# On the Generalized Hermite–Hadamard Inequalities via the Tempered Fractional Integrals

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**Abstract:** Integral inequality plays a critical role in both theoretical and applied mathematics fields. It is clear that inequalities aim to develop different mathematical methods (numerically or analytically) and to dedicate the convergence and stability of the methods. Unfortunately, mathematical methods are useless if the method is not convergent or stable. Thus, there is a present day need for accurate inequalities in proving the existence and uniqueness of the mathematical methods. Convexity play a concrete role in the field of inequalities due to the behaviour of its definition. There is a strong relationship between convexity and symmetry. Which ever one we work on, we can apply to the other one due to the strong correlation produced between them especially in recent few years. In this article, we first introduced the notion of $\lambda$-incomplete gamma function. Using the new notation, we established a few inequalities of the Hermite–Hadamard (HH) type involved the tempered fractional integrals for the convex functions which cover the previously published result such as Riemann integrals, Riemann–Liouville fractional integrals. Finally, three example are presented to demonstrate the application of our obtained inequalities on modified Bessel functions and $q$-digamma function.

**Keywords:** Hermite–Hadamard inequality; incomplete gamma functions; fractional integrals

## 1. Introduction

Let $\hbar : \mathbf{J} \subseteq \mathcal{R} \to \mathcal{R}$ be a convex function and $\rtimes_3, \rtimes_4 \in \mathbf{J}$ with $\rtimes_3 < \rtimes_4$. Then, the well known inequalities, namely, the Hermite–Hadamard inequalities [1], defined by

$$\hbar \left( \frac{\rtimes_3 + \rtimes_4}{2} \right) \leq \frac{1}{\rtimes_4 - \rtimes_3} \int_{\rtimes_3}^{\rtimes_4} \hbar(x) \, \mathrm{d}x \leq \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2}. \tag{1}$$

We recall that the Hermite–Hadamard inequalities are related to the integral mean of a convex function. This provides an estimate from both sides of the mean value and assures the integrability of a convex function. Several classical inequalities can be obtained with the help of Hadamard's inequality considering the use of peculiar convex functions $\hbar$. Moreover, these inequalities for convex functions have a very important role in both applied and pure mathematics. Typical applications of the classical inequalities are: probabilistic problems, decision making in structural engineering and fatigue life.

The right and left part inequality of the inequalities (1) are called trapezoidal and midpoint inequalities. Researchers have ben working on two types of inequalities (1). Many of them have been worked only on the trapezoidal type inequality [2–4] or the midpoint type inequality [5,6] while the others have been workingd on both of them at the same time [7–9]. Both trapezoidal and midpoint inequalities can be explained using the following definition

**Definition 1** ([7]). *Suppose* $\hbar : [\rtimes_3, \rtimes_4] \subset \mathcal{R} \to \mathcal{R}$ *is a twice differentiable function on an open interval* $(\rtimes_3, \rtimes_4)$ *with the second derivative bounded on the interval* $(\rtimes_3, \rtimes_4)$*; that is,* $\|\hbar''\|_\infty :=$ $\sup_{x \in (\rtimes_3, \rtimes_4)} |\hbar''(x)| < \infty$*, then, the trapezoidal and midpoint type inequalities are defined by*

$$\left| \int_{\rtimes_3}^{\rtimes_4} \hbar(x)\mathrm{d}x - \frac{\rtimes_4 - \rtimes_3}{2} [\hbar(\rtimes_3) + \hbar(\rtimes_4)] \right| \leq \frac{(\rtimes_4 - \rtimes_3)^3}{12} \|\hbar''\|_\infty, \tag{2}$$

*and*

$$\left| \int_{\rtimes_3}^{\rtimes_4} \hbar(x)\mathrm{d}x - (\rtimes_4 - \rtimes_3)\, \hbar\left( \frac{\rtimes_3 + \rtimes_4}{2} \right) \right| \leq \frac{(\rtimes_4 - \rtimes_3)^3}{24} \|\hbar''\|_\infty, \tag{3}$$

*respectively.*

From a complementary viewpoint to Ostrowski type inequalities [10], trapezoidal and midpoint type inequalities provides a priory error bounds in estimating the Riemann integralby a generalized midpoint and trapezoidal formula [7,11]. We know that the development of Ostrowski's inequality has experienced attractive growth in the past decade, with over two thousands papers on it. A large number of refinements, generalizations, and extensions in both discrete and integral cases have been discovered (see [8,9]). Generalised versions have been discussed, e.g., the corresponding versions on time scales, form–time differentiable functions, for multiple integrals or vector valued functions as well (see [7,12]). Numerous applications in numerical analysis, special functions, probability theory, and other fields have been also given (see [8]).

In [2], Dragomir and Agarwal proved the following trapezoidal type equality and inequality, respectively:

**Lemma 1.** *If* $\hbar : \mathbf{J}^\circ \subseteq \mathcal{R} \to \mathcal{R}$ *is an* $L^1$ *function on* $\mathbf{J}^\circ$*, where* $\rtimes_3, \rtimes_4 \in \mathbf{J}^\circ$ *with* $\rtimes_3 < \rtimes_4$*. Then, we have*

$$\frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{1}{\rtimes_4 - \rtimes_3} \int_{\rtimes_3}^{\rtimes_4} \hbar(x)\mathrm{d}x = \frac{\rtimes_4 - \rtimes_3}{2} \int_0^1 (1 - 2\bar{\chi})\hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\mathrm{d}\bar{\chi}. \tag{4}$$

**Theorem 1.** *Let* $\hbar : \mathbf{J}^\circ \subseteq \mathcal{R} \to \mathcal{R}$ *be an* $L^1$ *function on* $\mathbf{J}^\circ$ *and let* $|\hbar'|$ *be convex on* $[\rtimes_3, \rtimes_4]$*, where* $\rtimes_3, \rtimes_4 \in \mathbf{J}^\circ$ *with* $\rtimes_3 < \rtimes_4$*. Then, we have*

$$\left| \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{1}{\rtimes_4 - \rtimes_3} \int_{\rtimes_3}^{\rtimes_4} \hbar(x)\mathrm{d}x \right| \leq \frac{\rtimes_4 - \rtimes_3}{8} \left( |\hbar'(\rtimes_3)| + |\hbar'(\rtimes_4)| \right). \tag{5}$$

In [5], Kirmaci proved the following midpoint type equality and inequality, respectively:

**Lemma 2.** *Let* $\hbar : \mathbf{J}^\circ \subset \mathcal{R} \to \mathcal{R}$ *be an* $L^1$ *function on* $\mathbf{J}^\circ$*,* $\rtimes_3, \rtimes_4 \in \mathbf{J}^\circ$ *with* $\rtimes_3 < \rtimes_4$*. Then, we have*

$$\frac{1}{\rtimes_4 - \rtimes_3} \int_{\rtimes_3}^{\rtimes_4} \hbar(x)\mathrm{d}x - \hbar\left( \frac{\rtimes_3 + \rtimes_4}{2} \right)$$
$$= (\rtimes_4 - \rtimes_3) \left[ \int_0^{\frac{1}{2}} \bar{\chi}\hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\mathrm{d}\bar{\chi} + \int_{\frac{1}{2}}^1 (\bar{\chi} - 1)\,\hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\mathrm{d}\bar{\chi} \right]. \tag{6}$$

**Theorem 2.** *Let $\hbar : \mathbf{J}^{\circ} \subseteq \mathcal{R} \to \mathcal{R}$ be an $L^1$ function on $\mathbf{J}^{\circ}$, $\rtimes_3, \rtimes_4 \in \mathbf{J}^{\circ}$ with $\rtimes_3 < \rtimes_4$ and $|\hbar'|$ be convex on $[\rtimes_3, \rtimes_4]$. Then, we have*

$$\left| \frac{1}{\rtimes_4 - \rtimes_3} \int_{\rtimes_3}^{\rtimes_4} \hbar(x)\mathrm{d}x - \hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right) \right| \leq \frac{(\rtimes_4 - \rtimes_3)}{8} \left(|\hbar'(\rtimes_3)| + |\hbar'(\rtimes_4)|\right). \tag{7}$$

The subject of the fractional calculus (integrals and derivatives) is a popular topic due to its fundamental applications in dealing with the dynamics of the complex sytems. This subject is still being studied extensively by many authors, see, for instance, [3,13–28].

One of the most important applications of fractional integrals is the well known inequality of the Hermite–Hadamard type, see [3,7–9,11,26–34] for more detail.

First, let us recall the above definition of the Riemann-Liouville fractional integrals (left and right) which are defined by [15,17]:

$$\mathfrak{I}^{\nu}_{\rtimes_3^+} \hbar(x) = \frac{1}{\Gamma(\nu)} \int_{\rtimes_3}^{x} (x - \bar{\chi})^{\nu-1} \hbar(\bar{\chi})\mathrm{d}\bar{\chi}, \;\; x > a,$$

$$\mathfrak{I}^{\nu}_{\rtimes_4^-} \hbar(x) = \frac{1}{\Gamma(\nu)} \int_{x}^{\rtimes_4} (\bar{\chi} - x)^{\nu-1} \hbar(\bar{\chi})\mathrm{d}\bar{\chi}, \;\; x < b,$$

where the gamma function is defined as

$$\Gamma(\nu) = \int_{0}^{\infty} \bar{\chi}^{\nu-1} e^{-\bar{\chi}}\mathrm{d}\bar{\chi}, \;\; \nu > 0.$$

Now, let us recall the basic expressions of Hermite–Hadamard inequality for fractional integrals is proved by Sarikaya et al. in [3] as follows.

**Theorem 3.** *If $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ is an $L^1$ function with $\rtimes_3 < \rtimes_4$. Then, we have*

$$\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right) \leq \frac{\Gamma(\nu+1)}{2(\rtimes_4 - \rtimes_3)^{\nu}} \left[\mathfrak{I}^{\nu}_{a+}\hbar(\rtimes_4) + \mathfrak{I}^{\nu}_{b-}\hbar(\rtimes_3)\right] \leq \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} \tag{8}$$

*with $\nu > 0$.*

Meanwhile, in [3], Sarikaya et al. established the following trapezoidal type equality and inequality for Riemann-Liouville integral, respectively:

**Lemma 3.** *If $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ is an $L^1$ function with $\rtimes_3 < \rtimes_4$. Then, we have*

$$\frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{\Gamma(\nu+1)}{2(\rtimes_4 - \rtimes_3)^{\nu}} \left[\mathfrak{I}^{\nu}_{a+}\hbar(\rtimes_4) + \mathfrak{I}^{\nu}_{b-}\hbar(\rtimes_3)\right]$$

$$= \frac{\rtimes_4 - \rtimes_3}{2} \int_{0}^{1} \left[(1 - \bar{\chi})^{\nu} - \bar{\chi}^{\nu}\right] \hbar'\left(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4\right) \mathrm{d}\bar{\chi}. \tag{9}$$

**Theorem 4.** *Let $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ be an $L^1$ function and $|\hbar'|$ be convex on $[\rtimes_3, \rtimes_4]$ with $\rtimes_3 < \rtimes_4$. Then, we have*

$$\left| \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{\Gamma(\nu+1)}{2(\rtimes_4 - \rtimes_3)^{\nu}} \left[\mathfrak{I}^{\nu}_{a+}\hbar(\rtimes_4) + \mathfrak{I}^{\nu}_{b-}\hbar(\rtimes_3)\right] \right|$$

$$\leq \frac{\rtimes_4 - \rtimes_3}{2(\nu+1)} \left(1 - \frac{1}{2^{\nu}}\right) \left[\hbar'(\rtimes_3) + \hbar'(\rtimes_4)\right]. \tag{10}$$

On the other hand, the equality and inequalities of midpoint type are pointed out in Remarks 10–12.

Now, we recall the basic definitions and new notations of tempered fractional operators.

**Definition 2** ([35,36])**.** *Let* $[\varkappa_3, \varkappa_4]$ *be a real interval and* $\lambda \geq 0, \nu > 0$. *Then for a function* $\hbar \in L^1[\varkappa_3, \varkappa_4]$, *the left and right tempered fractional integral, respectively, defined by*

$$\varkappa_3^{\tau} \mathfrak{I}_{\xi}^{(\nu,\lambda)} \hbar(\xi) = \frac{1}{\Gamma(\nu)} \int_{\varkappa_3}^{\xi} (\xi - \bar{\chi})^{\nu-1} e^{-\lambda(\xi - \bar{\chi})} \hbar(\bar{\chi}) d\bar{\chi}, \ \xi \in [\varkappa_3, \varkappa_4],$$

*and*

$$\varkappa_4^{\tau} \mathfrak{I}_{\xi}^{(\nu,\lambda)} \hbar(\xi) = \frac{1}{\Gamma(\nu)} \int_{\xi}^{\varkappa_4} (\bar{\chi} - \xi)^{\nu-1} e^{-\lambda(\bar{\chi} - \xi)} \hbar(\bar{\chi}) d\bar{\chi}, \ \xi \in [\varkappa_3, \varkappa_4].$$

We recall that several researchers the Riemann-Liouville fractionals integrals and provided important generalizations of Hermite–Hadamard type inequalities utilising these type of integrals for various type of convex functions, see, for instance, [3,19,20]. There is a strong relationship between convexity and symmetry. Which ever one we work on we can apply to the other due to the strong correlation produced between them especially in recent years (see [37]).

In this article, we followed the Sarikaya et al. [3] and Sarikaya and Yildirim [6] technique to establish a few inequalities of Hermite–Hadamard type (including both trapezoidal and midpoint type) which involved the tempered fractional integrals and the notion of $\lambda$-incomplete gamma function for convex functions. During the research, we found that our findings generalise the previous findings in the literature and this fact can be observed in Remarks 2–12.

The rest of this article is designed as: In Section 2.1, we obtain the inequalities of trapezoidal- and midpoint-type using integrals starting from the endpoints of the given interval, and in Section 2.2, we obtain analogous results using integrals starting from the midpoint of the given interval and some other relevant findings. Section 3, includes the application of our obtained results in special functions. The discussion on the proposed findings and concluding remarks are given in Section 4.

## 2. Hermite–Hadamard Inequalities Involving Beta Function

First of all we define the new incomplete gamma function:

**Definition 3.** *For the real numbers* $\nu > 0$ *and* $x, \lambda \geq 0$, *we define the* $\lambda$-*Incomplete gamma function by*

$$\gamma_\lambda(\nu, x) = \int_0^x \bar{\chi}^{\nu-1} e^{-\lambda t} d\bar{\chi}.$$

*If* $\lambda = 1$, *it reduces to the incomplete gamma function* [38]:

$$\gamma(\nu, x) = \int_0^x \bar{\chi}^{\nu-1} e^{-\bar{\chi}} d\bar{\chi}, \quad \nu > 0.$$

**Remark 1.** *For the real numbers* $\nu > 0$ *and* $x, \lambda \geq 0$, *we have*

(i)  $\gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, 1) = \int_0^1 \bar{\chi}^{\nu-1} e^{-\lambda(\varkappa_4 - \varkappa_3)\bar{\chi}} d\bar{\chi} = \frac{1}{(\varkappa_4 - \varkappa_3)^\nu} \gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)$.

(ii)  $\int_0^1 \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, x) dx = \frac{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^\nu} - \frac{\gamma_\lambda(\nu+1, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu+1}}$.

**Proof.**

(i)  The first item's proof follows from the Definition 3 and changing the variable $u := (\varkappa_4 - \varkappa_3)\bar{\chi}$.

(ii)  From the Definition 3, we have

$$\int_0^1 \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, x) dx = \int_0^1 \int_0^x y^{\nu-1} e^{-\lambda y} dy \, dx.$$

By changing the order of the integration, we get

$$\int_0^1 \gamma_{\lambda(\rtimes_4-\rtimes_3)}(\nu, x)dx = \int_0^1 \int_y^1 y^{\nu-1}e^{-\lambda y}dy\,dx$$

$$= \int_0^1 (1-y)y^{\nu-1}e^{-\lambda y}dy$$

$$= \int_0^1 y^{\nu-1}e^{-\lambda y}dy - \int_0^1 y^{\nu}e^{-\lambda y}dy.$$

Making the use of Remark 1 (i) we get

$$\int_0^1 \gamma_{\lambda(\rtimes_4-\rtimes_3)}(\nu, x)dx = \frac{\gamma_\lambda(\nu, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^\nu} - \frac{\gamma_\lambda(\nu+1, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^{\nu+1}}.$$

This ends the proof of the second item. □

In the next two subsections, we obtain some integral inequalities involved the $\lambda$-Incomplete gamma function.

### 2.1. Inequalities of $\left(\rtimes_3^+, \rtimes_4^-\right)$-Type

In this section, we prove a few inequalities of trapezoidal type or $\left(\rtimes_3^+, \rtimes_4^-\right)$-type.

**Proposition 1.** *Let $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ be a convex $L^1$ function on $[\rtimes_3, \rtimes_4]$ with $\rtimes_3 < \rtimes_4$. Then, we have*

$$\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right) \leq \frac{\Gamma(\nu)}{2(\rtimes_4 - \rtimes_3)^\nu \gamma_{\lambda(\rtimes_4-\rtimes_3)}(\nu, 1)} \left[{}_{\rtimes_3^+}^\tau \mathfrak{I}_{\rtimes_4}^{(\nu,\lambda)}\hbar(\rtimes_4) + {}_{\rtimes_4^-}^\tau \mathfrak{I}_{\rtimes_3}^{(\nu,\lambda)}\hbar(\rtimes_3)\right] \leq \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} \quad (11)$$

*for $\nu > 0, \lambda \geq 0$.*

**Proof.** The convexity of $\hbar$ allows us to write

$$\hbar\left(\frac{\bar{x} + \bar{y}}{2}\right) \leq \frac{\hbar(\bar{x}) + \hbar(\bar{y})}{2}, \tag{12}$$

or for $\bar{x} = \bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4$ and $\bar{y} = (1 - \bar{\chi})\rtimes_3 + \bar{\chi}\rtimes_4$ write

$$2\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right) \leq \hbar\left(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4\right) + \hbar\left((1 - \bar{\chi})\rtimes_3 + \bar{\chi}\rtimes_4\right), \quad \bar{\chi} \in [0, 1]. \tag{13}$$

Multiplying both sides of (13) by $\bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar{\chi}}$, and then integrating both sided with respect to $\bar{\chi}$ over $[0, 1]$ to get

$$2\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right)\int_0^1 \bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar{\chi}}d\bar{\chi}$$

$$\leq \int_0^1 \bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)t}\hbar\left(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4\right)d\bar{\chi}$$

$$+ \int_0^1 \bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar{\chi}}\hbar\left((1 - \bar{\chi})\rtimes_3 + \bar{\chi}\rtimes_4\right)d\bar{\chi}.$$

As consequence, we obtain

$$2\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right)\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)$$
$$\leq \frac{1}{(\rtimes_4 - \rtimes_3)^{\nu}}\int_{\rtimes_3}^{\rtimes_4}(\rtimes_4 - x)^{\nu - 1}e^{-\lambda(b-x)}\hbar(x)\,dx + \frac{1}{(\rtimes_4 - \rtimes_3)^{\nu}}\int_{\rtimes_3}^{\rtimes_4}(x - \rtimes_3)^{\nu - 1}e^{-\lambda(x - \rtimes_3)}\hbar(x)\,dx$$

and so we have proved the first part inequality.

We have to prove the other half of the inequality in (11). Since $\hbar$ is convex for every $\bar{\chi} \in [0, 1]$, we can find

$$\hbar(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4) + \hbar((1 - \bar{\chi})\rtimes_3 + \bar{\chi}\rtimes_4) \leq \hbar(\rtimes_3) + \hbar(\rtimes_4). \tag{14}$$

Then multiplying both sides of (14) by $\bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4 - \rtimes_3)\bar{\chi}}$ and integrating both sided with respect to $\bar{\chi}$ over $[0, 1]$ to get

$$\frac{1}{(\rtimes_4 - \rtimes_3)^{\nu}}\int_{\rtimes_3}^{\rtimes_4}(b - x)^{\nu - 1}e^{-\lambda(b-x)}\hbar(x)\,dx + \frac{1}{(\rtimes_4 - \rtimes_3)^{\nu}}\int_{\rtimes_3}^{\rtimes_4}(x - \rtimes_3)^{\nu - 1}e^{-\lambda(x - \rtimes_3)}\hbar(x)\,dx$$
$$\leq \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)[\hbar(\rtimes_3) + \hbar(\rtimes_4)]$$

and thus we have proved the second part inequality. This rearranges to the required result. □

**Remark 2.** *Inequalities* (11) *become the inequalities* (1) *for* $\lambda = 0$ *and* $\nu = 1$.

**Remark 3.** *Inequalities* (11) *become the inequalities* (8) *for* $\lambda = 0$.

Now, we give an identity which use to assist us in proving our next results.

**Lemma 4.** *If* $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ *is an* $L^1$ *function. Then, we have*

$$\frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{\Gamma(\nu)}{2(\rtimes_4 - \rtimes_3)^{\nu}\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)}\left[{}_{\rtimes_3^+}\mathfrak{I}_{\rtimes_4}^{(\nu,\lambda)}\hbar(\rtimes_4) + {}_{\rtimes_4^-}\mathfrak{I}_{\rtimes_3}^{(\nu,\lambda)}\hbar(\rtimes_3)\right]$$
$$= \frac{\rtimes_4 - \rtimes_3}{2}\int_0^1\left[\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi})\right]\hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\,d\bar{\chi} \tag{15}$$

*for* $\nu > 0$, $\lambda \geq 0$.

**Proof.** Applying integration by parts for the right part of (15) to get

$$\Delta_1 = \int_0^1\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi})\hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\,d\bar{\chi}$$
$$= \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)\frac{\hbar(\rtimes_4)}{\rtimes_4 - \rtimes_3} - \frac{1}{\rtimes_4 - \rtimes_3}\int_0^1(1 - \bar{\chi})^{\nu - 1}e^{-\lambda(\rtimes_4 - \rtimes_3)(1 - \bar{\chi})}\hbar(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\,d\bar{\bar{\chi}}$$
$$= \hbar(\rtimes_4)\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1) - \frac{1}{(\rtimes_4 - \rtimes_3)^{\nu+1}}\int_{\rtimes_3}^{\rtimes_4}(x - \rtimes_3)^{\nu - 1}e^{-\lambda(x - \rtimes_3)}\hbar(x)\,dx.$$

And similarly, we obtain

$$\Delta_2 = \int_0^1\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi})\hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\,d\bar{\chi}$$
$$= -\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)\frac{\hbar(\rtimes_3)}{\rtimes_4 - \rtimes_3} + \frac{1}{\rtimes_4 - \rtimes_3}\int_0^1\bar{\chi}^{\nu - 1}e^{-\lambda(\rtimes_4 - \rtimes_3)t}\hbar(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4)\,d\bar{\chi}$$
$$= -\hbar(\rtimes_3)\gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1) + \frac{1}{(\rtimes_4 - \rtimes_3)^{\nu+1}}\int_{\rtimes_3}^{\rtimes_4}(b - x)^{\nu - 1}e^{-\lambda(b-x)}\hbar(x)\,dx.$$

If we subtract $\Delta_2$ from $\Delta_1$ and multiply by $\dfrac{\rtimes_4 - \rtimes_3}{2}$, we obtain (15). □

**Remark 4.** *Identity* (15) *becomes the identity* (4) *for* $\lambda = 0$ *and* $\nu = 1$.

**Remark 5.** *Identity* (15) *becomes the identity* (9) *for* $\lambda = 0$.

In the extensions of our results, we have:

**Theorem 5.** *Let* $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ *be an* $L^1$ *function and* $|\hbar'|$ *be convex on* $[\rtimes_3, \rtimes_4]$ *with* $\rtimes_3 < \rtimes_4$. *Then, we have*

$$\left| \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{\Gamma(\nu)}{2(\rtimes_4 - \rtimes_3)^\nu \, \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)} \left[ {}_{\rtimes_3^+}^{\tau}\mathfrak{I}_{\rtimes_4}^{(\nu,\lambda)} \hbar(\rtimes_4) + {}_{\rtimes_4^-}^{\tau}\mathfrak{I}_{\rtimes_3}^{(\nu,\lambda)} \hbar(\rtimes_3) \right] \right|$$
$$\leq \left[ \frac{|\hbar'(\rtimes_3)| + |\hbar'(\rtimes_4)|}{2} \right] L_1^{(\nu,\lambda)}(\rtimes_3, \rtimes_4), \quad (16)$$

*where*

$$L_1^{(\nu,\lambda)}(\rtimes_3, \rtimes_4) := (\rtimes_4 - \rtimes_3) \int_0^{\frac{1}{2}} \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right] d\bar{\chi}.$$

**Proof.** Making the use of Lemma 4 and the convexity of $|\hbar'|$, we deduce

$$\left| \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{\Gamma(\nu)}{2(\rtimes_4 - \rtimes_3)^\nu \, \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)} \left[ {}_{\rtimes_3^+}^{\tau}\mathfrak{I}_{\rtimes_4}^{(\nu,\lambda)} \hbar(\rtimes_4) + {}_{\rtimes_4^-}^{\tau}\mathfrak{I}_{\rtimes_3}^{(\nu,\lambda)} \hbar(\rtimes_3) \right] \right|$$
$$\leq \frac{(\rtimes_4 - \rtimes_3)}{2} \int_0^1 \left| \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right| \left| \hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4) \right| d\bar{\chi}$$
$$\leq \frac{(\rtimes_4 - \rtimes_3) L_1^{(\nu,\lambda)}(\rtimes_3, \rtimes_4)}{2} \frac{1}{\rtimes_4 - \rtimes_3} \left[ |\hbar'(\rtimes_3)| + |\hbar'(\rtimes_4)| \right]$$
$$\left[ \frac{|\hbar'(\rtimes_3)| + |\hbar'(\rtimes_4)|}{2} \right] L_1^{(\nu,\lambda)}(\rtimes_3, \rtimes_4),$$

where we have used the inequality in Appendix A. Thus our proof is done. □

**Remark 6.** *Inequality* (16) *becomes the inequality* (5) *for* $\lambda = 0$ *and* $\nu = 1$.

**Remark 7.** *Inequality* (16) *becomes the inequality* (10) *for* $\lambda = 0$.

**Theorem 6.** *Let* $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ *be an* $L^1$ *function and* $|\hbar'|^\varrho, \varrho > 1$ *be convex on* $[\rtimes_3, \rtimes_4]$ *with* $\rtimes_3 < \rtimes_4$. *Then, we have*

$$\left| \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2} - \frac{\Gamma(\nu)}{2(\rtimes_4 - \rtimes_3)^\nu \, \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1)} \left[ {}_{\rtimes_3^+}^{\tau}\mathfrak{I}_{\rtimes_4}^{(\nu,\lambda)} \hbar(\rtimes_4) + {}_{\rtimes_4^-}^{\tau}\mathfrak{I}_{\rtimes_3}^{(\nu,\lambda)} \hbar(\rtimes_3) \right] \right|$$
$$\leq \left( \frac{|\hbar'(\rtimes_3)|^\varrho + |\hbar'(\rtimes_4)|^\varrho}{2} \right)^{\frac{1}{\varrho}} L_2^{(\nu,\lambda)}(\rtimes_3, \rtimes_4), \quad (17)$$

*where*

$$L_\varrho^{(\nu,\lambda)}(\rtimes_3, \rtimes_4) := (\rtimes_4 - \rtimes_3) \left( \int_0^1 \left| \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right|^{\bar{\varrho}} d\bar{\chi} \right)^{\frac{1}{\bar{\varrho}}}.$$

**Proof.** By making the use of Lemma 4, Hölder's inequality and the convexity of $|\hbar'|^\varrho$, we deduce

$$
\left| \frac{\hbar(\varkappa_3) + \hbar(\varkappa_4)}{2} - \frac{\Gamma(\nu)}{2(\varkappa_4 - \varkappa_3)^\nu \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, 1)} \left[ {}_{\varkappa_3^+}^\tau \mathfrak{J}_{\varkappa_4}^{(\nu,\lambda)} \hbar(\varkappa_4) + {}_{\varkappa_4^-}^\tau \mathfrak{J}_{\varkappa_3}^{(\nu,\lambda)} \hbar(\varkappa_3) \right] \right|
$$

$$
\leq \frac{\varkappa_4 - \varkappa_3}{2} \left( \int_0^1 \left| \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, \bar{\chi}) \right|^{\bar{\varrho}} d\bar{\chi} \right)^{\frac{1}{\bar{\varrho}}} \left( \int_0^1 \left| \hbar'((1 - \bar{\chi})\varkappa_3 + \bar{\chi}\varkappa_4) \right|^\varrho d\bar{\chi} \right)^{\frac{1}{\varrho}}
$$

$$
\leq \frac{\varkappa_4 - \varkappa_3}{2} \left( \int_0^1 \left| \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\varkappa_4 - \varkappa_3)}(\nu, \bar{\chi}) \right|^{\bar{\varrho}} d\bar{\chi} \right)^{\frac{1}{\bar{\varrho}}} \left( \int_0^1 \left[ (1 - \bar{\chi}) \left| \hbar'(\varkappa_3) \right|^\varrho + \bar{\chi} \left| \hbar'(\varkappa_4) \right|^\varrho \right] d\bar{\chi} \right)^{\frac{1}{\varrho}}
$$

$$
= \left( \frac{\left| \hbar'(\varkappa_3) \right|^\varrho + \left| \hbar'(\varkappa_4) \right|^\varrho}{2} \right)^{\frac{1}{\varrho}} L_2^{(\nu,\lambda)}(\varkappa_3, \varkappa_4),
$$

which completes the proof of (17). □

**Remark 8.** *Inequality* (17) *becomes the inequality* (3) *for* $\lambda = 0$ *and* $\nu = 1$.

**Remark 9.** *Inequality* (17) *becomes the inequality* (2.7) *in* ([21], Theorem 8) *for* $\lambda = 0$.

*2.2. Inequalities of $\left( \frac{\varkappa_3 + \varkappa_4}{2} \right)$-Type*

　　In this section, a few inequalities of midpoint type or $\left( \frac{\varkappa_3 + \varkappa_4}{2} \right)$-type will be proved.

**Proposition 2.** *Let* $\hbar : [\varkappa_3, \varkappa_4] \to \mathcal{R}$ *be a positive convex* $L^1$ *function on* $[\varkappa_3, \varkappa_4]$ *with* $\varkappa_3 < \varkappa_4$. *Then, we have*

$$
\hbar\left( \frac{\varkappa_3 + \varkappa_4}{2} \right) \leq \frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)} \left\{ {}_{\left( \frac{\varkappa_3 + \varkappa_4}{2} \right)^+}^\tau \mathfrak{J}_{\varkappa_4}^{(\nu,2\lambda)} \hbar(\varkappa_4) + {}_{\left( \frac{\varkappa_3 + \varkappa_4}{2} \right)^-}^\tau \mathfrak{J}_{\varkappa_3}^{(\nu,2\lambda)} \hbar(\varkappa_3) \right\}
$$
$$
\leq \frac{\hbar(\varkappa_3) + \hbar(\varkappa_4)}{2}. \quad (18)
$$

**Proof.** Making the use of (12) for $\bar{x} = \frac{\bar{\chi}}{2}\varkappa_3 + \frac{2 - \bar{\chi}}{2}\varkappa_4$ and $\bar{y} = \frac{2 - \bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4$ to get

$$
2\hbar\left( \frac{\varkappa_3 + \varkappa_4}{2} \right) \leq \hbar\left( \frac{\bar{\chi}}{2}\varkappa_3 + \frac{2 - \bar{\chi}}{2}\varkappa_4 \right) + \hbar\left( \frac{2 - \bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4 \right), \quad \bar{\chi} \in [0, 1].
$$

　　We multiply both sides by $\bar{\chi}^{\nu-1} e^{-\lambda(\varkappa_4 - \varkappa_3)\bar{\chi}}$ and then integrating with respect to $\bar{\chi}$ over $\bar{\chi} \in [0, 1]$ to deduce

$$
2\hbar\left( \frac{\varkappa_3 + \varkappa_4}{2} \right) \int_0^1 \bar{\chi}^{\nu-1} e^{-\lambda(\varkappa_4 - \varkappa_3)\bar{\chi}} d\bar{\chi} \leq \int_0^1 \bar{\chi}^{\nu-1} e^{-\lambda(\varkappa_4 - \varkappa_3)\bar{\chi}} \hbar\left( \frac{\bar{\chi}}{2}\varkappa_3 + \frac{2 - \bar{\chi}}{2}\varkappa_4 \right) d\bar{\chi}
$$
$$
+ \int_0^1 \bar{\chi}^{\nu-1} e^{-\lambda(\varkappa_4 - \varkappa_3)\bar{\chi}} \hbar\left( \frac{2 - \bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4 \right) d\bar{\chi}.
$$

　　By making the use of $u := \frac{\bar{\chi}}{2}\varkappa_3 + \frac{2 - \bar{\chi}}{2}\varkappa_4$, $v := \frac{2 - \bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4$ and Remark 1 (i) in the last inequality, we get

$$
2\hbar\left( \frac{\varkappa_3 + \varkappa_4}{2} \right) \frac{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^\nu} \leq \frac{2^\nu}{(\varkappa_4 - \varkappa_3)^\nu} \int_{\frac{\varkappa_3 + \varkappa_4}{2}}^{\varkappa_4} (\varkappa_4 - u)^{\nu-1} e^{-2\lambda(\varkappa_4 - u)} \hbar(u) du
$$
$$
+ \frac{2^\nu}{(\varkappa_4 - \varkappa_3)^\nu} \int_{\varkappa_3}^{\frac{\varkappa_3 + \varkappa_4}{2}} (v - \varkappa_3)^{\nu-1} e^{-2\lambda(v - \varkappa_3)} \hbar(v) dv,
$$

or equivalently,

$$\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right) \le \frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda\left(\nu, \rtimes_4 - \rtimes_3\right)}\left\{\left(\frac{\rtimes_3+\rtimes_4}{2}\right)^+ {}^\tau\mathfrak{I}_{\rtimes_4}^{(\nu,2\lambda)}\hbar(\rtimes_4) + \left(\frac{\rtimes_3+\rtimes_4}{2}\right)^- {}^\tau\mathfrak{I}_{\rtimes_3}^{(\nu,2\lambda)}\hbar(\rtimes_3)\right\},$$

and this gives the proof of the first part inequality.

On the other hand, we have from the convexity of $\hbar$ that

$$\hbar\left(\frac{\bar\chi}{2}\rtimes_3 + \frac{2 - \bar\chi}{2}\rtimes_4\right) \le \frac{\bar\chi}{2}\hbar(\rtimes_3) + \frac{2 - \bar\chi}{2}\hbar(\rtimes_4),$$

and

$$\hbar\left(\frac{2 - \bar\chi}{2}\rtimes_3 + \frac{\bar\chi}{2}\rtimes_4\right) \le \frac{2 - \bar\chi}{2}\hbar(\rtimes_3) + \frac{\bar\chi}{2}\hbar(\rtimes_4).$$

Adding these to get

$$\hbar\left(\frac{\bar\chi}{2}\rtimes_3 + \frac{2 - \bar\chi}{2}\rtimes_4\right) + \hbar\left(\frac{2 - \bar\chi}{2}\rtimes_3 + \frac{\bar\chi}{2}\rtimes_4\right) \le \hbar(\rtimes_3) + \hbar(\rtimes_4).$$

We multiply both sides by $\bar\chi^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar\chi}$ and then integrating with respect to $\bar\chi$ over $\bar\chi \in [0,1]$ to deduce

$$\int_0^1 \bar\chi^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar\chi}\hbar\left(\frac{\bar\chi}{2}\rtimes_3 + \frac{2 - \bar\chi}{2}\rtimes_4\right)d\bar\chi + \int_0^1 \bar\chi^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar\chi}\hbar\left(\frac{2 - \bar\chi}{2}\rtimes_3 + \frac{\bar\chi}{2}\rtimes_4\right)d\bar\chi$$

$$\le \left[\hbar(\rtimes_3) + \hbar(\rtimes_4)\right]\int_0^1 \bar\chi^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar\chi}d\bar\chi.$$

Again, by making the use of $u := \frac{\bar\chi}{2}\rtimes_3 + \frac{2-\bar\chi}{2}\rtimes_4$, $v := \frac{2-\bar\chi}{2}\rtimes_3 + \frac{\bar\chi}{2}\rtimes_4$ and Remark 1 (ii) in the last inequality, we get

$$\frac{2^\nu\Gamma(\nu)}{(\rtimes_4 - \rtimes_3)^\nu}\left\{\left(\frac{\rtimes_3+\rtimes_4}{2}\right)^+ {}^\tau\mathfrak{I}_{\rtimes_4}^{(\nu,2\lambda)}\hbar(\rtimes_4) + \left(\frac{\rtimes_3+\rtimes_4}{2}\right)^- {}^\tau\mathfrak{I}_{\rtimes_3}^{(\nu,2\lambda)}\hbar(\rtimes_3)\right\} \le \frac{\left[\hbar(\rtimes_3) + \hbar(\rtimes_4)\right]}{(\rtimes_4 - \rtimes_3)^\nu}\gamma_\lambda\left(\nu, \rtimes_4 - \rtimes_3\right),$$

and this ends the proof of the second inequality. Therefore, the inequalities (18) is proved. $\square$

**Remark 10.** *If in Proposition 2, we set*

1. $\lambda = 0$, *then inequalities* (18) *become the following inequalities*

$$\hbar\left(\frac{\rtimes_3 + \rtimes_4}{2}\right) \le \frac{2^{\nu-1}\Gamma(\nu+1)}{(\rtimes_4 - \rtimes_3)^\nu}\left\{\mathfrak{I}^\nu_{\left(\frac{\rtimes_3+\rtimes_4}{2}\right)^+}\hbar(\rtimes_4) + \mathfrak{I}^\nu_{\left(\frac{\rtimes_3+\rtimes_4}{2}\right)^-}\hbar(\rtimes_3)\right\} \le \frac{\hbar(\rtimes_3) + \hbar(\rtimes_4)}{2},$$

   *which is done by Sarikaya and Yildirim in [6].*

2. $\lambda = 0$ *and* $\nu = 1$, *then inequalities* (18) *become the inequalities* (1).

**Lemma 5.** *If* $\hbar : [\rtimes_3, \rtimes_4] \to \mathcal{R}$ *is an* $L^1$ *function. Then, we have*

$$\frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda\left(\nu, \rtimes_4 - \rtimes_3\right)}\left\{\left(\tfrac{\rtimes_3+\rtimes_4}{2}\right)^+\mathfrak{J}_{\rtimes_4}^{(\nu,2\lambda)}\hbar(\rtimes_4) + \left(\tfrac{\rtimes_3+\rtimes_4}{2}\right)^-\mathfrak{J}_{\rtimes_3}^{(\nu,2\lambda)}\hbar(\rtimes_3)\right\} - \hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right)$$

$$= \frac{(\rtimes_4-\rtimes_3)^{\nu+1}}{4\gamma_\lambda\left(\nu, \rtimes_4-\rtimes_3\right)}\left\{\int_0^1 \gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,\bar{\chi}\right)\hbar'\left(\frac{\bar{\chi}}{2}\rtimes_3 + \frac{2-\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}\right.$$

$$\left. - \int_0^1 \gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,\bar{\chi}\right)\hbar'\left(\frac{2-\bar{\chi}}{2}\rtimes_3 + \frac{\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}\right\}, \quad (19)$$

*where* $\gamma_\lambda\left(\nu,\bar{\chi}\right)$ *is as before.*

**Proof.** By making the use of integrating by parts and Remark 1 (i), we get

$$\epsilon_1 := \int_0^1 \gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,\bar{\chi}\right)\hbar'\left(\frac{\bar{\chi}}{2}\rtimes_3 + \frac{2-\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}$$

$$= -\frac{2\gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,\bar{\chi}\right)}{\rtimes_4-\rtimes_3}\hbar\left(\frac{\bar{\chi}}{2}\rtimes_3 + \frac{2-\bar{\chi}}{2}\rtimes_4\right)\Bigg|_0^1 + \frac{2}{\rtimes_4-\rtimes_3}\int_0^1 \bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar{\chi}}\hbar\left(\frac{\bar{\chi}}{2}\rtimes_3 + \frac{2-\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}$$

$$= -\frac{2\gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,1\right)}{\rtimes_4-\rtimes_3}\hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right) + \frac{2^{\nu+1}\Gamma(\nu)}{(\rtimes_4-\rtimes_3)^{\nu+1}}\int_{\frac{\rtimes_3+\rtimes_4}{2}}^{\rtimes_4} (\rtimes_4-u)^{\nu-1}e^{-2\lambda(\rtimes_4-u)}\hbar(u)du$$

$$= -\frac{2\gamma_\lambda\left(\nu, \rtimes_4-\rtimes_3\right)}{(\rtimes_4-\rtimes_3)^{\nu+1}}\hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right) + \frac{2^{\nu+1}\Gamma(\nu)}{(\rtimes_4-\rtimes_3)^{\nu+1}}\left(\tfrac{\rtimes_3+\rtimes_4}{2}\right)^+\mathfrak{J}_{\rtimes_4}^{(\nu,2\lambda)}\hbar(\rtimes_4).$$

Analogously, we have

$$\epsilon_2 := \int_0^1 \gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,\bar{\chi}\right)\hbar'\left(\frac{2-\bar{\chi}}{2}\rtimes_3 + \frac{\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}$$

$$= \frac{2\gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,\bar{\chi}\right)}{\rtimes_4-\rtimes_3}\hbar\left(\frac{2-\bar{\chi}}{2}\rtimes_3 + \frac{\bar{\chi}}{2}\rtimes_4\right)\Bigg|_0^1 - \frac{2}{\rtimes_4-\rtimes_3}\int_0^1 \bar{\chi}^{\nu-1}e^{-\lambda(\rtimes_4-\rtimes_3)\bar{\chi}}\hbar\left(\frac{2-\bar{\chi}}{2}\rtimes_3 + \frac{\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}$$

$$= \frac{2\gamma_{\lambda(\rtimes_4-\rtimes_3)}\left(\nu,1\right)}{\rtimes_4-\rtimes_3}\hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right) - \frac{2^{\nu+1}\Gamma(\nu)}{(\rtimes_4-\rtimes_3)^{\nu+1}}\int_{\rtimes_3}^{\frac{\rtimes_3+\rtimes_4}{2}} (v-\rtimes_3)^{\nu-1}e^{-2\lambda(v-\rtimes_3)}\hbar(v)dv$$

$$= \frac{2\gamma_\lambda\left(\nu, \rtimes_4-\rtimes_3\right)}{(\rtimes_4-\rtimes_3)^{\nu+1}}\hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right) - \frac{2^{\nu+1}\Gamma(\nu)}{(\rtimes_4-\rtimes_3)^{\nu+1}}\left(\tfrac{\rtimes_3+\rtimes_4}{2}\right)^-\mathfrak{J}_{\rtimes_3}^{(\nu,2\lambda)}\hbar(\rtimes_3).$$

Consequently, we have

$$\frac{(\rtimes_4-\rtimes_3)^{\nu+1}}{4\gamma_\lambda\left(\nu, \rtimes_4-\rtimes_3\right)}\left(\epsilon_1 - \epsilon_2\right) = \frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda\left(\nu, \rtimes_4-\rtimes_3\right)}\left\{\left(\tfrac{\rtimes_3+\rtimes_4}{2}\right)^+\mathfrak{J}_{\rtimes_4}^{(\nu,2\lambda)}\hbar(\rtimes_4) + \left(\tfrac{\rtimes_3+\rtimes_4}{2}\right)^-\mathfrak{J}_{\rtimes_3}^{(\nu,2\lambda)}\hbar(\rtimes_3)\right\} - \hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right).$$

This ends the complete proof of Lemma 5. □

**Remark 11.** *If in Lemma 5, we set*

1. $\lambda = 0$, *then equality* (19) *becomes the following equality*

$$\frac{2^{\nu-1}\Gamma(\nu+1)}{(\rtimes_4-\rtimes_3)^\nu}\left\{\mathfrak{J}_{\left(\frac{\rtimes_3+\rtimes_4}{2}\right)^+}^\nu\hbar(\rtimes_4) + \mathfrak{J}_{\left(\frac{\rtimes_3+\rtimes_4}{2}\right)^-}^\nu\hbar(\rtimes_3)\right\} - \hbar\left(\frac{\rtimes_3+\rtimes_4}{2}\right)$$

$$= \frac{\rtimes_4-\rtimes_3}{4}\left\{\int_0^1 \bar{\chi}^\nu\hbar'\left(\frac{\bar{\chi}}{2}\rtimes_3 + \frac{2-\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi} - \int_0^1 \bar{\chi}^\nu\hbar'\left(\frac{2-\bar{\chi}}{2}\rtimes_3 + \frac{\bar{\chi}}{2}\rtimes_4\right)d\bar{\chi}\right\},$$

2. $\lambda = 0$ and $\nu = 1$, then equality (19) becomes the following equality

$$\frac{1}{\varkappa_4 - \varkappa_3} \int_{\varkappa_3}^{\varkappa_4} \hbar(x)\mathrm{d}x - \hbar\left(\frac{\varkappa_3 + \varkappa_4}{2}\right)$$

$$= \frac{\varkappa_4 - \varkappa_3}{4}\left\{\int_0^1 \bar{\chi}\hbar'\left(\frac{\bar{\chi}}{2}\varkappa_3 + \frac{2-\bar{\chi}}{2}\varkappa_4\right)\mathrm{d}\bar{\chi} - \int_0^1 \bar{\chi}\hbar'\left(\frac{2-\bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4\right)\mathrm{d}\bar{\chi}\right\},$$

*these are both done by Sarikaya and Yildirim in [6].*

**Theorem 7.** *If $\hbar : [\varkappa_3, \varkappa_4] \to \mathcal{R}$ is an $L^1$ function. Then, for the convexity of $|\hbar'|^{\varrho}, \varrho \geq 1$ on $[\varkappa_3, \varkappa_4]$ with $\varkappa_3 < \varkappa_4$, we have*

$$\left|\frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}\left\{{}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^+}^{\tau}\mathfrak{I}_{\varkappa_4}^{(\nu,2\lambda)}\hbar(\varkappa_4) + {}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^-}^{\tau}\mathfrak{I}_{\varkappa_3}^{(\nu,2\lambda)}\hbar(\varkappa_3)\right\} - \hbar\left(\frac{\varkappa_3 + \varkappa_4}{2}\right)\right|$$

$$\leq \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}\mathcal{A}^{1-\frac{1}{\varrho}}(\nu,\lambda)\left\{\left(\frac{\mathcal{B}(\nu,\lambda)}{4}|\hbar'(\varkappa_3)|^{\varrho} + \frac{\mathcal{C}(\nu,\lambda)}{4}|\hbar'(\varkappa_4)|^{\varrho}\right)^{\frac{1}{\varrho}}\right.$$

$$\left. + \left(\frac{\mathcal{C}(\nu,\lambda)}{4}|\hbar'(\varkappa_3)|^{\varrho} + \frac{\mathcal{B}(\nu,\lambda)}{4}|\hbar'(\varkappa_4)|^{\varrho}\right)^{\frac{1}{\varrho}}\right\}, \quad (20)$$

*where*

$$\mathcal{A}(\nu,\lambda) = \frac{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu}} - \frac{\gamma_\lambda(\nu+1, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu+1}}, \quad \mathcal{B}(\nu,\lambda) = \frac{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu}} - \frac{\gamma_\lambda(\nu+1, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu+1}},$$

$$\mathcal{C}(\nu,\lambda) = \frac{\gamma_\lambda(\nu+2, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu+2}} - \frac{4\gamma_\lambda(\nu+1, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu+1}} + \frac{3\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu}}.$$

**Proof.** At first, we let $\varrho = 1$. Then by making the use of Lemma 5, Remark 1 (ii) and the convexity of $|\hbar'|^{\varrho}$, we have

$$\left|\frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}\left\{{}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^+}^{\tau}\mathfrak{I}_{\varkappa_4}^{(\nu,2\lambda)}\hbar(\varkappa_4) + {}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^-}^{\tau}\mathfrak{I}_{\varkappa_3}^{(\nu,2\lambda)}\hbar(\varkappa_3)\right\} - \hbar\left(\frac{\varkappa_3 + \varkappa_4}{2}\right)\right|$$

$$\leq \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}\int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}(\nu,\bar{\chi})\left\{\left|\hbar'\left(\frac{\bar{\chi}}{2}\varkappa_3 + \frac{2-\bar{\chi}}{2}\varkappa_4\right)\right| + \left|\hbar'\left(\frac{2-\bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4\right)\right|\right\}\mathrm{d}\bar{\chi}$$

$$\leq \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}\left(|\hbar'(\varkappa_3)| + |\hbar'(\varkappa_4)|\right)\int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}(\nu,\bar{\chi})\,\mathrm{d}\bar{\chi}$$

$$= \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}\left[\frac{\gamma_\lambda(\nu, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu}} - \frac{\gamma_\lambda(\nu+1, \varkappa_4 - \varkappa_3)}{(\varkappa_4 - \varkappa_3)^{\nu+1}}\right]\left(|\hbar'(\varkappa_3)| + |\hbar'(\varkappa_4)|\right).$$

For $\varrho > 1$, we use the Lemma 5, power-mean inequality, Remark 1 (ii) and the convexity of $|\hbar'|^\varrho$ to get

$$\left| \frac{2^{\nu-1}\Gamma(\nu)}{\gamma_\lambda\,(\nu, \varkappa_4 - \varkappa_3)} \left\{ {}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^+}^\tau \mathcal{J}_{\varkappa_4}^{(\nu,2\lambda)}\hbar(\varkappa_4) + {}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^-}^\tau \mathcal{J}_{\varkappa_3}^{(\nu,2\lambda)}\hbar(\varkappa_3) \right\} - \hbar\left(\frac{\varkappa_3 + \varkappa_4}{2}\right) \right|$$

$$\leq \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda\,(\nu, \varkappa_4 - \varkappa_3)} \left( \int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}\,(\nu,\bar{\chi})\,\mathrm{d}\bar{\chi} \right)^{1-\frac{1}{\varrho}} \left\{ \left( \int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}\,(\nu,\bar{\chi}) \left| \hbar'\left(\frac{\bar{\chi}}{2}\varkappa_3 + \frac{2-\bar{\chi}}{2}\varkappa_4\right) \right|^\varrho \mathrm{d}\bar{\chi} \right)^{\frac{1}{\varrho}} \right.$$

$$\left. + \left( \int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}\,(\nu,\bar{\chi}) \left| \hbar'\left(\frac{2-\bar{\chi}}{2}\varkappa_3 + \frac{\bar{\chi}}{2}\varkappa_4\right) \right|^\varrho \mathrm{d}\bar{\chi} \right)^{\frac{1}{\varrho}} \right\}$$

$$\leq \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda\,(\nu, \varkappa_4 - \varkappa_3)} \mathcal{A}^{1-\frac{1}{\varrho}}(\nu,\lambda) \left\{ \left( \int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}\,(\nu,\bar{\chi}) \left[ \frac{\bar{\chi}}{2}|\hbar'(\varkappa_3)|^\varrho + \frac{2-\bar{\chi}}{2}|\hbar'(\varkappa_4)|^\varrho \right] \mathrm{d}\bar{\chi} \right)^{\frac{1}{\varrho}} \right.$$

$$\left. + \left( \int_0^1 \gamma_{\lambda(\varkappa_4-\varkappa_3)}\,(\nu,\bar{\chi}) \left[ \frac{2-\bar{\chi}}{2}|\hbar'(\varkappa_3)|^\varrho + \frac{\bar{\chi}}{2}|\hbar'(\varkappa_4)|^\varrho \right] \mathrm{d}\bar{\chi} \right)^{\frac{1}{\varrho}} \right\}$$

$$= \frac{(\varkappa_4 - \varkappa_3)^{\nu+1}}{4\gamma_\lambda\,(\nu, \varkappa_4 - \varkappa_3)} \mathcal{A}^{1-\frac{1}{\varrho}}(\nu,\lambda)$$

$$\times \left\{ \left( \frac{\mathcal{B}(\nu,\lambda)}{4}|\hbar'(\varkappa_3)|^\varrho + \frac{\mathcal{C}(\nu,\lambda)}{4}|\hbar'(\varkappa_4)|^\varrho \right)^{\frac{1}{\varrho}} + \left( \frac{\mathcal{C}(\nu,\lambda)}{4}|\hbar'(\varkappa_3)|^\varrho + \frac{\mathcal{B}(\nu,\lambda)}{4}|\hbar'(\varkappa_4)|^\varrho \right)^{\frac{1}{\varrho}} \right\},$$

where we have used the identities in Appendix B. Thus our proof is done. $\qquad\square$

**Remark 12.** *If in Theorem 7, we set*

1. $\lambda = 0$, *then inequality* (20) *becomes the following inequality*

$$\left| \frac{2^{\nu-1}\Gamma(\nu+1)}{(\varkappa_4 - \varkappa_3)^\nu} \left\{ \mathcal{J}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^+}^\nu \hbar(\varkappa_4) + \mathcal{J}_{\left(\frac{\varkappa_3+\varkappa_4}{2}\right)^-}^\nu \hbar(\varkappa_3) \right\} - \hbar\left(\frac{\varkappa_3 + \varkappa_4}{2}\right) \right|$$

$$\leq \frac{\varkappa_4 - \varkappa_3}{4(\nu+1)} \left[ \frac{1}{2(\nu+2)} \right]^{\frac{1}{\varrho}} \left\{ \left( (\nu+1)|\hbar'(\varkappa_3)|^\varrho + (\nu+3)|\hbar'(\varkappa_4)|^\varrho \right)^{\frac{1}{\varrho}} \right.$$

$$\left. + \left( (\nu+3)|\hbar'(\varkappa_3)|^\varrho + (\nu+1)|\hbar'(\varkappa_4)|^\varrho \right)^{\frac{1}{\varrho}} \right\},$$

*which is done by Sarikaya and Yildirim in [6].*

2. $\lambda = 0$ *and* $\nu = \varrho = 1$, *then inequality* (20) *becomes the inequality* (7).

## 3. Examples

There are many applications to demonstrate the use of integral inequalities, especially applications on special means of the real numbers [2,5,8,39]. In this section, we present some examples to demonstrate the applications of our proposed results on modified Bessel functions and *q*-digamma functions.

**Example 1.** *Consider the function* $\mathcal{I}_{\bar{\varrho}} : \mathcal{R} \to [1, \infty)$, *defined by*

$$\mathcal{I}_{\bar{\varrho}}(z) = 2^{\bar{\varrho}}\Gamma(\bar{\varrho}+1)z^{-\upsilon}I_{\bar{\varrho}}(z), \quad z \in \mathcal{R}.$$

*Here, we consider the modified Bessel function of the first kind $\mathcal{I}_{\bar{\varrho}}$, defined by [40]:*

$$\mathcal{I}_{\bar{\varrho}}(z) = \sum_{n \geq 0} \frac{\left(\frac{z}{2}\right)^{\bar{\varrho}+2n}}{n!\Gamma(\bar{\varrho}+n+1)}.$$

*The first order derivative formula of $\mathcal{I}_{\bar{\varrho}}(z)$ is given by [40]:*

$$\mathcal{I}'_{\bar{\varrho}}(z) = \frac{z}{2(\bar{\varrho}+1)}\mathcal{I}_{\bar{\varrho}+1}(z). \tag{21}$$

*By making the use of Remark 2 and identity (21), we can deduce*

$$\left|\frac{\mathcal{I}_{\bar{\varrho}}(\bowtie_4) - \mathcal{I}_{\bar{\varrho}}(\bowtie_3)}{\bowtie_4 - \bowtie_3}\right| \leq \frac{\bowtie_3 \mathcal{I}_{\bar{\varrho}+1}(\bowtie_3) + \bowtie_4 \mathcal{I}_{\bar{\varrho}+1}(\bowtie_4)}{4(\bar{\varrho}+1)}$$

*for $\bar{\varrho} > -1$, $\bowtie_3, \bowtie_4 \in \mathcal{R}$ with $0 < \bowtie_3 < \bowtie_4$. Specifically, for $\mathcal{I}_{-\frac{1}{2}}(z) = \cosh(z)$ and $\mathcal{I}_{\frac{1}{2}}(z) = \frac{\sinh(z)}{z}$, we get*

$$\left|\frac{\cosh(\bowtie_4) - \cosh(\bowtie_3)}{\bowtie_4 - \bowtie_3}\right| \leq \frac{\sinh(\bowtie_3) + \sinh(\bowtie_4)}{2}.$$

**Example 2.** *Here, we consider the modified Bessel function of the second kind $\mathcal{K}_{\bar{\varrho}}$, defined by [40]:*

$$\mathcal{K}_{\bar{\varrho}}(z) = \frac{\pi}{2}\frac{\mathcal{I}_{-\bar{\varrho}}(z) + \mathcal{I}_{\bar{\varrho}}(z)}{\sin(\bar{\varrho}\pi)}.$$

*Let $\hbar_{\bar{\varrho}}(z) := -\left(\frac{\mathcal{K}_{\bar{\varrho}}(z)}{z^{\bar{\varrho}}}\right)'$ with $\bar{\varrho} \in \mathcal{R}$. Consider the integral representation [40]:*

$$\mathcal{K}_{\bar{\varrho}}(z) = \int_0^\infty e^{-z\cosh\xi}\cosh(\bar{\varrho}\xi)d\xi, \quad z > 0.$$

*It is clear that $z \mapsto \mathcal{K}_{\bar{\varrho}}(z)$ is a completely monotonic function on an interval $(0, \infty)$ for all $\bar{\varrho} \in \mathcal{R}$. Since the product of two completely monotonic functions is also completely monotonic, so $z \mapsto \hbar_{\bar{\varrho}}(z)$ is a strictly completely monotonic function on the same interval for all $\bar{\varrho} > 1$. Therefore, the function*

$$\hbar_p(z) = -\left(\frac{\mathcal{K}_{\bar{\varrho}}(z)}{z^{\bar{\varrho}}}\right)' = \frac{\mathcal{K}_{\bar{\varrho}+1}(z)}{z^{\bar{\varrho}}} \tag{22}$$

*is strictly completely monotonic on an interval $(0, \infty)$ for all $\bar{\varrho} > 1$ and thus $\hbar_{\bar{\varrho}}$ is a convex function. Then, by making the use of Remark 2 and identity (2), we can deduce*

$$\left|\frac{\bowtie_3^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}}(\bowtie_4) - \bowtie_4^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}}(\bowtie_3)}{\bowtie_4 - \bowtie_3}\right| \leq \frac{\bowtie_4^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}(\bowtie_3) + \bowtie_3^{\bar{\varrho}}\mathcal{K}_{\bar{\varrho}+1}(\bowtie_4)}{2}$$

*for all $\bar{\varrho} > 1$ and $\bowtie_3, \bowtie_4 \in \mathcal{R}$ with $0 < \bowtie_3 < \bowtie_4$.*

**Example 3.** *Consider the q-digamma function $\Psi_\varrho$, defined by*

$$\Psi_\varrho(z) = -\ln(1-\varrho) + \ln(\varrho)\sum_{\ell=0}^\infty \frac{\varrho^{\ell+z}}{1-\varrho^{\ell+z}}$$

$$= -\ln(1-\varrho) + \ln(\varrho)\sum_{\ell=1}^\infty \frac{\varrho^{\ell z}}{1-\varrho^{\ell z}}$$

*for $0 < \varrho < 1$, and*

$$\Psi_\varrho(z) = -\ln(\varrho - 1) + \ln(\varrho) \left( z - \frac{1}{2} - \sum_{\ell=0}^{\infty} \frac{\varrho^{-(\ell+z)}}{1 - \varrho^{-(\ell+z)}} \right)$$

$$= -\ln(\varrho - 1) + \ln(\varrho) \left( z - \frac{1}{2} - \sum_{\ell=1}^{\infty} \frac{\varrho^{-\ell z}}{1 - \varrho^{-\ell z}} \right)$$

*for $\varrho > 1$ and $z > 0$.*

*From this definitions, we see that $z \mapsto \Psi'_\varrho(z)$ is a completely monotonic function on an interval $(0, \infty)$ for all $\varrho > 0$, and consequently, $z \mapsto \Psi'_\varrho(z)$ is convex on the same interval.*

*Let $\hbar_\varrho(z) := \Psi'_\varrho(z)$ with $\varrho > 0$. Thus, $\hbar'_\varrho(z) := \Psi''_\varrho(z)$ is completely monotonic on the interval $(0, \infty)$. Then, from Remark 2, we have*

$$\Psi'_\varrho \left( \frac{\bowtie_3 + \bowtie_4}{2} \right) \leq \left| \frac{\Psi_\varrho(\bowtie_4) - \Psi_\varrho(\bowtie_3)}{\bowtie_4 - \bowtie_3} \right| \leq \frac{\Psi'_\varrho(\bowtie_3) + \Psi'_\varrho(\bowtie_4)}{2}. \tag{23}$$

*Combining the inequalities (3) and (23), and (7) and (23), we get*

$$\left| \frac{\Psi'_\varrho(\bowtie_3) + \Psi'_\varrho(\bowtie_4)}{2} - \frac{\Psi_\varrho(\bowtie_4) - \Psi_\varrho(\bowtie_3)}{\bowtie_4 - \bowtie_3} \right| \leq \frac{\bowtie_4 - \bowtie_3}{8} \left( \left| \Psi''_\varrho(\bowtie_3) \right| + \left| \Psi''_\varrho(\bowtie_4) \right| \right),$$

*and*

$$\left| \frac{\Psi_\varrho(\bowtie_4) - \Psi_\varrho(\bowtie_3)}{\bowtie_4 - \bowtie_3} - \Psi'_\varrho \left( \frac{\bowtie_3 + \bowtie_4}{2} \right) \right| \leq \frac{\bowtie_4 - \bowtie_3}{8} \left( \left| \Psi''_\varrho(\bowtie_3) \right| + \left| \Psi''_\varrho(\bowtie_4) \right| \right),$$

*respectively.*

## 4. Conclusions

In this article, we introduced an extension of the well known incomplete gamma function, namely the $\lambda$-incomplete gamma function to connect with the model of tempered fractional integrals. In view of this, we considered the integral inequalities of Hermite–Hadamard type in the context of tempered fractional integrals. Integral inequalities form a crucial branch of analysis and were combined with various types of fractional integrals but we had never seen this before with tempered fractional integrals. For this reason, we studied the inequality of Hermite–Hadamard type and related inequalities via the tempered fractional integrals which generalized the previous results obtained in [2,3,5,6].

## Appendix A

By using the properties of modulus and the convexity of $|\hbar'|$, we can deduce:

$$\int_0^1 \left| \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right| \left| \hbar'(\bar{\chi}\rtimes_3 + (1 - \bar{\chi})\rtimes_4) \right| d\bar{\chi}$$

$$\leq \int_0^{\frac{1}{2}} \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right] \left[ \bar{\chi} \left| \hbar'(\rtimes_3) \right| + (1 - \bar{\chi}) \left| \hbar'(\rtimes_4) \right| \right] d\bar{\chi}$$

$$+ \int_{\frac{1}{2}}^1 \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) \right] \left[ \bar{\chi} \left| \hbar'(\rtimes_3) \right| + (1 - \bar{\chi}) \left| \hbar'(\rtimes_4) \right| \right] d\bar{\chi}$$

$$= \left| \hbar'(\rtimes_3) \right| \left\{ \int_0^{\frac{1}{2}} \bar{\chi} \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right] d\bar{\chi} \right.$$

$$+ \int_{\frac{1}{2}}^1 \bar{\chi} \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) \right] d\bar{\chi} \right\}$$

$$+ \left| \hbar'(\rtimes_4) \right| \left\{ \int_0^{\frac{1}{2}} (1 - \bar{\chi}) \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right] d\bar{\chi} \right.$$

$$+ \int_{\frac{1}{2}}^1 (1 - \bar{\chi}) \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, 1 - \bar{\chi}) \right] d\bar{\chi} \right\}$$

$$= \left[ \left| \hbar'(\rtimes_3) \right| + \left| \hbar'(\rtimes_4) \right| \right] \int_0^{\frac{1}{2}} \left[ \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, (1 - \bar{\chi})) - \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \right] d\bar{\chi}$$

$$= \frac{L_1^{(\nu, \lambda)}(\rtimes_3, \rtimes_4)}{\rtimes_4 - \rtimes_3} \left[ \left| \hbar'(\rtimes_3) \right| + \left| \hbar'(\rtimes_4) \right| \right].$$

## Appendix B

By changing the order of the integration (just like Remark 1 (ii)), we have

$$\int_0^1 \frac{\bar{\chi}}{2} \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \, d\bar{\chi} = \int_0^1 \frac{\bar{\chi}}{2} \left( \int_0^{\bar{\chi}} y^{\nu-1} e^{-\lambda(\rtimes_4 - \rtimes_3)y} dy \right) d\bar{\chi}$$

$$= \int_0^1 y^{\nu-1} e^{-\lambda(\rtimes_4 - \rtimes_3)y} \left( \int_y^1 \frac{\bar{\chi}}{2} d\bar{\chi} \right) dy = \frac{1}{4} \int_0^1 \left( 1 - y^2 \right) y^{\nu-1} e^{-\lambda(\rtimes_4 - \rtimes_3)y} dy$$

$$= \frac{1}{4} \left( \frac{\gamma_\lambda(\nu, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^\nu} - \frac{\gamma_\lambda(\nu + 1, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^{\nu+1}} \right) = \frac{\mathcal{B}(\nu, \lambda)}{4},$$

and analogously,

$$\int_0^1 \frac{2 - \bar{\chi}}{2} \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \, d\bar{\chi} = \int_0^1 \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \, d\bar{\chi} - \int_0^1 \frac{\bar{\chi}}{2} \gamma_{\lambda(\rtimes_4 - \rtimes_3)}(\nu, \bar{\chi}) \, d\bar{\chi}$$

$$= \mathcal{A}(\nu, \lambda) - \mathcal{B}(\nu, \lambda)$$

$$= \frac{1}{4} \left( \frac{\gamma_\lambda(\nu + 2, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^{\nu+2}} - \frac{4\gamma_\lambda(\nu + 1, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^{\nu+1}} + \frac{3\gamma_\lambda(\nu, \rtimes_4 - \rtimes_3)}{(\rtimes_4 - \rtimes_3)^\nu} \right)$$

$$= \frac{\mathcal{C}(\nu, \lambda)}{4}.$$

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