mathematics

Article

Existence Results for Langevin Equation Involving Atangana-Baleanu Fractional Operators

Dumitru Baleanu [1,2,3], Rahmat Darzi [4] and Bahram Agheli [5,*]

1 Department of Mathematics, Cankaya University, 06790 Ankara, Turkey; dumitru@cankaya.edu.tr or Baleanu@mail.cmuh.org.tw
2 Institute of Space Sciences, 077125 Magurele-Bucharest, Romania
3 Department of Medical Research, China Medical University, Taichung 40447, Taiwan
4 Department of Mathematics, Neka Branch, Islamic Azad University, Neka, Iran; r.darzi@iauneka.ac.ir
5 Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran
* Correspondence: b.agheli@qaemiau.ac.ir

Received: 28 January 2020; Accepted: 25 February 2020; Published: 12 March 2020

Abstract: A new form of nonlinear Langevin equation (NLE), featuring two derivatives of non-integer orders, is studied in this research. An existence conclusion due to the nonlinear alternative of Leray-Schauder type (LSN) for the solution is offered first and, following that, the uniqueness of solution using Banach contraction principle (BCP) is demonstrated. Eventually, the derivatives of non-integer orders are elaborated in Atangana-Baleanu sense.

Keywords: Atangana-Baleanu fractional derivative; Langevin equation; Leray-Schauder nonlinear; existence results

MSC: 26A33, 47E05

1. Introduction

In the early twentieth century, the nonlinear Langevin equation (NLE) was offered by Paul Langevin [1], who was an outstanding French scholar in physics. Using the Langevien equation, this scientist prepared a detailed and accurate account of Brownian motion.

In oscillating domains, the Langevin differential equation is utilized as a model for the explanations of physical phenomena. Several significant areas in which this equation can be applied are analyzing the stock market [2], photo-electron counting [3], modeling evacuation processes [4], self-organization in complex systems [5], studying the fluid suspensions [6], protein dynamics [7], deuteron cluster dynamics [8], and anomalous transport [9].

The generalized nonlinear Langevin equation (GNLE) can be used to formulate a large number of various problems featuring molecular motion in condensed matter. It is possible to obtain the GNLE not only in the context of the Zwanzig-Mori projection operator technique [10] but also within the framework of the recurrent relation approach from the equation of motion. It is well-known that this approach is utilized in giving an account of phenomena of dynamic nature, ordinary and anomalous (such as anomalous diffusion) transport in physical, chemical, and even biophysical complex systems [10]. An important characteristics of the GNLE is that it involves an aftereffect function, which is named a memory function. In the case the memory function has the features of a delta function which is in correlation with "a white noise", the GNLE is shortened to the ordinary Langevin equation, which is related to the system without memory, and the time correlation function (TCF) corresponding to the momentum degree of freedom has simple exponential relaxation. Thus, a formalism based on GLE (and/or similarly, on the related generalized Fokker-Planck equation) includes built-in memory effects, which characterize the system of interest [10]. Consequently, memory effects can appear in the velocity autocorrelation function (VACF) either when an oscillatory behavior is present or by the use of slightly declining correlations.

A particular model of the GLE is the fractional nonlinear Langevin equation (FNLE), which is another appealing subject. Many researchers studied generalized nonlinear Langevin equation with fractional derivative [11–19]. For instance, in [13], an application of fractional generalized nonlinear Langevin equation was studied by Eab and Lim. In this paper, fractional generalized nonlinear Langevin equation featuring an external force can be utilized to form the model of single-file diffusion. It has been shown that, for an external force that alters with power law, in the case a suitable selection of parameters related to fractional generalized Langevin equation are employed, the solution for such a fractional Langevin equation presents the correct short and long time behavior for the mean square displacement of single-file diffusion.

As examples for applications of the NLE, one may refer to modeling gait variability [11], financial markets [12], single file diffusion [13], motor control system modeling [11], described anomalous diffusion [14], and studies on Brownian fractional motion [15]. Several surveys including derivatives of real orders have been conducted by previous researchers in the FNLE. In a research on the Brownian motion featuring derivative of real order, Lutz [16] studied FNLE, which included one derivative of real order.

The NLE featuring two or three derivative with non-integer orders was studied and investigated in [17–21].

In most of these papers, there have been some debates on the distinctiveness and existence of the solution for the NLEs featuring derivatives of non integer orders. In obtaining conclusions, theorems such as Krasnosel'skii fixed point (FP), Schauder FP, Banach contraction principle (BCP). and LSN have been utilized.

In this paper, we explore whether there exists a new type NLE featuring two fractional orders (FNLE) as:

$$\begin{cases} {}^{\text{ABD}}D^{\beta}\big({}^{\text{ABD}}D^{\theta}+\gamma\big)z(t)=h\big(t,z(t)\big), \\ z(0)=\alpha_1, \\ z'(0)=\alpha_2, \end{cases} \tag{1}$$

where $t \in (0,1)$; $0 < \theta,\beta \leq 1$, $\gamma \in R$, and ${}^{\text{ABD}}D^{\theta}$ show the θth Atangana-Baleanu derivatives; $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is considered as a function with two contentious and differentiable features; $h(0, z(0)) = 0$; and $\frac{\partial}{\partial t}h\,(t, z(t))\,\big|_{t=0} = 0$. In FNLE in Equation (1), $z(t)$ is the particle position, function h is the force acting on the particle from molecules of the fluid encircling the fractional Brownian particle, γ is a damping or viscosity term, and α_1 and α_2 are the initial positions.

The existence of a conclusion due to the nonlinear alternative of Leray-Schauder type (LSN) for the solution is offered first and, following that, the uniqueness of solution using Banach contraction principle (BCP) is demonstrated. Table 1 is list of abbreviations in which we used in the next sections.

Table 1. List of abbreviations.

NLE	Nonlinear Langevin equation
FNLE	Fractional nonlinear Langevin equation
LSN	Leray-Schauder type
BCP	Banach contraction principle
GLE	Generalized Langevin equation
FP	Fixed point
ABD	Atangana-Baleanu's derivative
RLI	Reimann-Liouville's integral
ABI	Atangana-Baleanu's integral
IVP	Initial value problem

2. Preliminaries

In this section, we define and offer ABD, ABI, and some important features that are necessary for the next sections.

Definition 1. *The Atangana-Baleanu's derivative (ABD) [22] of order θ ($\theta \in (0,1]$) and $t > a$ is stated as*

$$ {}^{ABD}D^{\theta} z(t) = \frac{B(\theta)}{1-\theta} \int_a^t E_{\theta}\Big((t-v)^{\theta} \frac{\theta}{\theta - 1}\Big) z'(v) dv, \tag{2} $$

in which E_{θ} is the one-parameter Mittag-Leffler function and $B(\theta) = 1 - \theta + \frac{\theta}{\Gamma(\theta)}$ is called the normalization function featuring $B(0) = B(1) = 1$.

Definition 2. *The Cauchy formula for repeated integration is stated as*

$$ \begin{aligned} I^k\left(z(t)\right) &= \int_x^{t_k} \int_x^{t_{k-1}} \int_x^{t_{k-2}} \cdots \int_x^{t_1} z(t_0) dt_0, dt_1, \ldots, d(t_{k-2}) d(t_{k-1}) dv \\ &= \frac{1}{k!} \int_x^t z(v)(t-v)^{k-1} dv, \end{aligned} $$

in which $t > x$ and $k \geq 1$.

Definition 3. *The Reimann-Liouville's integral (RLI) [23] of order θ is stated as*

$$ {}^{RLI}I^{\theta}\left(z(t)\right) = \frac{1}{\Gamma(\theta)} \int_a^t (t-v)^{\theta-1} z(v) dv, \;\; \theta > 0, \;\; t > a. \tag{3} $$

Lemma 1. *Suppose η, $\theta \geq 0$ and $n \in \mathbb{N}$; then, the following relations hold [23]:*

(1) $D^{\eta}\, {}^{RLI}I^{\eta}\left(z(t)\right) = z(t)$.

(2) $D^{\eta}\, {}^{RLI}I^{\theta}\left(z(t)\right) = D^{\eta-\theta} z(t)$, *for* $\eta \geq \theta$.

(3) $D^{\eta}\, {}^{RLI}I^{\theta}\left(z(t)\right) = {}^{RLI}I^{\theta-\eta} z(t)$, *for* $\theta \geq \eta$.

In these relations, D^{η} is the Caputo derivative of order η.

Definition 4. *The Atangana-Baleanu's integral (ABI) [22] of order $\theta \in (k, k+1]$ and $k \in \mathbb{Z}^+$ is defined by*

$$ {}^{ABI}I^{\theta}\left(z(t)\right) = \frac{\theta}{B(\theta)}\, {}^{RLI}I^{\theta}\left(z(t)\right) + \frac{1-\theta}{B(\theta)} z(t), \tag{4} $$

in which $t > a$ and $\phi(\theta)$ is called the normalization function featuring $B(0) = B(1) = 1$.

Remark 1. *The Atangana-Baleanu's integral of order θ ($\theta \in (k, k+1]$), $k \in \mathbb{Z}^+$ and $t > a$, is stated as*

$$ {}^{ABI}I^{\theta}\left(z(t)\right) = \frac{\theta - k}{B(\theta - k)}\, {}^{RLI}I^{\theta} z(t) + \frac{1 + k - \theta}{B(\theta - k)}\, {}^{RLI}I^{k} z(t). \tag{5} $$

Property 1. *For θ ($k \in \mathbb{Z}^+$), $k \in \mathbb{Z}^+$ and $t > a$, the following properties are satisfied [23–25]:*

(a) ${}^{ABD}D^{\theta}\, {}^{ABI}I^{\theta}\left(z(t)\right) = z(t)$.

(b) ${}^{ABI}I^{\theta}\, {}^{ABD}D^{\theta}\left(z(t)\right) = z(t) - \sum\limits_{d=0}^{k} z^{(d)}(a) \frac{(t-a)^d}{d!}$.

Property 2. *The ABD [26] of order $\theta(Re(\theta) > 0)$ and $t > a$ of a general power function is stated as*

$$ {}^{ABD}D^{\theta}\left((z-t)^{\rho}\right) = \frac{B(\theta)}{1-\theta} \sum_{k=0}^{\infty} \left(\frac{\theta}{\theta - 1}\right)^k \frac{\Gamma(\theta+1)}{\Gamma(\theta + k\theta + 1)} (z-t)^{\rho + k\theta}. \tag{6} $$

Theorem 1 ([27]). *Let X be a Banach space, $B \subset X$ be a convex set, and U be open in B with $0 \in U$. Let $\phi : \overline{U} \to B$ be a continuous and compact mapping. Then, either*

(1) the mapping ϕ has a FP in U; or

(2) there exists a point $u \in \partial U$ such that $u = \mu\phi u$ for some $0 < \mu < 1$.

3. Main Results

The uniqueness and existence for solutions of the FNLE in Equation (1) is argued above. Initially, the FNLE in Equation (1) is converted to a FP based on Lemma 2. In Theorems 2 and 3, the uniqueness and existence conclusions of the fixed problem are acquired. Suppose $B = (C[0,1];\mathbb{R})$ and it is considered as the BS of all continuous functions of $I = [0,1]$ within $\mathbb{R}$ in which

$$\|z\|_0 = sup|z(\tau)|, \quad \tau \in I.$$

Lemma 2. *$z(t)$ is a solution of the FNLE in Equation (1) if and only if $z(t)$ is a solution of the integral equation*

$$\begin{aligned} z(t) =& P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \\ &+ P_3 \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} h\big(v, z(v)\big) dv - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv + \phi(t), \end{aligned} \tag{7}$$

in which

$$\phi(t) = \frac{\mu_0}{MB(\theta)} \left(\frac{\theta}{\Gamma(\theta+1)} t - \theta + 1 \right) + \frac{\nu_0}{M},$$

$$\mu_0 = \left(\gamma + \frac{\theta - 1}{B(\theta)} \right) \alpha_1 + \alpha_2 \quad \nu_0 = \frac{1}{M} \Big(\alpha_1 - \mu_0 \frac{\theta - 1}{B(\theta)} \Big), \quad M = 1 + \gamma \frac{1-\theta}{B(\theta)},$$

$$P_1 = \frac{\theta(1-\beta)}{MB(\theta)B(\beta)}, \quad P_2 = \frac{1-\theta}{MB(\theta)}, \quad P_3 = \frac{\theta}{MB(\theta)}, \quad P_4 = \gamma \frac{\theta}{MB(\theta)}.$$

Proof. Consider for the FNLE in Equation (1) a solution be $z(t)$, $\hbar = h(t, z(t))$ and $\hbar(0) = 0$, then

$$\begin{aligned} &{}^{\text{ABD}}D^{\beta}({}^{\text{ABD}}D^{\theta} + \gamma) z(t) = h(t, z(t)), \\ &{}^{\text{ABD}}D^{\theta} z(t) = \frac{1-\beta}{B(\beta)} \hbar + \frac{\beta}{B(\beta)} I^{\beta} \hbar - \gamma z(t) + \mu_0, \end{aligned}$$

$$\begin{aligned} z(t) = \frac{1-\beta}{B(\beta)} \Big(\frac{1-\theta}{B(\theta)} \hbar + \frac{\theta}{B(\theta)} I^{\theta} \hbar \Big) + \frac{1-\theta}{B(\theta)} I^{\beta} \hbar + \frac{\theta}{B(\theta)} I^{\theta} I^{\beta} \hbar \\ - \gamma \Big(\frac{1-\theta}{B(\theta)} z(t) + \frac{\theta}{B(\theta)} I^{\theta} z(t) \Big) + \mu_0 \Big(\frac{1-\theta}{B(\theta)} + \frac{\theta}{B(\theta)} \frac{t}{\Gamma(\theta+1)} \Big) + \nu_0. \end{aligned}$$

Therefore, the general case of $z(t)$ is stated as

$$\begin{aligned} z(t) = & P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \\ &+ P_3 \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} h\big(v, z(v)\big) dv - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv + \phi(t). \end{aligned} \tag{8}$$

Replacing amounts of μ_0, ν_0 in Equation (8), we can gain the IE in Equation (7), which means a solution of the IE in Equation (7) also can be $z(t)$. With regard to the assumption that $z(t)$ is a solution of the IE in Equation (7), by utilizing Property 1, we can show that $z(t)$ is as a solution of the FNLE in Equation (1). □

The FNLE in Equation (1) is converted to a FP problem $z = \phi z$, by Lemma 2, in which $\phi : B \to B$ meet the following equation

$$\phi z(t) = P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \\ + P_3 \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} h\big(v, z(v)\big) dv - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv + \phi(t), \tag{9}$$

and we can view, the operator in Equation (9) has FPs if and only if the FNLE in Equation (1) has solutions.

Theorem 2. *Suppose that the following assumptions hold:*

(i) $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a continuously differentiable function.*
(ii) $|h(t,z)| \leq \varepsilon_1(t) + \varepsilon_2(t)|z|$, *where* $\varepsilon_1,\ \varepsilon_2 \in C[0,1]$ *are nonnegative functions.*

Then, the FNLE in Equation (1) *has at least one solution on* $[0,1]$ *provided*

$$\mathfrak{A} = \sup_{t\in I} \left\{ \int_0^t \Big(\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)} + \frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)} + \frac{|P_3|(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} \Big) \varepsilon_2(v) dv + \frac{|P_4|}{\Gamma(\theta+1)} \right\} < 1,$$

and

$$0 < \mathfrak{B} = \sup_{t\in I} \left\{ \int_0^t \Big(\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)} + \frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)} + \frac{|P_3|(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} \Big) \varepsilon_1(v) dv + |\phi(t)| \right\} < \infty.$$

Proof. Consider $Q_r = \{v \in B : \ \|v\|_* < r\}$ featuring $r = \frac{\mathfrak{B}}{1-\mathfrak{A}} > 0$. $\forall v \in Q_r$, we gain

$$\begin{aligned}
\|z(t)\|_0 \leq \sup_{t\in I} \Bigg| & P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \\
& + P_3 \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} h\big(v, z(v)\big) dv - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv + \phi(t) \Bigg| \\
\leq \sup_{t\in I} \Bigg\{ & |P_1| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |h\big(v, z(v)\big)| dv + |P_2| \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} |h\big(v, z(v)\big)| dv \\
& + |P_3| \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} |h\big(v, z(v)\big)| dv + |P_4| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |z(v)| dv + |\phi(t)| \Bigg\} \\
\leq \sup_{t\in I} \Bigg\{ & |P_1| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \big(\varepsilon_1(v) + \varepsilon_2(v)|z(v)|\big) dv \\
& + |P_2| \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \big(\varepsilon_1(v) + \varepsilon_2(v)|z(v)|\big) dv \\
& + |P_3| \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} \big(\varepsilon_1(v) + \varepsilon_2(v)|z(v)|\big) dv + |P_4| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |z(v)| dv + |\phi(t)| \Bigg\} \\
\leq \sup_{t\in I} \Bigg\{ & |P_1| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \varepsilon_1(v) dv + |P_2| \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \varepsilon_1(v) dv \\
& + |P_3| \int_0^t \frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)} \varepsilon_1(v) dv + |\phi(t)| \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{t\in I}\left(|P_1|\int_0^t\frac{(t-v)^{\theta-1}}{\Gamma(\theta)}\varepsilon_2(v)dv+|P_2|\int_0^t\frac{(t-v)^{\beta-1}}{\Gamma(\beta)}\varepsilon_2(v)dv\right.\\
&\left.+|P_3|\int_0^t\frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)}\varepsilon_2(v)dv+|\phi(t)|+\frac{|P_4|}{\Gamma(\theta+1)}\right)\|z\|_0\\
=\ &\mathfrak{B}+\mathfrak{A}\|z\|_0<r,
\end{aligned}
$$

which result in $\phi z\in\overline{Q}_r$. In Appendix A, we scrutinize that $\phi:\overline{Q}_r\to\overline{Q}_r$ can be continuous and compact mapping.

If we consider function z as a solution of the eigenvalue problem $z=\lambda\phi z$ with $\lambda\in(0,1)$, the expression below can easily be obtained:

$$
\begin{aligned}
\|z\|_0\le\ &\sup_{t\in I}\left|\lambda P_1\int_0^t\frac{(t-v)^{\theta-1}}{\Gamma(\theta)}h\big(v,z(v)\big)dv+\lambda P_2\int_0^t\frac{(t-v)^{\beta-1}}{\Gamma(\beta)}h\big(v,z(v)\big)dv\right.\\
&\left.+\lambda P_3\int_0^t\frac{(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)}h\big(v,z(v)\big)dv-\lambda P_4\int_0^t\frac{(t-v)^{\theta-1}}{\Gamma(\theta)}z(v)dv+\lambda\phi(t)\right|\\
\le\ &\sup_{t\in I}\left(\int_0^t\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)}+\frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)}+\frac{|P_3|(t-v)^{\theta+\beta-1}}{\Gamma(\theta+\beta)}\big|h\big(v,z(v)\big)\big|dv\right.\\
&\left.+|P_4|\int_0^t\frac{(t-v)^{\theta-1}}{\Gamma(\theta)}|z(v)|dv+|\phi(t)|\right)\\
\le\ &\mathfrak{B}+\mathfrak{A}\|z\|_0\le r,
\end{aligned}
$$

which shows that $t\notin\partial Q_r$. By Theorem 1, ϕ has a FP in $\overline{Q}_r$ that shows the FNLE in Equation (1) can have at least one solution on I.

The existence of the solution is gained with regard to a set of suppositions in Theorem 2. In the next theorem, we display uniqueness of the solution featuring a subset of it as a result of using BCP. □

Theorem 3. *Suppose we have the next suppositions:*

(i) $h:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is a continuously differentiable function.*
(ii) $|h(t,z_2)-h(t,z_1)|\le\mathfrak{g}(t)|z_2-z_1|$, *where* $\mathfrak{g}(t)\in C[0,1]$ *are nonnegative functions.*

Therefore, the FNLE in Equation (1) *has a unique solution on* I, *provided that*

$$
\mathfrak{C}=\sup_{t\in I}\left\{\int_0^t\left(\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)}+\frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)}+\frac{|P_3|(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)}\right)\mathfrak{g}(v)dv+\frac{|P_4|}{\Gamma(\theta+1)}\right\}<1,
$$

and

$$
0<\mathfrak{D}=\sup_{t\in I}\left|\int_0^t\left(\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)}+\frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)}+\frac{|P_3|(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)}\right)h(v,0)dv+\phi(t)\right|<\infty.
$$

Proof. Consider $Q_r=\{t\in B;\|z\|_*<r\}$, where $r=\frac{\mathfrak{D}}{1-\mathfrak{C}}$. $\forall v\in Q_r$, we get,

$$
\begin{aligned}
\|\phi z\|_0\le\ &\|\phi z-\phi 0\|_0+\|\phi 0\|_0\\
=\ &\sup_{t\in I}\left|P_1\int_0^t\frac{(t-v)^{\theta-1}}{\Gamma(\theta)}\Big(h\big(v,z(v)\big)-h(v,0)\Big)dv\right.\\
&+P_2\int_0^t\frac{(t-v)^{\beta-1}}{\Gamma(\beta)}\Big(h\big(v,z(v)\big)-h(v,0)\Big)dv
\end{aligned}
$$

$$
\begin{aligned}
&+ P_3 \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big(h\big(v,z(v)\big) - h(v,0)\Big)dv - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z(v)dv \Bigg| \\
&+ \sup_{t\in I} \Bigg| P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h(v,0)dv + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h(v,0)dv \\
&+ P_3 \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} h(v,0)dv + \phi(t) \Bigg| \\
\le\ & |P_1| \sup_{t\in I} \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \Big| h\big(v,z(v)\big) - h(v,0)\Big| dv \\
&+ |P_2| \sup_{t\in I} \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \Big| h\big(v,z(v)\big) - h(v,0)\Big| dv \\
&+ |P_3| \sup_{t\in I} \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big| h\big(v,z(v)\big) - h(v,0)\Big| dv + |P_4| \sup_{t\in I} \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |z(v)| dv \\
&+ \sup_{t\in I} \Bigg| \int_0^t \Big(\frac{P_1(t-v)^{\theta-1}}{\Gamma(\theta)} + \frac{P_2(t-v)^{\beta-1}}{\Gamma(\beta)} + \frac{P_1(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big) h(v,0)dv + \phi(t) \Bigg| \\
\le\ & \left(\sup_{t\in I} \int_0^t \Big(\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)} + \frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)} + \frac{|P_3|(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big) \mathfrak{g}(v)dv + \frac{|P_4|}{\Gamma(\theta+1)} \right) \|z\|_0 \\
&+ \sup_{t\in I} \Bigg| \int_0^t \Big(\frac{P_1(t-v)^{\theta-1}}{\Gamma(\theta)} + \frac{P_2(t-v)^{\beta-1}}{\Gamma(\beta)} + \frac{P_3(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big) h(v,0)dv + \phi(t) \Bigg| \\
\le\ & \mathfrak{C}r + \mathfrak{D},
\end{aligned}
$$

that is $\phi : \overline{Q}_r \to \overline{Q}_r$. Consider $z_1, z_2 \in \overline{Q}_r$. Thus,

$$
\begin{aligned}
\|\phi z_2 - \phi z_1\|_0 &= \sup_{t\in I} |\phi z_2(t) - \phi z_1(t)| \\
&= \sup_{t\in I} \Bigg| P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \Big(h\big(v,z_2(v)\big) - h\big(v,z_1(v)\big)\Big) dv \\
&\quad + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \Big(h\big(v,z_2(v)\big) - h\big(v,z_1(v)\big)\Big) dv \\
&\quad + P_3 \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big(h\big(v,z_2(v)\big) - h\big(v,z_1(v)\big)\Big) dv \\
&\quad - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \big(z_2(v) - z_1(v)\big) dv \Bigg| \\
&\le |P_1| \sup_{t\in I} \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \Big| h\big(v,z_2(v)\big) - h\big(v,z_1(v)\big)\Big| dv \\
&\quad + |P_2| \sup_{t\in I} \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \Big| h\big(v,z_2(v)\big) - h\big(v,z_1(v)\big)\Big| dv \\
&\quad + |P_3| \sup_{t\in I} \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big| h\big(v,z_2(v)\big) - h\big(v,z_1(v)\big| dv \\
&\quad + |P_4| \sup_{t\in I} \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |z_2(v) - z_1(v)| dv \\
&\le |P_1| \sup_{t\in I} \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \mathfrak{g}(v) |z_2(v) - z_1(v)| dv \\
&\quad + |P_2| \sup_{t\in I} \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \mathfrak{g}(v) |z_2(v) - z_1(v)| dv
\end{aligned}
$$

$$
\begin{aligned}
&+|P_3|\sup_{t\in I}\int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)}\mathfrak{g}(v)|z_2(v)-z_1(v)|dv \\
&+|P_4|\sup_{t\in I}\int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)}|z_2(v)-z_1(v)|dv\Bigg| \\
\le\ & |P_1|\|z_2-z_1\|_0\int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)}\mathfrak{g}(v)dv \\
&+|P_2|\|z_2-z_1\|_0\int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)}\mathfrak{g}(v)dv \\
&+|P_3|\|z_2-z_1\|_0\int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)}\mathfrak{g}(v)dv \\
&+|P_4|\|z_2-z_1\|_0\int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)}dv \\
\le\ & \Bigg(\sup_{t\in I}\int_0^t \Big(\frac{|P_1|(t-v)^{\theta-1}}{\Gamma(\theta)}+\frac{|P_2|(t-v)^{\beta-1}}{\Gamma(\beta)}+\frac{|P_3|(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)}\Big)\mathfrak{g}(v)dv \\
&+\frac{|P_4|}{\Gamma(\theta)}\Bigg)\|z_2-z_1\|_0 \\
\le\ & \mathfrak{C}\|z_2-z_1\|_0,
\end{aligned}
$$

since $\mathfrak{C} < 1$, and then ϕ is a contraction.

By the BCP, ϕ can have a unique FP, which means that there exists a unique solution for the FNLE in Equation (1). □

4. Examples

Example 1. *Consider the following fractional Langevin equation*

$$
\begin{cases}
{}^{ABD}D^{\beta}\big({}^{ABD}D^{\theta}+\gamma\big)z(t)=h\big(t,z(t)\big), \\
z(0)=\alpha_1, \\
z'(0)=\alpha_2,
\end{cases}
\tag{10}
$$

where $\theta=\beta=0.5$, $B(\theta)=1$, $\gamma=5$, $\alpha_1=\alpha_2=1$, $h\big(t,z(t)\big)=\frac{z(t)+1}{t^2+5}$.

Thus, we have $M=3.5$, $P_1=0.0714286$, $P_2=0.142857$, $P_3=0.142857$, *and* $P_4=0.714286$. *Thus,* $\mathfrak{A}=0.876727<1$ *and* $0<\mathfrak{B}=3.16906<\infty$. *As a result, Conditions (i) and (ii) of Theorem 2 hold. Hence, FLE in Equation (10) has a solution.*

Example 2. *Consider the fractional Langevin equation (Equation (10)), where* $\theta=\beta=0.5$, $B(\theta)=1$, $\gamma=5$, $\alpha_1=\alpha_2=1$, $g(t)=\frac{t}{t^2+3}$, *and* $h\big(t,z(t)\big)=\frac{z(t)+2}{t^2+5}$.

Thus, we have $M=3.5$, $\mu_0=5.5$, $\nu_0=1.07143$, $P_1=0.0714286$, $P_2=0.142857$, $P_3=0.142857$, $P_4=0.714286$, *and* $\phi(t)=0.886584t+1.09184$.

Thus, $\mathfrak{C}=0.870559<1$ *and* $0<\mathfrak{D}=2.91262<\infty$. *As a result, Conditions (i) and (ii) of Theorem 3 hold. Hence, FLE in Equation (10) has a unique solution.*

5. Conclusions

We obtain the uniqueness and existence of the initial value problem (IVP) of NLE featuring two different fractional orders in the Atangana-Baleanu sense. In addition, we use some relations and properties of the Atangana-Baleanu fractional integral and derivative and transform IVP to a FP. Moreover, we apply the LSN and BCP to prove uniqueness and existence of the solution, respectively.

Appendix A

Consider the operator $\phi : \overline{Q}_r \to \overline{Q}_r$ defined by Equations (3) and (4). The operator ϕ is investigated featuring two properties that are continuous and compact.

The first step. The compactness for ϕ.

Consider the bounded subset S of $\overline{Q}_r$. Therefore, there exists a constant L_1 so that $\|z\|_0 < L_1$, $\forall z \in S$.

- Function h is continuous; consider $L_2 = \sup_{t \in I \times [-L_1, L_1]} |h(t,z)|$.
- Function ϕ is continuous; consider $L_3 = \sup_{t \in I} |\phi(t)|$.

Then,

$$\begin{aligned}
|\phi z(t)| \le & \left| P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv + P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \right. \\
& \left. + P_3 \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} h\big(v, z(v)\big) dv - P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv + \phi(t) \right| \\
\le & |P_1| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |h\big(v, z(v)\big)| dv + |P_2| \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} |h\big(v, z(v)\big)| dv \\
& + |P_3| \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} |h\big(v, z(v)\big)| dv + |P_4| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} |z(v)| dv + |\phi(t)| \\
\le & \frac{|P_1| L_2}{\Gamma(\theta+1)} + \frac{|P_2| L_2}{\Gamma(\beta+1)} + \frac{|P_3| L_2}{\Gamma(\beta+\theta+1)} + \frac{|P_4| L_1}{\Gamma(\theta+1)} + L_3.
\end{aligned}$$

Then, uniformly bounded is confirmed for functions in $\phi(v)$.

In another view, suppose $\mathfrak{D}_1(t) = (t-v)^{\theta-1}$, $\mathfrak{D}_2(t) = (t-v)^{\beta-1}$ $\mathfrak{D}_3(t) = (t-v)^{\beta+\theta-1}$; then, $\mathfrak{D}_1(t), \mathfrak{D}_2(t), \mathfrak{D}_3(t)$ and $\psi(t)$ are with two properties: differentiable and all continuously functions. For all $t_1, t_2 \in I$, without loss of generality, let $t_1 < t_2$; there exist positive constants L_4, L_5, L_6, and L_7 such that

$$\begin{aligned}
&|\mathfrak{D}_1(t_2) - \mathfrak{D}_1(t_1)| = |g_1'(\mathfrak{C}_1)(t_2 - t_1)| \le L_4 |t_2 - t_1|; \quad \mathfrak{C}_1 \in [t_1, t_2], \\
&|\mathfrak{D}_2(t_2) - \mathfrak{D}_2(t_1)| = |g_2'(\mathfrak{C}_2)(t_2 - t_1)| \le L_5 |t_2 - t_1|; \quad \mathfrak{C}_2 \in [t_1, t_2], \\
&|\mathfrak{D}_3(t_2) - \mathfrak{D}_3(t_1)| = |g_3'(\mathfrak{C}_3)(t_2 - t_1)| \le L_6 |t_2 - t_1|; \quad \mathfrak{C}_3 \in [t_1, t_2], \\
&|\psi(t_2) - \psi(t_1)| = |\psi'(\mathfrak{C}_4)(t_2 - t_1)| \le L_7 |t_2 - t_1|; \quad \mathfrak{C}_4 \in [t_1, t_2].
\end{aligned}$$

Let

$$\omega = \frac{|P_1| L_2 + |P_4| L_1}{\Gamma(\theta+1)} (\theta L_4 + 1) + \frac{|P_2| L_2}{\Gamma(\beta+1)} (\beta L_5 + 1) + \frac{|P_3| L_2}{\Gamma(\beta+\theta+1)} \big((\beta+\theta+1) L_6 + 1\big) + L_7.$$

Then, for $z \in S$ and for all $\epsilon > 0$, there exists δ ($\delta \le (\frac{\epsilon}{\omega})^{\frac{1}{\lambda}}$) where $\lambda = \min\{\theta, \beta\}$ so that, when $|t_2 - t_1| \le \delta$, we have

$$\begin{aligned}
|\phi z(t_2) - \phi z(t_1)| \le & \left| P_1 \int_0^{t_2} \frac{(t_2-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv + P_2 \int_0^{t_2} \frac{(t_2-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \right. \\
& + P_3 \int_0^{t_2} \frac{(t_2-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} h\big(v, z(v)\big) dv - P_4 \int_0^{t_2} \frac{(t_2-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv + \phi(t_2) \\
& - P_1 \int_0^{t_1} \frac{(t_1-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z(v)\big) dv - P_2 \int_0^{t_1} \frac{(t_1-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z(v)\big) dv \\
& \left. - P_3 \int_0^{t_1} \frac{(t_1-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} h\big(v, z(v)\big) dv + P_4 \int_0^{t_1} \frac{(t_1-v)^{\theta-1}}{\Gamma(\theta)} z(v) dv - \phi(t_1) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq |P_1| \int_0^{t_1} \frac{(t_2-v)^{\theta-1}-(t_1-v)^{\theta-1}}{\Gamma(\theta)} |h(v,z(v))| dv \\
&\quad + |P_2| \int_0^{t_1} \frac{(t_2-v)^{\beta-1}-(t_1-v)^{\beta-1}}{\Gamma(\beta)} |h(v,z(v))| dv \\
&\quad + |P_3| \int_0^{t_1} \frac{(t_2-v)^{\beta+\theta-1}-(t_1-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} |h(v,z(v))| dv \\
&\quad - |P_4| \int_0^{t_1} \frac{(t_2-v)^{\theta-1}-(t_1-v)^{\theta-1}}{\Gamma(\theta)} |z(v)| dv \\
&\quad - |P_1| \int_{t_1}^{t_2} \frac{(t_2-v)^{\theta-1}}{\Gamma(\theta)} |h(v,z(v))| dv - |P_2| \int_{t_1}^{t_2} \frac{(t_2-v)^{\beta-1}}{\Gamma(\beta)} |h(v,z(v))| dv \\
&\quad - |P_3| \int_{t_1}^{t_2} \frac{(t_2-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} |h(v,z(v))| dv + |P_4| \int_{t_1}^{t_2} \frac{(t_2-v)^{\theta-1}}{\Gamma(\theta)} |z(v)| dv \\
&\quad + |\phi(t_2)-\phi(t_1)| \\
&\leq \frac{|P_1|L_2L_4}{\Gamma(\theta)}|t_2-t_1| + \frac{|P_2|L_2L_5}{\Gamma(\beta)}|t_2-t_1| + \frac{|P_3|L_2L_6}{\Gamma(\beta+\theta)}|t_2-t_1| \\
&\quad + \frac{|P_4|L_1L_4}{\Gamma(\theta)}|t_2-t_1| + L_7|t_2-t_1| + \frac{|P_1|L_2}{\Gamma(\theta+1)}|t_2-t_1|^{\theta} \\
&\quad + \frac{|P_2|L_2}{\Gamma(\beta+1)}|t_2-t_1|^{\beta} + \frac{|P_3|L_2}{\Gamma(\beta+\theta+1)}|t_2-t_1|^{\beta+\theta} + \frac{|P_4|L_1}{\Gamma(\theta+1)}|t_2-t_1|^{\theta} \\
&\leq \frac{|P_1|L_2}{\Gamma(\theta+1)}(\theta L_4+1)|t_2-t_1|^{\theta} + \frac{|P_2|L_2}{\Gamma(\beta+1)}(\beta L_5+1)|t_2-t_1|^{\beta} \\
&\quad + \frac{|P_3|L_2}{\Gamma(\beta+\theta+1)}((\theta+\beta)L_6+1)|t_2-t_1|^{\beta+\theta} + \frac{|P_4|L_1}{\Gamma(\theta+1)}(\theta L_4+1)|t_2-t_1|^{\theta} \\
&\quad + L_7|t_2-t_1|^{\theta} \\
&\leq \Bigg\{ \frac{|P_1|L_2+|P_4|L_1}{\Gamma(\theta+1)}(\theta L_4+1) + \frac{|P_2|L_2}{\Gamma(\beta+1)}(\beta L_5+1) \\
&\quad + \frac{|P_3|L_2}{\Gamma(\beta+\theta+1)}((\theta+\beta)L_6+1) + L_7 \Bigg\} |t_2-t_1|^{\lambda} \\
&= \omega |t_2-t_1|^{\lambda},
\end{aligned}
$$

where $\lambda = \min\{\theta, \beta\}$; this shows that in $\phi(v)$ all functions are equicontinuous. Suppose that set $\phi(v)$, due to Arzela-Ascoli theorem, $\phi(v)$ is considered as a relatively compact set; accordingly, the operator ϕ is compact.

The second step. Suppose that the operator ϕ is considered as continuous operator.

The function h has two properties: continuous and differentiable. Thus, $\forall z_1, z_2 \in S$, there is a positive constant L_8 such that

$$
|h(t,z_2(t)) - h(t,z_1(t))| \leq |f'(t,\rho)(z_2(t)-z_1(t))| \leq L_8|z_2(t)-z_1(t)|, \quad \rho \in [-L_1, L_1].
$$

For all $\epsilon > 0$, $\exists \delta\ (\delta = \frac{\epsilon}{\chi})$ where $\chi = \frac{|P_4|+|P_1L_8|}{\Gamma(\theta+1)} + \frac{|P_2L_8|}{\Gamma(\beta+1)} + \frac{|P_3L_8|}{\Gamma(\beta+\theta+1)}$, so that, when $\|z_2 - z_1\|_0 < \delta$, we have

$$
\begin{aligned}
\|\phi z_2 - \phi z_1\|_0 \leq \sup_{t\in I} \Bigg| & P_1 \int_0^{t} \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h(v,z_2(v)) dv + P_2 \int_0^{t} \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h(v,z_2(v)) dv \\
& + P_3 \int_0^{t} \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} h(v,z_2(v)) dv - P_4 \int_0^{t} \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z_2(v) dv
\end{aligned}
$$

$$
\begin{aligned}
&- P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} h\big(v, z_1(v)\big) dv - P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} h\big(v, z_1(v)\big) dv \\
&- P_3 \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} h\big(v, z_1(v)\big) dv + P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} z_1(v) dv \Bigg| \\
\leq & \sup_{t\in I} \Bigg| P_1 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \Big(h\big(v, z_2(v)\big) - h\big(v, z_1(v)\big) \Big) dv \\
&+ P_2 \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \Big(h\big(v, z_2(v)\big) - h\big(v, z_1(v)\big) \Big) dv \\
&+ P_3 \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big(h\big(v, z_2(v)\big) - h\big(v, z_1(v)\big) \Big) dv \\
&- P_4 \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \big(z_2(v) - z_1(v)\big) dv \Bigg| \\
\leq & \sup_{t\in I} |P_1| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \Big| h\big(v, z_2(v)\big) - h\big(v, z_1(v)\big) \Big| dv \\
&+ \sup_{t\in I} |P_2| \int_0^t \frac{(t-v)^{\beta-1}}{\Gamma(\beta)} \Big| h\big(v, z_2(v)\big) - h\big(v, z_1(v)\big) \Big| dv \\
&+ \sup_{t\in I} |P_2| \int_0^t \frac{(t-v)^{\beta+\theta-1}}{\Gamma(\beta+\theta)} \Big| h\big(v, z_2(v)\big) - h\big(v, z_1(v)\big) \Big| dv \\
&+ \sup_{t\in I} |P_4| \int_0^t \frac{(t-v)^{\theta-1}}{\Gamma(\theta)} \big| z_2(v) - z_1(v) \big| dv \\
\leq & |P_1| L_8 \sup_{t\in I} \big| z_2(t) - z_1(t) \big| \frac{t^{\theta}}{\Gamma(\theta+1)} \\
&+ |P_2| L_8 \sup_{t\in I} \big| z_2(t) - z_1(t) \big| \frac{t^{\beta}}{\Gamma(\beta+1)} \\
&+ |P_3| L_8 \sup_{t\in I} \big| z_2(t) - z_1(t) \big| \frac{t^{\beta+\theta}}{\Gamma(\beta+\theta+1)} \\
&+ |P_4| L_8 \sup_{t\in I} \big| z_2(t) - z_1(t) \big| \frac{t^{\theta}}{\Gamma(\theta+1)} \\
\leq & \left(\frac{|P_4| + |P_1 L_8|}{\Gamma(\theta+1)} + \frac{|P_2 L_8|}{\Gamma(\beta+1)} + \frac{|P_3 L_8|}{\Gamma(\beta+\theta+1)} \right) \|z_2 - z_1\|_0 \\
= & \chi \|z_2 - z_1\|_0;
\end{aligned}
$$

consequently, the operator ϕ is considered is a continuous operator.

Author Contributions: All authors contributed equally to this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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