Research Article

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Analytical mathematical schemes: Circular rod grounded via transverse Poisson's effect and extensive wave propagation on the surface of water

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Abstract: This article scrutinizes the efficacy of analytical mathematical schemes, improved simple equation and $\exp(-\Psi(\xi))$-expansion techniques for solving the well-known nonlinear partial differential equations. A longitudinal wave model is used for the description of the dispersion in the circular rod grounded via transverse Poisson's effect; similarly, the Boussinesq equation is used for extensive wave propagation on the surface of water. Many other such types of equations are also solved with these techniques. Hence, our methods appear easier and faster via symbolic computation.

Keywords: longitudinal wave equation, Boussinesq equation, improved simple equation method, $\exp(-\Psi(\xi))$-expansion method

1 Introduction

Many nonlinear physical phenomena in nature are described by nonlinear partial differential equations (PDEs). For deliberative speedy development of symbolic computation systems [1–5], the search for the exact solutions of nonlinear equations has attracted a lot of attention [6–9] as the exact solutions make it possible to research nonlinear physical phenomena comprehensively and facilitate testing the numerical schemes [10–14]. In the last two decades, various approaches have been proposed and applied to the nonlinear equations of PDEs, such as homogeneous balance method [15,16], extended tanh-function method [17–20], Jacobi elliptic function expansion method [21], simple equation method [22–24], (G/G')-expansion method [25–27], Hirotas bilinear method [28], Exp function method [29], general projective Riccati equation method [30], modified simple equation method [31–33], improved direct algebraic technique, [34,35], auxiliary scheme [36] and so on [37–44]. Recently, authors in [45] discussed the exact solutions of the longitudinal wave model by implementing the extended trial technique similarly in [46] derived the exact solutions of the Boussinesq equation via the (G/G')-expansion technique. But our central focus is to investigate solitary solutions of these models via improved simple equation and $\exp(-\Psi(\xi))$-expansion schemes. The derived solutions are more general and simple as compared to the previous solutions discussed in [45,46]. Hence, our investigated methods are fruitful tools for solving other complex nonlinear wave problems in mathematical physics.

The remnant structure of the article is arranged as follows: description of the proposed methods is given in Section 2. Applications are given in Section 3. Results and discussion and summary are illustrated in Sections 4 and 5, respectively.

2 Description of the proposed methods

Consider

$$Q_1(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0. \tag{1}$$

Let

$$u = V(\xi), \quad \xi = x + kt. \tag{2}$$

Substituting (2) in (1),

$$Q_2(V, V', V'', \ldots) = 0. \tag{3}$$

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2.1 Improved simple equation method

Let (3) have the solution:

$$V = \sum_{i=-N}^{N} A_i \Psi^i. \quad (4)$$

Let Ψ gratify,

$$\Psi' = c_0 + c_1\Psi + c_2\Psi^2 + c_3\Psi^3. \quad (5)$$

Substituting (4) along with (5) in (3), the required solution of (1) is obtained.

2.2 Exp$(-\Psi(\xi))$-expansion method

Let (3) have the solution:

$$V = a_N(\exp(-\Psi(\xi)))^N + \cdots,\ a_N \neq 0. \quad (6)$$

Suppose Ψ satisfies,

$$\Psi' = \exp(-\Psi(\xi)) + \mu\exp(\Psi(\xi)) + \lambda. \quad (7)$$

Substituting (6) with (7) in (3), the solution of (1) is obtained.

3 Applications

3.1 Applications of improved simple equation method

3.1.1 Longitudinal wave model

Longitudinal wave equation has the form [45]:

$$u_{tt} - a^2u_{xx} - \left(\frac{a}{2}u^2 + bu_{tt}\right)_{xx} = 0. \quad (8)$$

Substituting (2) in (8) by twice integrations with an integration constant equal to zero yields

$$2V(k^2 - a^2) - aV^2 - 2bk^2V'' = 0. \quad (9)$$

Let (9) have the solution:

$$V = A_2\Psi^2 + A_1\Psi + \frac{A_{-2}}{\Psi^2} + \frac{A_{-1}}{\Psi} + A_0. \quad (10)$$

Substituting (10) in (9) along with (5), the following system of equations is obtained.

Case I: $c_3 = 0$

Family I

$$A_0 = -\frac{2(abc_1^2 + 2abc_0c_2)}{bc_1^2 - 4bc_0c_2 + 1},\quad A_{-2} = 0,$$
$$A_{-1} = 0,\quad A_2 = \frac{12abc_2^2}{-bc_1^2 + 4bc_0c_2 - 1}, \quad (11)$$
$$A_1 = -\frac{12abc_1c_2}{bc_1^2 - 4bc_0c_2 + 1},\quad k = \frac{a}{\sqrt{bc_1^2 - 4bc_0c_2 + 1}}.$$

Substitute (11) in (10) with (5), then the solution of equation (1) is achieved as,

$$V_1 = -\frac{2(abc_1^2 + 2abc_0c_2)}{bc_1^2 - 4bc_0c_2 + 1} + \frac{12abc_1c_2}{bc_1^2 - 4bc_0c_2 + 1} \times \left(\frac{c_1 - \sqrt{4c_0c_2 - c_1^2}\tan\left(\frac{\sqrt{4c_0c_2 - c_1^2}}{2}(\xi + \xi_0)\right)}{2c_2}\right)$$
$$- \frac{12abc_2^2}{-bc_1^2 + 4bc_0c_2 - 1} \times \left(\frac{c_1 - \sqrt{4c_0c_2 - c_1^2}\tan\left(\frac{\sqrt{4c_0c_2 - c_1^2}}{2}\right)(\xi + \xi_0)}{2c_2}\right)^2, \quad (12)$$
$$\times\, 4c_0c_2 > c_1^2.$$

Family II

$$A_0 = -\frac{2(abc_1^2 + 2abc_0c_2)}{bc_1^2 - 4bc_0c_2 + 1},\quad A_{-2} = \frac{12abc_0^2}{-bc_1^2 + 4bc_0c_2 - 1},$$
$$A_{-1} = \frac{12abc_0c_1}{-bc_1^2 + 4bc_0c_2 - 1}, \quad (13)$$
$$A_2 = 0,\quad A_1 = 0,\quad k = \frac{a}{\sqrt{bc_1^2 - 4bc_0c_2 + 1}}.$$

Substituting (13) in (10),

$$V_2 = -\frac{2(abc_1^2 + 2abc_0c_2)}{bc_1^2 - 4bc_0c_2 + 1} - \frac{12abc_0c_1}{-bc_1^2 + 4bc_0c_2 - 1} \times \left(\frac{2c_2}{c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)}\right)$$
$$- \frac{12abc_0^2}{-bc_1^2 + 4bc_0c_2 - 1} \times \left(\frac{4c_2^2}{\left(c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)\right)^2}\right), \quad (14)$$
$$\times\, 4c_0c_2 > c_1^2.$$

Case II: $c_0 = c_3 = 0$

$$A_0 = 0, \quad A_{-2} = 0, \quad A_{-1} = 0, \quad A_2 = \frac{12abc_2^2}{bc_1^2 - 1}, \quad A_1 = \frac{12abc_1c_2}{bc_1^2 - 1}, \quad k = -\frac{a}{\sqrt{1 - bc_1^2}}. \tag{15}$$

Substituting (15) in (10),

$$V_3 = \frac{12abc_1c_2}{bc_1^2 - 1}\left(\frac{c_1 e^{c_1(\xi+\xi_0)}}{(1 - c_2 e^{c_1(\xi+\xi_0)})}\right) - \frac{12abc_2^2}{bc_1^2 - 1}\left(\frac{c_1^2 e^{2c_1(\xi+\xi_0)}}{(1 - c_2 e^{c_1(\xi+\xi_0)})^2}\right), \quad c_1 > 0. \tag{16}$$

$$V_4 = \frac{12abc_1c_2}{bc_1^2 - 1}\left(\frac{-c_1 e^{c_1(\xi+\xi_0)}}{(1 + c_2 e^{c_1(\xi+\xi_0)})}\right) - \frac{12abc_2^2}{bc_1^2 - 1}\left(\frac{c_1^2 e^{2c_1(\xi+\xi_0)}}{(1 + c_2 e^{c_1(\xi+\xi_0)})^2}\right), \quad c_1 < 0. \tag{17}$$

Case III: $c_1 = c_3 = 0$
Family I

$$A_0 = \frac{4abc_0c_2}{4bc_0c_2 - 1}, \quad A_{-2} = 0, \quad A_{-1} = 0, \quad A_2 = \frac{12abc_2^2}{4bc_0c_2 - 1}, \quad A_1 = 0, \quad k = -\frac{a}{\sqrt{1 - 4bc_0c_2}}. \tag{18}$$

Substituting (18) in (10),

$$V_5 = \frac{4abc_0c_2}{4bc_0c_2 - 1} + \frac{12ab}{4bc_0c_2 - 1} \times (c_0c_2 \tan\sqrt{c_0c_2}(\xi + \xi_0))^2, \quad c_2c_0 > 0, \quad b < 0. \tag{19}$$

$$V_6 = \frac{4abc_0c_2}{4bc_0c_2 - 1} - \frac{12ab}{4bc_0c_2 - 1} \times (-c_0c_2 \tanh\sqrt{-c_0c_2}(\xi + \xi_0))^2, \quad c_2c_0 < 0, \quad b > 0. \tag{20}$$

Family II

$$A_0 = \frac{4abc_0c_2}{4bc_0c_2 - 1}, \quad A_{-2} = \frac{12abc_0^2}{4bc_0c_2 - 1}, \quad A_{-1} = 0, \quad A_2 = 0, \quad A_1 = 0, \quad k = \frac{a}{\sqrt{1 - 4bc_0c_2}}. \tag{21}$$

Substituting (21) in (10),

$$V_7 = \frac{4abc_0c_2}{4bc_0c_2 - 1} + \frac{12abc_0^2}{4bc_0c_2 - 1} \times \left(\frac{c_2^2}{c_0c_2(\tan\sqrt{c_0c_2}(\xi + \xi_0))^2}\right), \quad c_0c_2 > 0, \quad b < 0. \tag{22}$$

$$V_8 = \frac{4abc_0c_2}{4bc_0c_2 - 1} - \frac{12abc_0^2}{4bc_0c_2 - 1} \times \left(\frac{c_2^2}{-c_0c_2(\tanh\sqrt{-c_0c_2}(\xi + \xi_0))^2}\right), \quad c_0c_2 < 0, \quad b > 0. \tag{23}$$

Family III

$$A_0 = -\frac{8abc_0c_2}{16bc_0c_2 - 1}, \quad A_{-2} = \frac{12abc_0^2}{16bc_0c_2 - 1}, \quad A_{-1} = 0, \quad A_2 = \frac{12abc_2^2}{16bc_0c_2 - 1}, \quad A_1 = 0, \quad k = \frac{a}{\sqrt{1 - 16bc_0c_2}} \tag{24}$$

Substituting (24) in (10),

$$V_9 = -\frac{8abc_0c_2}{16bc_0c_2 - 1} - \frac{12ab}{16bc_0c_2 - 1} \times (c_0c_2(\tan\sqrt{c_0c_2}(\xi + \xi_0))^2) + \frac{12abc_0c_2}{16bc_0c_2 - 1}\left(\frac{1}{(\tan\sqrt{c_0c_2}(\xi + \xi_0))^2}\right), \quad c_0c_2 > 0, \quad b < 0. \tag{25}$$

$$V_{10} = -\frac{8abc_0c_2}{16bc_0c_2 - 1} - \frac{12ab}{16bc_0c_2 - 1} \times (-c_0c_2(\tanh\sqrt{-c_0c_2}(\xi + \xi_0))^2) + \frac{12abc_0c_2}{16bc_0c_2 - 1}\left(\frac{1}{(\tanh\sqrt{-c_0c_2}(\xi + \xi_0))^2}\right), \quad c_0c_2 < 0, \quad b > 0. \tag{26}$$

3.1.2 Boussinesq equation

The Boussinesq model has the form [46]:

$$-u_{xx} + u_{tt} - 3(u^2)_{xx} - u_{xxxx} = 0. \tag{27}$$

Substituting (2) in (27), integrating twice with an integration constant of zero, yields

$$(k^2 - 1)V - V'' - 3V^2 = 0. \tag{28}$$

Let (10) be a solution of (28). Substituting (10) in (28) along with (5), the following system of equations is obtained.

Case I: $c_3 = 0$
Family I

$$A_0 = \frac{1}{3}(-c_1^2 - 2c_0c_2), \quad A_{-2} = 0, \quad A_{-1} = 0, \quad A_2 = -2c_2^2, \quad A_1 = -2c_1c_2, \quad k = \sqrt{-c_1^2 + 4c_0c_2 + 1}. \tag{29}$$

Substitute (29) in (10) with (5), then the solution of equation (27) is achieved as,

$$\begin{aligned} V_{11} &= \frac{1}{3}(-c_1^2 - 2c_0c_2) - 2c_1c_2 \\ &\times \left(\frac{c_1 - \sqrt{4c_0c_2 - c_1^2}\tan\left(\frac{\sqrt{4c_0c_2 - c_1^2}}{2}(\xi + \xi_0)\right)}{2c_2} \right) \\ &- 2c_2^2 \left(\frac{c_1 - \sqrt{4c_0c_2 - c_1^2}\tan\left(\frac{\sqrt{4c_0c_2 - c_1^2}}{2}(\xi + \xi_0)\right)}{2c_2} \right)^2, \\ &4c_0c_2 > c_1^2. \end{aligned} \tag{30}$$

Family II

$$\begin{aligned} A_0 &= -2c_0c_2, \quad A_{-2} = -2c_0^2, \quad A_{-1} = -2c_0c_1, \\ &A_2 = 0, \\ A_1 &= 0, \quad k = -\sqrt{c_1^2 - 4c_0c_2 + 1}. \end{aligned} \tag{31}$$

Substituting (31) in (10),

$$\begin{aligned} V_{12} &= -2c_0c_2 - 2c_0c_1 \\ &\times \left(\frac{2c_2}{c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)} \right) \\ &- 2c_0^2 \left(\frac{4c_2^2}{\left(c_1 - \sqrt{4c_2c_0 - c_1^2}\tan\left(\frac{1}{2}\sqrt{4c_2c_0 - c_1^2}(\xi + \xi_0)\right)\right)^2} \right), \\ &4c_0c_2 > c_1^2. \end{aligned} \tag{32}$$

Case II: $c_0 = c_3 = 0$,

$$\begin{aligned} A_0 &= 0, \quad A_{-2} = 0, \quad A_{-1} = 0, \quad A_2 = -2c_2^2, \\ A_1 &= -2c_1c_2, \quad k = \sqrt{c_1^2 + 1}. \end{aligned} \tag{33}$$

Substituting (33) in (10),

$$\begin{aligned} V_{13} &= -2c_1c_2\left(\frac{c_1 e^{c_1(\xi + \xi_0)}}{(1 - c_2 e^{c_1(\xi + \xi_0)})} \right) \\ &- 2c_2^2 \left(\frac{c_1^2 e^{2c_1(\xi + \xi_0)}}{(1 - c_2 e^{c_1(\xi + \xi_0)})^2} \right), \quad c_1 > 0. \end{aligned} \tag{34}$$

$$\begin{aligned} V_{14} &= -2c_1c_2\left(\frac{-c_1 e^{c_1(\xi + \xi_0)}}{(1 + c_2 e^{c_1(\xi + \xi_0)})} \right) \\ &- 2c_2^2 \left(\frac{c_1^2 e^{2c_1(\xi + \xi_0)}}{(1 + c_2 e^{c_1(\xi + \xi_0)})^2} \right), \quad c_1 < 0. \end{aligned} \tag{35}$$

Case III: $c_1 = c_3 = 0$

Family I

$$\begin{aligned} A_0 &= \tfrac{1}{3}(-2)c_0c_2, \quad A_{-2} = 0, \quad A_{-1} = 0, \\ A_2 &= -2c_2^2, \quad A_1 = 0, \quad k = -\sqrt{4c_0c_2 + 1}. \end{aligned} \tag{36}$$

Substituting (36) in (10),

$$\begin{aligned} V_{15} &= \tfrac{1}{3}(-2)c_0c_2 - 2(\sqrt{c_0c_2}\tan\sqrt{c_0c_2}(\xi + \xi_0))^2, \\ &c_2c_0 > 0. \end{aligned} \tag{37}$$

$$\begin{aligned} V_{16} &= \tfrac{1}{3}(-2)c_0c_2 - 2(\sqrt{-c_0c_2}\tanh\sqrt{-c_0c_2}(\xi + \xi_0))^2, \\ &c_2c_0 < 0. \end{aligned} \tag{38}$$

Family II

$$\begin{aligned} A_0 &= \frac{1}{3}(-2)c_0c_2, \quad A_{-2} = -2c_0^2, \quad A_{-1} = 0, \\ A_2 &= 0, \quad A_1 = 0, \quad k = -\sqrt{4c_0c_2 + 1}. \end{aligned} \tag{39}$$

Substituting (39) in (10),

$$\begin{aligned} V_{17} &= \frac{1}{3}(-2)c_0c_2 - 2c_0^2 \\ &\times \left(\frac{c_2^2}{c_0c_2(\tan\sqrt{c_0c_2}(\xi + \xi_0))^2} \right), \quad c_0c_2 > 0, \end{aligned} \tag{40}$$

$$\begin{aligned} V_{18} &= \tfrac{1}{3}(-2)c_0c_2 - 2c_0^2 \\ &\times \left(\frac{c_2^2}{-c_0c_2(\tanh\sqrt{c_0c_2}(\xi + \xi_0))^2} \right), \quad c_0c_2 < 0. \end{aligned} \tag{41}$$

Family III

$$\begin{aligned} A_0 &= \frac{4c_0c_2}{3}, \quad A_{-2} = -2c_0^2, \quad A_{-1} = 0, \\ A_2 &= -2c_2^2, \quad A_1 = 0, \quad k = -\sqrt{16c_0c_2 + 1}. \end{aligned} \tag{42}$$

Substituting (42) in (10),

$$\begin{aligned} V_{19} &= \frac{4c_0c_2}{3} - 2(c_0c_2(\tan\sqrt{c_0c_2}(\xi + \xi_0))^2) \\ &- 2\left(\frac{c_2c_0}{(\tan\sqrt{c_0c_2}(\xi + \xi_0))^2} \right), \\ &c_0c_2 > 0. \end{aligned} \tag{43}$$

$$\begin{aligned} V_{20} &= \frac{4c_0c_2}{3} + 2(c_0c_2(\tanh\sqrt{-c_0c_2})^2) \\ &- 2\left(\frac{c_2c_0}{(\tanh\sqrt{-c_0c_2}(\xi + \xi_0))^2} \right), \\ &c_0c_2 < 0. \end{aligned} \tag{44}$$

3.2 Applications of $\exp(-\Psi(\xi))$-expansion technique

3.2.1 Longitudinal wave model

Let the solution of (9) be:

$$V = a_0 + a_1\exp(-\Psi(\xi)) + a_2\exp(-2\Psi(\xi)) \tag{45}$$

Substituting equation (45) along with equation (7) in equation (9), we have:

$$a_1 = \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1}, \quad a_0 = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1}, \quad k = -\frac{a}{\sqrt{-b\lambda^2 + 4b\mu + 1}}, \quad a_2 = \frac{12ab}{b\lambda^2 - 4b\mu - 1}. \tag{46}$$

Substituting (46) in (45) yields

$$V = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1} + \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1}\exp(-\Psi(\xi)) + \frac{12ab}{b\lambda^2 - 4b\mu - 1}\exp(-2\Psi(\xi)). \tag{47}$$

For $\lambda^2 - 4\mu > 0$, $\mu \neq 0$

$$V_{21} = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1} + \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{2\mu}{\left(-\sqrt{\lambda^2 + (-4\mu)}\tanh\left(\frac{\sqrt{\lambda^2 + (-4\mu)}}{2}(\xi + \xi_0)\right) + (-\lambda)\right)}\right) + \frac{12ab}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{4\mu^2}{\left(-\sqrt{\lambda^2 + (-4\mu)}\tanh\left(\frac{\sqrt{\lambda^2 + (-4\mu)}}{2}(\xi + \xi_0)\right) + (-\lambda)\right)^2}\right), \quad b < 0. \tag{48}$$

For $\lambda^2 - 4\mu > 0$, $\mu = 0$,

$$V_{22} = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1} + \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{\lambda}{(\exp(\lambda(\xi + \xi_0)) + (-1))}\right) + \frac{12ab}{b\lambda^2 - 4b\mu - 1}\left(\frac{\lambda^2}{(\exp(\lambda(\xi + \xi_0)) + (-1))^2}\right), \quad b < 0. \tag{49}$$

For $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$V_{23} = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1} + \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{\lambda^2(\xi + \xi_0)}{2(\lambda(\xi + \xi_0) + 2)}\right) + \frac{12ab}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{\lambda^4(\xi + \xi_0)^2}{4(\lambda(\xi + \xi_0) + 2)^2}\right), \quad b > 0. \tag{50}$$

For $\lambda^2 - 4\mu = 0$, $\mu = \lambda = 0$,

$$V_{24} = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1} + \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1}\left(\frac{1}{(\xi + \xi_0)}\right) + \frac{12ab}{b\lambda^2 - 4b\mu - 1}\left(\frac{1}{(\xi + \xi_0)^2}\right), \quad b > 0. \tag{51}$$

For $\lambda^2 - 4\mu < 0$,

$$V_{25} = -\frac{12ab\mu}{-b\lambda^2 + 4b\mu + 1} + \frac{12ab\lambda}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{2\mu}{\left(\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + \xi_0)\right) - \lambda\right)}\right) + \frac{12ab}{b\lambda^2 - 4b\mu - 1} \times \left(\frac{4\mu^2}{\left(\sqrt{4\mu - \lambda^2}\tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + \xi_0)\right) - \lambda\right)^2}\right), \quad b > 0. \tag{52}$$

3.2.2 Boussinesq equation

Let equation (28) have the solution form as in equation (45). Substituting equation (45) along with equation (7) in equation (28), we have:

$$a_1 = -2\lambda, \quad a_0 = -2\mu, \quad k = \sqrt{\lambda^2 - 4\mu + 1}, \quad a_2 = -2. \tag{53}$$

Substituting (53) into (45) yields

$$V = -2\mu + -2\lambda\exp(-\Psi(\xi)) - 2\exp(-2\Psi(\xi)). \tag{54}$$

For $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$V_{26} = -2\mu - 2\lambda \times \left(\frac{2\mu}{\left(-\sqrt{\lambda^2 + (-4\mu)}\, \tanh\left(\frac{\sqrt{\lambda^2 + (-4\mu)}}{2} (\xi + \xi_0) \right) + (-\lambda) \right)} \right) - 2\left(\frac{4\mu^2}{\left(-\sqrt{\lambda^2 + (-4\mu)}\, \tanh\left(\frac{\sqrt{\lambda^2 + (-4\mu)}}{2} (\xi + \xi_0) \right) - \lambda \right)^2} \right). \tag{55}$$

For $\lambda^2 - 4\mu > 0$, $\mu = 0$,

$$V_{27} = -2\mu - 2\lambda \left(\frac{\lambda}{(\exp(\lambda(\xi + \xi_0)) + (-1))} \right) - 2\left(\frac{\lambda^2}{(\exp(\lambda(\xi + \xi_0)) + (-1))^2} \right). \tag{56}$$

For $\lambda^2 - 4\mu = 0$, $\mu \neq 0$, $\lambda \neq 0$,

$$V_{28} = -2\mu + -2\lambda \left(\frac{\lambda^2(\xi + \xi_0)}{2(\lambda(\xi + \xi_0) + 1 + 1)} \right) - 2\left(\frac{\lambda^4}{4(\lambda(\xi + \xi_0) + 1 + 1)^2} \right). \tag{57}$$

For $\lambda^2 - 4\mu = 0$, $\mu = \lambda = 0$,

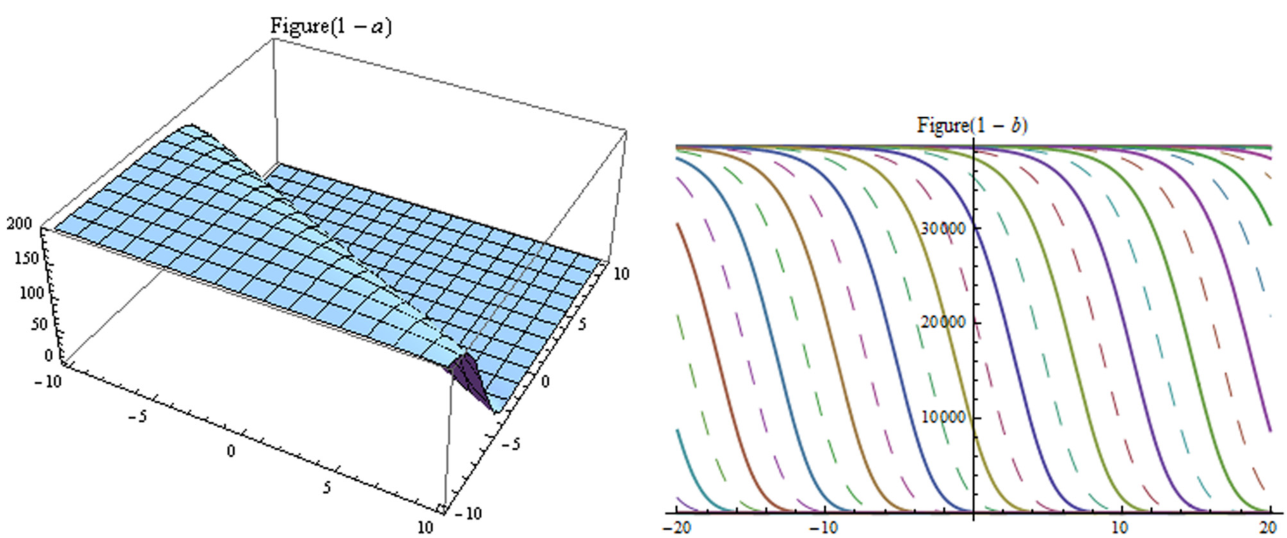

Figure 1: Exact traveling wave solution (17).

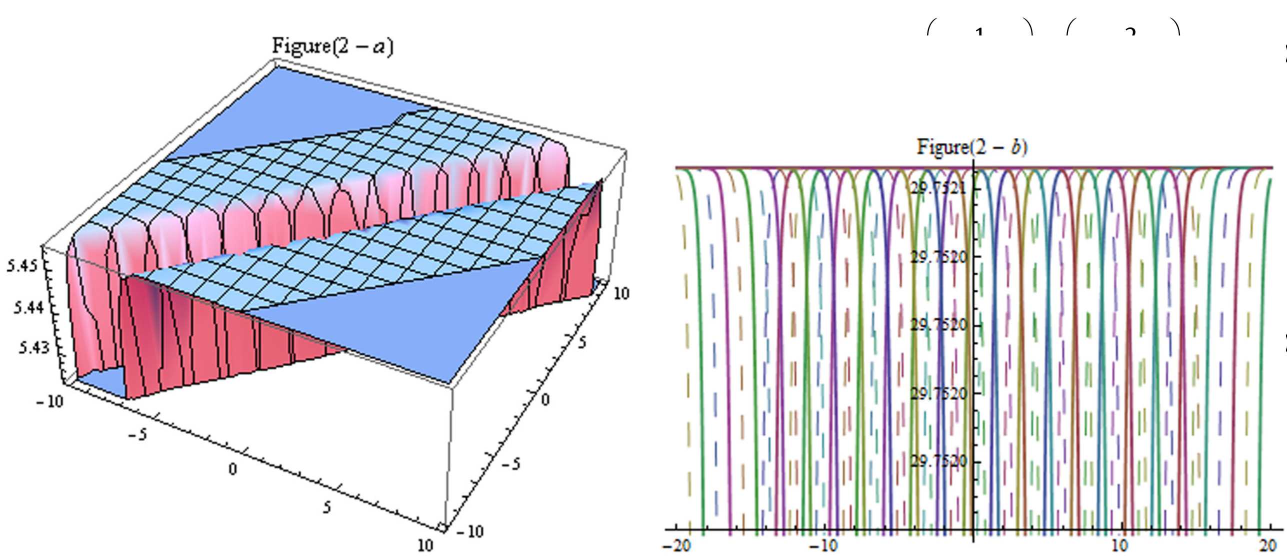

Figure 2: Exact traveling wave solution (20).

4 Results and discussion

Several researchers applied distinct schemes for the determination of exact solutions of longitudinal wave and Boussinesq equations. We have presented two mathematical methods for the construction of new wave exact solutions. With distinct values of parameters involving equations (4) and (6), several types of solitary wave solutions are achieved. However, some of our constructed solutions are likely the same as others. Solutions (14) and (16) are the same as solutions (13) and (15) in [47]. Solutions (48) and (49) have exactly the same form involving different parameters as solutions (36) and (37) in [48]. Our solutions (56) and (58) are approximately similar to solution $u_3(\eta)$ and $u_4(\eta)$ in [49], respectively. Our solutions (13) and (30) are approximately similar to solutions (2.54) and (2.63) in [50], respectively. The remaining discovered solutions are novel and more general when compared with the solutions obtained in [47–50].

Figures 1–7 are plotted by assigning these particular values to the parameters such that, solution V_4 at $a = -3$, $b = -5$, $c_1 = -0.5$, $c_2 = 1$, $\xi_0 = 0.5$, solution V_6 at $a = -3$, $b = 0.5$, $c_0 = 1$, $c_2 = -5$, $\xi = -0.5$ and solution V_{10} at $a = 5$, $b = 4$, $c_0 = -1$, $c_2 = 5$, $\xi_0 = -0.5$ and V_{12} at $c_0 = 1$, $c_1 = 1$, $c_2 = 1$, $\varepsilon = 0.5$ and V_{14} at $c_1 = -0.5$, $c_2 = 1$, $\xi_0 = 0.5$ and V_{16}

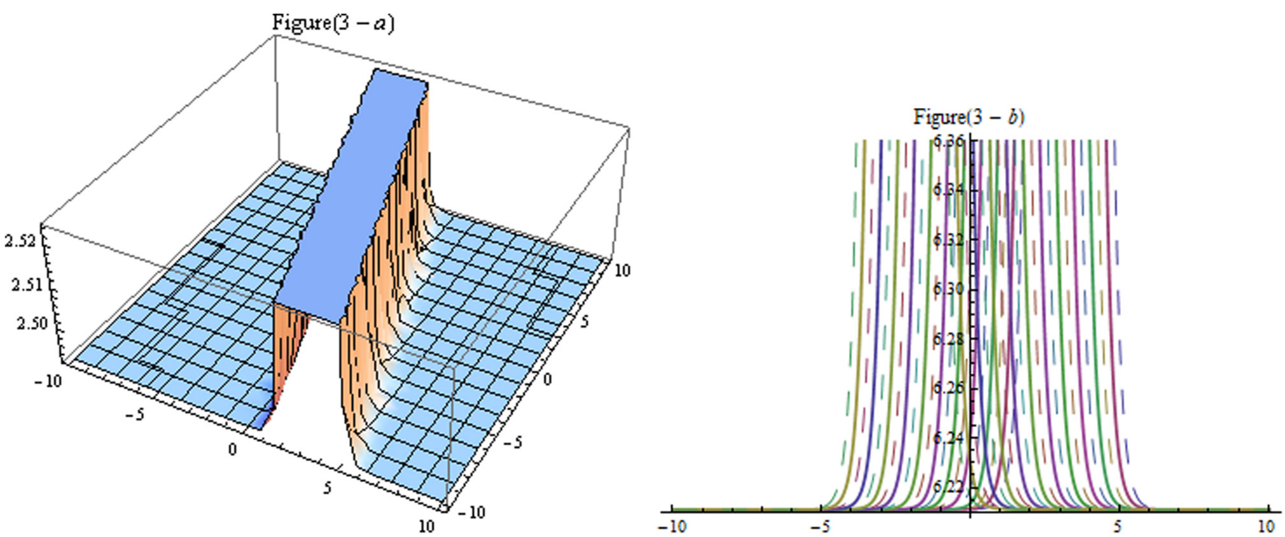

Figure 3: Exact traveling wave solution (26).

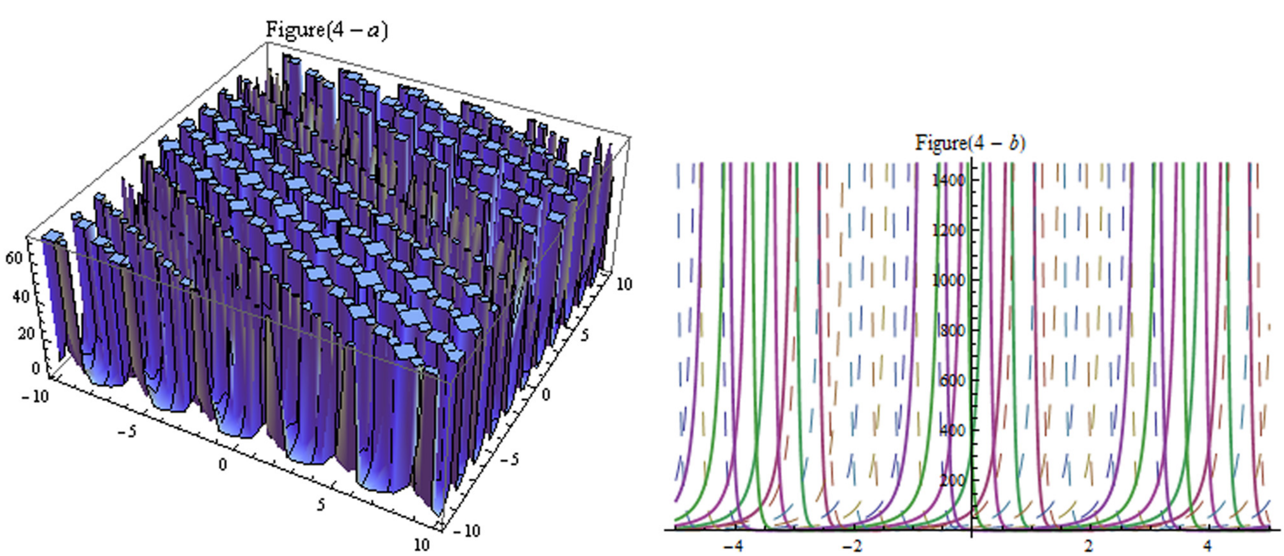

Figure 4: Exact traveling wave solution (32).

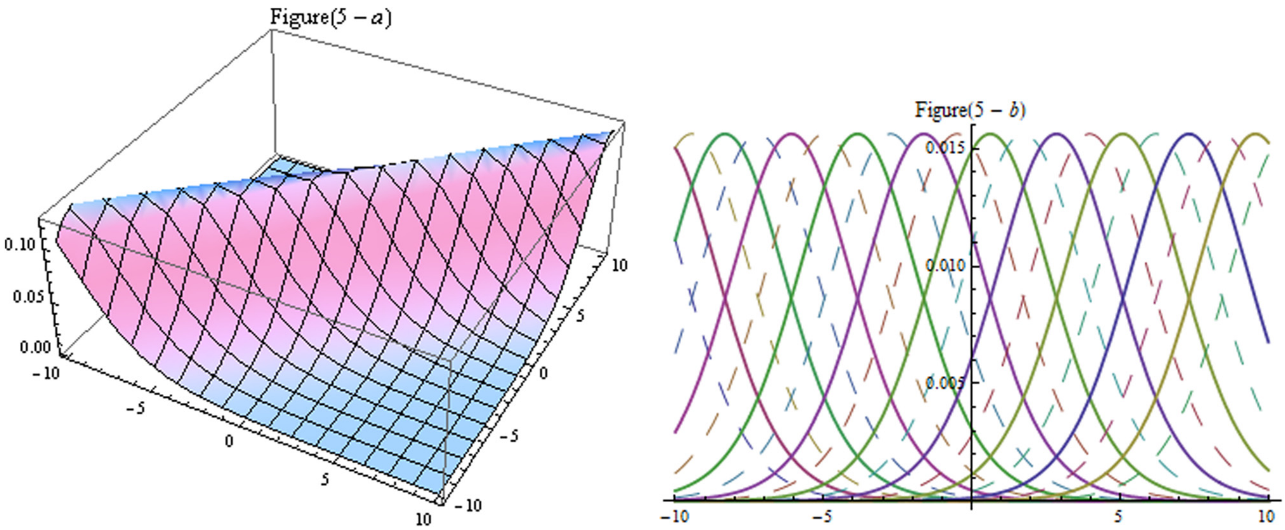

Figure 5: Exact traveling wave solution (35).

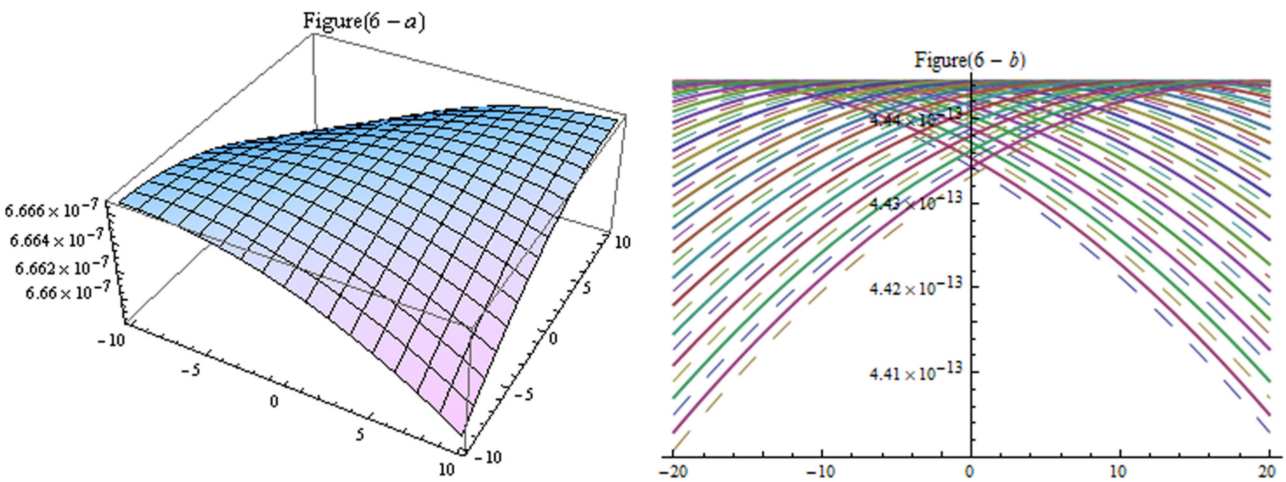

Figure 6: Exact traveling wave solution (38).

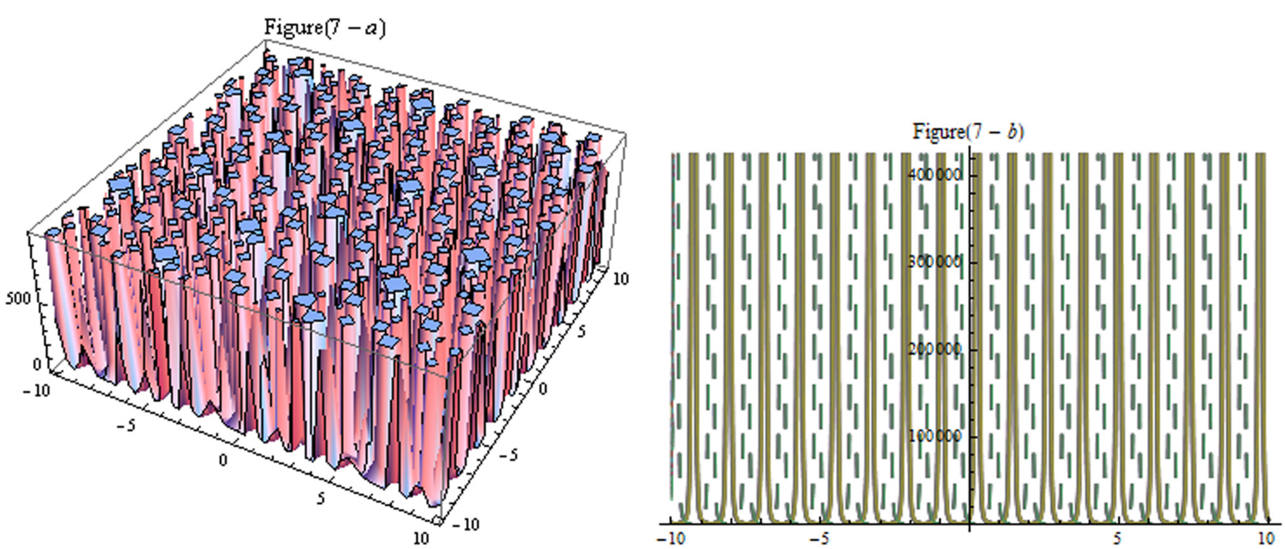

Figure 7: Exact traveling wave solution (52).

at $c_0 = 0.001$, $c_2 = -0.001$, $\xi_0 = -0.5$ and V_{25} at $a = 10$, $b_1 = 10$, $\lambda = 2$, $\mu = 8$, $\xi_0 = 0.5$, respectively. We discussed the obtained results and graphical representations of some solutions by giving the particular values of parameters. Hence, we have found that these methods provide a rich plate for solving nonlinear wave problems in mathematics, physics and engineering.

5 Conclusion

In this work, longitudinal wave and Boussinesq equations were successfully studied by adopting the two mathematical schemes, improved form of simple equation and $\exp(-\Psi(\xi))$, to generate the exact solutions. Furthermore, some solutions are plotted graphically to understand the dynamical behavior of the derived results. Hence, the proposed techniques are efficient algorithms for nonlinear PDEs in applied sciences.

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