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On a Fractional Operator Combining Proportional and Classical Differintegrals

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Abstract: The Caputo fractional derivative has been one of the most useful operators for modelling non-local behaviours by fractional differential equations. It is defined, for a differentiable function $f(t)$, by a fractional integral operator applied to the derivative $f'(t)$. We define a new fractional operator by substituting for this $f'(t)$ a more general proportional derivative. This new operator can also be written as a Riemann–Liouville integral of a proportional derivative, or in some important special cases as a linear combination of a Riemann–Liouville integral and a Caputo derivative. We then conduct some analysis of the new definition: constructing its inverse operator and Laplace transform, solving some fractional differential equations using it, and linking it with a recently described bivariate Mittag-Leffler function.

Keywords: fractional integrals; Caputo fractional derivatives; fractional differential equations; bivariate Mittag-Leffler functions

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1. Introduction

Much of applied mathematics is dedicated to the study of differential equations and their solutions. Almost any dynamic process in nature can be modelled by some ordinary or partial differential equation. When we allow the order of differentiation to be outside of the natural numbers, we obtain the very rich theory of fractional differential equations [1,2]. These have many useful applications due to the non-locality of fractional derivatives: many processes with non-local behaviours can be modelled most efficiently using fractional differential equations [3]. Analytical and numerical solution methods for fractional differential equations have been much studied in the literature [4–7].

There are many different ways of defining fractional derivatives and fractional integrals: Riemann–Liouville, Caputo, Marchaud, tempered, Hilfer, and Atangana–Baleanu, to name but a few [8–10]. These diverse definitions may be categorised into general classes according to their structure and properties [11].

Of particular interest for fractional differential equations is the so-called Caputo fractional derivative. In comparison with the classical Riemann–Liouville fractional derivative, the Caputo one requires more natural initial conditions when it is used for fractional differential equations [12]. These two are so fundamental that many other fractional derivatives are said to be of "Riemann–Liouville type" or "Caputo type" when they are derived from the corresponding fractional integral operators. The Riemann–Liouville fractional derivative is defined by taking a standard ($\mathbb{N}$-order) derivative of the fractional integral, while the Caputo derivative is defined by applying the fractional integral to a standard derivative of the function.

The so-called conformable derivative was presented in [13] in 2014 as a local, limit-based definition. It was first introduced as a conformable fractional derivative, but it lacks some of the desired properties for fractional derivatives [14]. This operator and its properties and applications have been intensely studied in other works, of which we mention [15] in particular. In [15], the following proportional derivative operator was defined:

$$^{P}D_{\alpha}f(t) = K_1(\alpha,t)f(t) + K_0(\alpha,t)f'(t), \tag{1}$$

where K_1 and K_0 are functions of $\alpha \in [0,1]$ and $t \in \mathbb{R}$ satisfying certain conditions, and where f is a differentiable function of $t \in \mathbb{R}$. This operator arises naturally in control theory, and it relates to the large and expanding theory of conformable derivatives.

Our aim in the current paper is to combine the ideas of the Caputo derivative and the proportional derivative in a new way, to create a hybrid fractional operator which may be expressed as a linear combination of the Caputo fractional derivative and the Riemann–Liouville fractional integral. We mention as a comparison the so-called fractal derivative (see for example [16] and the references therein), which in [17] was combined with Caputo-type integral transforms to create fractal fractional derivatives. In fact, for differentiable functions, the fractal derivative of [16] is a constant multiple of the conformable derivative [18]. The proportional derivative which we use here is more general.

The motivation, as usual when creating new types of fractional calculus, is to consider a more general context which allows for the modelling of real data from a wider variety of systems and processes. For our definition, in this paper we shall discover a connection, via an elementary fractional differential equation, to a bivariate Mittag-Leffler function which is emerging nowadays in various applications.

We organise the paper as follows. We construct the new operator in Section 2 and establish some of its important properties, such as the Laplace transform and the inverse operator. We solve some differential equations to find the eigenfunctions of the new operator in Section 3, discovering an unexpected relationship with some recently defined bivariate Mittag-Leffler functions. We present the conclusions in the last section.

2. The Hybrid Fractional Derivative Operator

2.1. Preliminaries

We recall [19] the definition of the Caputo derivative to order $\alpha \in (0,1)$, with initial point $t = 0$, of a differentiable function $f(t)$:

$${}^{C}_{0}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha}\,\mathrm{d}\tau. \tag{2}$$

This is one of the usual ways of extending the Riemann–Liouville integral, which is defined by

$${}^{RL}_{0}I^{\beta}_{t}f(t) = \frac{1}{\Gamma(\beta)}\int_0^t f(\tau)(t-\tau)^{\beta-1}\,\mathrm{d}\tau,$$

for $\beta > 0$ and $f(t)$ an integrable function. It is clear from the definitions that the Caputo derivative is

$${}^{C}_{0}D^{\alpha}_{t}f(t) = {}^{RL}_{0}I^{1-\alpha}_{t}f'(t),$$

which makes some sense as a definition of fractional derivatives. Other well-known properties of the Caputo derivative include [20]:

$$
\begin{aligned}
{}^{RL}_{0}I^{\alpha}_{t}\,{}^{C}_{0}D^{\alpha}_{t}f(t) &= f(t) - f(0); \\
{}^{C}_{0}D^{\alpha}_{t}\,{}^{RL}_{0}I^{\alpha}_{t}f(t) &= f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\lim_{t\to 0}{}^{RL}_{0}I^{\alpha}_{t}f(0); \\
\mathcal{L}\left[{}^{C}_{0}D^{\alpha}_{t}f(t)\right] &= s^{\alpha}\mathcal{L}\left[f(t)\right] - s^{\alpha-1}f(0),
\end{aligned}
$$

where $\mathcal{L}$ denotes the Laplace transform from a function of t to a function of s. We mention these properties as they will be important later in proving results about our new operators.

Furthermore, we recall from [15] the following general non-fractional differential operator, which has been called "proportional" or "conformable":

$$
{}^{P}D_{\alpha}f(t) = K_1(\alpha,t)f(t) + K_0(\alpha,t)f'(t),
$$

where K_0 and K_1 are functions of the variable t and the parameter $\alpha \in [0,1]$ which satisfy the following conditions for all $t \in \mathbb{R}$:

$$
\lim_{\alpha\to 0^+} K_0(\alpha,t) = 0; \quad \lim_{\alpha\to 1^-} K_0(\alpha,t) = 1; \quad K_0(\alpha,t) \neq 0, \alpha \in (0,1]; \tag{3}
$$

$$
\lim_{\alpha\to 0^+} K_1(\alpha,t) = 1; \quad \lim_{\alpha\to 1^-} K_1(\alpha,t) = 0; \quad K_1(\alpha,t) \neq 0, \alpha \in [0,1). \tag{4}
$$

This can be seen as a generalisation of the standard differentiation operator $Df(t) = f'(t)$, depending on an arbitrary parameter α, which is useful in control theory [15].

We shall also be interested in the particular case where the functions K_0 and K_1 are constant with respect to t, depending only on α. Let us denote this case by CP for "constant proportional":

$$
{}^{CP}D_{\alpha}f(t) = K_1(\alpha)f(t) + K_0(\alpha)f'(t).
$$

Remark 1. *We originally wrote this paper using the specific case*

$$
K_0(\alpha,t) = \alpha t^{1-\alpha}, \qquad K_1(\alpha,t) = (1-\alpha)t^{\alpha}, \tag{5}
$$

which is afforded special attention in [15]. However, we realised that this example will not be useful in applications, due to the lack of dimensional agreement in ${}^{P}D_{\alpha}f(t)$.

For physical consistency, the two terms $K_1(\alpha,t)f(t)$ *and* $K_0(\alpha,t)f'(t)$ *should have the same dimension. This means the dimension of* K_1 *should be* t *times the dimension of* K_0. *For the functions given by* (5), *we have*

$$
\dim\left[K_1(\alpha,t)f(t)\right] = \dim(t)^{\alpha}\cdot\dim(f), \qquad \dim\left[K_0(\alpha,t)f'(t)\right] = \dim(t)^{-\alpha}\cdot\dim(f),
$$

and so the dimensions do not agree. This is not an issue in mathematical analysis, but it is important when the operators will be used in applications.

2.2. The Main Definition

We propose a new type of fractional operator by starting from the Caputo fractional derivative (2), which is written as an integral formula, and substituting the expression (1) instead of $f'(\tau)$ in the integrand of this formula. Thus we obtain a hybrid fractional operator from combining the proportional and Caputo definitions:

$$
{}^{PC}_{0}D^{\alpha}_{t}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \Big(K_1(\alpha,\tau)f(\tau) + K_0(\alpha,\tau)f'(\tau)\Big)(t-\tau)^{-\alpha}\,\mathrm{d}\tau.
$$

In particular, an important special case is when K_0 and K_1 are independent of t as in the ${}^{CP}D_\alpha$ operator. We formalise the definitions of the new operators as follows.

Definition 1. *The proportional–Caputo hybrid operator may be defined in one of two possible ways. Either in the following general way:*

$$\begin{aligned}{}^{PC}_{\;\;0}D^\alpha_t f(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \Big(K_1(\alpha,\tau)f(\tau)+K_0(\alpha,\tau)f'(\tau)\Big)(t-\tau)^{-\alpha}\,\mathrm{d}\tau \\ &= {}^{RL}_{\;\;0}I^{1-\alpha}_t\Big(K_1(\alpha,t)f(t)+K_0(\alpha,t)f'(t)\Big) \\ &= \Big(K_1(\alpha,t)f(t)+K_0(\alpha,t)f'(t)\Big)*\left(\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right),\end{aligned} \tag{6}$$

or as the following simpler expression:

$$\begin{aligned}{}^{CPC}_{\quad 0}D^\alpha_t f(t) &= \frac{1}{\Gamma(1-\alpha)}\int_0^t \Big(K_1(\alpha)f(\tau)+K_0(\alpha)f'(\tau)\Big)(t-\tau)^{-\alpha}\,\mathrm{d}\tau \\ &= K_1(\alpha)\,{}^{RL}_{\;\;0}I^{1-\alpha}_t f(t)+K_0(\alpha)\,{}^{C}_{0}D^\alpha_t f(t),\end{aligned} \tag{7}$$

the latter being a simple linear combination of the Riemann–Liouville integral and the Caputo derivative. (Here PC stands for Proportional Caputo and CPC stands for Constant Proportional Caputo.)

In both of these formulae, the function space domain is given by requiring that f is differentiable and both f and f' are locally L^1 functions on the positive reals.

Proposition 1. *The PC and CPC operators are non-local and singular.*

Proof. Non-locality follows from the fact that these operators are defined by integrals: both ${}^{PC}_{\;\;0}D^\alpha_t f(t)$ and ${}^{CPC}_{\quad 0}D^\alpha_t f(t)$ depend on values of $f(\tau)$ for all τ between 0 and t.

These integrals are also singular, because they are defined, just like the Riemann–Liouville operators, using the function $(t-\tau)^{-\alpha}$ in the kernel. This function has an integrable singularity at the endpoint $\tau = t$ of the integral, since $0 < \alpha < 1$. □

Remark 2. *In the limiting cases $\alpha = 0$ and $\alpha = 1$, we recover the following special cases:*

$$\begin{aligned}\lim_{\alpha\to 0}{}^{PC}_{\;\;0}D^\alpha_t f(t) &= \lim_{\alpha\to 0}{}^{CPC}_{\quad 0}D^\alpha_t f(t) = \int_0^t f(\tau)\,\mathrm{d}\tau, \\ \lim_{\alpha\to 1}{}^{PC}_{\;\;0}D^\alpha_t f(t) &= \lim_{\alpha\to 1}{}^{CPC}_{\quad 0}D^\alpha_t f(t) = f'(t),\end{aligned}$$

where the $\alpha \to 1$ case follows from the fact that we are taking a $(1-\alpha)$th Riemann–Liouville derivative and the kernel function tends (in the sense of distributions) to the Dirac delta. This assumes that the limits in (3)–(4) are uniform in t, so that the limiting process is preserved in the integral expressions for the PC and CPC operators.

Thus, the new operators interpolate in some sense between the integral and the derivative of a function.

We now prove a result on Laplace transforms, which will be useful in many other derivations later, including the solution of some fractional differential equations. This theorem covers only the CPC operator; the Laplace transform for the PC operator would be more complicated, since it is not just a linear combination of Riemann–Liouville and Caputo differintegrals.

Theorem 1. *The Laplace transform of the CPC operator is given as follows:*

$$\mathcal{L}\Big[{}^{CPC}_{\quad 0}D^\alpha_t f(t)\Big] = \left[\frac{K_1(\alpha)}{s}+K_0(\alpha)\right]s^\alpha \widehat{f}(s) - K_0(\alpha)s^{\alpha-1}f(0), \tag{8}$$

where $f(t)$ is a differentiable function such that f and f' are locally L^1 on the positive reals and its Laplace transform $\widehat{f}(s)$ exists.

Proof. It is known [10] that the Laplace transforms of the Riemann–Liouville integral and the Caputo derivative are given by

$$\mathcal{L}\left[{}^{RL}_{0}I^{\alpha}_{t} f(t) \right] = s^{-\alpha}\widehat{f}(s), \qquad \mathcal{L}\left[{}^{C}_{0}D^{\alpha}_{t} f(t) \right] = s^{\alpha}\widehat{f}(s) - s^{\alpha-1}f(0),$$

for $0 < \alpha < 1$. Therefore, for the CPC operator the Laplace transform is

$$\begin{aligned} \mathcal{L}\left[{}^{CPC}_{0}D^{\alpha}_{t} f(t) \right] &= \mathcal{L}\left[K_1(\alpha)\, {}^{RL}_{0}I^{1-\alpha}_{t} f(t) + K_0(\alpha)\, {}^{C}_{0}D^{\alpha}_{t} f(t) \right] \\ &= K_1(\alpha)s^{-(1-\alpha)}\widehat{f}(s) + K_0(\alpha)\left[s^{\alpha}\widehat{f}(s) - s^{\alpha-1}f(0) \right] \\ &= \left[K_1(\alpha)s^{\alpha-1} + K_0(\alpha)s^{\alpha} \right]\widehat{f}(s) - K_0(\alpha)s^{\alpha-1}f(0), \end{aligned}$$

which is the desired result. □

3. The Corresponding Fractional Integral Operator

3.1. Inverting by Operational Calculus

Since both the PC and CPC fractional operators are given by the composition of a Riemann–Liouville fractional integral with proportional derivatives, namely

$$ {}^{PC}_{0}D^{\alpha}_{t} f(t) = {}^{RL}_{0}I^{1-\alpha}_{t}\left[{}^{P}D_{\alpha} f(t) \right] \quad \text{and} \quad {}^{CPC}_{0}D^{\alpha}_{t} f(t) = {}^{RL}_{0}I^{1-\alpha}_{t}\left[{}^{CP}D_{\alpha} f(t) \right], \tag{9}$$

it follows that to invert the fractional operators it will be sufficient to invert both the Riemann–Liouville integral and the proportional derivatives ${}^{P}D_{\alpha}$ and ${}^{CP}D_{\alpha}$. The Riemann–Liouville integral is inverted by the Riemann–Liouville derivative, and the inverse of the proportional derivative was constructed in ([15] (Lemma 1.9)). We give the latter result in the following Lemma.

Lemma 1 ([15])**.** *The inverse of the proportional derivative operator ${}^{P}D_{\alpha}$ is given by*

$${}^{P}_{a}I_{\alpha} f(t) = \int_a^t \exp\left[-\int_u^t \frac{K_1(\alpha,s)}{K_0(\alpha,s)}\,\mathrm{d}s \right] \frac{f(u)}{K_0(\alpha,u)}\,\mathrm{d}u,$$

and this satisfies the following inversion relations:

$${}^{P}D_{\alpha}\, {}^{P}_{a}I_{\alpha} f(t) = f(t), \qquad {}^{P}_{a}I_{\alpha}\, {}^{P}D_{\alpha} f(t) = f(t) - \exp\left(-\int_a^t \frac{K_1(\alpha,s)}{K_0(\alpha,s)}\,\mathrm{d}s \right) f(a). \tag{10}$$

In particular, for the constant-coefficient operator ${}^{CP}D_{\alpha}$, the integral formula is

$${}^{CP}_{a}I_{\alpha} f(t) = \frac{1}{K_0(\alpha)} \int_a^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-u) \right] f(u)\,\mathrm{d}u,$$

and the inversion relations are

$${}^{CP}D_{\alpha}\, {}^{CP}_{a}I_{\alpha} f(t) = f(t), \qquad {}^{CP}_{a}I_{\alpha}\, {}^{CP}D_{\alpha} f(t) = f(t) - \exp\left(-\frac{K_1(\alpha)}{K_0(\alpha)}(t-a) \right) f(a). \tag{11}$$

Note that, if $f(a) = 0$, then the operators ${}^{P}D_\alpha$, ${}^{P}_{a}I_\alpha$ and ${}^{CP}D_\alpha$, ${}^{CP}_{a}I_\alpha$ form pairs of two-sided inverses to each other.

Proposition 2. *The inverse operators to the fractional PC and CPC derivatives (6)–(7) are given by:*

$$ {}^{PC}_{0}I^{\alpha}_{t} f(t) = \int_0^t \exp\left[-\int_u^t \frac{K_1(\alpha,s)}{K_0(\alpha,s)}\,\mathrm{d}s \right] \frac{{}^{RL}_{0}D^{1-\alpha}_{u} f(u)}{K_0(\alpha,u)}\,\mathrm{d}u; \tag{12} $$

$$ {}^{CPC}_{0}I^{\alpha}_{t} f(t) = \frac{1}{K_0(\alpha)} \int_0^t \exp\left[-\frac{K_1(\alpha)}{K_0(\alpha)}(t-u) \right] {}^{RL}_{0}D^{1-\alpha}_{u} f(u)\,\mathrm{d}u. \tag{13} $$

These satisfy the following inversion relations:

$$ {}^{PC}_{0}D^{\alpha}_{t}\,{}^{PC}_{0}I^{\alpha}_{t} f(t) = f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}^{RL}_{0}I^{\alpha}_{t} f(t), $$

$$ {}^{PC}_{0}I^{\alpha}_{t}\,{}^{PC}_{0}D^{\alpha}_{t} f(t) = f(t) - \exp\left(-\int_0^t \frac{K_1(\alpha,s)}{K_0(\alpha,s)}\,\mathrm{d}s \right) f(0), $$

and, similarly,

$$ {}^{CPC}_{0}D^{\alpha}_{t}\,{}^{CPC}_{0}I^{\alpha}_{t} f(t) = f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}^{RL}_{0}I^{\alpha}_{t} f(t), $$

$$ {}^{CPC}_{0}I^{\alpha}_{t}\,{}^{CPC}_{0}D^{\alpha}_{t} f(t) = f(t) - \exp\left(-\frac{K_1(\alpha)}{K_0(\alpha)} t \right) f(0). $$

Proof. The definitions (12) and (13) can be written as operational compositions ${}^{PC}_{0}I^{\alpha}_{t} = {}^{P}_{0}I_\alpha \circ {}^{RL}_{0}D^{1-\alpha}_{t}$ and ${}^{CP}_{0}I^{\alpha}_{t} = {}^{CPC}_{0}I_\alpha \circ {}^{RL}_{0}D^{1-\alpha}_{t}$, so the inversion relations follow from composition of operators and the known inversion relations for the constituent parts of each operator:

$$ \begin{aligned} \left({}^{PC}_{0}D^{\alpha}_{t} \circ {}^{PC}_{0}I^{\alpha}_{t} \right) f(t) &= \left({}^{RL}_{0}I^{1-\alpha}_{t} \circ {}^{P}D_\alpha \right) \circ \left({}^{P}_{0}I_\alpha \circ {}^{RL}_{0}D^{1-\alpha}_{t} \right) f(t) \\ &= \left({}^{RL}_{0}I^{1-\alpha}_{t} \circ {}^{RL}_{0}D^{1-\alpha}_{t} \right) f(t) \\ &= f(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \lim_{t\to 0} {}^{RL}_{0}I^{\alpha}_{t} f(t); \\ \left({}^{PC}_{0}I^{\alpha}_{t} \circ {}^{PC}_{0}D^{\alpha}_{t} \right) f(t) &= \left({}^{P}_{0}I_\alpha \circ {}^{RL}_{0}D^{1-\alpha}_{t} \right) \circ \left({}^{RL}_{0}I^{1-\alpha}_{t} \circ {}^{P}D_\alpha \right) f(t) \\ &= \left({}^{P}_{0}I_\alpha \circ {}^{P}D_\alpha \right) f(t) \\ &= f(t) - \exp\left(-\int_0^t \frac{K_1(\alpha,s)}{K_0(\alpha,s)}\,\mathrm{d}s \right) f(0), \end{aligned} $$

and similarly for the CPC operators. Here we have used the composition expressions (9) for the PC and CPC derivatives, the inversion relations [10] for the Riemann–Liouville differintegrals, and the inversion relations (10) and (11) for the D_α and I_α operators. □

3.2. Inverting by Laplace Transform

An alternative way of inverting at least the CPC fractional operator is to use the Laplace transform and the result of Theorem 1. The following derivation using the Laplace transform is not rigorous, but it will provide us with an answer which we will then prove rigorously. In order to derive an appropriate expression for the inverse, we assume that $f(0) = 0$ and rewrite (8) as

$$\begin{aligned}\mathcal{L}\left[{}^{CPC}_{\ 0}D^{\alpha}_{t}f(t)\right] &= \left[\frac{K_1(\alpha)}{s} + K_0(\alpha)\right] s^{\alpha}\widehat{f}(s)\\ &= K_0(\alpha)\left[1 + \frac{K_1(\alpha)}{K_0(\alpha)}s^{-1}\right] s^{\alpha}\widehat{f}(s).\end{aligned}$$

Therefore, writing ${}^{CPC}_{\ 0}D^{\alpha}_{t}f(t) = g(t)$, we have

$$\begin{aligned}\widehat{f}(s) &= \left(K_0(\alpha)\left[1 + \frac{K_1(\alpha)}{K_0(\alpha)}s^{-1}\right]s^{\alpha}\right)^{-1}\widehat{g}(s)\\ &= \frac{1}{K_0(\alpha)}s^{-\alpha}\sum_{n=0}^{\infty}\left[\frac{-K_1(\alpha)}{K_0(\alpha)}s^{-1}\right]^n\widehat{g}(s)\\ &= \sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}s^{-\alpha-n}\widehat{g}(s).\end{aligned} \tag{14}$$

(This series converges only under the condition $\left|\frac{K_1(\alpha)}{K_0(\alpha)}s^{-\alpha}\right| < 1$, but we are performing only a formal derivation here. The series we will find in the t-domain will be convergent everywhere.) From here, there are two possible ways to proceed in order to write $f(t)$ in terms of $g(t)$.

One of them is to use the fact that the Laplace transform of the Riemann–Liouville fractional integral ${}^{RL}_{\ 0}I^{\alpha}_{t}g(t)$ is precisely $s^{-\alpha}\widehat{g}(s)$ for any positive number α. From (14), we therefore have the following series formula, following in the footsteps of [9,21] and related works:

$$f(t) = \sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}\,{}^{RL}_{\ 0}I^{\alpha+n}_{t}g(t).$$

The second method is to think of the right-hand side of (14) as a product of $\widehat{g}(s)$ with a function given by a power series, and then find the inverse Laplace transform of this power series in order to get a convolution expression for $f(t)$. We have

$$\begin{aligned}\widehat{f}(s) &= \left[\sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}s^{-\alpha-n}\right]\widehat{g}(s)\\ &= \mathcal{L}\left[\sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}\cdot\frac{t^{\alpha+n-1}}{\Gamma(\alpha+n)}\right]\widehat{g}(s)\\ &= \mathcal{L}\left[\frac{t^{\alpha-1}}{K_0(\alpha)}\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}t\right)^n\frac{1}{\Gamma(n+\alpha)}\right]\widehat{g}(s)\\ &= \mathcal{L}\left[\frac{t^{\alpha-1}}{K_0(\alpha)}E_{1,\alpha}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}t\right)\right]\widehat{g}(s),\end{aligned}$$

where we make use of the Mittag-Leffler type function $E_{\alpha,\beta}$ which is defined for $\alpha > 0$ by $E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n\alpha+\beta)}$.

Therefore, we find the following alternative expression for the inverse of the CPC derivative.

Theorem 2. *The inverse operator of the CPC fractional derivative is given by:*

$$\begin{aligned}{}^{CPC}_{0}I^{\alpha}_{t}f(t)&=\frac{1}{K_0(\alpha)}\int_0^t(t-\tau)^{\alpha-1}E_{1,\alpha}\left(-\frac{K_1(\alpha)}{K_0(\alpha)}(t-\tau)\right)f(\tau)\,\mathrm{d}\tau\\&=\sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t).\end{aligned}$$

This satisfies the following inversion relations:

$$\begin{aligned}{}^{CPC}_{0}D^{\alpha}_{t}\,{}^{CPC}_{0}I^{\alpha}_{t}f(t)&=f(t)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\lim_{t\to0}{}^{RL}_{0}I^{\alpha}_{t}f(t);\\{}^{CPC}_{0}I^{\alpha}_{t}\,{}^{CPC}_{0}D^{\alpha}_{t}f(t)&=f(t)-\exp\left(\frac{-K_1(\alpha)}{K_0(\alpha)}t\right)f(0).\end{aligned}$$

Proof. The equivalence of these two expressions for the fractional integral operator is clear from the series formula approach of [9,21]. In the case where both $f(t)$ and ${}^{CPC}_{0}I^{\alpha}_{t}f(t)$ are zero at $t=0$, the above work with Laplace transforms shows that ${}^{CPC}_{0}I^{\alpha}_{t}$ is precisely the two-sided inverse of ${}^{CPC}_{0}D^{\alpha}_{t}$. In general, we can use the composition properties of Riemann–Liouville fractional integrals and derivatives to prove the inversion results from the series formula:

$$\begin{aligned}&{}^{CPC}_{0}D^{\alpha}_{t}\,{}^{CPC}_{0}I^{\alpha}_{t}f(t)\\&=\left[K_1(\alpha)\,{}^{RL}_{0}I^{1-\alpha}_{t}+K_0(\alpha)\,{}^{C}_{0}D^{\alpha}_{t}\right]\sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t)\\&=\sum_{n=0}^{\infty}\frac{K_1(\alpha)^{n+1}}{K_0(\alpha)^{n+1}}(-1)^n\,{}^{RL}_{0}I^{1-\alpha}_{t}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t)+\sum_{n=0}^{\infty}\frac{K_1(\alpha)^{n}}{K_0(\alpha)^{n}}(-1)^n\,{}^{C}_{0}D^{\alpha}_{t}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t)\\&=-\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n+1}{}^{RL}_{0}I^{n+1}_{t}f(t)+\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n}{}^{C}_{0}D^{\alpha}_{t}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t).\end{aligned}$$

From standard properties of Riemann–Liouville operators, we have

$${}^{C}_{0}D^{\alpha}_{t}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t)={}^{RL}_{0}I^{1-\alpha}_{t}\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{RL}_{0}I^{\alpha+n}_{t}f(t)={}^{RL}_{0}I^{1-\alpha}_{t}\,{}^{RL}_{0}I^{\alpha+n-1}_{t}f(t).$$

For any $n\geq1$, this is simply equal to ${}^{RL}_{0}I^{n}_{t}f(t)$, while for $n=0$ we have

$${}^{C}_{0}D^{\alpha}_{t}\,{}^{RL}_{0}I^{\alpha}_{t}f(t)={}^{RL}_{0}I^{1-\alpha}_{t}\,{}^{RL}_{0}D^{1-\alpha}_{t}f(t)=f(t)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\lim_{t\to0}{}^{RL}_{0}I^{\alpha}_{t}f(t).$$

Therefore,

$$\begin{aligned}&{}^{CPC}_{0}D^{\alpha}_{t}\,{}^{CPC}_{0}I^{\alpha}_{t}f(t)\\&=-\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n+1}{}^{RL}_{0}I^{n+1}_{t}f(t)\\&\qquad+\sum_{n=1}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n}{}^{RL}_{0}I^{n}_{t}f(t)+f(t)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\lim_{t\to0}{}^{RL}_{0}I^{\alpha}_{t}f(t)\\&=f(t)-\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\lim_{t\to0}{}^{RL}_{0}I^{\alpha}_{t}f(t),\end{aligned}$$

as stated. For the other direction of composition,

$$
\begin{aligned}
&{}^{CPC}_{0}I^{\alpha}_{t}\,{}^{CPC}_{0}D^{\alpha}_{t}f(t)\\
&=\sum_{n=0}^{\infty}\frac{\big(-K_1(\alpha)\big)^n}{K_0(\alpha)^{n+1}}\,{}^{RL}_{0}I^{\alpha+n}_{t}\Big[K_1(\alpha)\,{}^{RL}_{0}I^{1-\alpha}_{t}f(t)+K_0(\alpha)\,{}^{C}_{0}D^{\alpha}_{t}f(t)\Big]\\
&=\sum_{n=0}^{\infty}\frac{K_1(\alpha)^{n+1}}{K_0(\alpha)^{n+1}}(-1)^n\,{}^{RL}_{0}I^{\alpha+n}_{t}\,{}^{RL}_{0}I^{1-\alpha}_{t}f(t)+\sum_{n=0}^{\infty}\frac{K_1(\alpha)^{n}}{K_0(\alpha)^{n}}(-1)^n\,{}^{RL}_{0}I^{\alpha+n}_{t}\,{}^{C}_{0}D^{\alpha}_{t}f(t)\\
&=\sum_{n=0}^{\infty}\frac{K_1(\alpha)^{n+1}}{K_0(\alpha)^{n+1}}(-1)^n\,{}^{RL}_{0}I^{n+1}_{t}f(t)+\sum_{n=0}^{\infty}\frac{K_1(\alpha)^{n}}{K_0(\alpha)^{n}}(-1)^n\,{}^{RL}_{0}I^{n+1}_{t}f'(t)\\
&=-\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n+1}{}^{RL}_{0}I^{n+1}_{t}f(t)+\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n}{}^{RL}_{0}I^{n}_{t}\big(f(t)-f(0)\big)\\
&=f(t)-\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n}{}^{RL}_{0}I^{n}_{t}f(0)\\
&=f(t)-\sum_{n=0}^{\infty}\left(\frac{-K_1(\alpha)}{K_0(\alpha)}\right)^{n}\frac{t^n}{n!}f(0)=f(t)-\exp\left(\frac{-K_1(\alpha)}{K_0(\alpha)}t\right)f(0).
\end{aligned}
$$

Thus, both inversion relations are as stated. □

Remark 3. *The inversion relations in Theorem 2 and those in Proposition 2 for the CPC operators are identical. This indicates that both ways of defining the CPC fractional integral ${}^{CPC}_{0}I^{\alpha}_{t}$ are the same, which can be confirmed for certain by expanding the exponential function in (13) as a power series.*

The PC fractional integral ${}^{PC}_{0}I^{\alpha}_{t}$ cannot be written in a similar form to Theorem 2, because the integral formula (12) is not directly a convolution.

Theorem 3. *The CPC fractional integral operator ${}^{CPC}_{0}I^{\alpha}_{t}$ is a special case of the Prabhakar integral operator due to [22], namely with the four parameters of Prabhakar being respectively 1, α, 1, and $\frac{-K_1(\alpha)}{K_0(\alpha)}$.*

Proof. The Prabhakar fractional integral is defined [22,23] by:

$$
{}^{\mathcal{P}}_{a}I^{\mu,\nu,\rho,\sigma}_{t}f(t)=\int_a^t (t-\tau)^{\nu-1}E^{\rho}_{\mu,\nu}\big(\sigma(t-\tau)^{\mu}\big)f(\tau)\,\mathrm{d}\tau,
$$

for $\mu>0$ and $\nu>0$. Putting $\mu=1$, $\nu=\alpha$, $\rho=1$, and $\sigma=\frac{-K_1(\alpha)}{K_0(\alpha)}$, we obtain immediately

$$
{}^{CPC}_{0}I^{\alpha}_{t}f(t)={}^{\mathcal{P}}_{a}I_t^{1,\alpha,1,\frac{-K_1(\alpha)}{K_0(\alpha)}}f(t).
$$

□

The class of Prabhakar operators is large enough to include several important types of fractional calculus within it [24]. It is interesting to see that the hybrid CPC operator is also a special case falling within the Prabhakar class. The general PC operator, however, seems to have a different type of behaviour from Prabhakar.

4. Eigenfunctions of the CPC Operator

In this section, we solve some differential equations with our new CPC derivative, applying Laplace transform methods and using Theorem 1.

Example 1. *Let us try to solve the following simple fractional differential equation:*

$$ {}^{CPC}_{\ \ 0}D^{\alpha}_{t} f(t) = 0, \qquad f(0) = A. \tag{15} $$

Applying the Laplace transform to both sides, using (8) *and the condition* $f(0) = A$*, we find*

$$ \left[\frac{K_1(\alpha)}{s} + K_0(\alpha)\right] s^{\alpha} \widehat{f}(s) - K_0(\alpha) s^{\alpha-1} A = 0, $$

and therefore

$$ \widehat{f}(s) = \frac{K_0(\alpha) s^{\alpha-1} A}{K_1(\alpha) s^{\alpha-1} + K_0(\alpha) s^{\alpha}} = \frac{A}{s + \frac{K_1(\alpha)}{K_0(\alpha)}}. $$

Taking the inverse Laplace transform, we find

$$ f(t) = A \exp\left(-\frac{K_1(\alpha)}{K_0(\alpha)} t\right). $$

Therefore, the set of functions with zero CPC derivative is a specific set of exponential functions. This is an unexpected result, since exponential functions do not usually have the property of differentiating to zero.

In classical calculus, exponential functions serve the role of eigenfunctions with respect to the usual derivative operator. A natural question to ask is then, what are the eigenfunctions of the operators studied in this paper?

Example 2. *Let us try to solve, for an arbitrary constant* λ*,*

$$ {}^{CPC}_{\ \ 0}D^{\alpha}_{t} f(t) = \lambda f(t), \qquad f(0) = 1. \tag{16} $$

Applying the Laplace transform to both sides, using (8) *and the condition* $f(0) = 1$*, we find*

$$ \left[\frac{K_1(\alpha)}{s} + K_0(\alpha)\right] s^{\alpha} \widehat{f}(s) - K_0(\alpha) s^{\alpha-1} = \lambda \widehat{f}(s), $$

and therefore

$$\begin{aligned}
\widehat{f}(s) &= \frac{K_0(\alpha)s^{\alpha-1}}{K_1(\alpha)s^{\alpha-1}+K_0(\alpha)s^{\alpha}-\lambda} \\
&= \frac{s^{-1}}{1-\frac{\lambda}{K_0(\alpha)}s^{-\alpha}+\frac{K_1(\alpha)}{K_0(\alpha)}s^{-1}} \\
&= s^{-1}\left(1-\frac{s^{-\alpha}-K_1(\alpha)s^{-1}}{K_0(\alpha)}\right)^{-1} \\
&= s^{-1}\sum_{n=0}^{\infty}\left[\frac{\lambda s^{-\alpha}-K_1(\alpha)s^{-1}}{K_0(\alpha)}\right]^n \\
&= s^{-1}\sum_{n=0}^{\infty}\frac{1}{K_0(\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}\left[\lambda s^{-\alpha}\right]^{n-k}\left[-K_1(\alpha)s^{-1}\right]^k \\
&= s^{-1}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(-K_1(\alpha)\right)^k\lambda^{n-k}}{K_0(\alpha)^n}\binom{n}{k}s^{-\alpha(n-k)-k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(-K_1(\alpha)\right)^k\lambda^{n-k}}{K_0(\alpha)^n}\binom{n}{k}s^{-\alpha n+\alpha k-k-1}.
\end{aligned}$$

Taking the inverse Laplace transform term by term, we find

$$f(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left(-K_1(\alpha)\right)^k\lambda^{n-k}}{K_0(\alpha)^n}\binom{n}{k}\frac{t^{\alpha n-\alpha k+k}}{\Gamma(\alpha n-\alpha k+k+1)}.$$

Relabelling as $l = n-k$ enables the double sum over n and k to be rearranged as an independent double sum over k and l both going from 0 to ∞:

$$\begin{aligned}
f(t) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left(-K_1(\alpha)\right)^k\lambda^{l}}{K_0(\alpha)^{k+l}}\frac{(k+l)!}{k!l!}\cdot\frac{t^{\alpha l+k}}{\Gamma(\alpha l+k+1)} \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(k+l)!}{k!l!}\left[\frac{-K_1(\alpha)}{K_0(\alpha)}t\right]^k\left[\frac{\lambda}{K_0(\alpha)}t^{\alpha}\right]^l\frac{1}{\Gamma(\alpha l+k+1)}.
\end{aligned}$$

This series can be written in terms of the bivariate Mittag-Leffler function which was defined very recently in [25]:

$$f(t) = E^{1}_{\alpha,1,1}\left(\frac{\lambda}{K_0(\alpha)}t^{\alpha},\frac{-K_1(\alpha)}{K_0(\alpha)}t\right). \tag{17}$$

The above example is important because the differential equation (15) should give the exponential-type function for our operator. For the original Caputo derivative, the corresponding differential equation

$${}^{C}_{0}D^{\alpha}_{t}f(t) = f(t), \qquad f(0) = 1,$$

has its solution given by the celebrated Mittag-Leffler function:

$$f(t) = E_{\alpha}(t^{\alpha}).$$

Therefore, the function we discovered in the above example is the equivalent of the Mittag-Leffler function for our new CPC fractional derivative. We note also that, by putting $\lambda = 0$, we would recover from (17) the exponential function that arose as a solution in the previous Example.

The bivariate Mittag-Leffler function which we find emerging here is already known [25] to arise naturally from the modelling of certain real-world systems. Motivated by this new connection, we hope that dynamical systems using our hybrid fractional operators may be useful in fitting efficiently to different types of real data, e.g., in mathematical biology systems.

5. Conclusions

We presented two new fractional derivatives to the literature in this work, which are closely related to each other and may be expressed as a combination (or hybridisation) of existing fractional operators in several different ways. They were first formulated by taking the Caputo fractional derivative and replacing the simple derivative by a derivative of proportional type. They can also be written as a composition of a Riemann–Liouville fractional integral with this proportional-type derivative. One of them, which we called CPC as opposed to PC, is a linear combination of a Riemann–Liouville integral with a Caputo derivative.

In studying these operators, we learned that the CPC type derivative is usually easier to handle than the PC type derivative. We calculated its Laplace transform, and found two different (equivalent) formulae for its inverse operator, as compared with just one formula for the inverse of the PC derivative.

There is a deep connection between fractional calculus and Mittag-Leffler functions, and this was emphasised here when we came to solve some differential equations using the CPC derivative. The solution, calculated using Laplace transform methods, is expressible in terms of a new bivariate Mittag-Leffler function which was defined very recently and is already discovering various applications.

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References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: San Diego, CA, USA, 2006.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
3. Sun, H.G.; Zhang, Y.; Baleanu, D.; Chen, W.; Chen, Y.Q. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *64*, 213–231.
4. Akgül, A. A novel method for a fractional derivative with non-local and non-singular kernel. *Chaos, Solitons Fractals* **2018**, *114*, 478–482.
5. Akgül, E.K. Solutions of the linear and nonlinear differential equations within the generalized fractional derivatives. *Chaos* **2019**, *29*, 023108.
6. Diethelm, K.; Ford, N.J. Multi-order fractional differential equations and their numerical solution. *Appl. Math. Comput.* **2004**, *154*, 621–640.
7. Fernandez, A.; Baleanu, D.; Fokas, A.S. Solving PDEs of fractional order using the unified transform method. *Appl. Math. Comput.* **2018**, *339C*, 738–749.
8. Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel, theory and application to heat transfer model. *Therm. Sci.* **2016**, *20*, 763-769.
9. Fernandez, A.; Özarslan, M.A.; Baleanu, D. On fractional calculus with general analytic kernels. *Appl. Math. Comput.* **2019**, *354*, 248–265.
10. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon & Breach Science Publishers: Yverdon, Switzerland, 1993. Originally in Russian: Nauka i Tekhnika, Minsk, 1987.
11. Baleanu, D.; Fernandez, A. On fractional operators and their classifications. *Mathematics* **2019**, *7*, 830.
12. Diethelm, K. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*; Springer: Berlin, Germany, 2010.
13. Khalil, R.; Al Horani, M.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70.
14. Hilfer, R.; Luchko, Y. Desiderata for Fractional Derivatives and Integrals. *Mathematics* **2019**, *7*, 149.

15. Anderson, D.R.; Ulness, D.J. Newly Defined Conformable Derivatives. *Adv. Dyn. Syst. Appl.* **2015**, *10*, 109–137.
16. Cai, W.; Chen, W.; Xu, W. Characterizing the creep of viscoelastic materials by fractal derivative models. *Int. J. Non-Linear Mech.* **2016**, *87*, 58–63.
17. Atangana, A. Fractal-fractional differentiation and integration: Connecting fractal calculus and fractional calculus to predict complex system. *Chaos Solitons Fractals* **2017**, *102*, 396–406.
18. Liang, Y.; Su, N.; Chen, W. A time-space Hausdorff derivative model for anomalous transport in porous media. *Fract. Calc. Appl. Anal.* **2019**, *22*, 1517–1536.
19. Caputo, M. Linear models of dissipation whose Q is almost frequency independent: II. *Geophys. J. R. Astron. Soc.* **1967**, *13*, 529–539.
20. Baleanu, D.; Diethelm, K.; Scalas, E.; Trujillo, J.J. *Fractional Calculus: Models and Numerical Methods*; World Scientific Publishing: Singapore, 2012.
21. Baleanu, D.; Fernandez, A. On some new properties of fractional derivatives with Mittag-Leffler kernel. *Commun. Nonlinear Sci. Numer. Simul.* **2018**, *59*, 444–462.
22. Prabhakar, T.R. A singular integral equation with a generalized Mittag Leffler function in the kernel. *Yokohama Math. J.* **1971**, *19*, 7–15.
23. Kilbas, A.A.; Saigo, M.; Saxena, R.K. Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transform. Spec. Funct.* **2004**, *15*, 31–49.
24. Fernandez, A.; Baleanu, D.; Srivastava, H.M. Series representations for fractional-calculus operators involving generalised Mittag-Leffler functions. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *67*, 517–527.
25. Fernandez, A.; Kürt, C.; Özarslan, M.A. A naturally emerging bivariate Mittag-Leffler function and associated fractional-calculus operators. *arXiv* **2020**, arXiv:2002.12171.