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Post Quantum Integral Inequalities of Hermite-Hadamard-Type Associated with Co-Ordinated Higher-Order Generalized Strongly Pre-Invex and Quasi-Pre-Invex Mappings

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Abstract: By using the contemporary theory of inequalities, this study is devoted to proposing a number of refinements inequalities for the Hermite-Hadamard's type inequality and conclude explicit bounds for two new definitions of (p_1p_2, q_1q_2)-differentiable function and (p_1p_2, q_1q_2)-integral for two variables mappings over finite rectangles by using pre-invex set. We have derived a new auxiliary result for (p_1p_2, q_1q_2)-integral. Meanwhile, by using the symmetry of an auxiliary result, it is shown that novel variants of the the Hermite-Hadamard type for (p_1p_2, q_1q_2)-differentiable utilizing new definitions of generalized higher-order strongly pre-invex and quasi-pre-invex mappings. It is to be acknowledged that this research study would develop new possibilities in pre-invex theory, quantum mechanics and special relativity frameworks of varying nature for thorough investigation.

Keywords: quantum calculus; post quantum calculus Hermite-Hadamard type inequality; strongly pre-invex mappings; co-ordinated generalized higher-order strongly pre-invex mappings; co-ordinated generalized higher-order strongly quasi-pre-invex mappings

1. Introduction

In the study of quantum calculus, it is the non-limited analysis of calculus and it is also recognized as q-calculus. We get the initial mathematical formulas in q-calculus as q reaches 1^-. The commencement of the analysis of q-calculus was initiated by Euler (1707–1783). Subsequently, Jackson [1] was launched the idea of q-integrals in systematic way. The aforementioned results lead to an intensive investigation on q-calculus in the twentieth century. The idea of q-calculus is used in numerous areas in mathematics and physics such as number theory, orthogonal polynomials, combinatory, basic hypergeometric functions, mechanics, and quantum and relativity theory. Tariboon et al. [2,3] discover-able of q-derivatives on $[\xi_1, \xi_2]$ of $\mathbb{R}$, unifies and modify numerous new concepts of classical convexity. From the last few years, the topic of q-calculus has become an interesting topic for many researchers and several new results have been established in the literature, see for instance [4–10] and the references cited therein. In addition, recent developments in the framework of the above concepts have been identified by Tunç and Göv [11–13], named as ((p,q)-derivative and (p,q)-integral over $[\phi,\psi]$ of $\mathbb{R}$. Kunt et al. [14] is obtained generalized (p,q)-Hermite-Hadamard inequalities on the finite interval and some relevant results that are linked to (p,q)-midpoint type inequality. The (p,q)-calculus is currently being reviewed predominantly by many researchers, and a variety of refinements and generalizations can be identified in the research [15–17] and the references cited therein.

Integral inequalities are considered a fabulous tool for constructing the qualitative and quantitative properties in the field of pure and applied mathematics. The continuous growth of interest has occurred in order to meet the requirements in needs of fertile applications of these inequalities. Such inequalities had been studied by many researchers who in turn used various techniques for the sake of exploring and offering these inequalities [18–32].

Recalling the definition of convex function as follows:

Let $\Psi : K \subseteq \mathbb{R} \to \mathbb{R}$. The function Ψ said to be convex on $K(convex\, set)$, if the inequality

$$\Psi(\tau\xi_1 + (1-\tau)\xi_2) \le \tau\Psi(\xi_1) + (1-\tau)\Psi(\xi_2),$$

holds for all $\xi_1, \xi_2 \in J$ and $\tau \in [0,1]$.

For convex functions, many equality or inequalities have been established by many authors; for example, Hardy type inequality, Ostrowski type inequality, Olsen type inequality and Gagliardo–Nirenberg type inequality but the most celebrated and significant inequality is the Hermite–Hadamard type inequality [33], which is defined as:

$$\Psi\left(\frac{\xi_1+\xi_2}{2}\right) \le \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi(\tau)d\tau \le \frac{\Psi(\xi_1)+\Psi(\xi_2)}{2}, \tag{1}$$

If Ψ is a concave function, the two inequalities can be held in the reverse direction. These inequalities have been extensively improved and generalized. Since then numerous scientists have broadly used the ideas of (1) and received several new generalizations via convex functions and their refinements, we refer the reader to References [34–36].

Because of recent progress in convexity, Hanson [37] was contemplated and investigated the notion of invex functions,which is generalization of convex functions. The definition of pre-invex functions was introduced by Weir et al. [38] and implemented in non-linear programming to define adequate optimal conditions and duality. The well-known C condition was introduced by Mohan et al. [39].

The notion of strongly convex functions was contemplated and investigated by Polyak [40], which has a significant contribution in fitting most machine learning models that involves solving some sort of optimization problem and concerned areas. A solution is focused on the strongly convex functions of nonlinear complementarity problem [41]. Zu and Marcotte [42] explored the convergence by applying the strong convex functional principle of iterative techniques for addressing variational inequalities and equilibrium issues. The novel and innovative application of the characterization of the inner product space was discovered by Nikodem and Pales in [43] with the help of strongly convex functions. The convergence of stochastic gradient descent for the class of functions satisfying the Polyak-Lojasiewicz condition depends on strongly-convex functions as well as a broad range of non-convex functions including those used in machine learning applications [44]. For more features and utilities of the strongly convex functions, see References [45–50] and the references therein.

During this whole paper we are going to take $\mathcal{S} = [\xi_1, \xi_2] \subseteq \mathbb{R}$ and let $\mathcal{H} = [\xi_1, \xi_2] \times [\xi_3, \xi_4] \subseteq \mathbb{R}^2$ with constants $q, q_k \in (0,1)$, $k = 1,2$

Noor et al. [51] has proven the quantum integral inequalities of Hermite-Hadamard for pre-invex functions.

Theorem 1. *Let $\Psi : \mathcal{S} \to \mathbb{R}$ is a pre-invex function on $\mathcal{S}$. Then, we get*

$$\Psi\left(\frac{2\xi_1+\eta(\xi_2,\xi_1)}{1+q}\right) \leq \frac{1}{\eta(\xi_2,\xi_1)}\int_{\xi_1}^{\xi_1+\eta(\xi_2,\xi_1)} \Psi(s)_{\xi_1}d_q s \leq \frac{q\Psi(\xi_1)+\Psi(\xi_2)}{1+q}. \quad (2)$$

The following Hermite-Hadamard type inequalities for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$ have been introduced by Dragomir [52]

Theorem 2. *Assume that $\Psi : \mathcal{H} \to \mathbb{R}$ is co-ordinated convex function on $\mathcal{H}$ with $\xi_1 < \xi_2$ and $\xi_3 < \xi_4$. Then, one has the inequalities:*

$$\begin{aligned}
&\Psi\left(\frac{\xi_1+\xi_2}{2},\frac{\xi_3+\xi_4}{2}\right) \leq \frac{1}{2}\left[\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi\left(s,\frac{\xi_3+\xi_4}{2}\right)ds + \frac{1}{\xi_3-\xi_4}\int_{\xi_3}^{\xi_4}\Psi\left(\frac{\xi_1+\xi_2}{2},t\right)dt\right] \\
&\leq \frac{1}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Psi(s,t)dsdt \leq \frac{1}{4}\left[\frac{1}{\xi_2-\xi_1}\left(\int_{\xi_1}^{\xi_2}\Psi(s,\xi_3)ds + \int_{\xi_1}^{\xi_2}\Psi(s,\xi_4)ds\right)\right. \\
&\left.+\frac{1}{\xi_4-\xi_3}\left(\int_{\xi_3}^{\xi_4}\Psi(\xi_1,t)dt + \int_{\xi_3}^{\xi_4}\Psi(\xi_2,t)dt\right)\right] \leq \frac{\Psi(\xi_1,\xi_3)+\Psi(\xi_2,\xi_3)+\Psi(\xi_1,\xi_4)+\Psi(\xi_2,\xi_4)}{4}.
\end{aligned}$$

In addition, Humaira et al. [53] showed that the q_1q_2-Hermite-Hadamard type inequalities were resolved by the use of quantum calculus for the co-ordinated convex functions.

Theorem 3. *Let $\Psi : \mathcal{H} \to \mathbb{R}$ is co-ordinated convex on $\mathcal{H}$, then the following inequalities holds*

$$\begin{aligned}
&\Psi\left(\frac{\xi_1 q_1+\xi_2}{1+q_1},\frac{\xi_3 q_2+\xi_4}{1+q_2}\right) \\
&\leq \frac{1}{2}\left[\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi\left(z,\frac{\xi_3 q_2+\xi_4}{1+q_2}\right){}_{\xi_1}\xi_{4q_1}z + \frac{1}{\xi_4-\xi_3}\int_{\xi_3}^{\xi_4}\Psi\left(\frac{\xi_1 q_1+\xi_2}{1+q_1},w\right){}_{\xi_3}dq_2 w\right] \\
&\leq \frac{1}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Psi(z,w)_{\xi_3}d_{q_2}w_{\xi_1}d_{q_1}z \leq \frac{q_2}{2(1+q_2)(\xi_2-\xi_1)}\int_{\xi_1}^{\xi_2}\Psi(z,\xi_3)_{\xi_1}d_{q_1}z \\
&+\frac{1}{2(1+q_2)(\xi_2-\xi_1)}\int_{\xi_1}^{\xi_2}\Psi(z,\xi_4)_{\xi_1}d_{q_1}z + \frac{q_1}{2(1+q_1)(\xi_4-\xi_3)}\int_{\xi_3}^{\xi_4}\Psi(\xi_1,w)_{\xi_3}d_{q_2}w \\
&+\frac{1}{2(1+q_1)(\xi_4-\xi_3)}\int_{\xi_3}^{\xi_4}\Psi(\xi_2,w)_{\xi_3}d_{q_2}w \leq \frac{q_1q_2\Psi(\xi_1,\xi_3)+q_1\Psi(\xi_1,\xi_4)+q_2\Psi(\xi_2,\xi_3)+\Psi(\xi_2,\xi_4)}{(1+q_1)(1+q_2)}.
\end{aligned}$$

Following forward with this propensity, we introduce two more general concepts of generalized higher-order strongly pre-invex and quasi-pre-invex mappings. Several novel versions of Hermite-Hadamard inequality are established that could be used to identify a uniform reflex (p_1p_2, q_1q_2)-integral. These innovations are a mixture of an identity-based auxiliary result that corresponds with co-ordenated generalized higher-order strongly pre-invex and quasi-pre-invex mappings. Furthermore there are mathematical approximations of the new Definitions 10 and 11 are introduced for (p_1p_2, q_1q_2)-differentiable function and (p_1p_2, q_1q_2)-integral for two variables mappings over finite rectangles by using pre-invex set. These new definitions will open new doors for pre-invexity and (p,q)-calculus for two variables functions over the finite rectangles in the plane $\mathbb{R}^2$. An intriguing feature of the present investigation is that it has a potential connection with quantum mechanics and the special theory of relativity, see [54].

2. Preliminaries

Throughout this paper, we are using continuous bifunctions $\eta(.,.), \eta_1(.,.), \eta_2(.,.) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ Mititelu [55] identified as follows the definition of invex sets

Definition 1. *If $\Omega_\eta \subset \mathbb{R}^n$, then $\Omega_\eta \subset \mathbb{R}^n$ is said to be invex set*

$$\xi_1 + v\eta(\xi_2, \xi_1) \in \Omega_\eta, \quad \forall \xi_1, \xi_2 \in \Omega_\eta, \ v \in [0, 1].$$

Often known as the η-connected set is the η-connected set. Suppose $\eta(\xi_1, \xi_2) = \xi_2 - \xi_1$ implies that reverse does not apply.

Weir and Mond [38], has been introduced the definition of pre-invex mapping.

Definition 2. *Let a mapping $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is called pre-invex if*

$$\Psi\left(\xi_1 + v\eta(\xi_2, \xi_1)\right) \leq (1 - v)\Psi(\xi_1) + v\Psi(\xi_2)$$

for all $\xi_1, \xi_2 \in \Omega_\eta, \ v \in [0, 1]$.

For information see [4,37–39,55].

Noor et al. [56] promoted the idea of strongly pre-invex mappings.

Definition 3. *Let a mapping $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is called strongly pre-invex for some $\mu > 0$ if*

$$\Psi\left(\xi_1 + v\eta(\xi_2, \xi_1)\right) \leq (1 - v)\Psi(\xi_1) + v\Psi(\xi_2) - \mu v(1 - v)\|\eta(\xi_2, \xi_1)\|^2$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $v \in [0, 1]$.

Humaira et al.[57] established a completely new definitions which mixes the above mentioned pre-invex mappings with highly pre-invex mappings

Definition 4. *Let a mapping $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is called generalized higher-order strongly pre-invex for some $\mu \geq 0$ and $\theta > 0$ if*

$$\Psi\left(\xi_1 + v\eta(\xi_2, \xi_1)\right) \leq (1 - v)\Psi(\xi_1) + v\Psi(\xi_2) - \mu v(1 - v)\|\eta(\xi_2, \xi_1)\|^\theta$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $v \in [0, 1]$.

We are now discussing certain special cases.

(I). As we choose $\mu = 0$, in the Definition 4, then the Definition 2 is retrieved.
(II). As we choose $\theta = 2$, in the Definition 4 then Definition 4 will be convert into below Definition, witch is describe as

Definition 5. *Let a mapping $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is called generalized strongly pre-invex for some $\mu \geq 0$ if*

$$\Psi\left(\xi_1 + v\eta(\xi_2, \xi_1)\right) \leq (1 - v)\Psi(\xi_1) + v\Psi(\xi_2) - \mu v(1 - v)\|\eta(\xi_2, \xi_1)\|^2$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $v \in [0, 1]$.

(III). If we choose $\eta(\xi_2, \xi_1) = \xi_2 - \xi_1$, then we obtain the following Definition

Definition 6. *Let a mapping $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ is called generalized higher-order strongly convex for some $\mu \geq 0$ and $\theta > 0$ if*

$$\Psi\left((1 - v)\xi_1 + \tau\xi_2\right) \leq (1 - v)\Psi(\xi_1) + v\Psi(\xi_2) - \mu v(1 - v)\|\xi_2 - \xi_1\|^\theta$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $v \in [0, 1]$.

Definition 7. *Let a mapping* $\Psi : \Omega_\eta \subset \mathbb{R}^n \to \mathbb{R}$ *is called generalized higher-order strongly quasi-pre-invex for some* $\mu \geq 0$ *and* $\theta > 0$ *if*

$$\Psi\left(\xi_1 + v\eta(\xi_2, \xi_1)\right) \leq \max(\Psi(\xi_1), \Psi(\xi_2)) - \mu v(1-v)\|\eta(\xi_2, \xi_1)\|^\theta$$

for all $\xi_1, \xi_2 \in \Omega_\eta$, $v \in [0,1]$.

Let us recall some simple concepts and characteristics of (p,q) derivatives and (p,q) -integral .

Definition 8 ([11])**.** *Let* $\Psi : \mathcal{S} \to \mathbb{R}$ *is a continuous mapping, the* (p,q)*-derivative of* Ψ *at* $\tau \in [\xi_1, \xi_2]$ *is describe as*

$${}_{\xi_1}D_{p,q}\Psi(v) = \frac{\Psi(pv + (1-p)\xi_1) - \Psi(qv + (1-q)\xi_1)}{(p-q)(v-\xi_1)}, \quad v \neq \xi_1.$$

Since for a continuous mapping $\Psi : \mathcal{S} \to \mathbb{R}$*, we define* ${}_{\xi_1}D_{p,q}\Psi(\xi_1) = \lim_{v \to \xi_1} D_{p,q}\Psi(v)$.

Definition 9 ([11])**.** *Let a mapping* $\Psi : \mathcal{S} \to \mathbb{R}$ *be continuous, the* (p,q)*-integral on* $\mathcal{S}$ *is describe as*

$$\int_{\xi_1}^{v} \Psi(x)_{\xi_1} d_{p,q}x = (p-q)(v-\xi_1) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \Psi\left(\frac{q^n}{p^{n+1}}v + \left(1 - \frac{q^n}{p^{n+1}}\right)\xi_1\right),$$

holds for $v \in \mathcal{S}$. If $\xi_3 \in (\xi_1, v)$, then the (p,q)-definite integral on $[\xi_3, v]$ is expressed as

$$\int_{\xi_3}^{v} \Psi(x)_{\xi_1} d_{p,q}x = \int_{\xi_1}^{v} \Psi(x)_{\xi_1} d_{p,q}x - \int_{\xi_1}^{\xi_3} \Psi(x)_{\xi_1} d_{p,q}x.$$

Now we develop the theory of (p_1p_2, q_1q_2)-differentiable function and (p_1p_2, q_1q_2)-integral for two variables mappings over finite rectangles by using pre-invex set and let $\mathcal{U} = [\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subseteq \mathbb{R}^2$ with constants $0 < q_k < p_k \leq 1, 1 \leq k \leq 2$

Definition 10. *Let a mapping* $\Psi : \mathcal{U} \to \mathbb{R}$ *be continuous in each variable; the partial* (p_1, q_1)*-,* (p_2, q_2)*- and* (p_1p_2, q_1q_2)*-derivatives at* $(s,t) \in \mathcal{U}$ *are, respectively, describe as:*

$$\begin{aligned}
\frac{{}_{\xi_1}\partial_{p_1,q_1}\Psi(s,t)}{{}_{\xi_1}\partial_{p_1,q_1}s} &= \frac{\Psi(\xi_1 + p_1\eta_1(s,\xi_1), t) - \Psi(\xi_1 + q_1\eta_1(s,\xi_1), t)}{(p_1 - q_1)\eta_1(s,\xi_1)}, \quad s \neq \xi_1 \\
\frac{{}_{\xi_3}\partial_{p_2,q_2}\Psi(s,t)}{{}_{\xi_3}\partial_{p_2,q_2}t} &= \frac{\Psi(s, \xi_3 + p_2\eta_2(\xi_3, t)) - \Psi(s, \xi_3 + q_2\eta_2(t, \xi_3))}{(p_2 - q_2)\eta_2(t,\xi_3)}, \quad t \neq \xi_3 \\
\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(s,t)}{{}_{\xi_1}\partial_{p_1,q_1}s\,{}_{\xi_3}\partial_{p_2,q_2}t} &= \frac{\Psi(\xi_1 + q_1\eta_1(s,\xi_1), \xi_3 + q_2\eta_2(t,\xi_3))}{(p_1-q_1)(p_2-q_2)\eta_1(s,\xi_1)\eta_2(t,\xi_3)} \\
&\quad - \frac{\Psi(\xi_1 + q_1\eta_1(s,\xi_1), \xi_3 + p\eta_2(t,\xi_3))}{(p_1-q_1)(p_2-q_2)\eta_1(s,\xi_1)\eta_2(t,\xi_3)} \\
&\quad - \frac{\Psi(\xi_1 + p_1\eta_1(s,\xi_1), \xi_3 + q_2\eta_2(t,\xi_3))}{(p_1-q_1)(p_2-q_2)\eta_1(s,\xi_1)\eta_2(t,\xi_3)} \\
&\quad + \frac{\Psi(\xi_1 + p_1\eta_1(s,\xi_1), \xi_3 + p_2\eta_2(t,\xi_3))}{(p_1-q_1)(p_2-q_2)\eta_1(s,\xi_1)\eta_2(t,\xi_3)}, \quad s \neq \xi_1, \quad t \neq \xi_3.
\end{aligned}$$

As well, higher-order partial derivatives are also presented.

Definition 11. *Let a mapping $\Psi : \mathcal{U} \to \mathbb{R}$ be continuous in each variable, then the definite (p_1p_2, q_1q_2)-integral on $\mathcal{U}$ is described as:*

$$\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Psi(z,w)_{\xi_1}d_{p_1,q_1}z_{\xi_3}d_{p_2,q_2}w = (p_1-q_1)(p_2-q_2)\eta_1(s,\xi_1)\eta_2(t,\xi_3) \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^{n+1}}\frac{q_2^m}{p_2^{m+1}}\Psi\left(\xi_1+\frac{q_1^n}{p_1^{n+1}}\eta_1(s,\xi_1),\xi_3+\frac{q_1^m}{p_1^{m+1}}\eta_2(t,\xi_3)\right) \tag{3}$$

for $(s,t) \in \mathcal{U}$.

3. Auxiliary Result

The following lemma plays a key role in establishing the main results of this paper. The identification is stated as follows.

Lemma 1. *Let $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a mapping such that (p_1p_2, q_1q_2)-derivatives exist on Λ^o (the interior of Λ) and $0 < q_k < p_k \le 1$ where $1 \le k \le 2$. Moreover, if $\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}$ be continuous and integrable on $[\xi_1,\xi_2]\times[\xi_3,\xi_4] \subset \triangle^o$, then:*

$$\Gamma_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi) := \frac{1}{(p_1+q_1)(p_2+q_2)}\left[\begin{array}{c} q_1q_2\Psi(\xi_1,\xi_3) \\ +p_2q_1\Psi(\xi_1,\xi_3+\eta_2(\xi_4,\xi_3)) \\ +p_1q_2\Psi(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3) \\ +p_1p_2\Psi(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3+\eta_2(\xi_4,\xi_3)) \end{array}\right]$$
$$-\frac{1}{p_1(p_2+q_2)\eta_1(\xi_2,\xi_1)}\left[\begin{array}{c} q_2\int_{\xi_1}^{\xi_1+p_1\eta_1(\xi_2,\xi_1)}\Psi(s,\xi_3)_{\xi_1}d_{p_1,q_1}s \\ +p_2\int_{\xi_1}^{\xi_1+p_1\eta_1(\xi_2,\xi_1)}\Psi(s,\xi_3+\eta_2(\xi_4,\xi_3))_{\xi_1}d_{p_1,q_1}s \end{array}\right]$$
$$-\frac{1}{p_2(p_1+q_1)\eta_2(\xi_4,\xi_3)}\left[\begin{array}{c} q_1\int_{\xi_3}^{\xi_3+p_2\eta_2(\xi_4,\xi_3)}\Psi(\xi_1,t)_{\xi_3}d_{p_2,q_2}t \\ +p_1\int_{\xi_3}^{\xi_3+p_2\eta_2(\xi_4,\xi_3)}\Psi(\xi_1+\eta_1(\xi_2,\xi_1),t)_{\xi_3}d_{p_2,q_2}t \end{array}\right]$$
$$+\frac{1}{p_1\eta_1(\xi_2,\xi_1)p_2\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+p_1\eta_1(\xi_2,\xi_1)}\int_{\xi_3}^{\xi_3+p_2\eta_2(\xi_4,\xi_3)}\Psi(s,t)_{\xi_3}d_{p_2,q_2}t_{\xi_1}d_{p_1,q_1}s$$

$$= \int_0^1\int_0^1(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho) \times \frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_1,q_2}\rho}{}_0d_{p_1,q_1}\varrho_0d_{p_2,q_2}\rho.$$

Where $\mathcal{K} = \frac{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}{(p_1+q_1)(p_2+q_2)}$

Proof. Utilizing Definitions 10 and 11 we obtain

$$\int_0^1\int_0^1(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_1,q_2}\rho}\,{}_0d_{p_1,q_1}\varrho_0d_{p_2,q_2}\rho$$

$$=\frac{1}{(p_1-q_1)(p_2-q_2)\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1\frac{(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)}{\varrho\rho}$$

$$\times(\Psi(\xi_1+\varrho q_1\eta_1(\xi_2,\xi_1),\xi_3+\rho q_2\eta_2(\xi_4,\xi_3))-\Psi(\xi_1+\varrho q_1\eta_1(\xi_2,\xi_1),\xi_3+\rho p_2\eta_2(\xi_4,\xi_3))$$

$$-\Psi(\xi_1+\varrho p_1\eta_1(\xi_2,\xi_1),\xi_3+\rho q_2\eta_2(\xi_4,\xi_3))+\Psi(\xi_1+\varrho p_1\eta_1(\xi_2,\xi_1),\xi_3+\rho p_2\eta_2(\xi_4,\xi_3)))\,{}_0d_{p_1,q_1}\varrho_0d_{p_2,q_2}\rho$$

$$=\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=0}^\infty\left(1-(p_1+q_1)\frac{q_1^n}{p_1^{n+1}}\right)\left(1-(p_2+q_2)\frac{q_2^m}{p_2^{m+1}}\right)$$
$$\times\left[\begin{array}{c}\Psi\left(\xi_1+\frac{q_1^{n+1}}{p_1^{n+1}}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^{m+1}}{p_2^{m+1}}\eta_2(\xi_4,\xi_3)\right)-\Psi\left(\xi_1+\frac{q_1^{n+1}}{p_1^{n+1}}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\\ \\ -\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^{m+1}}{p_2^{m+1}}\eta_2(\xi_4,\xi_3)\right)+\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\end{array}\right]$$

$$=\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=1}^\infty\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$-\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=0}^\infty\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$-\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=1}^\infty\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$+\frac{1}{\eta_2(\xi_4,\xi_3)\eta_1(\xi_2,\xi_1)}\sum_{n=0}^\infty\sum_{m=0}^\infty\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$-\frac{(p_1+q_1)}{q_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$+\frac{(p_1+q_1)}{q_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=0}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$+\frac{(p_1+q_1)}{p_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=1}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$-\frac{(p_1+q_1)}{p_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$-\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$+\frac{(p_2+q_2)}{p_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=0}^\infty\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$+\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=1}^\infty\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$-\frac{(p_2+q_2)}{p_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=0}^\infty\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$+\frac{(p_1+q_1)(p_2+q_2)}{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right).$$

$$\frac{(p_1+q_1)}{p_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^\infty\sum_{m=1}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)$$
$$=\frac{(p_1+q_1)}{p_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left[-\sum_{n=0}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),d\xi_3+\eta_2(\xi_4,\xi_3)\right)\right.$$
$$\left.+\sum_{n=0}^\infty\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3\right)\right]$$

$$+\frac{(p_1+q_1)}{p_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right), \tag{4}$$

$$-\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$=\frac{(p_2+q_2)\Psi(\xi_1,\xi_3+\eta_2\left(\xi_4,\xi_3\right)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}$$
$$-\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1,\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$-\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right), \tag{5}$$

$$\frac{(p_2+q_2)}{p_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$=-\frac{(p_2+q_2)}{p_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left[-\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)\right.$$
$$\left.+\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1,\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)\right]$$
$$+\frac{(p_2+q_2)}{p_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right), \tag{6}$$

$$\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$=-\frac{(p_2+q_2)\Psi(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\eta_2\left(\xi_4,\xi_3\right))}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}$$
$$+\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$+\frac{(p_2+q_2)}{q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right), \tag{7}$$

$$\frac{(p_1+q_1)(p_2+q_2)}{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$=-\frac{(p_1+q_1)(p_2+q_2)\Psi(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\eta_2\left(\xi_4,\xi_3\right))}{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}$$
$$-\frac{(p_1+q_1)(p_2+q_2)}{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$-\frac{(p_1+q_1)(p_2+q_2)}{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right)$$
$$+\frac{(p_1+q_1)(p_2+q_2)}{q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1\left(\xi_2,\xi_1\right),\xi_3+\frac{q_2^m}{p_2^m}\eta_2\left(\xi_4,\xi_3\right)\right), \tag{8}$$

$$
\begin{aligned}
&-\frac{(p_1+q_1)(p_2+q_2)}{p_2q_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\\
&=\frac{(p_1+q_1)(p_2+q_2)}{p_2q_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\\
&-\frac{(p_1+q_1)(p_2+q_2)}{p_2q_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right), \qquad (9)\\
&-\frac{(p_1+q_1)(p_2+q_2)}{p_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\\
&=\frac{(p_1+q_1)(p_2+q_2)}{p_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\eta_2(\xi_4,\xi_3)\right)\\
&-\frac{(p_1+q_1)(p_2+q_2)}{p_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right). \qquad (10)
\end{aligned}
$$

Using (4)–(10) and simplifying, we get:

$$
\int_0^1\int_0^1(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_1,q_2}\rho}\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho
$$

$$
=\frac{1}{(p_1+q_1)(p_2+q_2)}\left[\begin{array}{c} q_1q_2\Psi(\xi_1,\xi_3)+p_2q_1\Psi(\xi_1,\xi_3+\eta_2(\xi_4,\xi_3))\\ +p_1q_2\Psi(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3)\\ +p_1p_2\Psi(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3+\eta_2(\xi_4,\xi_3))\end{array}\right]
$$

$$
\begin{aligned}
&-\frac{(p_1+q_1)(p_1-q_1)}{p_1q_1\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3\right)\\
&-\frac{p_2(p_1+q_1)(p_1-q_1)}{p_1q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{n=0}^{\infty}\frac{q_1^n}{p_1^n}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\eta_2(\xi_4,\xi_3)\right)\\
&-\frac{(p_2+q_2)(p_2-q_2)}{p_2q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_3,\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\\
&-\frac{p_1(p_2+q_2)(p_2-q_2)}{p_2q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\sum_{m=0}^{\infty}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right)\\
&+\frac{(p_1+q_1)(p_2+q_2)(p_1-q_1)(p_2-q_2)}{p_1p_2q_1q_2\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q_1^n}{p_1^n}\frac{q_2^m}{p_2^m}\Psi\left(\xi_1+\frac{q_1^n}{p_1^n}\eta_1(\xi_2,\xi_1),\xi_3+\frac{q_2^m}{p_2^m}\eta_2(\xi_4,\xi_3)\right).
\end{aligned}
$$

$$= \frac{1}{q_1 q_2 \eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)} \begin{bmatrix} q_1 q_2 \Psi(\xi_1,\xi_3) \\ +p_2 q_1 \Psi(\xi_1,\xi_3+\eta_2(\xi_4,\xi_3)) \\ +p_1 q_2 \Psi(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3) \\ +p_1 p_2 \Psi(\xi_1+\eta_1(\xi_2,\xi_1),\xi_3+\eta_2(\xi_4,\xi_3)) \end{bmatrix}$$
$$- \frac{(p_1+q_1)}{p_1 q_1 \eta_1(\xi_2,\xi_1)^2 \eta_2(\xi_4,\xi_3)} \int_{\xi_1}^{\xi_1+p_1\eta_1(\xi_2,\xi_1)} \Psi(s,\xi_3) \quad {}_{\xi_1}d_{p_1,q_1}s$$
$$- \frac{p_2(p_1+q_1)}{p_1 q_1 q_2 \eta_1(\xi_2,\xi_1)^2 \eta_2(\xi_4,\xi_3)} \int_{\xi_1}^{\xi_1+p_1\eta_1(\xi_2,\xi_1)} \Psi(s,\xi_3+\eta_2(\xi_4,\xi_3))_{\xi_1}d_{p_1,q_1}s$$
$$- \frac{(p_2+q_2)}{p_2 q_2 \eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)^2} \int_{\xi_3}^{\xi_3+p_2\eta_2(\xi_4,\xi_3)} \Psi(\xi_1,t)_{\xi_3}d_{p_2,q_2}t$$
$$- \frac{p_1(p_2+q_2)}{p_2 q_1 q_2 \eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)^2} \int_{\xi_3}^{\xi_3+p_2\eta_2(\xi_4,\xi_3)} \Psi(\xi_1+\eta_1(\xi_2,\xi_1),t)_{\xi_3}d_{p_2,q_2}t$$
$$+ \frac{(p_1+q_1)(p_2+q_2)}{p_1 p_2 q_1 q_2 \eta_1(\xi_2,\xi_1)^2 \eta_2(\xi_4,\xi_3)^2} \int_{\xi_1}^{\xi_1+p_1\eta_1(\xi_2,\xi_1)} \int_{\xi_3}^{\xi_3+p_2\eta_2(\xi_4,\xi_3)} \Psi(s,t)_{\xi_3}d_{p_2,q_2}t_{\xi_1}d_{p_1,q_1}s. \quad (11)$$

Multiplying both sides of (11) by $\frac{q_1 q_2 \eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}{(p_1+q_1)(p_2+q_2)}$, we get the desired equality. □

4. Main Results

Theorem 4. *Let $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a mixed partial (p_1p_2, q_1q_2)-differentiable mapping on Λ^o (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}$ being continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)] \times [\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)] \subset \Lambda^o$ for $\eta_1(\xi_2,\xi_1), \eta_2(\xi_4,\xi_3) > 0$ and $0 < q_k < p_k \leq 1$ where $1 \leq k \leq 2$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|$ is a higher-order generalized strongly pre-invex mapping on the co-ordinates, then the following inequality holds*

$$\left|\Pi_{p_1p_2,q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K} \begin{bmatrix} \mathcal{A}_{p_1,q_1}\mathcal{A}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{A}_{p_1,q_1}\mathcal{B}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{A}_{p_1,q_1}\mathcal{B}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{B}_{p_1,q_1}\mathcal{B}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mu_1(\mathcal{C}_{p_1,q_1})\mu_2(\mathcal{C}_{p_2,q_2})\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3) \end{bmatrix},$$

and $\mathcal{A}_{p_k,q_k}$,$\mathcal{B}_{p_k,q_k}$, $\mathcal{C}_{p_k,q_k}$ *are given by*

$$\mathcal{A}_{p_k,q_k} = \frac{q_k[(p_k^3 - 2 + 2p_k) + (2p_k^2 + 2)q_k + p_k q_k^2] + 2p_k^2 - 2p_k}{(p_k + q_k)^3(p_k^2 + p_k q_k + q_k^2)},$$

$$\mathcal{B}_{p_k,q_k} = \frac{q_k[(5p_k^3 - 4p_k^2 - 2p_k + 2) + (6p_k^2 - 4p_k - 2)q_k + (5p_k - 2)q_k^2 + 2q_k^3]}{(p_k + q_k)^3(p_k^2 + p_k q_k + q_k^2)}$$
$$+ \frac{2p_k^4 - 2p_k^3 - 2p_k^2 + 2p_k}{(p_k + q_k)^3(p_k^2 + p_k q_k + q_k^2)}$$
$$\mathcal{C}_{p_k,q_k} = \frac{2(p_k + q_k)(p_k^2 + q_k^2) - 2(p_k^2 + p_k q_k + q_k^2) + p_k q_k(p_k^3 + q_k^3)}{(p_k + q_k)(p_k^2 + q_k^2)(p_k^2 + p_k q_k + q_k^2)}.$$

and

$$\mathcal{K} = \frac{q_1 q_2 \eta_1(\xi_2, \xi_1)\eta_2(\xi_4, \xi_3)}{(p_1 + q_1)(p_2 + q_2)}$$

Proof. By the given supposition, utilizing Lemma 1 and the modulus property

$$\begin{aligned}\left|\Pi_{p_1 p_2, q_1 q_2}(\xi_1, \xi_2, \xi_3, \xi_4)(\Psi)\right| &\le \mathcal{K} \int_0^1 \int_0^1 \left|(1 - (p_1 + q_1)\varrho)\right|\left|(1 - (p_2 + q_2)\rho)\right| \\ &\times \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(\xi_1 + \varrho\eta_1(\xi_2,\xi_1), \xi_3 + \rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{q_1}\varrho\,{}_{\xi_4}\partial_{q_2}\rho}\right| {}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho.\end{aligned} \tag{12}$$

Calculate (12) by using (p_1, q_1)-integral , we get

$$\int_0^1 \left|(1 - (p_1 + q_1)\varrho)\right| \left\{\begin{array}{c} (1-\varrho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,,q_1q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ -\mu_1\varrho(1-\varrho)\eta_1^\sigma(\xi_2,\xi_1) \end{array}\right\} {}_0d_{p_1,q_1}\varrho$$
$$\begin{aligned}&= \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1, \xi_3 + \rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \int_0^1 (1-\varrho)\left|(1 - (p_1 + q_1)\varrho)\right|{}_0d_{p_1,q_1}\varrho \\ &+ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2, \xi_3 + \rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right| \int_0^1 \varrho\left|(1 - (p_1 + q_1)\varrho)\right|{}_0d_{p_1,q_1}\varrho \\ &- \mu_1\eta_1^\sigma(\xi_2, \xi_1) \int_0^1 \varrho(1-\varrho)\left|(1 - (p_1 + q_1)\varrho)\right|{}_0d_{p_1,q_1}\varrho\end{aligned}$$

Utilizing Definitions 10 and 11, we get

$$\begin{aligned}\mathcal{A}_{p_k,q_k} &= \int_0^1 (1-\varrho)\left|(1-(p_k+q_k)\varrho)\right|{}_0d_{p_k,q_k}\varrho \\ &= \frac{q_k[(p_k^3-2+2p_k)+(2p_k^2+2)q_k+p_kq_k^2]+2p_k^2-2p_k}{(p_k+q_k)^3(p_k^2+p_kq_k+q_k^2)},\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{p_k,q_k} &= \int_0^1 \varrho\left|(1-(p_k+q_k)\varrho)\right|{}_0d_{p_k,q_k}\varrho \\ &= \frac{q_k[(5p_k^3-4p_k^2-2p_k+2)+(6p_k^2-4p_k-2)q_k+(5p_k-2)q_k^2+2q_k^3]}{(p_k+q_k)^3(p_k^2+p_kq_k+q_k^2)} \\ &+ \frac{2p_k^4-2p_k^3-2p_k^2+2p_k}{(p_k+q_k)^3(p_k^2+p_kq_k+q_k^2)}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_{p_k,q_k} &= \int_0^1 \varrho(1-\varrho)\left|(1-(p_k+q_k)\varrho)\right|{}_0d_{p_k,q_k}\varrho \\ &= \frac{2(p_k+q_k)(p_k^2+q_k^2)-2(p_k^2+p_kq_k+q_k^2)+p_kq_k(p_k^3+q_k^3)}{(p_k+q_k)(p_k^2+q_k^2)(p_k^2+p_kq_k+q_k^2)}.\end{aligned}$$

$$\begin{aligned}&= \mathcal{A}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| + \mathcal{B}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ &-\mu_1\mathcal{C}_{p_1,q_1}\eta_1^\sigma(\xi_2,\xi_1).\end{aligned}$$

Putting above calculations into (12), we obtain

$$\leq \mathcal{K}\int_0^1\left|(1-(p_2+q_2)\rho)\right|\left\{\begin{array}{c}\mathcal{A}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{B}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ -\mu_1\mathcal{C}_{p_1,q_1}\eta_1^\sigma(\xi_2,\xi_1)\end{array}\right\}{}_0d_{q_2}\rho. \tag{13}$$

As well, calculate (13) by using the (p_2,q_2)-integral, we get

$$\leq \mathcal{K}\left[\begin{array}{c}\mathcal{A}_{p_1,q_1}\mathcal{A}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{A}_{p_1,q_1}\mathcal{B}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{A}_{p_1,q_1}\mathcal{B}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mathcal{B}_{p_1,q_1}\mathcal{B}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right| \\ +\mu_1(\mathcal{C}_{p_1,q_1})\mu_2(\mathcal{C}_{p_2,q_2})\eta_1^\sigma(\xi_2,\xi_1)\eta_2^\sigma(\xi_4,\xi_3)\end{array}\right].$$

Hence, we deduce the required result. □

Theorem 5. *Let $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a mixed partial (p_1p_2, q_1q_2)-differentiable mapping on Λ° (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}$ being continuous and integrable on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \Lambda^{\circ}$ for $\eta_1(\xi_2, \xi_1), \eta_2(\xi_4, \xi_3) > 0$ and $0 < q_k < p_k \leq 1$ where $1 \leq k \leq 2$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau}$ is a higher-order generalized strongly pre-invex mapping on co-ordinates for $\tau \geq 1$, then the following inequality holds*

$$\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_2)(\Psi)\right| \leq \mathcal{K}\left(\mathcal{D}_{p_1,q_1}\mathcal{D}_{p_2,q_2}\right)^{1-\frac{1}{\tau}}$$
$$\times\begin{bmatrix} \mathcal{A}_{p_1,q_1}\mathcal{A}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +\mathcal{A}_{p_1,q_1}\mathcal{B}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +\mathcal{B}_{p_1,q_1}\mathcal{A}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +\mathcal{B}_{p_1,q_1}\mathcal{B}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +\mu_1(\mathcal{C}_{p_1,q_1})\mu_2(\mathcal{C}_{p_2,q_2})\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3) \end{bmatrix}^{\frac{1}{\tau}},$$

and

$$\mathcal{D}_{p_k,q_k} = \frac{2(p_k+q_k-1)}{(p_k+q_k)^2}.$$

$\mathcal{K}$, $\mathcal{A}_{p_k,q_k}$, $\mathcal{B}_{p_k,q_k}$ and $\mathcal{C}_{p_k,q_k}$ are defined in Theorem 4.

Proof. By the given supposition, utilizing Lemma 1 and using the (p_1p_2, q_1q_2)-Hölder inequality,

$$\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\left(\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho\right)^{1-\frac{1}{\tau}}$$
$$\times\left\{\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)| \begin{bmatrix} (1-\varrho)(1-\rho)\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +(1-\varrho)\rho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +(1-\rho)\varrho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +\varrho\rho\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau} \\ +\mu_1\varrho(1-\varrho)\mu_2\rho(1-\rho)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3) \end{bmatrix}\right\}^{\frac{1}{\tau}} {}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho.$$

Considering Definitions 10 and 11, we get

$$\mathcal{D}_{p_k,q_k} = \int_0^1 |1-(p_k+q_k)\varrho|\,{}_0d_{p_k,q_k}\varrho = \frac{2(p_k+q_k-1)}{(p_k+q_k)^2},$$

and $\mathcal{A}_{p_k,q_k}$, $\mathcal{B}_{p_k,q_k}$ and $\mathcal{C}_{p_k,q_k}$, are defined in Theorem 4.

We observe that,

$$
\begin{aligned}
&\int_0^1 \int_0^1 (1-(p_1+q_1)\varrho)\,(1-(p_2+q_2)\rho)\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho \\
&= \left(\int_0^1 |(1-(p_1+q_1)\varrho)|\,{}_0d_{p_1,q_1}\varrho\right)\left(\int_0^1 |(1-(p_2+q_2)\rho)|\,{}_0d_{p_2,q_2}\rho\right) \\
&= \mathcal{D}_{p_1,q_1}\mathcal{D}_{p_2,q_2}, \\
&\int_0^1 \int_0^1 (1-\varrho)(1-\rho)\,|(1-(p_1+q_1)\varrho)|\,|(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho \\
&= \left(\int_0^1 (1-\varrho)\,|(1-(p_1+q_1)\varrho)|\,{}_0d_{p_1,q_1}\varrho\right)\left(\int_0^1 (1-\rho)\,|(1-(p_2+q_2)\rho)|\,{}_0d_{p_2,q_2}\rho\right) \\
&= \mathcal{A}_{p_1,q_1}\mathcal{A}_{p_2,q_2}, \\
&\int_0^1 \int_0^1 (1-\varrho)\rho\,|(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho \\
&= \left(\int_0^1 (1-\varrho)\,|(1-(p_1+q_1)\varrho)|\,{}_0d_{p_1,q_1}\varrho\right)\left(\int_0^1 \rho\,|(1-(p_2+q_2)\rho)|\,{}_0d_{p_2,q_2}\rho\right) \\
&= \mathcal{A}_{p_1,q_1}\mathcal{B}_{p_1,q_1}, \\
&\int_0^1 \int_0^1 \varrho(1-\rho)\,|(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho \\
&= \left(\int_0^1 \varrho\,|(1-(p_1+q_1)\varrho)|\,{}_0d_{p_1,q_1}\varrho\right)\left(\int_0^1 (1-\rho)\,|(1-(p_2+q_2)\rho)|\,{}_0d_{p_2,q_2}\rho\right) \\
&= \mathcal{B}_{p_1,q_1}\mathcal{A}_{p_2,q_2}, \\
&\int_0^1 \int_0^1 zw\,|(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho \\
&= \left(\int_0^1 \varrho\,|(1-(p_1+q_1)\varrho)|\,{}_0d_{p_1,q_1}\varrho\right)\left(\int_0^1 \rho\,|(1-(p_2+q_2)\rho)|\,{}_0d_{p_2,q_2}\rho\right) \\
&= \mathcal{B}_{p_1,q_1}\mathcal{B}_{p_2,q_2}, \\
&\int_0^1 \int_0^1 zw(1-\rho)(1-\varrho)\,|(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho \\
&= \left(\int_0^1\int_0^1 \varrho(1-\varrho)\,|(1-(p_1+q_1)\varrho)|\,{}_0d_{p_1,q_1}\varrho\right)\left(\int_0^1\int_0^1 \rho(1-\rho)\,|(1-(p_2+q_2)\rho)|\,{}_0d_{p_2,q_2}\rho\right) \\
&= \mathcal{C}_{p_1,q_1}\mathcal{C}_{p_2,q_2}.
\end{aligned}
$$

Utilizing the values of the above (p_1p_2,q_1q_2)-integrals, we get our required inequality. □

Theorem 6. *Let $\Psi:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a mixed partial (p_1p_2,q_1q_2)-differentiable mapping on Λ° (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}$ being continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\Lambda^{\circ}$ for $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$ and $0<q_k<p_k\leq 1$ where $1\leq k\leq 2$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}$ is higher-order generalized strongly pre-invex function on co-ordinates for $\frac{1}{\tau_1}+\frac{1}{\tau_2}=1$, then the following inequality holds*

$$
\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\left(\mathcal{E}_{p_1,q_1}\mathcal{E}_{p_2,q_2}\right)^{\frac{1}{\tau_2}}
\times\left[\begin{array}{c}
\mathcal{F}_{p_1,q_1}\mathcal{F}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1} \\
+\mathcal{F}_{p_1,q_1}\mathcal{G}_{p_1,q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1} \\
+\mathcal{G}_{p_1,q_1}\mathcal{F}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1} \\
+\mathcal{G}_{p_1,q_1}\mathcal{G}_{p_2,q_2}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\,{}_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1} \\
+\mu_1(\mathcal{C}_{p_1,q_1})\mu_2(\mathcal{C}_{p_2,q_2})\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3)
\end{array}\right]^{\frac{1}{\tau_1}},
$$

and

$$\mathcal{E}_{p_k,q_k}(\tau_2) = \frac{(p_k - q_k)(1 + (p_k + q_k - 1)^{\tau_2+1})}{(p_k + q_k)(p_k^{\tau_2+1} - q_k^{\tau_2+1})},$$

$$\mathcal{F}_{p_k,q_k}(\tau_1) = \frac{p_k + q_k - 1}{p_k + q_k},$$

$$\mathcal{G}_{p_k,q_k}(\tau_1) = \frac{1}{p_k + q_k}.$$

$\mathcal{K}$ and $\mathcal{C}_{p_k,q_k}$ are given by in Theorem 4.

Proof. By the given supposition, utilizing Lemma 1 and the (p_1p_2, q_1q_2)-Hölder inequality

$$\left| \Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)\,(\Psi) \right| \le \mathcal{K} \left(\int_0^1 \int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|^{\tau_2}\, {}_0d_{p_1,q_1}\varrho\, {}_0d_{p_2,q_2}\rho \right)^{\frac{1}{\tau_2}}$$

$$\times \left[\begin{array}{c} (1-\varrho)(1-\rho)\left| \frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\, {}_{\xi_3}\partial_{p_2,q_2}\rho} \right|^{\tau_1} \\ +(1-\varrho)\rho\left| \frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\, {}_{\xi_3}\partial_{p_2,q_2}\rho} \right|^{\tau_1} \\ +(1-\rho)\varrho\left| \frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\, {}_{\xi_3}\partial_{p_2,q_2}\rho} \right|^{\tau_1} \\ +\varrho\rho\left| \frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho\, {}_{\xi_3}\partial_{p_2,q_2}\rho} \right|^{\tau_1} \\ +\mu_1\varrho(1-\varrho)\mu_2\rho(1-\rho)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3) \end{array} \right]^{\frac{1}{\tau_1}}.$$

We calculate as,

$$\textstyle\int_0^1 |(1-(p_k+q_k)\varrho)|^{\tau_2}\, {}_0d_{p_k,q_k}\varrho = \int_0^{\frac{1}{p_k+q_k}} (1-(p_k+q_k)\varrho)^{\tau_2}\, {}_0d_{p_k,q_k}\varrho + \int_{\frac{1}{p_k+q_k}}^1 ((p_k+q_k)\varrho - 1)^{\tau_2}\, {}_0d_{p_k,q_k}\varrho. \quad (14)$$

Considering the first (p_k, q_k)-integral from (14) and substitute $1-(p_k+q_k)\varrho = \rho$, we obtain:

$$\int_0^{\frac{1}{p_k+q_k}} (1-(p_k+q_k)\varrho)^{\tau_2}\, {}_0d_{p_k,q_k}\varrho = \frac{1}{p_k+q_k}\int_0^1 \rho^{\tau_2}\, {}_0d_{p_k,q_k}\rho = \frac{p_k - q_k}{(p_k+q_k)(p_k^{\tau_2+1} - q_k^{\tau_2+1})}.$$

Considering the second (p_k, q_k)-integral from (14) and substitute $(p_k+q_k)\varrho - 1 = \rho$, we obtain:

$$\int_{\frac{1}{p_k+q_k}}^1 ((p_k+q_k)\varrho - 1)^{\tau_2}\, {}_0d_{p_k,q_k}\varrho = \frac{1}{p_k+q_k}\int_0^{p_k+q_k-1} \rho^{\tau_2}\, {}_0d_{p_k,q_k}\rho$$

$$= \frac{(p_k-q_k)(p_k+q_k-1)^{\tau_2+1}}{(p_k+q_k)(p_k^{\tau_2+1} - q_k^{\tau_2+1})}.$$

After this calculation, we get:

$$\mathcal{E}_{p_k,q_k}(\tau_2) = \int_0^1 |(1-(p_k+q_k)\varrho)|^{\tau_2} \, {}_0d_{p_k,q_k}\varrho = \frac{(p_k-q_k)(1+(p_k+q_k-1)^{\tau_2+1})}{(p_k+q_k)(p_k^{\tau_2+1}-q_k^{\tau_2+1})}.$$

Finally, we also have:

$$\mathcal{F}_{p_k,q_k}(\tau_1) = \int_0^1 (1-\varrho)_0 d_{p_k,q_k}\varrho = \frac{p_k+q_k-1}{p_k+q_k}$$
$$\mathcal{G}_{p_k,q_k}(\tau_1) = \int_0^1 \varrho_0 d_{p_k,q_k}\varrho = \frac{1}{p_k+q_k}.$$

Utilizing the values of the above (p_1p_2, q_1q_2)-integrals, we get our required inequality. □

Theorem 7. *Let $\Psi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a mixed partial (p_1p_2, q_1q_2)-differentiable mapping on Λ^o (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}$ being continuous and integrable on $[\xi_1, \xi_1+\eta_1(\xi_2,\xi_1)] \times [\xi_3, \xi_3+\eta_2(\xi_4,\xi_3)] \subset \Lambda^o$ for $\eta_1(\xi_2,\xi_1), \eta_2(\xi_4,\xi_3) > 0$ and $0 < q_k < p_k \leq 1$ where $1 \leq k \leq 2$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^\tau$ is a higher-order generalized strongly quasi-pre-invex mapping on co-ordinates for $\tau \geq 1$, then the following inequality holds*

$$\begin{aligned}
&\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\left(\mathcal{D}_{p_1,q_1}\mathcal{D}_{p_2,q_2}\right)^{1-\frac{1}{\tau}} \\
&\times \left\{ \begin{array}{c} \mathcal{D}_{p_1,q_1}\mathcal{D}_{p_2,q_2}\max\left(\begin{array}{c} \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^\tau, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^\tau, \\ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^\tau, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^\tau \end{array}\right) \\ +\mu_1(\mathcal{C}_{p_1,q_1})\mu_2(\mathcal{C}_{p_2,q_2})\eta_1^\sigma(\xi_2,\xi_1)\eta_2^\sigma(\xi_4,\xi_3) \end{array}\right\}^{\frac{1}{\tau}},
\end{aligned}$$

where $\mathcal{K}$, $\mathcal{C}_{p_k,q_k}$ and $\mathcal{D}_{p_k,q_k}$ are defined in Theorems 4 and 5

Proof. By the given supposition, utilizing Lemma 1 and using the (p_1p_2, q_1q_2)-Hölder inequality

$$\begin{aligned}
&\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)| \, {}_0d_{p_1,q_1}\varrho_0 d_{p_2,q_2}\rho \\
&\times \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1), \xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|.
\end{aligned}$$

$$\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\left(\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|\,{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho\right)^{1-\frac{1}{\tau}}$$

$$\times\left\{\begin{array}{c}\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)| \\ \times\left[\begin{array}{c}\max\left(\begin{array}{c}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau},\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau}, \\ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau},\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau}\end{array}\right) \\ +\mu_1\varrho(1-\varrho)\mu_2\rho(1-\rho)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3)\end{array}\right]{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho\end{array}\right\}^{\frac{1}{\tau}}.$$

and obtain required integrals $\mathcal{C}_{p_k,q_k}$ and $\mathcal{D}_{p_k,q_k}$, that have been calculated in Theorem 4 and Theorem 5. This complete our result. □

Theorem 8. *Let $\Psi:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a mixed partial (p_1p_2,q_1q_2)-differentiable mapping on Λ° (the interior of Λ) with $\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}$ being continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\Lambda^{\circ}$ for $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$ and $0<q_k<p_k\leq 1$ where $1\leq k\leq 2$. If $\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\varrho,\rho)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}$ is a higher-order generalized strongly quasi-pre-invex mapping on co-ordinates for $\frac{1}{\tau_1}+\frac{1}{\tau_2}=1$, then following inequality holds*

$$\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\left(\mathcal{E}_{p_1,q_1}\mathcal{E}_{p_2,q_2}\right)^{\frac{1}{\tau_2}}$$

$$\times\left\{\begin{array}{c}\max\left(\begin{array}{c}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1},\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}, \\ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1},\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}\end{array}\right) \\ +\mu_1(\mathcal{C}_{p_1,q_1})\mu_2(\mathcal{C}_{p_2,q_2})\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3)\end{array}\right\}^{\frac{1}{\tau_1}},$$

where $\mathcal{K}$, $\mathcal{C}_{p_k,q_k}$ and $\mathcal{E}_{p_k,q_k}$ are defined in Theorems 4 and 6.

Proof. By the given supposition, utilizing Lemma 1 and using the (p_1p_2,q_1q_2)-Hölder inequality, then we have

$$\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|$$
$$\times\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1+\varrho\eta_1(\xi_2,\xi_1),\xi_3+\rho\eta_2(\xi_4,\xi_3))}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|{}_0d_{p_1,q_1}\varrho\,{}_0d_{p_2,q_2}\rho.$$

$$\left|\Pi_{(p_1p_2,q_1q_2)}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \leq \mathcal{K}\left(\int_0^1\int_0^1 |(1-(p_1+q_1)\varrho)(1-(p_2+q_2)\rho)|^{\tau_2}\, {}_0d_{p_1,q_1}\varrho_0 d_{p_2,q_2}\rho\right)^{\frac{1}{\tau_2}}$$
$$\times\left\{\begin{array}{l}\max\left(\begin{array}{c}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_1,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1},\\ \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_3)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}, \left|\frac{{}_{\xi_1,\xi_3}\partial^2_{p_1p_2,q_1q_2}\Psi(\xi_2,\xi_4)}{{}_{\xi_1}\partial_{p_1,q_1}\varrho_{\xi_3}\partial_{p_2,q_2}\rho}\right|^{\tau_1}\end{array}\right)\\ +\mu_1\varrho(1-\varrho)\mu_2\rho(1-\rho)\eta_1^{\sigma}(\xi_2,\xi_1)\eta_2^{\sigma}(\xi_4,\xi_3)\end{array}\right\}^{\frac{1}{\tau_1}}.$$

and obtain required integrals $\mathcal{C}_{p_k,q_k}$ and $\mathcal{E}_{p_k,q_k}$, that have been calculated in Theorems 4 and 6. This complete our result. □

5. Conclusions

We conducted a preliminary attempt to develop a quantum formulation presumably for new Hermite-Hadamard type for proposing two new classes of higher-order generalized strongly pre-invex and quasi-pre-invex functions and presented their (p_1p_2, q_1q_2)-analogues by using the two parameters of deformation at the quantum scale based on special relativity. An auxiliary result was chosen because of its success in leading to the well-known Hermite-Hadamard type inequalities. An intriguing feature of (p_1p_2, q_1q_2)-analogues is that these simple rules lead to Einstein's velocity-addition formula in the macroscopic limit by inserting the condition of time in our computed results. Such a potential connection needs further investigation. We conclude that the results derived in this paper are general in character and give some contributions to inequality theory, some applications for establishing the uniqueness of solutions in quantum boundary value problems, quantum mechanics and special relativity theory. We can formulate the quantum version of the postulates for special relativity and derived the rules for determining relative velocity at the quantum scale by involving Dirac delta. This interesting aspect of time is worth further investigation. Finally, the innovative concept of higher-order generalized strongly convex functions have potential application in parallelogram law of L_p-spaces in functional analysis and opened new doors for futuristic research.

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