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Existence and uniqueness of positive solutions for a new class of coupled system via fractional derivatives

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Abstract

In this paper we study the existence of unique positive solutions for the following coupled system:

$$\begin{cases} D^{\alpha}_{0+}x(\tau)+f_1(\tau,x(\tau),D^{\eta}_{0+}x(\tau))+g_1(\tau,y(\tau))=0,\\ D^{\beta}_{0+}y(\tau)+f_2(\tau,y(\tau),D^{\gamma}_{0+}y(\tau))+g_2(\tau,x(\tau))=0,\\ \tau\in(0,1),\qquad n-1<\alpha,\beta<n;\\ x^{(i)}(0)=y^{(i)}(0)=0,\quad i=0,1,2,\ldots,n-2;\\ [D^{\xi}_{0+}y(\tau)]_{\tau=1}=k_1(y(1)),\qquad [D^{\zeta}_{0+}x(\tau)]_{\tau=1}=k_2(x(1)), \end{cases}$$

where the integer number $n>3$ and $1\leq\gamma\leq\xi\leq n-2$, $1\leq\eta\leq\zeta\leq n-2$, $f_1,f_2:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, $g_1,g_2:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ and $k_1,k_2:\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions, D^{α}_{0+} and D^{β}_{0+} stand for the Riemann–Liouville derivatives. An illustrative example is given to show the effectiveness of theoretical results.

Keywords: Fractional differential equation; Mixed monotone operator; Normal cone; Coupled system

1 Introduction

A lot of fractional differential equations and coupled systems have been studied widely; see [1–19, 24] and the references therein. As is well known, coupled systems with boundary conditions appear in the investigations of many problems such as mathematical biology (see [9, 30]), natural sciences and engineering; for example, we can see beam deformation and steady-state heat flow (see [25, 26]) and heat equations (see [18, 24]). So the subject of coupled systems is gaining much attention and importance. There are a large number of articles dealing with the existence or multiplicity of solutions or positive solutions for some nonlinear coupled systems with boundary conditions; for details, see [7, 8, 10, 11, 20, 21, 27, 29, 32, 33, 35–41].

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In [42] Zhang and Tian considered a unique positive solution for the following problem:

$$\begin{cases} D^{\alpha}_{0^+}w(\tau)+f(\tau,w(\tau),D^{\gamma}_{0^+}w(\tau))+g(\tau,w(\tau))=0, \quad \tau\in(0,1), n-1<\alpha<n;\\ w^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2;\\ [D^{\beta}_{0^+}w(\tau)]_{\tau=1}=k(w(1)), \end{cases} \tag{1}$$

where $n>3$, $1\leq\gamma\leq\beta\leq n-2$, $f:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, $g:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ and $k:\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions, $D^{\alpha}_{0^+}$ is the Riemann–Liouville fractional derivative and $w^{(i)}$ represents the ith (ordinary) derivative of w.

Continuing their work, we establish the existence of solutions for the following coupled system:

$$\begin{cases} D^{\alpha}_{0^+}x(\tau)+f_1(\tau,x(\tau),D^{\eta}_{0^+}x(\tau))+g_1(\tau,y(\tau))=0,\\ D^{\beta}_{0^+}y(\tau)+f_2(\tau,y(\tau),D^{\gamma}_{0^+}y(\tau))+g_2(\tau,x(\tau))=0, \quad \tau\in(0,1), n-1<\alpha,\beta<n;\\ x^{(i)}(0)=y^{(i)}(0)=0, \quad i=0,1,2,\dots,n-2;\\ [D^{\xi}_{0^+}y(\tau)]_{\tau=1}=k_1(y(1)), \qquad [D^{\zeta}_{0^+}x(\tau)]_{\tau=1}=k_2(x(1)), \end{cases} \tag{2}$$

where the integer number $n>3$ and $1\leq\gamma\leq\xi\leq n-2$, $1\leq\eta\leq\zeta\leq n-2$, $f_1,f_2:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, $g_1,g_2:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ and $k_1,k_2:\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions, $D^{\alpha}_{0^+}$ and $D^{\beta}_{0^+}$ stand for the Riemann–Liouville derivatives.

2 Preliminaries

Suppose $(E,\|\cdot\|)$ is a Banach space which is partially ordered by a cone $P\subseteq E$. We denote the zero element of E by θ. A cone P is called normal if there exists a constant $N>0$ such that $\theta\leq x\leq y$ implies $\|x\|\leq N\|y\|$.

Definition 2.1 ([22, 23]) $A:P\times P\to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, i.e., for $x_i,y_i\in P$ $(i=1,2)$, $x_1\leq x_2$, $y_1\geq y_2$ imply $A(x_1,y_1)\leq A(x_2,y_2)$. The element $x\in P$ is called a fixed point of A if $A(x,x)=x$.

An element $u^*\in D$ is called a fixed point of A if it satisfies $A(u^*,u^*)=u^*$. Let $h>\theta$, write $P_h=\{u\in E|\exists\lambda,\mu>0:\lambda h\leq u\leq\mu h\}$.

Let Φ be a class of functions $\varphi:(0,1)\to(0,1)$ with $\varphi(\tau)>\tau$ for $\tau\in(0,1)$.

Theorem 2.2 ([34]) *Let P be a normal cone in E, $\alpha\in(0,1)$ $A:P\to P$ is an increasing sub-homogeneous, $B:P\to P$ is a decreasing operator, $C:P\times P\to P$ is a mixed monotone operator and that satisfy the following conditions*:

$$B\left(\frac{1}{\tau}u\right)\geq\tau Bu, \qquad C\left(\tau u,\frac{1}{\tau}v\right)\geq\tau^{\alpha}C(u,v), \quad u,v\in P. \tag{3}$$

Assume that

(*i*) $\exists\, h_0\in P_h$ *such that* $Ah_0\in P_h$, $Bh_0\in P_h$, $C(h_0,h_0)\in P_h$;

(*ii*) $\exists\,\delta_0>0$ *with* $C(u,v)\geq\delta_0(Au+Bv)$ *for* $u,v\in P$.

Then

(1) $A:P_h\to P_h$, $B:P_h\to P_h$ *and* $C:P_h\times P_h\to P_h$;

(2) $\exists\, x_0, y_0 \in P_h$ *and* $r \in (0,1)$ *with*

$$rx_0 \le x_0 < y_0, x_0 \le Ax_0 + By_0 + C(x_0, y_0) \le Ay_0 + Bx_0 + C(y_0, x_0) \le y_0;$$

(3) *the equation* $Au + Bu + C(u,u) = u$ *has a unique solution* u^* *in* P_h;

(4) *for* $x_0, y_0 \in P_h$, *we can construct*

$$\begin{aligned} u_n &= Ax_{n-1} + By_{n-1} + C(x_{n-1}, y_{n-1}), \\ v_n &= Ay_{n-1} + Bx_{n-1} + C(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots \end{aligned}$$

and $u_n \to u^*$ *and* $v_n \to v^*$.

Definition 2.3 ([28, 31]) The Riemann–Liouville fractional derivative for a continuous function f is defined by

$$D^\alpha f(\tau) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{d\tau}\right)^n \int_0^\tau \frac{f(\rho)}{(t-\rho)^{\alpha-n+1}}\, d\rho \quad \bigl(n = [\alpha] + 1\bigr),$$

where the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.4 ([28, 31]) Let $[a, b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann–Liouville fractional order integral of a function $f \in L^1([a,b], \mathbb{R})$ is defined by

$$I_a^\alpha f(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau \frac{f(\rho)}{(\tau-\rho)^{1-\alpha}}\, d\rho,$$

whenever the integral exists.

Lemma 2.5 ([42]) *Let* $h \in C[0,1]$, *then the unique solution of the linear problem*

$$D_{0^+}^\alpha x(\tau) + h(\tau) = 0, \quad \tau \in (0,1), n-1 < \alpha \le n; \tag{4}$$

$$x^i(0) = 0, \quad i = 0, 1, 2, 3, \ldots, n-2; \tag{5}$$

$$\bigl[D_{0^+}^\beta x(\tau)\bigr]_{\tau=1} = k\bigl(x(1)\bigr), \quad 1 \le \beta \le n-2; \tag{6}$$

is given by

$$x(\tau) = \int_0^1 G(\tau,\rho) h(\rho)\, d\rho + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} k\bigl(x(1)\bigr) \tau^{\alpha-1},$$

where

$$G(\tau,\rho) = \begin{cases} \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\beta-1} - (\tau-\rho)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le \rho \le \tau \le 1; \\ \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le \tau \le \rho \le 1; \end{cases} \tag{7}$$

is the Green function.

Lemma 2.6 ([42]) *The Green function* (7) *has the following properties*:

$$0 \le \tau^{\alpha-1}(1-\rho)^{\alpha-\beta-1}\bigl[1-(1-\rho)^{\beta}\bigr] \le \Gamma(\alpha)G(\tau,\rho) \le \tau^{\alpha-1}(1-\rho)^{\alpha-\beta-1},$$

$$0 \le \tau^{\alpha-\gamma-1}(1-\rho)^{\alpha-\beta-1}\bigl[1-(1-\rho)^{\beta-\gamma}\bigr] \le \Gamma(\alpha-\gamma)D_{0^+}^{\gamma}G(\tau,\rho) \le \tau^{\alpha-\gamma-1}(1-\rho)^{\alpha-\beta-1}, \quad \tau,\rho \in [0,1].$$

Lemma 2.7 ([36]) $K_h = P_{h_1} \times P_{h_2}$, *where that* $K = P \times P$ *and* $h(\tau) = (h_1, h_2)$.

3 Main results

Let $E \times E \subset X \times X$ with $X = C[0,1]$ such that $E = \{x | x, D_{0^+}^{\eta}x, D_{0^+}^{\gamma}x \in X\}$ endowed with the norm $\|x\| = \max\{\max_{\tau\in[0,1]}|x(\tau)|, \max_{\tau\in[0,1]}D_{0^+}^{\eta}|x(\tau)|, \max_{\tau\in[0,1]}D_{0^+}^{\gamma}|x(\tau)|\}$. For $(x,y) \in E \times E$, let $\|(x,y)\| = \max\{\|x\|,\|y\|\}$. It is easy to see that $(E \times E, \|(x,y)\|)$ is a Banach space. Define $P = \{x \in E : x, D_{0^+}^{\eta}x, D_{0^+}^{\gamma}x \ge 0\}$, $K = P \times P$, then K is a normal cone equipped with the following partial order:

$$(x_1,y_1) \preceq (x_2,y_2) \quad \Leftrightarrow \quad x_1 \le x_2, y_1 \le y_2, \tag{8}$$

and

$$D_{0^+}^{\eta}x_1(\tau) \le D_{0^+}^{\eta}x_2(\tau), \qquad D_{0^+}^{\gamma}x_1(\tau) \le D_{0^+}^{\gamma}x_2(\tau),$$
$$D_{0^+}^{\eta}y_1(\tau) \le D_{0^+}^{\eta}y_2(\tau), \qquad D_{0^+}^{\gamma}y_1(\tau) \le D_{0^+}^{\gamma}y_2(\tau).$$

By Lemma 2.5 in [42], the unique positive solution for the problem (1) is given by

$$x(\tau) = \int_0^1 G(\tau,\rho)f\bigl(\rho,x(\rho),D_{0^+}^{\gamma}x(\rho)\bigr)\,d\rho + \int_0^1 G(\tau,\rho)g\bigl(\rho,x(\rho)\bigr)\,d\rho + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}k\bigl(x(1)\bigr)\tau^{\alpha-1},$$

where

$$G(\tau,\rho) = \begin{cases} \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\beta-1}-(\tau-\rho)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le \rho \le \tau \le 1; \\ \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \le \tau \le \rho \le 1, \end{cases} \tag{9}$$

is a Green function.

Assume that $f_1(\tau,x,y), f_2(\tau,x,y)$ are continuous, then $(x,y) \in X \times X$ is a solution of the system (2) if and only if (x,y) is a solution of the integral equations

$$\begin{cases} x(\tau) = \int_0^1 G_1(\tau,\rho)f_1(\rho,x(\rho),D_{0^+}^{\eta}x(\rho))\,d\rho \\ \qquad + \int_0^1 G_1(\tau,\rho)g_1(\rho,y(\rho))\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2(x(1))\tau^{\alpha-1}, \\ y(\tau) = \int_0^1 G_2(\tau,\rho)f_2(\rho,y(\rho),D_{0^+}^{\gamma}y(\rho))\,d\rho \\ \qquad + \int_0^1 G_2(\tau,\rho)g_2(\rho,x(\rho))\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1(y(1))\tau^{\beta-1}, \end{cases} \tag{10}$$

where

$$G_1(\tau,\rho)=\begin{cases}\frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\zeta-1}-(\tau-\rho)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le\rho\le\tau\le 1;\\ \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)}, & 0\le\tau\le\rho\le 1,\end{cases} \tag{11}$$

and

$$G_2(\tau,\rho)=\begin{cases}\frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}-(\tau-\rho)^{\beta-1}}{\Gamma(\beta)}, & 0\le\rho\le\tau\le 1;\\ \frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)}, & 0\le\tau\le\rho\le 1,\end{cases} \tag{12}$$

are Green functions.

Let us define the operators A_1, B_1, C_1, A_2, B_2, C_2 by

$$\begin{aligned}
&A_1(u)(\tau)=\int_0^1 G_1(\tau,\rho)g_1\bigl(\rho,v(\rho)\bigr)\,d\rho, \qquad A_2(v)(\tau)=\int_0^1 G_2(\tau,\rho)g_2\bigl(\rho,u(\rho)\bigr)\,d\rho,\\
&B_1(u)(\tau)=\frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\bigl(u(1)\bigr)\tau^{\alpha-1}, \qquad B_2(v)(\tau)=\frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\bigl(v(1)\bigr)\tau^{\beta-1},\\
&C_1(v,u)(\tau)=\int_0^1 G_1(\tau,\rho)f_1\bigl(\rho,v(\rho),D_{0^+}^{\eta}u(\rho)\bigr)\,d\rho,\\
&C_2(u,v)(\tau)=\int_0^1 G_2(\tau,\rho)f_2\bigl(\rho,u(\rho),D_{0^+}^{\gamma}v(\rho)\bigr)\,d\rho,
\end{aligned} \tag{13}$$

for $0\le\tau\le 1$.

Theorem 3.1 *Assume that*

(H_1) $f_1,f_2:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, $g_1,g_2:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ *and* $k_1,k_2:\mathbb{R}^+\to\mathbb{R}^+$ *are continuous, also*: $f_1(\tau,1,0)\not\equiv 0, f_2(\tau,1,0)\not\equiv 0$;

(H_2) $f_1(\tau,x,y)$ *and* $f_2(\tau,x,y)$ *are increasing respect to* $x\in\mathbb{R}^+$, *decreasing respect to* $y\in\mathbb{R}^+$, g_1, g_2 *are increasing respect to* y *for fixed* $0\le\tau\le 1$ *and* k_1, k_2 *are decreasing with* $k_1(y(1)),k_2(x(1))\neq 0$;

(H_3) $\exists\,\alpha_1,\alpha_2\in(0,1)$ *such that*

$$f_1\bigl(\tau,\lambda x,\lambda^{-1}y\bigr)\ge\lambda^{\alpha_1}f_1(\tau,x,y), \qquad f_2\bigl(\tau,\lambda x,\lambda^{-1}y\bigr)\ge\lambda^{\alpha_2}f_2(\tau,x,y), \tag{14}$$

and g_1, g_2, k_1, k_2 *satisfy*

$$g_i(\tau,\lambda x)\ge\lambda g_i(\tau,x), \qquad k_i\bigl(\lambda^{-1}x\bigr)\ge\lambda k_i(x), \quad i=1,2, \tag{15}$$

for $\lambda\in(0,1)$, $0\le\tau\le 1$, $x\in\mathbb{R}^+$;

(H_4) $g_i(\tau,0)\not\equiv 0$ *and there exist positive constants* δ_{11}, δ_{12}, δ_{21} *and* δ_{22} *such that*

$$\begin{aligned}
&f_i(\tau,x,y)\ge\delta_{i1}g_i(\tau,x),\\
&f_i(\tau,x,y)\ge\delta_{i2}\ge k_i(y), \quad (i=1,2), 0\le\tau\le 1, x,y\in\mathbb{R}^+.
\end{aligned}$$

Then

(1) $\exists\ (u_{01},u_{02}),(v_{01},v_{02})\in K\subset E\times E$ *and* $r\in(0,1)$ *such that*

$$r(v_{01},v_{02})\preceq(u_{01},u_{02})\prec(v_{01},v_{02}),$$

that is,

$$\begin{aligned}
&r(v_{01},v_{02})\leq(u_{01},u_{02})<(v_{01},v_{02}),\\
&r\big(D^{\gamma}_{0^+}v_{01},D^{\gamma}_{0^+}v_{02}\big)\leq\big(D^{\eta}_{0^+}u_{01},D^{\eta}_{0^+}u_{02}\big)<\big(D^{\gamma}_{0^+}v_{01},D^{\gamma}_{0^+}v_{02}\big),\\
&(u_{01},u_{02})\leq\bigg(\int_0^1 G_1(\tau,\rho)f_1\big(\rho,u_{01}(\rho),D^{\eta}_{0^+}u_{01}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 G_1(\tau,\rho)g_1\big(\rho,v_{01}(\rho)\big)\,d\rho+\frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\big(u_{01}(1)\big)\tau^{\alpha-1},\\
&\qquad\int_0^1 G_1(\tau,\rho)f_1\big(\rho,u_{02}(\rho),D^{\eta}_{0^+}u_{02}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 G_1(\tau,\rho)g_1\big(\rho,v_{02}(\rho)\big)\,d\rho+\frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\big(u_{02}(1)\big)\tau^{\alpha-1}\bigg),\\
&D^{\eta}_{0^+}\big((u_{01},u_{02})\big)\leq\bigg(\int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)f_1\big(\rho,u_{01}(\rho),D^{\eta}_{0^+}u_{01}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)g_1\big(\rho,v_{01}(\rho)\big)\,d\rho+\frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\big(u_{01}(1)\big)\tau^{\alpha-1},\\
&\qquad\int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)f_1\big(\rho,u_{02}(\rho),D^{\eta}_{0^+}u_{02}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)g_1\big(\rho,v_{02}(\rho)\big)\,d\rho+\frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\big(u_{02}(1)\big)\tau^{\alpha-1}\bigg),\\
&(v_{01},v_{02})\geq\bigg(\int_0^1 G_2(\tau,\rho)f_2\big(\rho,v_{01}(\rho),D^{\gamma}_{0^+}v_{01}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 G_2(\tau,\rho)g_2\big(\rho,u_{01}(\rho)\big)\,d\rho+\frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{01}(1)\big)\tau^{\beta-1},\\
&\qquad\int_0^1 G_2(\tau,\rho)f_2\big(\rho,v_{02}(\rho),D^{\gamma}_{0^+}v_{02}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 G_2(\tau,\rho)g_2\big(\rho,u_{02}(\rho)\big)\,d\rho+\frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{02}(1)\big)\tau^{\beta-1}\bigg),\\
&D^{\gamma}_{0^+}(v_{01},v_{02})\geq\bigg(\int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)f_2\big(\rho,v_{01}(\rho),D^{\gamma}_{0^+}v_{01}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)g_2\big(\rho,u_{01}(\rho)\big)\,d\rho+\frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{01}(1)\big)\tau^{\beta-1},\\
&\qquad\int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)f_2\big(\rho,v_{02}(\rho),D^{\gamma}_{0^+}v_{02}(\rho)\big)\,d\rho\\
&\qquad+\int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)g_2\big(\rho,u_{02}(\rho)\big)\,d\rho+\frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{02}(1)\big)\tau^{\beta-1}\bigg),
\end{aligned}$$

where $G_1(\tau,\rho)$, $G_2(\tau,\rho)$ *are defined by* (11) *and* (12), *respectively.*

(2) *The problem* (2) *has a unique positive solution* (u^*, v^*) *in* K_h, *with* $h(\tau) = (h_1(\tau), h_2(\tau)) = (\tau^{\alpha-1}, \tau^{\beta-1})$, $0 \leq \tau \leq 1$.

(3) *For* $(x_{01}, x_{02}), (y_{01}, y_{02}) \in P_h \times P_h$, *there are two iterative sequences* $\{(x_{n1}, x_{n2})\}$, $\{(y_{n1}, y_{n2})\}$ *for approximating* (x^*, y^*), *that is*, $(x_{n1}, x_{n2}) \to (x^*, y^*)$, $(y_{n1}, y_{n2}) \to (x^*, y^*)$, *where*

$$\begin{aligned}
\big(x_{n1}(\tau), x_{n2}(\tau)\big) = {}& \bigg(\int_0^1 G_1(\tau,\rho) f_1\big(\rho, x_{(n-1)1}(\rho), D_{0^+}^{\eta} x_{(n-1)1}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_1(\tau,\rho) g_1\big(\rho, y_{(n-1)1}(\rho)\big)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)} k_2\big(x_{(n-1)1}(1)\big)\tau^{\alpha-1}, \\
&\int_0^1 G_1(\tau,\rho) f_1\big(\rho, x_{(n-1)2}(\rho), D_{0^+}^{\eta} x_{(n-1)2}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_1(\tau,\rho) g_1\big(\rho, y_{(n-1)2}(\rho)\big)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)} k_2\big(x_{(n-1)2}(1)\big)\tau^{\alpha-1}\bigg), \\
\big(y_{n1}(\tau), y_{n2}(\tau)\big) = {}& \bigg(\int_0^1 G_2(\tau,\rho) f_2\big(\rho, y_{(n-1)1}(\rho), D_{0^+}^{\gamma} y_{(n-1)1}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_2(\tau,\rho) g_2\big(\rho, x_{(n-1)1}(\rho)\big)\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)} k_1\big(y_{(n-1)1}(1)\big)\tau^{\beta-1}, \\
&\int_0^1 G_2(\tau,\rho) f_2\big(\rho, y_{(n-1)2}(\rho), D_{0^+}^{\gamma} y_{(n-1)2}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_2(\tau,\rho) g_2\big(\rho, x_{(n-1)2}(\rho)\big)\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)} k_1\big(y_{(n-1)2}(1)\big)\tau^{\beta-1}\bigg), \\
& n = 1, 2, \ldots.
\end{aligned}$$

Proof By Lemma 2.6 we have

$$G_1(\tau,\rho), G_2(\tau,\rho), D_{0^+}^{\eta} G_1(\tau,\rho), D_{0^+}^{\gamma} G_1(\tau,\rho), D_{0^+}^{\eta} G_2(\tau,\rho), D_{0^+}^{\gamma} G_2(\tau,\rho) \geq 0. \tag{16}$$

Regarding (16) and (H_1) in (13) we get $A_1, A_2, B_1, B_2 : P \to P$ and $C_1, C_2 : P \times P \to P \times P$.

Obviously A_1, A_2 are increasing and sub-homogeneous, Because g_1, g_2 are increasing and sub-homogeneous. B_1, B_2 are decreasing (due to this fact, k_1 and k_2 are decreasing) and satisfy in conditions $B_i(\lambda^{-1}x) \geq \lambda B_i(x)$, $i = 1, 2$, by (15). For any $(u_1, v_1), (u_2, v_2) \in K$ with $(u_1, v_1) \preceq (u_2, v_2)$, considering that $f_1(\tau, x, y)$ and $f_2(\tau, x, y)$ are increasing in x and decreasing in y, we have

$$\begin{aligned}
&C_1(v_1, u_1) \leq C_1(v_2, u_1) \quad \text{for fixed } u_1 \quad \text{and} \quad C_1(v_1, u_1) \geq C_1(v_1, u_2) \quad \text{for fixed } v_1, \\
&C_2(u_1, v_1) \leq C_2(u_2, v_1) \quad \text{for fixed } v_1 \quad \text{and} \quad C_2(u_1, v_1) \geq C_2(u_1, v_2) \quad \text{for fixed } u_1,
\end{aligned}$$

also

$$C_1\big(\tau, \lambda x, \lambda^{-1} y\big) \geq \lambda^{\alpha_1} C_1(\tau, x, y), \qquad C_2\big(\tau, \lambda x, \lambda^{-1} y\big) \geq \lambda^{\alpha_2} C_2(\tau, x, y).$$

Set $A = (A_1, A_2) : K \to K$, $B = (B_1, B_2) : K \to K$, $C = (C_1, C_2) : K \times K \to K$. Then A, B, C satisfy Eq. (3) of Theorem 2.2, with replacing the cone K for the cone P.

From Lemma 2.7, we get $K_h = P_{h_1} \times P_{h_2}$, where $h(\tau) = (h_1(\tau), h_2(\tau)) = (\tau^{\alpha-1}, \tau^{\beta-1})$, also by condition (*i*) of Theorem 2.2, we need prove $A_{1_{h_1}}, B_{1_{h_1}} \in P_{h_1}$, $A_{2_{h_2}}, B_{2_{h_2}} \in P_{h_2}$ and $C_1(h_1, h_1) \in P_{h_1}$, $C_2(h_2, h_2) \in P_{h_2}$.

Indeed

$$\begin{aligned} A_{1_{h_1}}(\tau) &= \int_0^1 G_1(\tau,\rho) g_1\bigl(\rho, h_2(\rho)\bigr)\,d\rho = \int_0^1 G_1(\tau,\rho) g_1\bigl(\rho, \rho^{\alpha-1}\bigr)\,d\rho \\ &\geq \tau^{\alpha-1}\int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta}]}{\Gamma(\alpha)} g_1(\rho,0)\,d\rho > 0, \\ A_{1_{h_1}}(\tau) &= \int_0^1 G_1(\tau,\rho) g_1\bigl(\rho, h_2(\rho)\bigr)\,d\rho = \int_0^1 G_1(\tau,\rho) g_1\bigl(\rho, \rho^{\alpha-1}\bigr)\,d\rho \\ &\leq \tau^{\alpha-1}\int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)} g_1(\rho,1)\,d\rho. \end{aligned}$$

Let

$$\begin{aligned} a_{11} &:= \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta}]}{\Gamma(\alpha)} g_1(\rho,0)\,d\rho > 0. \\ a_{12} &:= \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)} g_1(\rho,1)\,d\rho. \end{aligned}$$

Then $a_{12} \geq a_{11} > 0$ and thus

$$a_{11} h(\tau) \leq A_{1_{h_1}}(\tau) \leq a_{12} h(\tau), \quad 0 \leq \tau \leq 1. \tag{17}$$

Also,

$$\begin{aligned} A_{2_{h_2}}(\tau) &= \int_0^1 G_1(\tau,\rho) g_2\bigl(\rho, h_1(\rho)\bigr)\,d\rho = \int_0^1 G_1(\tau,\rho) g_2\bigl(\rho, \rho^{\beta-1}\bigr)\,d\rho \\ &\geq \tau^{\beta-1}\int_0^1 \frac{(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi}]}{\Gamma(\beta)} g_2(\rho,0)\,d\rho > 0, \\ A_{2_{h_2}}(\tau) &= \int_0^1 G_1(\tau,\rho) g_2\bigl(\rho, h_1(\rho)\bigr)\,d\rho = \int_0^1 G_1(\tau,\rho) g_2\bigl(\rho, \rho^{\beta-1}\bigr)\,d\rho \\ &\leq \tau^{\beta-1}\int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)} g_2(\rho,1)\,d\rho. \end{aligned}$$

Let

$$\begin{aligned} a_{21} &:= \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi}]}{\Gamma(\beta)} g_1(\rho,0)\,d\rho > 0, \\ a_{22} &:= \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)} g_1(\rho,1)\,d\rho. \end{aligned}$$

Then $a_{22} \geq a_{21} > 0$ and thus

$$\begin{aligned} &a_{21} h(\tau) \leq A_{1_{h_1}}(\tau) \leq a_{22} h(\tau), \quad 0 \leq \tau \leq 1, \\ &B_1(u)(\tau) = \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)} k_2\bigl(u(1)\bigr)\tau^{\alpha-1}, \qquad B_2(v)(\tau) = \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)} k_1\bigl(v(1)\bigr)\tau^{\beta-1}, \end{aligned} \tag{18}$$

therefore

$$B_1(h_1)(\tau) = \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2(1)\tau^{\alpha-1}, \qquad B_2(h_2)(\tau) = \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1(1)\tau^{\beta-1}.$$

From $k_2(u(1)) \not\equiv 0$ and $k_1(v(1)) \not\equiv 0$ we get $B_{1_{h_1}} \in P_{h_1}$, $B_{2_{h_2}} \in P_{h_2}$. We have

$$\begin{aligned}
C_1(h_1,h_1)(\tau) &= \int_0^1 G_1(\tau,\rho)f_1\bigl(\rho,h(\rho),D_{0^+}^{\eta}h(\rho)\bigr)\,d\rho \\
&\le \int_0^1 \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)}f_1\left(\rho,\rho^{\alpha-1},\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\tau^{\alpha-\eta-1}\right)d\rho \\
&\le \tau^{\alpha-1}\int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)}f_1(\rho,1,0)\,d\rho, \\
C_1(h_1,h_1)(\tau) &= \int_0^1 G_1(\tau,\rho)f_1\bigl(\rho,h(\rho),D_{0^+}^{\eta}h(\rho)\bigr)\,d\rho \\
&\ge \int_0^1 \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta}]}{\Gamma(\alpha)}f_1\left(\rho,\rho^{\alpha-1},\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}(\tau)^{\alpha-\eta-1}\right)d\rho \\
&\ge \tau^{\alpha-1}\int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta}]}{\Gamma(\alpha)}f_1\left(\rho,0,\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\right)d\rho, \\
C_2(h_2,h_2)(\tau) &= \int_0^1 G_2(\tau,\rho)f_2\bigl(\rho,u(\rho),D_{0^+}^{\gamma}v(\rho)\bigr)\,d\rho \\
&\le \int_0^1 \frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)}f_2\left(\rho,\rho^{\beta-1},\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\tau^{\beta-\gamma-1}\right)d\rho \\
&\le \tau^{\beta-1}\int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)}f_2(\rho,1,0)\,d\rho, \\
C_2(h_2,h_2)(\tau) &= \int_0^1 G_2(\tau,\rho)f_2\bigl(\rho,u(\rho),D_{0^+}^{\gamma}v(\rho)\bigr)\,d\rho \\
&\ge \int_0^1 \frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi}]}{\Gamma(\beta)}f_2\left(\rho,\rho^{\beta-1},\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}(\tau)^{\beta-\gamma-1}\right)d\rho \\
&\ge \tau^{\beta-1}\int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)}f_2\left(\rho,0,\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\right)d\rho.
\end{aligned}$$

We can calculate that

$$\begin{aligned}
&D_{0^+}^{\eta}A_1(u)(\tau) = \int_0^1 D_{0^+}^{\eta}G_1(\tau,\rho)g_1\bigl(\rho,v(\rho)\bigr)\,d\rho, \\
&D_{0^+}^{\gamma}A_2\bigl(v(\tau)\bigr) = \int_0^1 D_{0^+}^{\gamma}G_2(\tau,\rho)g_2\bigl(\rho,u(\rho)\bigr)\,d\rho, \\
&D_{0^+}^{\eta}B_1(u)(\tau) = \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha-\eta)}k_2\bigl(u(1)\bigr)\tau^{\alpha-\eta-1}, \qquad D_{0^+}^{\gamma}B_2(v)(\tau) = \frac{\Gamma(\beta-\xi)}{\Gamma(\beta-\gamma)}k_1\bigl(v(1)\bigr)\tau^{\beta-\gamma-1}, \\
&D_{0^+}^{\eta}C_1(v,u)(\tau) = \int_0^1 D_{0^+}^{\eta}G_1(\tau,\rho)f_1\bigl(\rho,v(\rho),D_{0^+}^{\eta}u(\rho)\bigr)\,d\rho, \\
&D_{0^+}^{\gamma}C_2(u,v)(\tau) = \int_0^1 D_{0^+}^{\gamma}G_2(\tau,\rho)f_2\bigl(\rho,u(\rho),D_{0^+}^{\gamma}v(\rho)\bigr)\,d\rho,
\end{aligned}$$

also

$$
\begin{aligned}
D_{0^+}^{\eta}A_1(h)(\tau) &= \int_0^1 D_{0^+}^{\eta}G_1(\tau,\rho)g_1\left(\rho,\rho^{\alpha-1}\right)d\rho \\
&\geq \int_0^1 \frac{\tau^{\alpha-\eta-1}(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta-\eta}]}{\Gamma(\alpha-\eta)}g_1\left(\rho,\rho^{\alpha-1}\right)d\rho \\
&\geq \tau^{\alpha-\eta-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta-\eta}]}{\Gamma(\alpha)}g_1(\rho,0)\,d\rho, \\
D_{0^+}^{\eta}A_1(h)(\tau) &= \int_0^1 D_{0^+}^{\eta}G_1(\tau,\rho)g_1\left(\rho,\rho^{\alpha-1}\right)d\rho \\
&\leq \int_0^1 \frac{\tau^{\alpha-\eta-1}(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha-\eta)}g_1\left(\rho,\rho^{\alpha-1}\right)d\rho \\
&\leq \tau^{\alpha-\eta-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)}g_1(\rho,1)\,d\rho.
\end{aligned}
$$

Set $a'_{11} = \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta-\eta}]}{\Gamma(\alpha)}g_1(\rho,0)\,d\rho$ and $a'_{12} = \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)}g_1(\rho,1)\,d\rho$, we have

$$
a'_{11}D_{0^+}^{\eta}h \leq D_{0^+}^{\eta}A_1(h) \leq a'_{12}D_{0^+}^{\eta}h
$$

and by (17) and (18) we have $a'_{11}h \leq A_1(h) \leq a'_{12}h$. So $\min\{a_{11},a'_{11}\}h \preceq A_1(h) \preceq \max\{a_{12}, a'_{12}\}h$. Hence $A_1(h) \in P_h$.

Again we have

$$
\begin{aligned}
D_{0^+}^{\gamma}A_2(h)(\tau) &= \int_0^1 D_{0^+}^{\gamma}G_1(\tau,\rho)g_1\left(\rho,\rho^{\beta-1}\right)d\rho \\
&\geq \int_0^1 \frac{\tau^{\beta-\gamma-1}(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi-\gamma}]}{\Gamma(\beta-\gamma)}g_1\left(\rho,\rho^{\beta-1}\right)d\rho \\
&\geq \tau^{\beta-\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\int_0^1 \frac{(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi-\gamma}]}{\Gamma(\beta)}g_1(\rho,0)\,d\rho, \\
D_{0^+}^{\gamma}A_2(h)(\tau) &= \int_0^1 D_{0^+}^{\gamma}G_1(\tau,\rho)g_1\left(\rho,\rho^{\beta-1}\right)d\rho \\
&\leq \int_0^1 \frac{\tau^{\beta-\gamma-1}(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta-\gamma)}g_1\left(\rho,\rho^{\beta-1}\right)d\rho \\
&\leq \tau^{\beta-\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)}g_1(\rho,1)\,d\rho.
\end{aligned}
$$

Similarly we set $a'_{21} = \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi-\eta}]}{\Gamma(\beta)}g_1(\rho,0)\,d\rho$ and $a'_{22} = \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)}g_1(\rho,1)\,d\rho$, we have

$$
a'_{21}D_{0^+}^{\gamma}h \leq D_{0^+}^{\gamma}A_2(h) \leq a'_{22}D_{0^+}^{\gamma}h
$$

and by (17) we have $a'_{21}h \leq A_2(h) \leq a'_{22}h$. So $\min\{a_{21},a'_{21}\}h \preceq A_2(h) \preceq \max\{a_{21},a'_{22}\}h$, hence $A_2(h) \in P_h$.

Furthermore,

$$
B_1(h_1) = \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2(1)\tau^{\alpha-1} = \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2(1)h_1(\tau),
$$

$$\begin{aligned}
D_{0^+}^{\eta} B_1(h_1) &= \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha-\eta)} k_2(1)\tau^{\alpha-\eta-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)} k_2(1)\tau^{\alpha-\eta-1} \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)} \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)} k_2(1) D_{0^+}^{\eta} h_1(\tau),
\end{aligned}$$

therefore

$$\begin{aligned}
B_2(h_2) &= \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)} k_1(1)\tau^{\beta-1} = \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)} k_1(1) h_2(\tau), \\
D_{0^+}^{\gamma} B_2(h_2) &= \frac{\Gamma(\beta-\xi)}{\Gamma(\beta-\gamma)} k_1(1)\tau^{\beta-\gamma-1} \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} k_1(1)\tau^{\beta-\gamma-1} \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)} \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)} k_1(1) D_{0^+}^{\gamma} h_2(\tau),
\end{aligned}$$

from $k_2(u(1)) \not\equiv 0$ and $k_1(v(1)) \not\equiv 0$ we get $B_{1_{h_1}} \in P_{h_1}$, $B_{2_{h_2}} \in P_{h_2}$.

$$\begin{aligned}
D_{0^+}^{\eta} C_1(h_1,h_1)(\tau) &= \int_0^1 D_{0^+}^{\eta} G_1(\tau,\rho) f_1\bigl(\rho,\rho^{\alpha-1}, D_{0^+}^{\eta}\rho^{\alpha-1}\bigr)\,d\rho \\
&\leq \int_0^1 \frac{\tau^{\alpha-\eta-1}(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha-\eta)} f_1\biggl(\rho,\rho^{\alpha-1}, \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\tau^{\alpha-\eta-1}\biggr)\,d\rho \\
&\leq \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\tau^{\alpha-\eta-1} \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)} f_1(\rho,1,0)\,d\rho, \\
D_{0^+}^{\eta} C_1(h_1,h_1)(\tau) &= \int_0^1 D_{0^+}^{\eta} G_1(\tau,\rho) f_1\bigl(\rho,\rho^{\alpha-1}, D_{0^+}^{\eta}\rho^{\alpha-1}\bigr)\,d\rho \\
&\geq \int_0^1 \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta-\eta}]}{\Gamma(\alpha)} \\
&\quad \times f_1\biggl(\rho,\rho^{\alpha-1}, \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\tau^{\alpha-\eta-1}\biggr)\,d\rho \\
&\geq \tau^{\alpha-\eta-1} \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta-\eta}]}{\Gamma(\alpha)} f_1\biggl(\rho,0, \frac{\Gamma(\alpha)}{\Gamma(\alpha-\eta)}\biggr)\,d\rho, \\
D_{0^+}^{\gamma} C_2(h_2,h_2)(\tau) &= \int_0^1 D_{0^+}^{\gamma} G_2(\tau,\rho) f_2\bigl(\rho,\rho^{\beta-1}, D_{0^+}^{\gamma}\rho^{\beta-1}\bigr)\,d\rho \\
&\leq \int_0^1 \frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)} f_2\biggl(\rho,\rho^{\beta-1}, \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\tau^{\beta-\gamma-1}\biggr)\,d\rho \\
&\leq \tau^{\beta-1} \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)} f_2(\rho,1,0)\,d\rho, \\
D_{0^+}^{\gamma} C_2(h_2,h_2)(\tau) &= \int_0^1 D_{0^+}^{\gamma} G_2(\tau,\rho) f_2\bigl(\rho,\rho^{\beta-1}, D_{0^+}^{\gamma}\rho^{\beta-1}\bigr)\,d\rho \\
&\geq \int_0^1 \frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi-\gamma}]}{\Gamma(\beta)} \\
&\quad \times f_2\biggl(\rho,\rho^{\beta-1}, \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\tau^{\beta-\gamma-1}\biggr)\,d\rho \\
&\geq \tau^{\beta-1} \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)} f_2\biggl(\rho,0, \frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)}\biggr)\,d\rho.
\end{aligned}$$

Set

$$\begin{aligned} c_1 &= \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\alpha-\eta}]f_1(\rho,0,\frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma)})}{\Gamma(\alpha)}\,d\rho, \\ c_2 &= \int_0^1 \frac{(1-\rho)^{\alpha-\zeta-1}}{\Gamma(\alpha)} f_1(\rho,1,0)\,d\rho, \end{aligned} \tag{19}$$

and

$$\begin{aligned} c_3 &= \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\beta-\gamma}]f_2(\rho,0,\frac{\Gamma(\beta)}{\Gamma(\beta-\gamma)})}{\Gamma(\beta)}\,d\rho, \\ c_4 &= \int_0^1 \frac{(1-\rho)^{\beta-\xi-1}}{\Gamma(\beta)} f_2(\rho,1,0)\,d\rho. \end{aligned} \tag{20}$$

From (H_2) and (H_4), it is clear that

$$c_2 \geq c_1 \geq \delta_1 a_{11} > 0, \qquad c_4 \geq c_3 \geq \delta_1 a_{21} > 0.$$

Consequently,

$$c_1 h \preceq C_1(h,h) \preceq c_2 h, \qquad c_3 h \preceq C_2(h,h) \preceq c_4 h.$$

Next, we show the proof the condition (A_2) of Lemma 2.5. By (H_4),

$$\begin{aligned} C_1(y,x) &= \int_0^1 G_1(\tau,\rho) f_1\big(\rho, y(\rho), D_{0^+}^{\eta} x(\rho)\big)\,d\rho \\ &\geq \delta_{11} \int_0^1 G_1(\tau,\rho) g_1\big(\rho, y(\rho)\big)\,d\rho \\ &= \delta_{11} A_1(x), \\ D_{0^+}^{\eta} C_1(y,x) &= \int_0^1 D_{0^+}^{\eta} G_1(\tau,\rho) f_1\big(\rho, y(\rho), D_{0^+}^{\eta} x(\rho)\big)\,d\rho \\ &\geq \delta_{11} \int_0^1 D_{0^+}^{\eta} G_1(\tau,\rho) g_1\big(\rho, y(\rho)\big)\,d\rho \\ &= \delta_{11} D_{0^+}^{\eta} A_1(x). \end{aligned}$$

Then $C_1(y,x) \succeq \delta_{11} A_1(x)$.

$$\begin{aligned} C_2(x,y) &= \int_0^1 G_2(\tau,\rho) f_2\big(\rho, x(\rho), D_{0^+}^{\gamma} y(\rho)\big)\,d\rho \\ &\geq \delta_{21} \int_0^1 G_2(\tau,\rho) g_2\big(\rho, x(\rho)\big)\,d\rho \\ &= \delta_{21} A_2(y), \end{aligned}$$

$$\begin{aligned}
D^{\gamma}_{0^+}C_2(x,y) &= \int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)f_2\big(\rho,x(\rho),D^{\gamma}_{0^+}y(\rho)\big)\,d\rho \\
&\geq \delta_{21}\int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)g_2\big(\rho,x(\rho)\big)\,d\rho \\
&= \delta_{21}D^{\gamma}_{0^+}A_2(y).
\end{aligned}$$

Then $C_2(y,x) \succeq \delta_{21}A_2(y)$. From (H_4) and Lemma 2.6, we have

$$\begin{aligned}
C_1(y,x) &= \int_0^1 G_1(\tau,\rho)f_1\big(\rho,y(\rho),D^{\eta}_{0^+}x(\rho)\big)\,d\rho \\
&\geq \int_0^1 \frac{\tau^{\alpha-1}(1-\rho)^{\alpha-\zeta-1}[1-(1-\rho)^{\zeta}]}{\Gamma(\alpha)}f_1\big(\rho,y(\rho),D^{\eta}_{0^+}x(\rho)\big)\,d\rho \\
&\geq \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha}\right)\delta_{12} \\
&\geq \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha-\eta}\right)k_1\big(y(1)\big) \\
&= \frac{1}{\Gamma(\alpha-\zeta)}\left(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha-\eta}\right)B_2y, \\
D^{\eta}_{0^+}C_1(y,x) &= \int_0^1 D^{\eta}_{0^+}G(\tau,\rho)f_1\big(\rho,y(\rho),D^{\eta}_{0^+}x(\rho)\big)\,d\rho \\
&\geq \frac{\tau^{\alpha-\eta-1}}{\Gamma(\alpha-\eta)}\int_0^1 (1-\rho)^{\alpha-\zeta-1}\big(1-(1-\rho)^{\zeta-\eta}\big)f_1\big(\rho,y(\rho),D^{\eta}_{0^+}x(\rho)\big)\,d\rho \\
&\geq \frac{\tau^{\alpha-\eta-1}}{\Gamma(\alpha-\eta)}\left(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha-\eta}\right)k_1\big(y(1)\big) \\
&= \frac{1}{\Gamma(\alpha-\zeta)}\left(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha-\eta}\right)D^{\eta}_{0^+}B_2y.
\end{aligned}$$

That means $C_1(x,y) \succeq \frac{1}{\Gamma(\alpha-\zeta)}(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha-\eta})B_2y$. Let

$$\delta_1 = \min\left\{\delta_{12}, \frac{1}{\Gamma(\alpha-\zeta)}\left(\frac{1}{\alpha-\zeta}-\frac{1}{\alpha-\eta}\right)\right\},$$

then

$$\begin{aligned}
&C_1(x,y) \succeq \delta_1(A_1x+B_2y), \\
&\begin{aligned}
C_2(x,y) &= \int_0^1 G_2(\tau,\rho)f_2\big(\rho,x(\rho),D^{\gamma}_{0^+}y(\rho)\big)\,d\rho \\
&\geq \int_0^1 \frac{\tau^{\beta-1}(1-\rho)^{\beta-\xi-1}[1-(1-\rho)^{\xi}]}{\Gamma(\beta)}f_2\big(\rho,x(\rho),D^{\gamma}_{0^+}y(\rho)\big)\,d\rho \\
&\geq \frac{\tau^{\beta-1}}{\Gamma(\beta)}\left(\frac{1}{\beta-\xi}-\frac{1}{\beta}\right)\delta_{22} \\
&\geq \frac{\tau^{\beta-1}}{\Gamma(\beta)}\left(\frac{1}{\beta-\xi}-\frac{1}{\beta-\gamma}\right)k_2\big(x(1)\big) \\
&= \frac{1}{\Gamma(\beta-\xi)}\left(\frac{1}{\beta-\xi}-\frac{1}{\beta-\gamma}\right)B_1x,
\end{aligned}
\end{aligned}$$

$$\begin{aligned} D^{\gamma}_{0^+}C_2(x,y) &= \int_0^1 D^{\gamma}_{0^+}G_2(\tau,\rho)f_2\bigl(\rho,x(\rho),D^{\gamma}_{0^+}y(\rho)\bigr)\,d\rho \\ &\geq \frac{\tau^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\int_0^1 (1-\rho)^{\beta-\gamma-1}\bigl(1-(1-\rho)^{\xi-\gamma}\bigr)f_2\bigl(\rho,x(\rho),D^{\gamma}_{0^+}y(\rho)\bigr)\,d\rho \\ &\geq \frac{\tau^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}\biggl(\frac{1}{\beta-\xi}-\frac{1}{\beta-\gamma}\biggr)k_1\bigl(y(1)\bigr) \\ &= \frac{1}{\Gamma(\beta-\xi)}\biggl(\frac{1}{\beta-\xi}-\frac{1}{\beta-\gamma}\biggr)D^{\gamma}_{0^+}B_1x. \end{aligned}$$

That means $C_2(y,x) \succeq \frac{1}{\Gamma(\beta-\xi)}(\frac{1}{\beta-\xi}-\frac{1}{\beta-\gamma})B_1x$. Let

$$\delta_2 = \min\biggl\{\delta_{22}, \frac{1}{\Gamma(\beta-\xi)}\biggl(\frac{1}{\beta-\xi}-\frac{1}{\beta-\gamma}\biggr)\biggr\}.$$

Then we have

$$C_2(y,x) \succeq \delta_2(A_2x + B_1y).$$

We see that the conclusion (2) in Lemma 2.5 means that there exist $u_{01}, u_{02}, v_{01}, v_{02} \in P_h$ and $r \in (0,1)$ such that

(1) $\exists\ (u_{01}, v_{01}), (u_{02}, v_{02}) \in K \subset E \times E$ and $r \in (0,1)$ with

$$r(v_{01}, v_{02}) \preceq (u_{01}, u_{02}) \prec (v_{01}, v_{02}),$$

that is,

$$\begin{aligned} &r(v_{01}, v_{02}) \leq (u_{01}, u_{02}) < (v_{01}, v_{02}), \\ &r\bigl(D^{\gamma}_{0^+}v_{01}, D^{\gamma}_{0^+}v_{02}\bigr) \leq \bigl(D^{\eta}_{0^+}u_{01}, D^{\eta}_{0^+}u_{02}\bigr) < \bigl(D^{\gamma}_{0^+}v_{01}, D^{\gamma}_{0^+}v_{02}\bigr), \\ &(u_{01}, u_{02}) \leq \biggl(\int_0^1 G_1(\tau,\rho)f_1\bigl(\rho,u_{01}(\rho),D^{\eta}_{0^+}u_{01}(\rho)\bigr)\,d\rho \\ &\qquad + \int_0^1 G_1(\tau,\rho)g_1\bigl(\rho,v_{01}(\rho)\bigr)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\bigl(u_{01}(1)\bigr)\tau^{\alpha-1}, \\ &\qquad \int_0^1 G_1(\tau,\rho)f_1\bigl(\rho,u_{02}(\rho),D^{\eta}_{0^+}u_{02}(\rho)\bigr)\,d\rho \\ &\qquad + \int_0^1 G_1(\tau,\rho)g_1\bigl(\rho,v_{02}(\rho)\bigr)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\bigl(u_{02}(1)\bigr)\tau^{\alpha-1}\biggr), \\ &D^{\eta}_{0^+}\bigl((u_{01}, u_{02})\bigr) \leq \biggl(\int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)f_1\bigl(\rho,u_{01}(\rho),D^{\eta}_{0^+}u_{01}(\rho)\bigr)\,d\rho \\ &\qquad + \int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)g_1\bigl(\rho,v_{01}(\rho)\bigr)\,d\rho \\ &\qquad + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\bigl(u_{01}(1)\bigr)\tau^{\alpha-1}, \\ &\qquad \int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)f_1\bigl(\rho,u_{02}(\rho),D^{\eta}_{0^+}u_{02}(\rho)\bigr)\,d\rho \\ &\qquad + \int_0^1 D^{\eta}_{0^+}G_1(\tau,\rho)g_1\bigl(\rho,v_{02}(\rho)\bigr)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\bigl(u_{02}(1)\bigr)\tau^{\alpha-1}\biggr), \end{aligned}$$

$$
\begin{aligned}
(v_{01},v_{02}) \geq \Big(&\int_0^1 G_2(\tau,\rho)f_2\big(\rho,v_{01}(\rho),D_{0^+}^{\gamma}v_{01}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_2(\tau,\rho)g_2\big(\rho,u_{01}(\rho)\big)\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{01}(1)\big)\tau^{\beta-1}, \\
&\int_0^1 G_2(\tau,\rho)f_2\big(\rho,v_{02}(\rho),D_{0^+}^{\gamma}v_{02}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_2(\tau,\rho)g_2\big(\rho,u_{02}(\rho)\big)\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{02}(1)\big)\tau^{\beta-1}\Big), \\
D_{0^+}^{\gamma}(v_{01},v_{02}) \geq \Big(&\int_0^1 D_{0^+}^{\gamma}G_2(\tau,\rho)f_2\big(\rho,v_{01}(\rho),D_{0^+}^{\gamma}v_{01}(\rho)\big)\,d\rho \\
&+ \int_0^1 D_{0^+}^{\gamma}G_2(\tau,\rho)g_2\big(\rho,u_{01}(\rho)\big)\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{01}(1)\big)\tau^{\beta-1}, \\
&\int_0^1 D_{0^+}^{\gamma}G_2(\tau,\rho)f_2\big(\rho,v_{02}(\rho),D_{0^+}^{\gamma}v_{02}(\rho)\big)\,d\rho \\
&+ \int_0^1 D_{0^+}^{\gamma}G_2(\tau,\rho)g_2\big(\rho,u_{02}(\rho)\big)\,d\rho + \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(v_{02}(1)\big)\tau^{\beta-1}\Big),
\end{aligned}
$$

where $h(\tau) = (h_1(\tau),h_2(\tau)) = (\tau^{\alpha-1},\tau^{\beta-1})$, $0 \leq \tau \leq 1$, and $G_1(\tau,\rho)$, $G_2(\tau,\rho)$ are defined by (11) and (12), respectively.

(2) The problem (2) has a unique positive solution (u^*,v^*) in K_h;

(3) For $(x_{01},x_{02}),(y_{01},y_{02}) \in P_h \times P_h$, there are two iterative sequences $\{(x_{n1},x_{n2})\}$ and $\{(y_{n1},y_{n2})\}$ for approximating (x^*,y^*), that is, $(x_{n1},x_{n2}) \to (x^*,y^*)$ and $(y_{n1},y_{n2}) \to (x^*,y^*)$, where

$$
\begin{aligned}
\big(x_{n1}(\tau),x_{n2}(\tau)\big) = \Big(&\int_0^1 G_1(\tau,\rho)f_1\big(\rho,x_{(n-1)1}(\rho),D_{0^+}^{\eta}x_{(n-1)1}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_1(\tau,\rho)g_1\big(\rho,y_{(n-1)1}(\rho)\big)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\big(x_{(n-1)1}(1)\big)\tau^{\alpha-1}, \\
&\int_0^1 G_1(\tau,\rho)f_1\big(\rho,x_{(n-1)2}(\rho),D_{0^+}^{\eta}x_{(n-1)2}(\rho)\big)\,d\rho \\
&+ \int_0^1 G_1(\tau,\rho)g_1\big(\rho,y_{(n-1)2}(\rho)\big)\,d\rho + \frac{\Gamma(\alpha-\zeta)}{\Gamma(\alpha)}k_2\big(x_{(n-1)2}(1)\big)\tau^{\alpha-1}\Big), \\
\big(y_{n1}(\tau),y_{n2}(\tau)\big) = \Big(&\int_0^1 G_2(\tau,\rho)f_2\big(\rho,y_{(n-1)1}(\rho),D_{0^+}^{\gamma}y_{(n-1)1}(\rho)\big)\,d\rho, \\
&\int_0^1 G_2(\tau,\rho)g_2\big(\rho,x_{(n-1)1}(\rho)\big)\,d\rho, \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(y_{(n-1)1}(1)\big)\tau^{\beta-1}, \\
&\int_0^1 G_2(\tau,\rho)f_2\big(\rho,y_{(n-1)2}(\rho),D_{0^+}^{\gamma}y_{(n-1)2}(\rho)\big)\,d\rho, \\
&\int_0^1 G_2(\tau,\rho)g_2\big(\rho,x_{(n-1)2}(\rho)\big)\,d\rho, \frac{\Gamma(\beta-\xi)}{\Gamma(\beta)}k_1\big(y_{(n-1)2}(1)\big)\tau^{\beta-1}\Big), \\
&n = 1,2,\ldots.
\end{aligned}
$$

□

3.1 Example

Let us consider

$$\begin{cases} D_{0^+}^{\frac{7}{2}}x(\tau)+\tau^2+(y(\tau))^{\frac{1}{4}}+(x(\tau))^{\frac{1}{4}}+(D_{0^+}^{\frac{3}{2}}x(\tau)+1)^{-\frac{1}{2}}+1=0, \quad \tau\in(0,1),\\ D_{0^+}^{\frac{10}{3}}y(\tau)+\tau+\tau^3+\frac{y}{1+y}+\frac{x}{1+x}+\frac{1}{D_{0^+}^{\frac{5}{3}}y(\tau)+1}=0, \quad \tau\in(0,1),\\ x(0)=x'(0)=x''(0)=0,\\ y(0)=y'(0)=y''(0)=0,\\ [D_{0^+}^{\frac{8}{5}}x(\tau)]_{\tau=1}=(x(1))^{-\frac{1}{3}}+5, \qquad [D_{0^+}^{\frac{11}{6}}y(\tau)]_{\tau=1}=\frac{1}{1+y(1)^{\frac{1}{2}}}. \end{cases} \tag{21}$$

Let $g_1(\tau,y)=(x(\tau))^{\frac{1}{4}}+\tau^2$, $f_1(\tau,x,y)=(x(\tau))^{\frac{1}{4}}+(y(\tau)+1)^{-\frac{1}{2}}+1$ and $k_1(y)=\frac{1}{1+y^{\frac{1}{2}}}$, also $g_2(\tau,x)=\tau^3+\frac{x}{1+x}, f_2(\tau,x,y)=\tau+\frac{x}{1+x}+\frac{1}{y+1}$ and $k_2(x)=x^{-\frac{1}{3}}+5$.

Obviously, $g_1,g_2:[0,1]\times\mathbb{R}^+\to\mathbb{R}^+, f_1,f_2:[0,1]\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ and $k_1,k_2:\mathbb{R}^+\to\mathbb{R}^+$ are continuous. It is easy to check that $g_1(\tau,y)$, $g_2(\tau,x)$ are increasing in y, x, respectively, and $k_1(y)$, $k_2(x)$ are decreasing in $y,x\in\mathbb{R}^+$ (respectively) and $f_1(\tau,x,y), f_2(\tau,x,y)$ are increasing in x and decreasing in y for fixed $\tau\in(0,1)$. In addition, for any $\lambda\in(0,1)$ we get

$$\begin{aligned} g_1(\tau,\lambda y)&=\tau^2+\big(\lambda y(\tau)\big)^{\frac{1}{4}}\geq\lambda^{\frac{1}{4}}2+\lambda^{\frac{1}{4}}\big(y(\tau)\big)^{\frac{1}{4}}=\lambda^{\frac{1}{4}}g_1(\tau,y),\\ g_2(\tau,\lambda x)&=\tau^3+\frac{\lambda x}{1+\eta x}\geq\lambda\tau+\lambda\frac{x}{1+x}=\lambda g_2(\tau,x),\\ f_1\big(\tau,\lambda x,\lambda^{-1}y\big)&=\lambda^{\frac{1}{4}}\big(x(\tau)\big)^{\frac{1}{4}}+\lambda^{\frac{1}{2}}\big(y(\tau)+1\big)^{-\frac{1}{2}}+1\geq\lambda^{\frac{1}{2}}\Big(\big(x(\tau)\big)^{\frac{1}{4}}+\lambda^{\frac{1}{2}}\big(y(\tau)+1\big)^{-\frac{1}{2}}+1\Big)\\ &=\lambda^{\frac{1}{2}}f(\tau,x,y),\\ f_2\big(\tau,\lambda x,\lambda^{-1}y\big)&=\tau+\frac{\lambda x}{1+\lambda x}+\frac{1}{\lambda^{-1}y+1}\geq\tau+\frac{\lambda x}{1+x}+\frac{\lambda}{y+1}\geq\lambda f_2(\tau,x,y),\\ k_1\big(\lambda^{-1}y\big)&=\frac{1}{1+(\lambda^{-1}y)^{\frac{1}{2}}}\geq\frac{\lambda^{\frac{1}{2}}}{1+y^{\frac{1}{2}}}\geq\frac{\lambda}{1+y^{\frac{1}{2}}}=\lambda k_1(y).\\ k_2\big(\lambda^{-1}x\big)&=\big(\lambda^{-1}x\big)^{-\frac{1}{3}}+5\geq\lambda^{\frac{1}{3}}k_2(x). \end{aligned}$$

Besides, $g_1(\tau,0)=2\not\equiv 0$, $g_2(\tau,0)=\tau\not\equiv 0$ Moreover, set $\delta_1=\delta_2=1$,

$$\begin{aligned} f_1(\tau,x,y)&=\big(x(\tau)\big)^{\frac{1}{4}}+\big(y(\tau)+1\big)^{-\frac{1}{2}}+1\geq\big(x(\tau)\big)^{\frac{1}{4}}+\tau^2=\delta_1 g(\tau,x),\\ f_2(\tau,x,y)&=\tau+\frac{x}{1+x}+\frac{1}{y+1}\geq\tau^3+\frac{x}{1+x}=\delta_1 g(\tau,x). \end{aligned}$$

Then by Theorem 3.1 we deduce that (21) has a unique positive solution (x^*,y^*) in (P_{h_1},P_{h_2}), where $(h_1,h_2)=(\tau^{\frac{5}{2}},\tau^{\frac{7}{3}})$.

4 Conclusion

In this manuscript, we extend the existence and uniqueness of positive solutions from a class of fractional differential equations with nonlinear boundary conditions for a new class of coupled system of fractional derivatives.

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