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Quantum Analogs of Ostrowski-Type Inequalities for Raina's Function correlated with Coordinated Generalized Φ-Convex Functions

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Abstract: In this paper, the newly proposed concept of Raina's function and quantum calculus are utilized to anticipate the quantum behavior of two variable Ostrowski-type inequalities. This new technique is the convolution of special functions with hypergeometric and Mittag–Leffler functions, respectively. This new concept will have the option to reduce self-similitudes in the quantum attractors under investigation. We discuss the implications and other consequences of the quantum Ostrowski-type inequalities by deriving an auxiliary result for a q_1q_2-differentiable function by inserting Raina's functions. Meanwhile, we present a numerical scheme that can be used to derive variants for Ostrowski-type inequalities in the sense of coordinated generalized Φ-convex functions with the quantum approach. This new scheme of study for varying values of parameters with the involvement of Raina's function yields extremely intriguing outcomes with an illustrative example. It is supposed that this investigation will provide new directions for the capricious nature of quantum theory.

Keywords: quantum calculus; Ostrowski-type inequalities; coordinated generalized Φ-convex functions; Raina's function

1. Introduction

Quantum calculus is the non-limited analysis of calculus, and it is also recognized as q-calculus. We get the initial mathematical formulas in q-calculus as q reaches 1^-. The analysis of q-calculus was initiated by Euler (1707–1783). Subsequently, Jackson [1] launched the idea of q-integrals in a systematic way. The aforementioned results led to an intensive investigation on q-calculus in the Twentieth Century. The idea of q-calculus is used in numerous areas of mathematics and physics especially in orthogonal polynomials, number theory, hypergeometric functions, mechanics, and the theory of relativity. Tariboon et al. [2,3] discovered the idea of q-derivatives on $[\xi_1, \xi_2]$ of $\mathbb{R}$ and unified and modified numerous new concepts of classical convexity. In the last few years, the topic of q-calculus has become an interesting topic for many researchers, and several new results have been established in the literature; see for instance [4–9] and the references cited therein.

The feasibility of the new approach of consecutive articulation and the arrangement of issues of deciding the spatial stress-strain state, volume damageability state, and multicriteria states of deformable frameworks, all the while experiencing the activity of volume distortion under tension-compression or bending and local loading under contact connection with grinding created with the utilities of q-calculus based on several functions for bounding volumetric damageable areas in tribo-fatigue and mechanothermodynamic systems, have been established in the literature; see for instance [10–17] and the references cited therein.

Integral inequalities are considered a fabulous tool for constructing the qualitative and quantitative properties in the field of pure and applied mathematics. A continuous growth of interest has occurred in order to meet the requirements of the need for fertile applications of these inequalities. Such inequalities were studied by many researchers who in turn used various techniques for the sake of exploring and offering these inequalities [18–27]. In many practical problems, it is important to bound one quantity by another quantity. The classical inequalities such as Ostrowski's inequality are very useful for this purpose. In recent years, many authors proved numerous inequalities associated with the functions of bounded variation, Lipschitzian, monotone, absolutely continuous, convex functions, s-convex and h-convex functions, and n-times differentiable mappings with error estimates with some special means, and some numerical quadrature was done. For the latest consequences, modifications, counterparts, generalizations, and novelty of Ostrowski-type inequalities, see [28–31].

The following integral inequality was presented by Ostrowski [32].

Theorem 1. *Suppose that a function $\Psi : \mathcal{W} \subseteq \mathbb{R} \to \mathbb{R}$ is continuous on $[\xi_1, \xi_2]$ and differentiable on (ξ_1, ξ_2), whose derivative $\Psi' : (\xi_1, \xi_2) \to \mathbb{R}$ is bounded on (ξ_1, ξ_2), i.e., $|\Psi(\tau)| \le \mathcal{M}$. Then, one has the inequality:*

$$\left|\Psi(\varrho) - \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \Psi(\tau)\, d\tau\right| \le \left[\frac{1}{4} + \frac{\left(\varrho - \frac{\xi_1+\xi_2}{2}\right)^2}{(\xi_2-\xi_1)^2}\right](\xi_2 - \xi_1)\,\mathcal{M}, \tag{1}$$

for all $\varrho \in [\xi_1, \xi_2]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

The inequality (1) can be written in equivalent form as:

$$\left|\Psi(\varrho) - \frac{1}{\xi_2 - \xi_1}\int_{\xi_1}^{\xi_2} \Psi(\tau)\, d\tau\right| \le \frac{\mathcal{M}}{\xi_2 - \xi_1}\left[\frac{(\varrho-\xi_1)^2 + (\xi_2-\varrho)^2}{2}\right].$$

In [33], the classical Ostrowski-type inequality for coordinated convex functions was established via the following equality:

Theorem 2. *Suppose that a function $\Psi : \mathcal{W} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is a second-order partial derivative over $\mathcal{W}^o$, and let $[\xi_1, \xi_2] \times [\xi_3, \xi_4] \subseteq \mathcal{W}^o$ with $\xi_1, \xi_2, \xi_3, \xi_4 \in \mathcal{W}^o$ such that $\xi_1 < \xi_2$, $\xi_3 < \xi_4$. If $\frac{\partial^2 \Psi}{\partial z\, \partial w} \in L([\xi_1, \xi_2] \times [\xi_3, \xi_4])$, then one has the equality:*

$$\begin{aligned}
&\Psi(\varrho, \rho) + \frac{1}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4} \Psi(u,v)dudv - \mathcal{A} \\
&= \frac{(\varrho-\xi_1)^2(\rho-\xi_3)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z \partial w}\Psi(z\varrho + (1-z)\xi_1, w\rho + (1-w)\xi_3)dzdw \\
&- \frac{(\varrho-\xi_1)^2(\xi_4-\rho)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z \partial w}\Psi(z\varrho + (1-z)\xi_1, w\rho + (1-w)\xi_4)dzdw
\end{aligned}$$

$$-\frac{(\xi_2-\varrho)^2(\rho-\xi_3)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Psi(z\varrho+(1-z)\xi_2, w\rho+(1-w)\xi_3)dzdw$$
$$+\frac{(\xi_2-\varrho)^2(\xi_4-\rho)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Psi(z\varrho+(1-z)\xi_2, w\rho+(1-w)\xi_4)dzdw \quad (2)$$

for all $(\varrho,\rho)\in\mathcal{W}$,

$$\mathcal{A}=\frac{1}{\xi_4-\xi_3}\int_{\xi_3}^{\xi_4}\Psi(\varrho,v)dv+\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi(u,\rho)du.$$

Noor et al. [34] proposed the quantum estimates for Ostrowski-type inequalities based on the convexity function of one variable, which are associated with the equality below.

Theorem 3. *Suppose that a function* $\Psi:\mathcal{W}\to\mathbb{R}$ *is continuous with* $q\in(0,1)$. *If* ${}_{\xi_1}D_q\Psi$ *is an integrable function on* $\mathcal{W}^{\circ}$, $\xi_1,\xi_2\in\mathcal{W}^{\circ}$ *such that* $\xi_1<\xi_2$, *then one has the equality:*

$$\Psi(\varrho)-\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi(u)\,{}_{\xi_1}d_qu=\frac{q(\varrho-\xi_1)^2}{\xi_2-\xi_1}\int_0^1 z_{\xi_1}D_q\Psi\left((1-z)\,\xi_1+z\varrho\right){}_0d_qz$$
$$+\frac{q(\xi_2-\varrho)^2}{\xi_2-\xi_1}\int_0^1 z_{\xi_1}D_q\Psi\left((1-z)\,\xi_2+z\varrho\right){}_0d_qz.$$

The following inequality of the q-Hermite–Hadamard type for coordinated convex functions on a rectangle from the plane $\mathbb{R}^2$, see [35].

Theorem 4. *Let* $\Psi:\mathcal{W}=[\xi_1,\xi_2]\times[\xi_3,\xi_4]\subseteq\mathbb{R}^2\to\mathbb{R}$ *be convex on the coordinates on* $\mathcal{W}$ *with* $q_1,q_2\in(0,1)$ *and* $\xi_1<\xi_2$, $\xi_3<\xi_4$. *Then, one has the inequalities:*

$$\Psi\left(\frac{q_1\xi_1+\xi_2}{1+q_1},\frac{q_2\xi_3+\xi_4}{1+q_2}\right)$$
$$\leq\frac{1}{2}\left[\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Psi\left(z,\frac{q_2\xi_3+\xi_4}{1+q_2}\right){}_0d_{q_1}z+\frac{1}{\xi_3-\xi_4}\int_{\xi_3}^{\xi_4}\Psi\left(\frac{q_1\xi_1+\xi_2}{1+q_1},w\right){}_0d_{q_2}w\right]$$
$$\leq\frac{1}{(\xi_1-\xi_2)(\xi_3-\xi_4)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Psi(z,w)_0d_{q_1}z_0d_{q_2}w$$
$$\leq\frac{1}{4}\left[\frac{1}{(1+q_1)(\xi_2-\xi_1)}\left(q_2\int_{\xi_1}^{\xi_2}\Psi(z,\xi_3)_0d_{q_1}z+\int_{\xi_1}^{\xi_2}\Psi(z,\xi_4)_0d_{q_1}z\right)\right.$$
$$\left.+\frac{1}{(1+q_2)(\xi_3-\xi_4)}\left(q_1\int_{\xi_3}^{\xi_3}\Psi(\xi_1,w)_0d_{q_2}w+\int_{\xi_3}^{\xi_4}\Psi(\xi_2,w)_0d_{q_2}w\right)\right]$$
$$\leq\frac{q_1q_2\Psi(\xi_1,\xi_3)+q_2\Psi(\xi_2,\xi_3)+q_1\Psi(\xi_1,\xi_4)+\Psi(\xi_2,\xi_4)}{(1+q_1)(1+q_2)}. \quad (3)$$

For several recent results on different types of inequalities for functions that satisfy different kinds of convexity on the coordinates on the rectangle from the plane $\mathbb{R}^2$, we refer the reader to [36–40].

Our present paper was inspired by the above-mentioned literature, and the principal intention of this research is to introduced the idea of a new class of a coordinated generalized Φ-convex set and a coordinated generalized Φ-convex function by using Raina's function and presenting some preliminaries related to quantum calculus. q-calculus for functions of one and two variables over finite rectangles in the plane will be introduced. Moreover, we derive an identity for q_1q_2 differentiable by involving Raina's functions. Applying this new identity, we develop some new quantum analogs of Ostrowski inequalities for a coordinated generalized Φ-convex function. Furthermore, we derive some special cases (hypergeometric and Mittag–Leffler functions), on the specific values of Raina's function parameters. The ideas and techniques of the paper may open a new venue for further research in this field.

2. Preliminaries

In this section, we recall some previously known concepts and also introduce the notion of a coordinated generalized Φ-convex set and coordinated generalized Φ-convex function by using Raina's function.

Suppose that J is a finite interval of real numbers. A function $\Psi : J \to \mathbb{R}$ is said to be convex if,

$$\Psi(\tau x + (1-\tau)\bar{z}) \leq \tau\Psi(x) + (1-\tau)\Psi(\bar{z}) \tag{4}$$

holds for all $x, \bar{z} \in J$ and $\tau \in [0,1]$.

A modification for convex functions on $\mathcal{W}$, which are also known as coordinated convex functions, was introduced by Dragomir [41], which is stated below:

Definition 1. *Suppose that a function $\Psi : \mathcal{W} := [\xi_1, \xi_2] \times [\xi_3, \xi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is said to be convex on the coordinates on Δ with $\xi_1 < \xi_2$ and $\xi_3 < \xi_4$ if the partial functions:*

$$\begin{aligned} &\Psi\left(\tau x + (1-\tau)\bar{z}, \gamma y + (1-\gamma)\bar{w}\right) \\ &\leq \tau\gamma\Psi(x,y) + \tau(1-\gamma)\Psi(x,\bar{w}) + (1-\tau)\gamma\Psi(\bar{z},y) + (1-\tau)(1-\gamma)\Psi(\bar{z},\bar{w}) \end{aligned} \tag{5}$$

hold for all $\tau, \gamma \in [0,1]$ and $(x,y), (\bar{z},\bar{w}) \in \Delta$.

R.K.Raina, in [42], introduced the following class:

$$\mathcal{F}^{\nu}_{\alpha,\lambda}(s) = \mathcal{F}^{\nu(0),\nu(1),\ldots}_{\alpha,\lambda}(s) = \sum_{l=0}^{\infty} \frac{\nu(l)}{\Gamma(\alpha l + \lambda)} s^l, \tag{6}$$

where $\alpha, \lambda > 0$, $|s| < \mathbb{R}$ and:

$$\nu = (\nu(0), \nu(1), \ldots \nu(l), \ldots)$$

is a bounded sequence of $\mathbb{R}^+$. Moreover, taking $\alpha = 1, \lambda = 0$ in (6) and:

$$\nu(l) = \frac{(\phi)_l (\psi)_l}{(\varphi)_l} \text{ for } l = 0,1,2,3\ldots,$$

where ϕ, ψ, and φ are parameters and may be real or complex values (provided that $\varphi = 0, -1, -2, \ldots$), and the symbol $(b)_l$ denotes the quantity:

$$(b)_l = \frac{\Gamma(b+l)}{\Gamma(b)} = b(b+1)\ldots(b+l-1), \quad l = 0,1,2,\ldots,$$

while its domain is restrict as $|s| \leq 1$ (with $s \in \mathbb{C}$), then we obtain the following hypergeometric function,

$$\mathcal{F}^{\nu}_{\alpha,\lambda}(s) = F(\phi, \psi; \varphi; s) = \sum_{l=0}^{\infty} \frac{(\phi)_l (\psi)_l}{l! (\varphi)_l} s^l.$$

Moreover, if $\nu = (1,1,\ldots)$ with $\alpha = \phi$, $(Re(\phi) > 0)$, $\lambda = 1$, and its domain is restricted as $s \in \mathbb{C}$ in (6), then we obtain the following Mittag–Leffler function:

$$E_{\varphi}(s) = \sum_{l=0}^{\infty} \frac{1}{\Gamma(1+\varphi l)} s^l.$$

Finally, we introduce a new definition that combines the coordinated convex function and Raina's function described above.

Definition 2. *Let $\alpha, \lambda > 0$ and $\nu = (\nu(0), \nu(1), \ldots \nu(l), \ldots)$ be a bounded sequence of $\mathbb{R}^+$. A nonempty set $\mathcal{W}$ is called a coordinated generalized Φ-convex set where:*

$$\Psi\left(\bar{z} + \tau \mathcal{F}^{\nu}_{\alpha,\lambda}(x - \bar{z}), \varpi + \gamma \mathcal{F}^{\nu}_{\alpha,\lambda}(y - \varpi)\right) \in \mathcal{W}$$

holds for all $\tau, \gamma \in [0,1], (x,y), (\bar{z}, \varpi) \in \mathcal{W}$, and $\mathcal{F}^{\nu}_{\alpha,\lambda}(.)$ is Raina's function.

Definition 3. *Let $\alpha, \lambda > 0$ and $\nu = (\nu(0), \nu(1), \ldots \nu(l), \ldots)$ be a bounded sequence of $\mathbb{R}^+$. If a function $\Psi : \mathcal{W} := [\xi_1, \xi_2] \times [\xi_3, \xi_4] \subseteq \mathbb{R}^2 \to \mathbb{R}$ is satisfied, the expiration below:*

$$\begin{aligned}&\Psi\left(\bar{z} + \tau \mathcal{F}^{\nu}_{\alpha,\lambda}(x - \bar{z}), \varpi + \gamma \mathcal{F}^{\nu}_{\alpha,\lambda}(y - \varpi)\right) \\ &\leq \tau\gamma\Psi(x,y) + \tau(1-\gamma)\Psi(x,\varpi) + (1-\tau)\gamma\Psi(\bar{z},y) + (1-\tau)(1-\gamma)\Psi(\bar{z},\varpi)\end{aligned} \tag{7}$$

holds for all $\tau, \gamma \in [0,1]$ and $(x,y), (\bar{z}, \varpi) \in \mathcal{W}$, then Ψ is called a coordinated generalized Φ-convex function.

Remark 1. *Letting $\mathcal{F}^{\nu}_{\alpha,\lambda}(x - \bar{z}) = x - \bar{z} > 0$ and $\mathcal{F}^{\nu}_{\alpha,\lambda}(y - \varpi) = y - \varpi > 0$ in Definition 7, we get Definition 5.*

This is reminiscent of some basic concepts and characteristics in the q-analog for single and double variables.

Let $\mathcal{W} = [\xi_1, \xi_2] \subseteq \mathbb{R}$, and let $\mathcal{B} = [\xi_1, \xi_2] \times [\xi_3, \xi_4] \subseteq \mathbb{R}^2$ with constants $q, q_k \in (0,1)$, $k = 1,2$.

Tariboon and Ntouyas [2,3] established the idea of the q-derivative, q-integral, and properties for the finite interval, which was presented as:

Definition 4. *Let a function $\Psi : \mathcal{W} \to \mathbb{R}$ be continuous and $s \in \mathcal{W}$. Then, one has the q-derivative of Ψ on $\mathcal{W}$ at s defined as:*

$$_{\xi_1}D_q\Psi(s) = \frac{\Psi(s) - \Psi(qs + (1-q)\xi_1)}{(1-q)(s - \xi_1)}, \quad s \neq \xi_1. \tag{8}$$

It is obvious that:

$$\lim_{s \to \xi_1} {}_{\xi_1}D_q\Psi(s) = {}_{\xi_1}D_q\Psi(\xi_1),$$

and we call the function Ψ q-differentiable over $\mathcal{W}$; moreover, $_{\xi_1}D_q\Psi(s)$ exists $\forall\, s \in \mathcal{W}$.

Note that if $\xi_1 = 0$ in (8), then $_0D_q\Psi = D_q\Psi$, where $D_q\Psi$ is the well-defined q-derivative of $\Psi(s)$, explained as:

$$D_q\Psi(s) = \frac{\Psi(s) - \Psi(qs)}{(1-q)(s)}.$$

Definition 5. *Let a function $\Psi : \mathcal{W} \to \mathbb{R}$ be continuous, and it is denoted by $_{\xi_1}D_q^2\Psi$, given that $_{\xi_1}D_q^2\Psi$ is q-differentiable from $\mathcal{W} \to \mathbb{R}$ defined by:*

$$_{\xi_1}D_q^2\Psi = {}_{\xi_1}D_q\left({}_{\xi_1}D_q\Psi\right).$$

In addition, the higher order q-differentiable is described as $_{\xi_1}D_q^n\Psi$: $\mathcal{W} \to \mathbb{R}$.

Definition 6. *Let a function $\Psi : \mathcal{W} \to \mathbb{R}$ be continuous. Then, the q-integral on $\mathcal{W}$ is described as:*

$$\int_{\xi_1}^{s} \Psi(z)\,_{\xi_1}d_qz = (1-q)(s - \xi_1)\sum_{n=0}^{\infty} q^n\Psi(q^ns + (1-q^n)\xi_1), \quad \forall s \in V. \tag{9}$$

Furthermore, if $\xi_1 = 0$ in (9), then we have one of formulae of the q-integral, which is pointed out as:

$$\int_0^s \Psi(z)_0 d_q z = (1-q)s \sum_{n=0}^{\infty} q^n \Psi(q^n s).$$

Theorem 5. *Let a function $\Psi : \mathcal{W} \to \mathbb{R}$ be continuous, then one has that the following properties hold:*

$$\begin{aligned}
&(i) \quad {}_{\xi_1}D_q \int_{\xi_1}^{s} \Psi(z)_{\xi_1} d_q z = \Psi(s);\\
&(ii) \quad \int_{\xi_1}^{s} {}_{\xi_1}D_q \Psi(z)_{\xi_1} d_q z = \Psi(s);\\
&(iii) \quad \int_{\xi_2}^{s} {}_{\xi_1}D_q \Psi(z)_{\xi_1} d_q z = \Psi(s) - \Psi(\xi_2), \quad \xi_2 \in (\xi_1, s).
\end{aligned}$$

Theorem 6. *Let a function $\Psi : \mathcal{W} \to \mathbb{R}$ be continuous and $a \in \mathbb{R}$, then we have the following properties:*

$$\begin{aligned}
&(i) \quad \int_{\xi_1}^{s} [\Psi_1(z) + \Psi_2(z)]\, {}_{\xi_1}d_q z = \int_{\xi_1}^{s} \Psi_1(z)_{\xi_1} d_q z + \int_{\xi_1}^{s} \Psi_2(z)_{\xi_1} d_q z;\\
&(ii) \quad \int_{\xi_1}^{s} (a\Psi_1(z))_{\xi_1} d_q z = a \int_{\xi_1}^{s} \Psi_1(z)_{\xi_1} d_q z.
\end{aligned}$$

The theory of quantum integral inequalities for two variable functions was developed by Humaira et al. [35].

Definition 7. *Let a function of two variables $\Psi : \mathcal{B} \to \mathbb{R}$ be continuous. Then, the partial q_1-derivative, q_2-derivative, and q_1q_2-derivative at $(z,w) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$ are, respectively, defined as:*

$$\begin{aligned}
\frac{{}_{\xi_1}\partial_{q_1} \Psi(z,w)}{{}_{\xi_1}\partial_{q_1} z} &= \frac{\Psi(z,w) - \Psi(q_1 z + (1-q_1)\xi_1, w)}{(1-q_1)(z-\xi_1)}, \quad z \neq \xi_1,\\
\frac{{}_{\xi_3}\partial_{q_2} \Psi(z,w)}{{}_{\xi_3}\partial_{q_2} w} &= \frac{\Psi(z,w) - \Psi(z, q_2 w + (1-q_2)\xi_3)}{(1-q_2)(w-\xi_3)}, \quad w \neq \xi_3,\\
\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2} \Psi(z,w)}{{}_{\xi_1}\partial_{q_1} z\, {}_{\xi_3}\partial_{q_2} w} &= \frac{1}{(1-q_1)(1-q_2)(z-\xi_1)(w-\xi_3)}\\
&\times \Big[\Psi(q_1 z + (1-q_1)\xi_1, q_2 w + (1-q_2)\xi_3) - \Psi(q_1 z + (1-q_1)\xi_1, w)\\
&- \Psi(z, q_2 w + (1-q_2)\xi_3) + \Psi(z,w)\Big], \quad z \neq \xi_1, \quad w \neq \xi_3.
\end{aligned}$$

The function Ψ: $\mathcal{B} \to \mathbb{R}$ is called partially q_1-, q_2-, and q_1q_2-differentiable on $[\xi_1, \xi_2] \times [\xi_3, \xi_4]$ if $\frac{{}_{\xi_1}\partial_{q_1} \Psi(z,w)}{{}_{\xi_1}\partial_{q_1} z}$, $\frac{{}_{\xi_3}\partial_{q_2} \Psi(z,w)}{{}_{\xi_3}\partial_{q_2} w}$, and $\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1 q_2} \Psi(z,w)}{{}_{\xi_1}\partial_{q_1} z\, {}_{\xi_3}\partial_{q_2} w}$ exist for all $(z,w) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Definition 8. *Let a function of two variables $\Psi : \mathcal{B} \to \mathbb{R}$ be continuous. Then, the definite q_1q_2-integral on $[\xi_1, \xi_2] \times [\xi_3, \xi_4]$ is described as:*

$$\begin{aligned}
&\int_{\xi_3}^{t} \int_{\xi_1}^{s} \Psi(z,w)_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w\\
&= (1-q_1)(1-q_2)(s-\xi_1)(t-\xi_3) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_1^n q_2^m \Psi\left(q_1^n s + (1-q_1^n)\,\xi_1, q_2^m t + (1-q_2^m)\,\xi_3\right)
\end{aligned}$$

for $(s,t) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Theorem 7. *Let a function of two variables $\Psi : \mathcal{B} \to \mathbb{R}$ be continuous, then the following properties holds:*

$$
\begin{aligned}
&(i) \quad \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} s_{\xi_3}\partial_{q_2} t} \int_{\xi_4}^{t}\int_{\xi_1}^{s} \Psi(z,w)_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w = \Psi(s,t);\\
&(ii) \quad \int_{\xi_3}^{t}\int_{\xi_1}^{s} \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\, {}_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w = \Psi(s,t);\\
&(iii) \quad \int_{t_1}^{t}\int_{s_1}^{s} \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Psi(z,w)}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\, {}_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w;\\
&= \Psi(s,t) - \Psi(s,t_1) - \Psi(s_1,t) + \Psi(s_1,t_1), \quad (s_1,t_1) \in (\xi_1,s)\times(\xi_4,t).
\end{aligned}
$$

Theorem 8. *Suppose that $\Psi_1, \Psi_2 : \mathcal{B} \to \mathbb{R}$ are continuous mappings of two variables. Then, the following properties hold for $(s,t) \in [\xi_1,\xi_2]\times[\xi_3,\xi_4]$,*

$$
\begin{aligned}
&(i) \quad \int_{\xi_3}^{t}\int_{\xi_1}^{s} [\Psi_1(z,w)+\Psi_2(z,w)]\, {}_{\xi_1} d_{q_1} z_{\xi_4} d_{q_2} w\\
&= \int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi_1(z,w)_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w + \int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi_2(z,w)_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w;\\
&(ii) \quad \int_{\xi_3}^{t}\int_{\xi_1}^{s} a\Psi(z,w)_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w = a \int_{\xi_3}^{t}\int_{\xi_1}^{s} \Psi(z,w)_{\xi_1} d_{q_1} z_{\xi_3} d_{q_2} w.
\end{aligned}
$$

3. A Key Lemma

We first establish the following identity, which is helpful for proving our main results.

Lemma 1. *For $\varrho, \rho \in \mathcal{Q}$, and assume that a function $\Psi : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^{\circ}$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1, \xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)] \times [\xi_3, \xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)] \subseteq \mathcal{Q}^{\circ}$ where $\alpha, \lambda > 0$, and $\nu = (\nu(0), \ldots, \nu(l))$ are the bounded sequence of positive real numbers with $0 < q_1, q_2 < 1$, then one has the equality:*

$$
\begin{aligned}
&\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi) = -\frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\\
&\times \int_0^1\int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0 d_{q_1} z_0 d_{q_2} w\\
&- \frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\\
&\times \int_0^1\int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1), \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))_0 d_{q_1} z_0 d_{q_2} w\\
&- \frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\\
&\times \int_0^1\int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0 d_{q_1} z_0 d_{q_2} w
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\\
&\times\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))_0d_{q_1}z_0d_{q_2}w,
\end{aligned}
$$

where:

$$
\begin{aligned}
&\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)=\Psi(\varrho,\rho)+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\left[\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(\varrho,v)_0d_{q_2}v+\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(\varrho,v)_0d_{q_2}v\right]\\
&+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)}\left[\int_{\xi_1}^{\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)}\Psi(u,\rho)_0d_{q_1}u+\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\Psi(u,\rho)_0d_{q_1}u\right]-\mathcal{T},
\end{aligned}
$$

and:

$$
\begin{aligned}
\mathcal{T} \;=\;& \frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)}\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v\\
&+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)}\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v\\
&+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v\\
&+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v.
\end{aligned}
$$

Proof. Consider:

$$
\begin{aligned}
&-\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&-\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))_0d_{q_1}z_0d_{q_2}w\\
&-\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&-\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))_0d_{q_1}z_0d_{q_2}w. \qquad (10)
\end{aligned}
$$

By the definition of partial q_1q_2-derivatives and definite q_1q_2-integrals, we have:

$$
\begin{aligned}
&\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
=\;& \frac{1}{(1-q_1)(1-q_2)\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\\
&\times\Bigg[\int_0^1\int_0^1\Psi(\xi_1+zq_1\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+wq_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&-\int_0^1\int_0^1\Psi(\xi_1+zq_1\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&-\int_0^1\int_0^1\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+wq_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&+\int_0^1\int_0^1\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z\;_0d_{q_2}w\Bigg]\\
&=\frac{1}{q_1q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&-\frac{1}{q_1\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&-\frac{1}{q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)).
\end{aligned}\tag{11}
$$

We observe that:

$$
\begin{aligned}
&\frac{1}{q_1q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&=-\frac{\Psi(\varrho,\rho)}{q_1q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}-\frac{1}{q_1q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}q^n\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\rho)\\
&-\frac{1}{q_1q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{m=0}^{\infty}q_2^m(\varrho,\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&+\frac{1}{q_1q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))
\end{aligned}\tag{12}
$$

$$
\begin{aligned}
&-\frac{1}{q_1\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)),\\
&=\frac{1}{q_1\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{m=0}^{\infty}q_2^m\Psi(\varrho,\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&-\frac{1}{q_1\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))
\end{aligned}\tag{13}
$$

and:

$$
\begin{aligned}
&-\frac{1}{q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\\
&=\frac{1}{q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}q_1^n\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\rho)\\
&-\frac{1}{q_2\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Psi(\xi_1+q_1^n\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+q_2^m\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)).
\end{aligned}\tag{14}
$$

Utilizing (12)–(14) in (11), we get:

$$\int_0^1 \int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3))_0 d_{q_1} z_0 d_{q_2} w$$
$$= -\frac{\Psi(\varrho,\rho)}{q_1 q_2 \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} - \frac{(1-q_2)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)}{q_1 q_2 \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)^2} \sum_{m=0}^{\infty} q_2^m(\varrho, \xi_3 + q_2^m \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3))$$
$$- \frac{(1-q_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)}{q_1 q_2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \sum_{n=0}^{\infty} q^n \Psi(\xi_1 + q_1^n \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \rho)$$
$$+ \frac{(1-q_1)(1-q_2)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)}{q_1 q_2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right]^2} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Psi(\xi_1 + q_1^n \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_3 + q_2^m \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)).$$

Furthermore:

$$\int_0^1 \int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3))_0 d_{q_1} z_0 d_{q_2} w$$
$$= -\frac{\Psi(\varrho,\rho)}{q_1 q_2 \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} - \frac{1}{q_1 q_2 \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)^2} \int_{\xi_3}^{\xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \Psi(\varrho, v)_0 d_{q_2} v$$
$$- \frac{1}{q_1 q_2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \int_{\xi_1}^{\xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)} \Psi(u,\rho)\, {}_0 d_{q_1} u$$
$$+ \frac{1}{q_1 q_2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right]^2} \int_{\xi_1}^{\xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)} \int_{\xi_3}^{\xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \Psi(u,v)_0 d_{q_1} u_0 d_{q_2} v. \quad (15)$$

Multiplying both sides of Equality (15) by $\frac{q_1 q_2 [\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)]^2 [\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)}$, then we acquire:

$$\frac{q_1 q_2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \int_0^1 \int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3))_0 d_{q_1} z_0 d_{q_2} w$$
$$= -\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \Psi(\varrho,\rho) - \frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \int_{\xi_3}^{\xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \Psi(\varrho, v)_0 d_{q_2} v$$
$$- \frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)}{(\xi_2 - \xi_1)(\xi_4 - \xi_3))} \int_{\xi_3}^{\xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \Psi(u,\rho)_0 d_{q_1} u$$
$$+ \frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \int_{\xi_1}^{\xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)} \int_{\xi_3}^{\xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)} \Psi(u,v)_0 d_{q_1} u_0 d_{q_2} v. \quad (16)$$

Similarly, we calculate the remainingintegrals:

$$\frac{q_1 q_2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \int_0^1 \int_0^1 zw \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4))_0 d_{q_1} z_0 d_{q_2} w$$
$$= -\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \rho)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \Psi(\varrho,\rho) - \frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \int_{\xi_4}^{\xi_4 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4)} \Psi(\varrho, v)_0 d_{q_2} v$$
$$+ \frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \rho)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3))} \int_{\xi_1}^{\xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)} \Psi(u,\rho)_0 d_{q_1} u$$
$$+ \frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \int_{\xi_1}^{\xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)} \int_{\xi_4}^{\xi_4 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4)} \Psi(u,v)_0 d_{q_1} u_0 d_{q_2} v. \quad (17)$$

$$\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\Psi(\varrho,\rho)-\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(\varrho,v)_0d_{q_2}v$$
$$-\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3))}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\Psi(u,\rho)_0d_{q_1}u$$
$$+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v. \tag{18}$$

$$\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{(\xi_2-\xi_1)\,(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\Psi(\varrho,\rho)$$
$$-\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(\varrho,v)_0d_{q_2}v$$
$$-\frac{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3))}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\Psi(u,\rho)_0d_{q_1}u$$
$$+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v. \tag{19}$$

From (16)–(19) in (10), we derive the desired result of Lemma 1. □

Corollary 1. *Setting $q_1,q_2\to 1^-$ in Lemma 1, we obtain the following new equality:*

$$O(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)$$
$$=-\frac{[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))dzdw$$
$$-\frac{[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))dzdw$$
$$-\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))dzdw$$
$$-\frac{[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))dzdw,$$

$$O(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)=\Psi(\varrho,\rho)+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\left[\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(\varrho,v)dv+\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(\varrho,v)dv\right]$$
$$+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)}\left[\int_{\xi_1}^{\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)}\Psi(u,\rho)du+\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\Psi(u,\rho)du\right]-\mathcal{A}\,,$$

and:

$$\begin{aligned}\mathcal{A} &= \frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)}\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(u,v)dzdw \\ &+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)}\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(u,v)dzdw \\ &+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_3}^{\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)}\Psi(u,v)dzdw \\ &+\frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2)}\int_{\xi_4}^{\xi_4+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4)}\Psi(u,v)dzdw.\end{aligned}$$

4. Main Results

Theorem 9. *For $\varrho,\rho \in \mathcal{Q}$, and assume that a function $\Psi : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^{\circ}$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)]\times[\xi_3,\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)] \subseteq \mathcal{Q}^{\circ}$ where $\alpha,\lambda>0$ and $\nu=(\nu(0),\dots,\nu(l))$ are the bounded sequence of positive real numbers with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|$ is generalized Φ-convex on the coordinates on $[\xi_1,\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)]\times[\xi_3,\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)]$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\le\mathcal{M}$, then the following inequality holds:*

$$\begin{aligned}&\left|\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \\ &\le q_1q_2\mathcal{M}\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2}{(1+q_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)}\right]\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{(1+q_2)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\right].\end{aligned} \tag{20}$$

Proof. Taking the absolute value on both sides of (10) and utilizing the coordinated generalized Φ-convex of $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|$, we get the following inequality:

$$\begin{aligned}&\left|\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \\ &= -\frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\left|\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \\ &-\frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w \\ &-\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \\ &-\frac{q_1q_2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)]^2[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w.\end{aligned}$$

Taking the first integral:

$$\begin{aligned}
&\left|\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \\
\le\ &\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\int_0^1\int_0^1 z^2w^2{}_0d_{q_1}z_0d_{q_2}w+\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\xi_3)\right|\int_0^1\int_0^1 z^2w\left(1-w\right){}_0d_{q_1}z_0d_{q_2}w \\
&+\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1,\rho)\right|\int_0^1\int_0^1 w^2z\left(1-z\right){}_0d_{q_1}z_0d_{q_2}w+\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1,\xi_3)\right|\int_0^1\int_0^1 zw\left(1-z\right)\left(1-w\right){}_0d_{q_1}z_0d_{q_2}w.
\end{aligned} \tag{21}$$

By using Definition 8, we get:

$$\begin{aligned}
\int_0^1\int_0^1 z^2w^2{}_0d_{q_1}z_0d_{q_2}w &= \frac{1}{\left(1+q_1^2+q_1\right)\left(1+q_2^2+q_2\right)}, \\
\int_0^1\int_0^1 z^2w\left(1-w\right){}_0d_{q_1}z_0d_{q_2}w &= \frac{q_2^2}{\left(1+q_2\right)\left(1+q_1^2+q_1\right)\left(1+q_2^2+q_2\right)}, \\
\int_0^1\int_0^1 w^2z\left(1-z\right){}_0d_{q_1}z_0d_{q_2}w &= \frac{q_1^2}{\left(1+q_1\right)\left(1+q_1^2+q_1\right)\left(1+q_2^2+q_2\right)}, \\
\int_0^1\int_0^1 zw\left(1-z\right)\left(1-w\right){}_0d_{q_1}z_0d_{q_2}w &= \frac{q_1^2q_2^2}{\left(1+q_1\right)\left(1+q_2\right)\left(1+q_1^2+q_1\right)\left(1+q_2^2+q_2\right)}
\end{aligned}$$

and:

$$\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\le\mathcal{M},\varrho,\rho\in\mathcal{Q}.$$

Hence, from (21), we get:

$$\left|\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w\le\frac{1}{(1+q_1)(1+q_2)}\mathcal{M}. \tag{22}$$

Analogously, we also have:

$$\left|\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w\le\frac{1}{(1+q_1)(1+q_2)}\mathcal{M}, \tag{23}$$

$$\left|\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w\le\frac{1}{(1+q_1)(1+q_2)}\mathcal{M}, \tag{24}$$

$$\left|\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w\le\frac{1}{(1+q_1)(1+q_2)}\mathcal{M}. \tag{25}$$

Now, by making use of the inequalities (22)–(25) and the fact that:

$$\begin{aligned}
&\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2 \\
&+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2 \\
&=\left[\left[\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\varrho-\xi_1\right)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_2-\varrho\right)\right]^2\right]\left[\left[\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\rho-\xi_3\right)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_4-\rho\right)\right]^2\right],
\end{aligned}$$

we get the inequality (20). This completes the proof. □

Corollary 2. *Setting $q_1, q_2 \to 1^-$ in Theorem 9, we obtain the following new inequality:*

$$O(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi) \le \mathcal{M}\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2}{2\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)}\right]\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{2\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\right].$$

Theorem 10. *For $\varrho,\rho \in \mathcal{Q}$, and assume that a function $\Psi : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1, \xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)] \times [\xi_3, \xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)] \subseteq \mathcal{Q}^o$ where $\alpha,\lambda > 0$ and $\nu = (\nu(0),\ldots,\nu(l))$ are the bounded sequence of positive real numbers with $0 < q_1, q_2 < 1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^{\tau}$ is generalized Φ-convex on the coordinates on $[\xi_1, \xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)] \times [\xi_3, \xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)]$, $\sigma,\tau > 1$, $\frac{1}{\tau}+\frac{1}{\sigma}=1$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right| \le \mathcal{M}$, then the following inequality holds:*

$$\begin{aligned}
&\left|\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \le q_1q_2\mathcal{M}\sqrt[\sigma]{\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)} \\
&\times\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)}\right]\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\right]. \qquad (26)
\end{aligned}$$

Proof. Taking the absolute value on both sides of (10) and applying the Hölder inequality for double integrals, we have that the inequality holds:

$$\begin{aligned}
&\left|\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \\
\le\ &\left(\int_0^1\int_0^1 z^{\sigma}w^{\sigma}\,{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\sigma}}\left[\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\right. \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}} \\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}} \\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)} \\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}} \\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)} \\
&\left.\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}}\right].
\end{aligned}$$

Using the coordinated generalized Φ-convex of $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$, we get that the following inequality holds:

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^\nu_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^\nu_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w\\
\le\ & \left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|^\tau\int_0^1\int_0^1 zw_0d_{q_1}z_0d_{q_2}w+\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\xi_3)\right|^\tau\int_0^1\int_0^1 z(1-w)\,{}_0d_{q_1}z_0d_{q_2}w\\
&+\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1,\rho)\right|^\tau\int_0^1\int_0^1 w(1-z)\,{}_0d_{q_1}z_0d_{q_2}w+\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1,\xi_3)\right|^\tau\int_0^1\int_0^1 (1-z)(1-w)\,{}_0d_{q_1}z_0d_{q_2}w.
\end{aligned} \tag{27}$$

By using Definition 8, we get:

$$\begin{aligned}
&\int_0^1\int_0^1 zw_0d_{q_1}z_0d_{q_2}w=\frac{1}{(1+q_1)(1+q_2)},\int_0^1\int_0^1 z(1-w)\,{}_0d_{q_1}z_0d_{q_2}w=\frac{q_2}{(1+q_1)(1+q_2)},\\
&\int_0^1\int_0^1 w(1-z)\,{}_0d_{q_1}z_0d_{q_2}w=\frac{q_1}{(1+q_1)(1+q_2)},\int_0^1\int_0^1 (1-z)(1-w)\,{}_0d_{q_1}z_0d_{q_2}w=\frac{q_1q_2}{(1+q_1)(1+q_2)}
\end{aligned}$$

and:

$$\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\le\mathcal{M},\varrho,\rho\in\mathcal{Q}.$$

Hence, from (27), we get:

$$\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^\nu_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^\nu_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w\le\mathcal{M}^\tau.$$

Similarly, we also have the following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^\nu_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^\nu_{\alpha,\lambda}(\rho-\xi_4))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w\le\mathcal{M}^\tau,$$

$$\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^\nu_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^\nu_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w\le\mathcal{M}^\tau,$$

$$\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^\nu_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^\nu_{\alpha,\lambda}(\rho-\xi_4))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w\le\mathcal{M}^\tau$$

and using the fact that:

$$\int_0^1\int_0^1 z^\sigma w^\sigma{}_0d_{q_1}z_0d_{q_2}w=\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)$$

and the above inequalities in (27), we get (26). This completes the proof of the theorem. □

Corollary 3. *Setting $q_1,q_2\to 1^-$ in Theorem 10, we obtain the following new inequality:*

$$\begin{aligned}
&|O(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)|\\
\le\ & q_1q_2\mathcal{M}\sqrt[\sigma]{\frac{1}{(1+\sigma)^2}}\left[\frac{\left[\mathcal{F}^\nu_{\alpha,\lambda}(\varrho-\xi_1)\right]^2+\left[\mathcal{F}^\nu_{\alpha,\lambda}(\xi_2-\varrho)\right]^2}{\mathcal{F}^\nu_{\alpha,\lambda}(\xi_2-\xi_1)}\right]\left[\frac{\left[\mathcal{F}^\nu_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^\nu_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^\nu_{\alpha,\lambda}(\xi_4-\xi_3)}\right].
\end{aligned}$$

Theorem 11. *For $\varrho, \rho \in \mathcal{Q}$, and assume that a function $\Psi : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1, \xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)] \times [\xi_3, \xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)] \subseteq \mathcal{Q}^o$ where $\alpha, \lambda > 0$ and $\nu = (\nu(0), \ldots, \nu(l))$ are the bounded sequence of positive real numbers with $0 < q_1, q_2 < 1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\right|^{\tau}$ is generalized Φ-convex on the coordinates on $[\xi_1, \xi_1 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)] \times [\xi_3, \xi_3 + \mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)]$, $\tau \geq 1$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\varrho, \rho)\right| \leq \mathcal{M}$, then the following inequality holds:*

$$\begin{aligned}
&\left|\Omega_{q_1q_2}(\xi_1, \xi_2, \xi_3, \xi_4)(\Psi)\right| \\
\leq\ & \frac{q_1q_2\mathcal{M}}{(1+q_1)(1+q_2)}\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2 + \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \varrho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)}\right]\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right]^2 + \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)}\right].
\end{aligned}$$

Proof. Taking the absolute value on both sides of (10) and applying the power mean inequality for double integrals, we have that the inequality holds:

$$\begin{aligned}
&\left|\Omega_{q_1q_2}(\xi_1, \xi_2, \xi_3, \xi_4)(\Psi)\right| \leq \left(\int_0^1\int_0^1 zw\,{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{1-\frac{1}{\tau}}\left[\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)}\right. \\
&\times\left(\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}} \\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \\
&\times\left(\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}} \\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \\
&\times\left(\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_2), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}} \\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2 - \xi_1)\,\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4 - \xi_3)} \\
&\left.\times\left(\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_2), \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4))\right|^{\tau}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\tau}}\right].
\end{aligned}$$

By a similar argument as in Theorem 9 that $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\right|^\tau$ is coordinated generalized Φ-convex on $\mathcal{Q}$:

$$\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w \le \frac{1}{(1+q_1)(1+q_2)}\mathcal{M}^\tau,$$
$$\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w \le \frac{1}{(1+q_1)(1+q_2)}\mathcal{M}^\tau,$$
$$\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w \le \frac{1}{(1+q_1)(1+q_2)}\mathcal{M}^\tau$$

and:

$$\int_0^1\int_0^1 zw\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|^\tau {}_0d_{q_1}z_0d_{q_2}w \le \frac{1}{(1+q_1)(1+q_2)}\mathcal{M}^\tau.$$

Now, by utilizing the above inequalities and the fact that:

$$\int_0^1\int_0^1 zw_0d_{q_1}z_0d_{q_2}w = \frac{1}{(1+q_1)(1+q_2)},$$

in (28), this completes the proof of the theorem. $\square$

Corollary 4. *Setting $q_1, q_2 \to 1^-$ in Theorem 10, we obtain the following new inequality:*

$$|O(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)| \le \frac{\mathcal{M}}{4}\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)}\right]\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\right].$$

Theorem 12. *For $\varrho,\rho \in \mathcal{Q}$, and assume that a function $h : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)]\times[\xi_3,\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)] \subseteq \mathcal{Q}^o$ where $\alpha,\lambda>0$ and $\nu=(\nu(0),\ldots,\nu(l))$ are the bounded sequence of positive real numbers with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\right|^\tau$ is generalized Φ-concave on the coordinates on $[\xi_1,\xi_1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)]\times[\xi_3,\xi_3+\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)]$, $\sigma,\tau>1$, $\frac{1}{\tau}+\frac{1}{\sigma}=1$, then the following inequality holds:*

$$\left|\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Psi)\right| \le \frac{q_1q_2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\xi_1)\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\xi_3)}\sqrt[\sigma]{\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)}$$
$$\times\left[\begin{array}{c}\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left\{\begin{array}{c}\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau\\+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_4}{1+q_2}\right)\right|^\tau\end{array}\right\}\\ \\+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left\{\begin{array}{c}\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau\\+\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w}\Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1},\frac{q_2\rho+\xi_4}{1+q_2}\right)\right|^\tau\end{array}\right\}\end{array}\right].$$

Proof. Taking the absolute value on both sides of (10) and applying the Hölder inequality for double integrals, we have that the inequality holds:

$$
\begin{aligned}
&\left|\Omega_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right| \leq \left(\int_0^1\int_0^1 z^\sigma w^\sigma{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{\sigma}}\left[\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_2-\xi_1\right)\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_4-\xi_3\right)}\right.\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{\tau}}\\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_2-\xi_1\right)\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_4-\xi_3\right)}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|^\tau{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{\tau}}\\
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_2-\xi_1\right)\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_4-\xi_3\right)}\\
&\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{\tau}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{q_1q_2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_2-\varrho)\right]^2\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\xi_4-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_2-\xi_1\right)\mathcal{F}^{\nu}_{\alpha,\lambda}\left(\xi_4-\xi_3\right)}\\
&\left.\times\left(\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_2),\xi_4+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_4))\right|^\tau{}_0d_{q_1}z_0d_{q_2}w\right)^{\frac{1}{\tau}}\right].
\end{aligned} \tag{28}
$$

Since $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$ is generalized Φ-concave on the coordinates on $\mathcal{Q}$, so an application of (4) with the inequalities in reversed direction gives us the following inequalities:

$$
\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3))\right|^\tau{}_0d_{q_1}z_0d_{q_2}w\\
&\leq\frac{1}{2}\left[\begin{array}{c}\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\xi_1+z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-\xi_1),\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau{}_0d_{q_1}z\\+\int_0^1\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\xi_3+w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-\xi_3)\right)\right|^\tau{}_0d_{q_2}w\end{array}\right]\\
&\leq\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau,
\end{aligned} \tag{29}
$$

$$\int_0^1\int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4)) \right|^{\tau} {}_0d_{q_1}z_0d_{q_2}w$$
$$\leq \frac{1}{2}\left[\begin{array}{c} \int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\xi_1 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_1), \frac{q_2\rho+\xi_4}{1+q_2}\right) \right|^{\tau} {}_0d_{q_1}z \\ + \int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1}, \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4)\right) \right|^{\tau} {}_0d_{q_2}w \end{array} \right]$$
$$\leq \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} h\left(\frac{q_1\varrho + \xi_1}{1+q_1}, \frac{q_2\rho + \xi_4}{1+q_2}\right) \right|^{\tau}, \tag{30}$$

$$\int_0^1\int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_2), \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)) \right|^{\tau} {}_0d_{q_1}z_0d_{q_2}w$$
$$\leq \frac{1}{2}\left[\begin{array}{c} \int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_2), \frac{q_2\rho+\xi_3}{1+q_2}\right) \right|^{\tau} {}_0d_{q_1}z \\ + \int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1}, \xi_3 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_3)\right) \right|^{\tau} {}_0d_{q_2}w \end{array} \right]$$
$$\leq \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\frac{q_1\varrho + \xi_2}{1+q_1}, \frac{q_2\rho + \xi_3}{1+q_2}\right) \right|^{\tau} \tag{31}$$

and:

$$\int_0^1\int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_2), \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4)) \right|^{\tau} {}_0d_{q_1}z_0d_{q_2}w$$
$$\leq \frac{1}{2}\left[\begin{array}{c} \int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\xi_2 + z\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho - \xi_2), \frac{q_2\rho+\xi_4}{1+q_2}\right) \right|^{\tau} {}_0d_{q_1}z \\ + \int_0^1 \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1}, \xi_4 + w\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho - \xi_4)\right) \right|^{\tau} {}_0d_{q_2}w \end{array} \right]$$
$$\leq \left| \frac{\partial^2_{q_1,q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\left(\frac{q_1\varrho + \xi_2}{1+q_1}, \frac{q_2\rho + \xi_4}{1+q_2}\right) \right|^{\tau}. \tag{32}$$

By making use of (29)–(32) in (28), the proof of Theorem 12 is complete. □

5. Quantum Estimates Using the Hypergeometric and Mittag–Leffler Functions

As stated in the Preliminaries Section, for suitable values of parameters α, λ, and ν in Raina's function (6), by using the new form of Raina's function (6), we can establish results for the hypergeometric function and the Mittag–Leffler function as special cases.

5.1. For the Hypergeometric Function

Taking $\alpha = 1$, $\lambda = 0$, and:

$$\nu(l) = \frac{(\phi)_l (\psi)_l}{(\varphi)_l} \text{ for } l = 0, 1, 2, 3 \ldots,$$

then from Lemma 1, Theorems 9–12, the following results hold.

Lemma 2. *Suppose that a function $\Psi : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, and let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1, \xi_1 + F(\phi, \psi; \varphi; \xi_2 - \xi_1)] \times [\xi_3, \xi_3 + F(\phi, \psi; \varphi; \xi_4 - \xi_3)] \subseteq \mathcal{Q}^o$ with $0 < q_1, q_2 < 1$, then one has the equality:*

$$
\begin{aligned}
&\Upsilon_{q_1 q_2}\left(\xi_1, \xi_2, \xi_3, \xi_4\right)(\Psi) = -\frac{q_1 q_2 [F(\phi, \psi; \varphi; \varrho-\xi_1)]^2 [F\left(\phi, \psi; \varphi; \rho-\xi_3\right)]^2}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \\
&\times \int_0^1 \int_0^1 zw \frac{\partial^2_{q_1, q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + zF\left(\phi, \psi; \varphi; \varrho-\xi_1\right), \xi_3 + wF\left(\phi, \psi; \varphi; \rho-\xi_3\right))_0 d_{q_1} z_0 d_{q_2} w \\
&- \frac{q_1 q_2 [F(\phi, \psi; \varphi; \varrho-\xi_1)]^2 [F(\phi, \psi; \varphi; \xi_4-\rho)]^2}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \\
&\times \int_0^1 \int_0^1 zw \frac{\partial^2_{q_1, q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_1 + zF\left(\phi, \psi; \varphi; \varrho-\xi_1\right), \xi_4 + wF(\phi, \psi; \varphi; \rho-\xi_4))_0 d_{q_1} z_0 d_{q_2} w \\
&- \frac{q_1 q_2 [F(\phi, \psi; \varphi; \xi_2-\varrho)]^2 [F\left(\phi, \psi; \varphi; \rho-\xi_3\right)]^2}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \\
&\times \int_0^1 \int_0^1 zw \frac{\partial^2_{q_1, q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_2 + zF(\phi, \psi; \varphi; \varrho-\xi_2), \xi_3 + wF\left(\phi, \psi; \varphi; \rho-\xi_3\right))_0 d_{q_1} z_0 d_{q_2} w \\
&- \frac{q_1 q_2 [F(\phi, \psi; \varphi; \xi_2-\varrho)]^2 [F(\phi, \psi; \varphi; \xi_4-\rho)]^2}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \\
&\times \int_0^1 \int_0^1 zw \frac{\partial^2_{q_1, q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\xi_2 + zF(\phi, \psi; \varphi; \varrho-\xi_2), \xi_4 + wF(\phi, \psi; \varphi; \rho-\xi_4))_0 d_{q_1} z_0 d_{q_2} w,
\end{aligned}
$$

where:

$$
\begin{aligned}
&\Upsilon_{q_1 q_2}\left(\xi_1, \xi_2, \xi_3, \xi_4\right)(\Psi) = \\
&\Psi\left(\varrho, \rho\right) + \frac{1}{F(\phi, \psi; \varphi; \xi_4-\xi_3)} \left[\int_{\xi_3}^{\xi_3+F(\phi,\psi;\varphi;\rho-\xi_3)} \Psi(\varrho, v)_0 d_{q_2} v + \int_{\xi_4}^{\xi_4+F(\phi,\psi;\varphi;\rho-\xi_4)} \Psi(\varrho, v)_0 d_{q_2} v\right] \\
&+ \frac{1}{F(\phi, \psi; \varphi; \xi_2-\xi_1)} \left[\int_{\xi_1}^{\xi_1+F(\phi,\psi;\varphi;\varrho-\xi_1)} \Psi(u, \rho)_0 d_{q_1} u + \int_{\xi_2}^{\xi_2+F(\phi,\psi;\varphi;\varrho-\xi_2)} \Psi(u, \rho)_0 d_{q_1} u\right] - \mathcal{C}
\end{aligned}
$$

and:

$$
\begin{aligned}
\mathcal{C} &= \frac{1}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \int_{\xi_1}^{\xi_1+F(\phi,\psi;\varphi;\varrho-\xi_1)} \int_{\xi_3}^{\xi_3+F(\phi,\psi;\varphi;\rho-\xi_3)} \Psi(u, v)_0 d_{q_1} u_0 d_{q_2} v \\
&+ \frac{1}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \int_{\xi_1}^{\xi_1+F(\phi,\psi;\varphi;\varrho-\xi_1)} \int_{\xi_4}^{\xi_4+F(\phi,\psi;\varphi;\rho-\xi_4)} \Psi(u, v)_0 d_{q_1} u_0 d_{q_2} v \\
&+ \frac{1}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \int_{\xi_2}^{\xi_2+F(\phi,\psi;\varphi;\varrho-\xi_2)} \int_{\xi_3}^{\xi_3+F(\phi,\psi;\varphi;\rho-\xi_3)} \Psi(u, v)_0 d_{q_1} u_0 d_{q_2} v \\
&+ \frac{1}{F\left(\phi, \psi; \varphi; \xi_2-\xi_1\right) F\left(\phi, \psi; \varphi; \xi_4-\xi_3\right)} \int_{\xi_2}^{\xi_2+F(\phi,\psi;\varphi;\varrho-\xi_2)} \int_{\xi_4}^{\xi_4+F(\phi,\psi;\varphi;\rho-\xi_4)} \Psi(u, v)_0 d_{q_1} u_0 d_{q_2} v.
\end{aligned}
$$

Theorem 13. *For $\varrho, \rho \in \mathcal{Q}$, and assume that a function $\Psi : \mathcal{Q} \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the second-order partial $q_1 q_2$-derivatives over $\mathcal{Q}^{\circ}$, then let the second-order partial $q_1 q_2$-derivatives be continuous and integrable over $[\xi_1, \xi_1 + F(\phi, \psi; \varphi; \xi_2-\xi_1)] \times [\xi_3, \xi_3 + F(\phi, \psi; \varphi; \xi_4-\xi_3)] \subseteq \mathcal{Q}^{\circ}$ with $0 < q_1, q_2 < 1$. If $\left|\frac{\partial^2_{q_1, q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi\right|^{\tau}$ is generalized Φ-convex on the coordinates on $[\xi_1, \xi_1 + F(\phi, \psi; \varphi; \xi_2-\xi_1)] \times [\xi_3, \xi_3 + F(\phi, \psi; \varphi; \xi_4-\xi_3)]$, $\sigma, \tau > 1$, $\frac{1}{\tau} + \frac{1}{\sigma} = 1$ and $\left|\frac{\partial^2_{q_1, q_2}}{\partial_{q_1} z \partial_{q_2} w} \Psi(\varrho, \rho)\right| \leq \mathcal{M}$, then the following inequality holds:*

$$\left|\Upsilon_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right| \leq q_1q_2\mathcal{M}\sqrt[\sigma]{\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)}$$
$$\times\left[\frac{\left[F\left(\phi,\psi;\varphi;\varrho-\xi_1\right)\right]^2+\left[F\left(\phi,\psi;\varphi;\xi_2-\varrho\right)\right]^2}{F\left(\phi,\psi;\varphi;\xi_2-\xi_1\right)}\right]\left[\frac{\left[F\left(\phi,\psi;\varphi;\rho-\xi_3\right)\right]^2+\left[F\left(\phi,\psi;\varphi;\xi_4-\rho\right)\right]^2}{F\left(\phi,\psi;\varphi;\xi_4-\xi_3\right)}\right].$$

Theorem 14. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $\Psi:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+F(\phi,\psi;\varphi;\xi_2-\xi_1)]\times[\xi_3,\xi_3+F(\phi,\psi;\varphi;\xi_4-\xi_3)]\subseteq\mathcal{Q}^o$ with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$ is generalized Φ-convex on the coordinates on $[\xi_1,\xi_1+F(\phi,\psi;\varphi;\xi_2-\xi_1)]\times[\xi_3,\xi_3+F(\phi,\psi;\varphi;\xi_4-\xi_3)]$, $\tau\geq1$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\leq\mathcal{M}$, then the following inequality holds*

$$\left|\Upsilon_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right| \leq \frac{q_1q_2\mathcal{M}}{(1+q_1)(1+q_2)}$$
$$\times\left[\frac{\left[F\left(\phi,\psi;\varphi;\varrho-\xi_1\right)\right]^2+\left[F\left(\phi,\psi;\varphi;\xi_2-\varrho\right)\right]^2}{F\left(\phi,\psi;\varphi;\xi_2-\xi_1\right)}\right]\left[\frac{\left[F\left(\phi,\psi;\varphi;\rho-\xi_3\right)\right]^2+\left[F\left(\phi,\psi;\varphi;\xi_4-\rho\right)\right]^2}{F\left(\phi,\psi;\varphi;\xi_4-\xi_3\right)}\right].$$

Theorem 15. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $h:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+F(\phi,\psi;\varphi;\xi_2-\xi_1)]\times[\xi_3,\xi_3+F(\phi,\psi;\varphi;\xi_4-\xi_3)]\subseteq\mathcal{Q}^o$ with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$ is generalized Φ-concave on the coordinates on $[\xi_1,\xi_1+F(\phi,\psi;\varphi;\xi_2-\xi_1)]\times[\xi_3,\xi_3+F(\phi,\psi;\varphi;\xi_4-\xi_3)]$, $\sigma,\tau>1$, $\frac{1}{\tau}+\frac{1}{\sigma}=1$, then the following inequality holds:*

$$\left|\Upsilon_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right| \leq \frac{q_1q_2}{F(\phi,\psi;\varphi;\xi_2-\xi_1)F(\phi,\psi;\varphi;\xi_4-\xi_3)}\sqrt[\sigma]{\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)}$$
$$\times\left[\begin{array}{c}[F(\phi,\psi;\varphi;\varrho-\xi_1)]^2\left\{\begin{array}{c}[F(\phi,\psi;\varphi;\rho-\xi_3)]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau\\+[F(\phi,\psi;\varphi;\xi_4-\rho)]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_4}{1+q_2}\right)\right|^\tau\end{array}\right\}\\ \\+[F(\phi,\psi;\varphi;\xi_2-\varrho)]^2\left\{\begin{array}{c}[F(\phi,\psi;\varphi;\rho-\xi_3)]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau\\+[F(\phi,\psi;\varphi;\xi_4-\rho)]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1},\frac{q_2\rho+\xi_4}{1+q_2}\right)\right|^\tau\end{array}\right\}\end{array}\right].$$

5.2. For the Mittag–Leffler Function

Taking $\nu=(1,1,\ldots)$ with $\alpha=\phi$, $Re(\phi)>0$ and $\lambda=1$, then from Lemma 1, Theorems 9–12, the following results hold.

Lemma 3. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $\Psi:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]\subseteq\mathcal{Q}^o$ with $0<q_1,q_2<1$, then one has the equality:*

$$
\begin{aligned}
&\Pi_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\\
&=-\frac{q_1q_2[E_\varphi(\varrho-\xi_1)]^2[E_\varphi(\rho-\xi_3)]^2}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+zE_\varphi(\varrho-\xi_1),\xi_3+wE_\varphi(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&-\frac{q_1q_2[E_\varphi(\varrho-\xi_1)]^2[E_\varphi(\xi_4-\rho)]^2}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_1+zE_\varphi(\varrho-\xi_1),\xi_4+wE_\varphi(\rho-\xi_4))_0d_{q_1}z_0d_{q_2}w\\
&-\frac{q_1q_2[E_\varphi(\xi_2-\varrho)]2[E_\varphi(\rho-\xi_3)]^2}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+zE_\varphi(\varrho-\xi_2),\xi_3+wE_\varphi(\rho-\xi_3))_0d_{q_1}z_0d_{q_2}w\\
&-\frac{q_1q_2[E_\varphi(\xi_2-\varrho)]^2[E_\varphi(\xi_4-\rho)]^2}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\xi_2+zE_\varphi(\varrho-\xi_2),\xi_4+wE_\varphi(\rho-\xi_4))_0d_{q_1}z_0d_{q_2}w,
\end{aligned}
$$

where:

$$
\begin{aligned}
&\Pi_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)=\Psi\left(\varrho,\rho\right)+\frac{1}{E_\varphi(\xi_4-\xi_3)}\left[\int_{\xi_3}^{\xi_3+E_\varphi(\rho-\xi_3)}\Psi(\varrho,v)_0d_{q_2}v+\int_{\xi_4}^{\xi_4+E_\varphi(\rho-\xi_4)}\Psi(\varrho,v)_0d_{q_2}v\right]\\
&+\frac{1}{E_\varphi(\xi_2-\xi_1)}\left[\int_{\xi_1}^{\xi_1+E_\varphi(\varrho-\xi_1)}\Psi(u,\rho)_0d_{q_1}u+\int_{\xi_2}^{\xi_2+E_\varphi(\varrho-\xi_2)}\Psi(u,\rho)_0d_{q_1}u\right]-\mathcal{D}
\end{aligned}
$$

and:

$$
\begin{aligned}
\mathcal{D}=&\frac{1}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_1+E_\varphi(\varrho-\xi_1)}\int_{\xi_3}^{\xi_3+E_\varphi(\rho-\xi_3)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v\\
&+\frac{1}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_1+E_\varphi(\varrho-\xi_1)}\int_{\xi_4}^{\xi_4+E_\varphi(\rho-\xi_4)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v\\
&+\frac{1}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+E_\varphi(\varrho-\xi_2)}\int_{\xi_3}^{\xi_3+E_\varphi(\rho-\xi_3)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v\\
&+\frac{1}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\int_{\xi_2}^{\xi_2+E_\varphi(\varrho-\xi_2)}\int_{\xi_4}^{\xi_4+E_\varphi(\rho-\xi_4)}\Psi(u,v)_0d_{q_1}u_0d_{q_2}v.
\end{aligned}
$$

Theorem 16. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $\Psi:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^\circ$, then let the second-order partial $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]\subseteq\mathcal{Q}^\circ$ where $\alpha,\lambda>0$, and $\nu=(\nu(0),\ldots,\nu(l))$ are the bounded sequence of positive real numbers with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|$ is generalized Φ-convex on the coordinates on $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\le\mathcal{M}$, then the following inequality holds:*

$$
\left|\Pi_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right|\le q_1q_2\mathcal{M}\left[\frac{\left[E_\varphi\left(\varrho-\xi_1\right)\right]^2+\left[E_\varphi\left(\xi_2-\varrho\right)\right]^2}{\left(1+q_1\right)E_\varphi\left(\xi_2-\xi_1\right)}\right]\left[\frac{\left[E_\varphi\left(\rho-\xi_3\right)\right]^2+\left[E_\varphi\left(\xi_4-\rho\right)\right]^2}{\left(1+q_2\right)E_\varphi\left(\xi_4-\xi_3\right)}\right].
$$

Theorem 17. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $\Psi:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^\circ$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]\subseteq\mathcal{Q}^\circ$ with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$ is generalized Φ-convex on the coordinates on $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]$, $\sigma,\tau>1$, $\frac{1}{\tau}+\frac{1}{\sigma}=1$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\le\mathcal{M}$, $\varrho,\rho\in\mathcal{Q}$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\Pi_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right| \\
\leq\ & q_1q_2\mathcal{M}\sqrt[\sigma]{\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)}\left[\frac{\left[E_\varphi\left(\varrho-\xi_1\right)\right]^2+\left[E_\varphi\left(\xi_2-\varrho\right)\right]^2}{E_\phi\left(\xi_2-\xi_1\right)}\right]\left[\frac{\left[E_\phi\left(\rho-\xi_3\right)\right]^2+\left[E_\varphi\left(\xi_4-\rho\right)\right]^2}{E_\varphi\left(\xi_4-\xi_3\right)}\right].
\end{aligned}
$$

Theorem 18. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $\Psi:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]\subseteq\mathcal{Q}^o$ with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$ is generalized Φ-convex on the coordinates on $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]$, $\tau\geq1$ and $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi(\varrho,\rho)\right|\leq\mathcal{M}$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\Pi_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right|\leq\frac{q_1q_2\mathcal{M}}{(1+q_1)(1+q_2)} \\
&\times\left[\frac{\left[E_\varphi\left(\varrho-\xi_1\right)\right]^2+\left[E_\varphi\left(\xi_2-\varrho\right)\right]^2}{E_\phi\left(\xi_2-\xi_1\right)}\right]\left[\frac{\left[E_\phi\left(\rho-\xi_3\right)\right]^2+\left[E_\varphi\left(\xi_4-\rho\right)\right]^2}{E_\varphi\left(\xi_4-\xi_3\right)}\right].
\end{aligned}
$$

Theorem 19. *For $\varrho,\rho\in\mathcal{Q}$, and assume that a function $\Psi:\mathcal{Q}\subseteq\mathbb{R}^2\to\mathbb{R}$ is the second-order partial q_1q_2-derivatives over $\mathcal{Q}^o$, then let the second-order partial q_1q_2-derivatives be continuous and integrable over $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]\subseteq\mathcal{Q}^o$ with $0<q_1,q_2<1$. If $\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\right|^\tau$ is generalized Φ-concave on the coordinates on $[\xi_1,\xi_1+E_\varphi(\xi_2-\xi_1)]\times[\xi_3,\xi_3+E_\varphi(\xi_4-\xi_3)]$, $\sigma,\tau>1$, $\frac{1}{\tau}+\frac{1}{\sigma}=1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\Pi_{q_1q_2}\left(\xi_1,\xi_2,\xi_3,\xi_4\right)(\Psi)\right|\leq\frac{q_1q_2}{E_\varphi(\xi_2-\xi_1)E_\varphi(\xi_4-\xi_3)}\sqrt[\sigma]{\left(\frac{1-q_1}{1-q_1^{\sigma+1}}\right)\left(\frac{1-q_2}{1-q_2^{\sigma+1}}\right)} \\
&\times\left[\begin{array}{c}
\left[E_\varphi(\varrho-\xi_1)\right]^2\left\{\begin{array}{c}\left[E_\varphi(\rho-\xi_3)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau\\+\left[E_\varphi(\xi_4-\rho)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_1}{1+q_1},\frac{q_2\rho+\xi_4}{1+q_2}\right)\right|^\tau\end{array}\right\}\\
\\
+\left[E_\varphi(\xi_2-\varrho)\right]^2\left\{\begin{array}{c}\left[E_\varphi(\rho-\xi_3)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1},\frac{q_2\rho+\xi_3}{1+q_2}\right)\right|^\tau\\+\left[E_\varphi(\xi_4-\rho)\right]^2\left|\frac{\partial^2_{q_1,q_2}}{\partial_{q_1}z\partial_{q_2}w}\Psi\left(\frac{q_1\varrho+\xi_2}{1+q_1},\frac{q_2\rho+\xi_4}{1+q_2}\right)\right|^\tau\end{array}\right\}
\end{array}\right].
\end{aligned}
$$

6. Example

Example 1. *Let a function $\Psi(\varrho,\rho)=\varrho\rho$ be the second-order partial $\frac{1}{2},\frac{1}{3}$-derivatives over $\mathcal{Q}^o$, and let the second-order partial $\frac{1}{2},\frac{1}{3}$-derivatives be continuous and integrable over $[0,\mathcal{F}^\nu_{\alpha,\lambda}(1-0)]\times[0,\mathcal{F}^\nu_{\alpha,\lambda}(2-0)]\subseteq\mathcal{Q}^o$ where $\alpha,\lambda>0$ and $\nu=(\nu(0),\ldots,\nu(l))$ are the bounded sequence of positive real numbers. If $\left|\frac{\partial^2_{\frac{1}{2},\frac{1}{3}}}{\partial_{\frac{1}{2}}z\partial_{\frac{1}{3}}w}\Psi\right|$ is generalized Φ-convex on the coordinates on $[0,\mathcal{F}^\nu_{\alpha,\lambda}(1-0)]\times[0,\mathcal{F}^\nu_{\alpha,\lambda}(2-0)]$ and $\mathcal{M}=0.68$, then all assumptions of Theorem 9 are satisfied.*

Clearly,

$$\begin{aligned}
&|\Omega_{\frac{1}{2}\frac{1}{3}}(0,1,0,2)(\Psi)| = |\varrho\rho + \frac{\varrho}{\mathcal{F}^{\nu}_{\alpha,\lambda}(2-0)}\left[\int_{0}^{\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-0)} v_0 d_{\frac{1}{3}} v + \int_{2}^{2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-2)} v_0 d_{\frac{1}{3}} v\right] \\
&+\frac{\rho}{\mathcal{F}^{\nu}_{\alpha,\lambda}(1-0)}\left[\int_{0}^{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-0)} u_0 d_{\frac{1}{2}} u + \int_{1}^{1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-1)} u_0 d_{\frac{1}{2}} u\right] - \mathcal{A}| \\
&\leq \frac{1}{12}\mathcal{M}\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-0)\right]^2 + \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(1-\varrho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(1-0)}\right]\left[\frac{\left[\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-0)\right]^2 + \left[\mathcal{F}^{\nu}_{\alpha,\lambda}(2-\rho)\right]^2}{\mathcal{F}^{\nu}_{\alpha,\lambda}(2-0)}\right]
\end{aligned} \tag{33}$$

and:

$$\begin{aligned}
\mathcal{A} &= \frac{1}{\mathcal{F}^{\nu}_{\alpha,\lambda}(1-0)\mathcal{F}^{\nu}_{\alpha,\lambda}(2-0)}\left[\int_{0}^{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-0)}\int_{0}^{\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-0)} uv_0 d_{\frac{1}{2}} u_0 d_{\frac{1}{3}} + \int_{0}^{\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-0)}\int_{2}^{2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-2)} uv_0 d_{\frac{1}{2}} u_0 d_{\frac{1}{3}} v\right. \\
&\left.+\int_{1}^{1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-1)}\int_{0}^{\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-0)} uv_0 d_{\frac{1}{2}} u_0 d_{\frac{1}{3}} v + \int_{1}^{1+\mathcal{F}^{\nu}_{\alpha,\lambda}(\varrho-1)}\int_{2}^{2+\mathcal{F}^{\nu}_{\alpha,\lambda}(\rho-2)} uv_0 d_{\frac{1}{2}} u_0 d_{\frac{1}{3}} v\right],
\end{aligned}$$

where $\mathcal{F}^{\nu}_{\alpha,\lambda}(.)$ is Raina's function and having the property $\mathcal{F}^{\nu}_{\alpha,\lambda}(a,b) = b - a > 0$.

We get our required inequality by using Definition 8, the above property, and taking suitable choice of $\varrho, \rho \in \mathcal{Q}$ in (33).

7. Conclusions

The endurance of any area of research, pure and applied mathematics, relies on the capability of the specialists progressing in the direction of yet-to-be-addressed inquiries and to update the existing hypothesis and practice. The idea of quantum calculus and special functions has seen numerous variations since it was originated by Leibniz and Newton. Several generalizations are predominantly because of the fact that analysts might want to explore a new scheme of study, and they have to comprehend its tendency and dissect and anticipate it well. The prediction requires its utilities in the real world. Quantum theory known as the theory with no limits is frequently used to find consequences in various scientific studies. Over the idea of the quantum and special function, a novel study was proposed in the past [43] wherein the ideas of quantum and special functions were joined to acquire the new outcomes. We suggested in this paper coordinated generalized Φ-convexity with new quantum estimates of Ostrowski-type variants to be arrived at utilizing the innovative technique. We considered various cases in the present research study. The first was associated with a useful identity, and the second one was related to the main results correlated with the coordinated generalized Φ-convexity with new quantum estimates of Ostrowski-type variants, while the last one was quantum estimates associated with hypergeometric and Mittag–Leffler functions. Our consequences are helpful for resolving integral equations' construction for the system of interacting n bodies subject to mixed boundary conditions; see [10–14]. We omitted their proof, and the details are left to the interested reader. For each section, we derived an innovative numerical scheme that required all precision necessities and being simultaneously simpler to execute. To observe both of these novelties of the quantum and Raina's function alongside new recommended numerical schemes under certain conditions, we thought about some intrigued readers. We exhibited several special cases for changing the parametric values of Raina's function. These new investigations will be displayed in future research work being handled by the authors of the present paper.

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