) - f(x,\tilde{u},h_{\tilde{u}})\right| \le \frac{1}{50}|u - \tilde{u}| + \frac{1}{50}|h_u - h_{\tilde{u}}|. \tag{19}$$

From (19), one has $L_f = \frac{1}{50}$, $M_f = \frac{1}{50}$, $\theta = \frac{1}{2}$. Also

$$\left|f\left(x,u(x),h_u(x)\right)\right| \le \frac{1}{10} + \frac{1}{50}\left|u(x)\right| + \frac{1}{50}\left|h_u(x)\right|.$$

Thus $a_f = \frac{1}{10}$, $b_f = c_f = \frac{1}{50}$, and then $D_\theta = \frac{1}{2}$, $\bar{D}_\theta = \frac{1}{2}$, $T = 1$, and $(D_\theta + \bar{D}_\theta T)\frac{L_f}{1-M_f} = \frac{1}{49} < 1$. Hence the conditions of Theorem 1 are satisfied, so (18) has a unique solution. Further, $\frac{D_\theta L_f}{1-M_f} = \frac{1}{98} < 1$, therefore the conditions of Theorem 3 also hold. Thus the results of Theorem 3 hold. Further, to verify Theorem 4, we see that $\Omega = 1$, $\Omega\frac{L_f}{1-M_f} = 0.0204 < 1$. Hence the solution of the given problem is Ulam– Hyers stable and, consequently, generalized Ulam–Hyers stable.

*Example* 2 Here to strengthen our analysis, we investigate another problem:

$$\begin{cases} {}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{99}{100}} u(x) = \frac{1}{80+x^4} + \frac{u(x)}{1+|{}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{99}{100}} u(x)|} + \frac{\exp(-3x)\cos|{}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{99}{100}}|}{200+4x^2}, \quad x \in [0,1], \\ u(0) = 1. \end{cases} \tag{20}$$

Clearly, from (20), we have $T = 1$ and

$$f\left(x,u(x),h_u(x)\right) = \frac{1}{80 + x^4} + \frac{u(x)}{1 + |{}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{99}{100}} u(x)|} + \frac{\exp(-3x)\cos|{}_0^{\mathrm{CF}}\mathbf{D}_x^{\frac{99}{100}}|}{200 + 4x^2}$$

is continuous for all $x \in [0,1]$. Further, for $u, \tilde{u}, h_u, h_{\tilde{u}} \in \mathrm{R}$, one has

$$\left|f(x,u,h_u) - f(x,\tilde{u},h_{\tilde{u}})\right| \le \frac{1}{120}|u - \tilde{u}| + \frac{1}{200}|h_u - h_{\tilde{u}}|. \tag{21}$$

From (21), we take $L_f = \frac{1}{120}$, $M_f = \frac{1}{200}$, $\theta = \frac{99}{100}$. Also

$$\left|f\left(x,u(x),h_u(x)\right)\right| \le \frac{1}{80} + \frac{1}{120}\left|u(x)\right| + \frac{1}{200}\left|h_u(x)\right|.$$

Thus $a_f = \frac{1}{80}$, $b_f = \frac{1}{120}$, $c_f = \frac{1}{200}$, and then $D_\theta = \frac{1}{100}$, $\bar{D}_\theta = \frac{99}{100}$ with $T = 1$, and $(D_\theta + \bar{D}_\theta T)\frac{L_f}{1-M_f} = \frac{99}{5970} < 1$. Hence the conditions of Theorem 1 are satisfied, so (20) has a unique solution. Further, $\frac{D_\theta L_f}{1-M_f} = \frac{1}{11940} < 1$. Therefore the conditions of Theorem 3 also hold. Thus the results of Theorem 3 hold. Further, to verify Theorem 4, we see that $\Omega = 1$, $\Omega \frac{L_f}{1-M_f} = 0.0083752 < 1$. Hence the solution of the given problem is Ulam–Hyers stable and, consequently, generalized Ulam–Hyers stable.

## 6 Conclusion

The existence theory of solutions to nonsingular kernel-type FODEs has been framed. For the said theory, we have applied the usual Banach and Krasnoselskii fixed point theorems. Also some appropriate results about Ulam–Hyers and generalized Ulam–Hyers stability have been established by using the tools of nonlinear analysis. The obtained results have been testified by two interesting examples. To the best of our knowledge, the said results are new for FODEs involving CFFD. In the future, the above theory and analysis can be extended to more complicated and applicable problems of FODEs involving CFFD.

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The authors declare that they have no competing interests.

**Authors' contributions**
All authors equally contributed to this manuscript, read and approved the final version.

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