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Article

A Novel Analytical Technique of the Fractional Bagley-Torvik Equations for Motion of a Rigid Plate in Newtonian Fluids

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Abstract: The current paper is concerned with a modified Homotopy perturbation technique. This modification allows achieving an exact solution of an initial value problem of the fractional differential equation. The approach is powerful, effective, and promising in analyzing some classes of fractional differential equations for heat conduction problems and other dynamical systems. To crystallize the new approach, some illustrated examples are introduced.

Keywords: Bagley–Torvik equation; caputo sense; Riemann–Liouville integral; fractional differential equation

Mathematics Subject Classification (2020): 26A33, 34A08, 47B32, 35F10, 34B10, 74540

1 Introduction

Some important problems in dynamical systems, diffusion wave, heat conduction, cellular systems, oil industries, signal processing, control theory, fluid mechanics and other areas of science and engineering would be more accurate to be modeled by utilizing fractional differential equations; for instance, see Podlubny [1]. Managing numerical techniques to accomplish an approximate solution of the fractional differential equations have been studied by various researchers [2,3]. Throughout this work, the Homotopy perturbation method (HPM) will be modified and implemented to acquire a numerical approximation of the Bagley–Torvik equation (BTE), considering Caputo's sense. BTE governs the motion of a rigid plate submerged in a Newtonian fluid of infinite extent and connected by a massless spring of stiffness K to a fixed point, as was modeled by Bagley et al. [4]. It may be formulated as follows:

$$au''(t) + bD^{\alpha}u(t) + cu(t) = f(t), \ \ p-1 < \alpha < p, \ \ p \in \mathbb{N}, \tag{1}$$

with initial conditions

$$\frac{\partial^k u(0)}{\partial t^k} = h_k, \ \ k = 0, \ 1, \ldots, \ \ p-1, \tag{2}$$

herein, $f(t)$ is a continuous function, $a = M, b = 2a\sqrt{\eta\rho},\ c = K, \eta$ is the viscosity, ρ is the fluid density, and M is the mass of the rigid plate.

Al-Mdallal et al. [5] found out the existence and uniqueness of the BTE with a Dirichlet boundary condition. It is very difficult to obtain, analytically, the solution of the fractional differential equations. Therefore, many pieces of research utilized widely approximate and numerical methods; for instance, see [6–11]. A lot of numerical processes have been proposed of the BTE by many researchers. Ray et al. [12] suggested the Adomian decomposition method (ADM) in solving the initial value problem of the Bagley–Torvik type. Others introduced fractional linear multi-step methods and a predictor-corrector method of an Adams type. The latter method is based on the finite difference methods to analyze the initial value problem of the BTE; for instance, see Diethelm et al. [13]. Saadatmandi et al. [14] tried to solve the fractional differential equations using the Legendre operational matrix method. However, Yuzbasi [15] implemented a mixture of collocation points and first-order Bessel functions and he named it the Bessel-collocation method for the boundary value problem of the BTE.

A lot of work nowadays aims to deduce a method to achieve the exact solutions of the linear as well as nonlinear fractional differential equations. One of these methods is the Homotopy perturbation method (HPM). It was first initiated by He [16–22] for solving linear, nonlinear differential and integral equations with integer or fractional order. It's not saying that it always gets the exact solution. However, if the exact solution is not reached, an accurate approximation for the solution is deduced. Some modifications have been made by utilizing the HPM; for instance; see Siddiqui et al. [23]. Wazwaz [24] has proposed a useful alteration of the Adomian decomposition method by putting a new operator in the picture to solve a class of differential equations called the Lane-Emden equations. These methods may be used effectively to solve equations with many types of linearities, such as the Klein-Gordon equation.

Lately, El-Dib et al. [25] introduced a merging between a power series formula and the Homotopy technique to find an exact solution of linear as well as nonlinear differential equations. In the current method, the solution of the functional equations is considered as a summation of an infinite series. In most cases, this series converges to the exact solution, and therefore, the modification is done. As we cut off the series at its first order term and equate it to zero then all the higher-order terms, which are dependent on the first-order term, will be vanished. In other words, the whole series will fall directly into exactly one term which will be the zero-order term and it will be the exact solution itself. All of that is done with the aid of the initial guess which has been formulated as a power series in the undetermined coefficients and obtaining these coefficients will lead to the exact solution.

In this work, the same technique as was given by El-Dib et al. [25] will be used after minor modifications. These modifications enable us to solve fractional differential equations. A class of fractional differential equations that is famously named as the Bagley-Torvik fractional differential equation is tested [26]. To clarify the organization of the paper, the rest of the manuscript is organized as follows: Section 2 is devoted to introducing the essential relations and preliminaries. The convergence and the error analysis are displayed in Section 3. The algorithm of the solution, using the proposed method, is presented in Section 4. Some illustrated examples are acquainted within Section 5. Finally, a summarize of the outlines of the manuscript is introduced as concluding remarks in Section 6.

2 Essential Relations and Preliminaries

Throughout this section, the fundamentals of the fractional calculus are presented. The fractional calculus refers to the analysis of integrals and derivatives with different possibilities of having, a real or complex number, order to the integral, or the differential operator. It is a generalization and the unification of the differentiation with integer-order and integration with n-fold. There were existing wide varieties of books and research papers that study and analyze fractional calculus. Furthermore, a lot of definitions of the integration and differentiation of the fractional-order, such as Riemann–Liouville, Caputo, Grunwald–Letnikov, and the generalized function approach definitions, for instance, see Podlubny [1] and West [27]. To serve the aim of this article, Caputo's definition of fractional differentiation is implemented here. Making use of the Caputo's approach advantage in which initial conditions for fractional differential equations with Caputo's derivatives uses the same traditional form that the integer-order differential equations use.

Definition 2.1 Caputo defined the fractional-order derivative as follows:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(p-\alpha)}\int_{a}^{t}\frac{f^{(p)}(\tau)}{(t-\tau)^{\alpha+1-p}}d\tau, \quad (p-1 < Re(\alpha) \leq p, \quad p \in \mathbb{N}), \tag{3}$$

in which the constant α is the derivative order. It may be real or, in some cases, even complex and the parameter α is the function initial value. Here α is assuming as real and positive.

Considering the Caputo's derivative, one gets

$$D^{\alpha}C = 0, \quad (C \text{ is a constant}) \tag{4}$$

$$D^{\alpha}t^{\beta} = \begin{cases} 0, & (\beta \leq p-1) \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha} & (\beta > p-1) \end{cases}, \quad p \in \mathbb{N}. \tag{5}$$

Both of the fractional differentiation and integer-order differentiation have a common property of being linear operators. Therefore, one can write

$$D^{\alpha}(\beta f(t) + \mu g(t)) = \beta D^{\alpha}f(t) + \mu D^{\alpha}g(t), \tag{6}$$

where β, μ are constants.

For building our results, introducing the Riemann–Liouville fractional integral operator as follows:

Definition 2.2 The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a mapping $f \in C_{\mu}$, $\mu \geq -1$, is referred to as

$$J^{\alpha}f(x) = \int_{0}^{x}(x-t)^{\alpha-1}f(t)dt, \quad \alpha > 0, \quad x > 0. \tag{7}$$

The characteristics of the integration operator J^{α} would be seen by West et al. [27] and here only some of them will be mentioned as follows:

For $f \in C_{\mu}$, $\mu \geq -1, \quad \alpha, \quad \beta \geq 0, \quad \gamma \geq -1:$

$$J^{0}f(x) = f(x), \quad J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma},$$

$$J^{\alpha}J^{\beta}f(x) = J^{\alpha+\beta}f(x), \quad J^{\alpha}J^{\beta}f(x) = J^{\beta}J^{\alpha}f(x). \tag{8}$$

Furthermore, two of its basic properties are needed here as follows:

If $p-1 < \alpha \leq p, \quad p \in \mathbb{N}$ and $f \in C_{\mu}^{p}$, $\mu \geq -1$, then

$$D^{\alpha}J^{\alpha}f(x) = f(x), \quad J^{\alpha}D^{\alpha}f(x) = f(x) - \sum_{i=0}^{p-1}f^{(i)}(0)\frac{x^{i}}{i!}, \quad x > 0. \tag{9}$$

For further, a deeper study in the mathematical characteristics of the fractional derivatives and integrals, you might find what you need in West et al. [27].

3 Convergence and Error Analysis

This section is devoted to demonstrating the uniqueness of the solution of the BTE.

Definition 3.1 Let $H = C[a, b]$ be the set of all continuous functions defined on the closed interval $[a, b]$. The distance between any two arbitrary functions $f_1(t),\ f_2(t) \in H$ is defined in the form $d(f_1(t), f_2(t)) = \max_{a \le t \le b} |f_1(t) - f_2(t)|$. It is known that (H, d) is a complete metric space and the following properties are satisfied:

1) $d(f_1,\ f_2) = 0 \quad \leftrightarrow \quad f_1 = f_2\ \ \forall\ f_1,\ f_2\ \ \in H$

2) $d(f_1 + f_3,\ f_2 + f_3) = d(f_1, f_2)\ \ \forall\ f_1,\ f_2,\ f_3 \in H$

3) $d(f_1 + f_3,\ f_2 + f_4) \le d(f_1, f_2) + d(f_3, f_4) \quad \forall f_1,\ f_2,\ f_3,\ f_4 \in H$

Now, consider $u(x)$ is a bounded function for all $x \in R$. Moreover, suppose that the linear and nonlinear operators $L\ and\ K$ satisfy Lipchitz conditions, where

$$d(L(t_1, u(t_1)), L(t_2, u(t_2))) \le M_1 d(u(t_1), u(t_2)),\ \ M_1 \ge 0, \tag{10}$$

$$d(K(t_1, u(t_1)), K(t_2, u(t_2)) \le M_2 d(u(t_1), u(t_2)),\ \ M_2 \ge 0, \tag{11}$$

Let $m(t) = (M_1 + M_2)\dfrac{t^\alpha}{\Gamma(\alpha + 1)}$.

For the equation

$$a_1 u''(t) + D^\alpha u(t) + c_1 u(t) = \ b_1 f(t),\ \ p - 1 < \alpha < p,\ \ p \in \mathbb{N}, \tag{12}$$

with initial conditions

$$\frac{\partial^k u(0)}{\partial t^k} = h_k,\ \ k = 0,\ 1,\ \ldots,\ \ p - 1. \tag{13}$$

One can write Eq. (12) in the form

$$u(t) = a_1 J_t^\alpha(u''(t)) + c_1 J_t^\alpha(u(t)) + b_1 J_t^\alpha(f(t)) + \sum_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{14}$$

then suppose that

$$u(t) = J_t^\alpha L(u(t)) + J_t^\alpha k(u(t)) + J_t^\alpha F(t) + \sum_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{15}$$

where

$$L(u(t)) = a_1 u''(t),\ \ a_1 = \frac{-a}{b},$$

$$K(u(t)) = c_1 u(t),\ \ c_1 = \frac{-c}{b},$$

$$F(t) = b_1 f(t),\ b_1 = \frac{-1}{b}. \tag{16}$$

Theorem 3.1

Assuming that inequalities (10) and (11) hold such that $0 < m(t) < 1$, then there exists a unique solution to the equation

$$D^{\alpha}u(t) = L(u(t)) + k(u(t)) + F(t) + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{17}$$

with initial conditions

$$\frac{\partial^k u(0)}{\partial t^k} = h_k, \quad k = 0,\ 1, \quad \ldots, \quad p-1, \tag{18}$$

3.1 Proof

Let u *and* u^* be two different solutions for Eq. (17), then one can write

$$u(t) = J_t^{\alpha}L(u(t)) + J_t^{\alpha}k(u(t)) + J_t^{\alpha}F(t) + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{19}$$

$$u^*(t) = J_t^{\alpha}L(u^*(t))) + J_t^{\alpha}k(u^*(t))) + J_t^{\alpha}F(t) + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{20}$$

and,

$$d(u,u^*) = d(J_t^{\alpha}L(u(t)) + J_t^{\alpha}k(u(t)) + J_t^{\alpha}F(t) + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{21}$$

$$J_t^{\alpha}L(u^*(t))) + J_t^{\alpha}k(u^*(t))) + J_t^{\alpha}F(t) + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}) \tag{22}$$

$$= \max_{a\le t\le b}\left|\left[J_t^{\alpha}L(u(t)) + J_t^{\alpha}k(u(t))\right] - \left[J_t^{\alpha}L(u^*(t))) + J_t^{\alpha}k(u^*(t))\right]\right| \tag{23}$$

$$= \max_{a\le t\le b}\left|J_t^{\alpha}L(u(t)) - J_t^{\alpha}L(u^*(t)) + J_t^{\alpha}k(u(t)) - J_t^{\alpha}k(u^*(t))\right| \tag{24}$$

$$\le \max_{a\le t\le b}\left|J_t^{\alpha}L(u(t)) - J_t^{\alpha}L(u^*(t))\right| + \max_{a\le t\le b}\left|J_t^{\alpha}k(u(t)) - J_t^{\alpha}k(u^*(t))\right| \tag{25}$$

$$= d\left(J_t^{\alpha}L(u(t), J_t^{\alpha}L(u^*(t)\right) + d\left(J_t^{\alpha}K(u(t), J_t^{\alpha}K(u^*(t)\right). \tag{26}$$

Now, using Lipschitz condition, one gets

$$d(u,u^*) \le M_1 d\left(J_t^{\alpha}u(t), J_t^{\alpha}u^*(t)\right) + M_2 d\left(J_t^{\alpha}u(t), J_t^{\alpha}u^*(t)\right)$$

$$= (M_1 + M_2)J_t^{\alpha}d(u(t), u^*(t)) = (M_1 + M_2)d(u,u^*)\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$= m(t).d(u,u^*), \tag{27}$$

then

$$d(u,u^*) \le m(t).d(u,u^*). \tag{28}$$

Accordingly, it's clear that $(m(t)-1)d(u,u^*) \geq 0$, since $d(u,u^*)$ can't be negative then it is either positive or zero, However; to get a unique solution $d(u,u^*)$ must be equal to zero, then $m(t)-1$ must be negative, so set $0 < m(t) < 1$, then $d(u,u^*) = 0$, which implies that $u = u^*$, under the condition that $0 < m(t) < 1$ or $0 < \frac{M_1 + M_2}{\Gamma(\alpha+1)} t^{\alpha} < 1$ which is equivalent to $0 < t < \sqrt[\alpha]{\frac{\Gamma(\alpha+1)}{M_1+M_2}}$.

As a special case of the theorem for the non-fractional differential equation, put $\alpha = 2$ to get the condition for solving the equation

$$\frac{\partial^2 u(t)}{dt^2} = L(u(t)) + k(u(t)) + f(t), \tag{29}$$

with initial conditions

$$u(0) = A, \quad \frac{\partial u(0)}{dt} = B. \tag{30}$$

Let $m(t) = \frac{M_1 + M_2}{\Gamma(\alpha+1)} t^2$, it is found that a unique solution for this problem under the condition that $0 < m(t) < 1$ is obtained, which results in $|t| < \sqrt{\frac{\Gamma(\alpha+1)}{M_1+M_2}}$.

Lemma 3.1

Consider a series solution of Eq. (12) is in the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\alpha n}$ and consider the fractional derivative operator D^{α} as defined in Eq. (3), one finds

$$D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] = \sum_{n=p}^{\infty} a_n \frac{\Gamma(1+n)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, \quad p-1 < \alpha < p. \tag{31}$$

3.2 *Proof*

Taking the L.H.S.

$$D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] = D^{\alpha}\left[a_0 + a_1 t^{\alpha} + a_2 t^{2\alpha} + a_3 t^{3\alpha} + \ldots\right]. \tag{32}$$

Using the Caputo fractional derivative definition, it follows

$$= \left[a_0 \frac{\Gamma(1)}{\Gamma(1-\alpha)} t^{-\alpha} + a_1 \frac{\Gamma(2)}{\Gamma(2-\alpha)} t^{1-\alpha} + a_2 \frac{\Gamma(3)}{\Gamma(3-\alpha)} t^{2-\alpha} + a_3 \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + \ldots\right],$$

$$= \sum_{n=p}^{\infty} a_n \frac{\Gamma(1+n)}{\Gamma(n+1-\alpha)} t^{n-\alpha} = R.H.S. \tag{33}$$

Lemma 3.2

Consider a series solution of Eq. (12) is in the form $u(t) = \sum_{n=0}^{\infty} a_n t^{\alpha n}$ and consider D^{α} and I^{α} be the fractional derivative and integral operators respectively, one gets

$$I^{\alpha}D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] = \sum_{n=p}^{\infty} a_n t^n + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}, \quad p-1 < \alpha < p. \tag{34}$$

3.3 *Proof*

Taking the L.H.S. and using the proved lemma above, one finds

$$I^{\alpha}D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] = I^{\alpha}\left[\sum_{n=p}^{\infty} a_n \frac{\Gamma(1+n)}{\Gamma(n+1-\alpha)} t^{n-\alpha}\right]. \tag{35}$$

Using the Caputo fractional integral operator definition

$$I^{\alpha}D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] = \sum_{n=p}^{\infty} a_n t^n + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!} = R.H.S. \tag{36}$$

4 An Algorithm of the Proposed Solution

Starting with the fundamentals of the Homotopy technique as given in Eq. (12), one can be decomposed into L and N, to refer to the linear and nonlinear parts, respectively. For instance, see [16–22], therefore, one gets

$$L(u) + \delta N(u) = f(t), \; \delta \in [0, 1] \tag{37}$$

Construct a Homotopy in the form

$$H(u, \delta) = (1-\delta)(L(u) - L(U)) + \delta[N(u) - f(t)] = 0, \; \delta \in (0, 1], \tag{38}$$

$$H(u, \delta) = L(u) - L(U) + \delta[L(U) + N(u) - f(t)] = 0, \quad \delta \in (0, 1], \tag{39}$$

where $U(t)$ is the initial approximation for the solution of Eq. (12), where

$$H(u, 0) = L(u) - L(U) = 0,$$

$$H(y, 1) = N(u) - f(t) = 0. \tag{40}$$

Use the Homotopy parameter δ to expand $u(t)$

$$u(t) = u_0(t) + \delta u_1(t) + \delta^2 u_2(t) + \ldots. \tag{41}$$

Suppose that

$$L(u(t)) = D^{\alpha}(u(t)),$$

$$N(u(t)) = a_1 u''(t) + c_1 u(t). \tag{42}$$

Take the initial approximation as a series in t as

$$U(t) = \sum_{n=0}^{\infty} a_n t^{\alpha n}, \tag{43}$$

where $p-1 < \alpha < \delta, \quad \delta \in \mathbb{N}$.

From Eqs. (43), (41) and (42) into Eq. (39), one gets

$$D^{\alpha}u_0(t) + \delta D^{\alpha}u_1(t) + \ldots - D^{\alpha}[U(t)] + \tag{44}$$

$$\delta\Big[D^{\alpha}U(t) + a_1\Big(u_0^{''}(t) + \delta u_1^{''}(t) + \ldots\Big) + c_1(u_0(t) + \delta u_1(t) + \ldots) - f(t)\Big] = 0.$$

Comparing the coefficients of powers of δ

$$\delta^0 : D^{\alpha}u_0(t) = D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right], \quad \text{integrating } \alpha - \textit{times} \text{ to get} \tag{45}$$

$$: u_0(t) = \sum_{n=p}^{\infty} a_n t^{\alpha n} + \sum\nolimits_{k=0}^{p-1} h_k \frac{t^k}{k!}, \tag{46}$$

$$\delta^1 : D^{\alpha}u_1(t) = -D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] - a_1 u_0^{''}(t) - c_1 u_0(t) - f(t), \tag{47}$$

$$\delta^2 : u_2(t) = a_1 u_1^{''}(t) + c_1 u_1(t), \tag{48}$$

$$\delta^3 : u_3(t) = a_1 u_2^{''}(t) + c_1 u_2(t), \tag{49}$$

$\vdots$

$$\delta^n : u_n(t) = a_1 u_{n-1}^{''}(t) + c_1 u_{n-1}(t), \quad n \geq 2, \quad n \text{ is an integer} \tag{50}$$

Equating $u_1(t) = 0$ in Eq. (47) will result in all u_n's ($n \geq 2,\ n$ is an integer) will be equal to zero.

$$u_1(t) = -D^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] - a_1 u_0^{''}(t) - c_1 u_0(t) - f(t) = 0$$

This expression will be expanded and the coefficients of powers of t will be compared to calculate the undetermined coefficients $a_n^{'}s,$ and substitute into Eq. (46) to get the zero-order exact solution of Eq. (12).

5 Illustrated Examples

A numerical proof Section with various examples, at which the proved algorithm presented in Section 4, is applied to some BTE's. The obtained results, here, show that the method is accurate, easier, and reliable.

Example 5.1

In this example, presenting the inhomogeneous BTE of initial value type; for instance, see [28,29], therefore, one finds

$$D^2u(t) + D^{\frac{3}{2}}u(t) + u(t) = 1 + t, \tag{51}$$

with the initial conditions (I.C.)

$$u(0) = u'(0) = 1. \tag{52}$$

For his purpose, set

$$L(u) = D^{\frac{3}{2}}u(t), N(u) = u''(t) + u(t) - 1 - t, \tag{53}$$

$$U(t) = \sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n}, \tag{54}$$

Substituting into the Homotopy in Eq. (39), then applying the recurrence relation in Eq. (50), one gets

$$\delta^0 : D^{\frac{3}{2}}u_0(t) = D^{\frac{3}{2}}\left[\sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n}\right], \text{ applying the integral operator } J^{\frac{3}{2}} \text{ to get} \tag{55}$$

$$: u_0(t) = \sum_{n=2}^{\infty} a_n t^{\frac{3}{2}n} + \sum\nolimits_{k=0}^{1} h_k \frac{t^k}{k!} = \sum_{n=2}^{\infty} a_n t^{\frac{3}{2}n} + 1 + t, \tag{56}$$

$$\delta^1 : D^{\frac{3}{2}}u_1(t) = -D^{\frac{3}{2}}\left[\sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n}\right] - u_0''(t) - u_0(t), \tag{57}$$

$$\delta^2 : u_2(t) = u_1''(t) + u_1(t), \tag{58}$$

$$\delta^3 : u_3(t) = u_2''(t) + u_2(t), \tag{59}$$

$\vdots$

$$\delta^n : u_n(t) = u_{n-1}''(t) + u_{n-1}(t), n \geq 2, \quad n \text{ is an integer} \tag{60}$$

Equating $u_1(t) = 0$ in Eq. (57) will result in all u_n's ($n \geq 2$, n is an integer) will be equal to zero, one gets

$$D^{\frac{3}{2}}u_1(t) = -D^{\frac{3}{2}}\left[\sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n}\right] - u_0''(t) - u_0(t) = 0, \tag{61}$$

$$\begin{aligned} u_1(t) = &-\sum_{n=2}^{\infty} a_n t^{\frac{3}{2}n} - \sum_{n=2}^{\infty} \left(\frac{3}{2}n\right)\left(\frac{3}{2}n - 1\right) a_n t^{\frac{3}{2}n-2+\frac{3}{2}} \frac{\Gamma\left(\frac{3}{2}n - 2 + 1\right)}{\Gamma\left(\frac{3}{2}n - 2 + \frac{3}{2} + 1\right)} \\ &- \sum_{n=1}^{\infty} a_n t^{\frac{3}{2}n+\frac{3}{2}} \frac{\Gamma\left(\frac{3}{2}n + 1\right)}{\Gamma\left(\frac{3}{2}n + \frac{3}{2} + 1\right)} = 0 \end{aligned} \tag{62}$$

Comparing the coefficients of powers of t to get the undetermined coefficients $a_n's$, which will be

$$a_2 = a_3 = a_4 = \ldots = 0 \tag{63}$$

and substitute into Eq. (56) to get the zero-order exact solution of Eq. (51) in the form

$$u(t) = 1 + t. \tag{64}$$

Example 5.2

This example will also discuss the initial value problem within inhomogeneous BTE; for instance, see [29,30], therefore, one finds

$$D^2u(t) + 0.5D^{\frac{1}{2}}u(t) + u(t) = 3 + t^2\left(\frac{1}{\Gamma(2.5)}t^{-0.5} + 1\right), \tag{65}$$

$$I.C.\ \ u(0) = 1,\ \ u'(0) = 0,\ u^{'}(0) = 1. \tag{66}$$

For this objective, set

$$L(u) = D^{\frac{1}{2}}u(t), N(u) = 2u''(t) + 2u(t) - 6 - 2t^2\left(\frac{1}{\Gamma(2.5)}t^{-0.5} + 1\right), \tag{67}$$

$$U(t) = \sum_{n=0}^{\infty} a_n t^{\frac{1}{2}n}, \tag{68}$$

Substituting into the Homotopy in Eq. (39), then applying the recurrence relation in Eq. (50), one gets

$$\delta^0 : D^{\frac{1}{2}}u_0(t) = D^{\frac{1}{2}}\left[\sum_{n=0}^{\infty} a_n t^{\frac{1}{2}n}\right],\ \text{applying the integral operator } J^{\frac{1}{2}} \text{ to get} \tag{69}$$

$$: u_0(t) = \sum_{n=1}^{\infty} a_n t^{\frac{1}{2}n} + \sum\nolimits_{k=0}^{1} h_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} a_n t^{\frac{1}{2}n} + 1, \tag{70}$$

$$\delta^1 : D^{\frac{1}{2}}u_1(t) + D^{\frac{1}{2}}\left[\sum_{n=0}^{\infty} a_n t^{\frac{1}{2}n}\right] + 2u_0^{''}(t) + 2u_0(t) - 6 - 2t^2\left(\frac{1}{\Gamma(2.5)}t^{-0.5} + 1\right) = 0, \tag{71}$$

$$\delta^2 : u_2(t) = 2u_1^{''}(t) + 2u_1(t), \tag{72}$$

$$\delta^3 : u_3(t) = 2u_2^{''}(t) + 2u_2(t), \tag{73}$$

$\vdots$

$$\delta^n : u_n(t) = 2u_{n-1}^{''}(t) + 2u_{n-1}(t),\ \ n \geq 2,\ \ n \text{ is an integer} \tag{74}$$

Equate $u_1(t) = 0$ into Eq. (71) will result in all u_n's ($n \geq 2, n$ is an integer) will be equal to zero. Therefore, one finds

$$u_1(t) = -\sum_{n=1}^{\infty} a_n t^{\frac{1}{2}n} - 2\sum_{n=1}^{\infty}\left(\frac{1}{2}n\right)\left(\frac{1}{2}n - 1\right)a_n t^{\frac{1}{2}n-2+\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}n - 2 + 1\right)}{\Gamma\left(\frac{1}{2}n - 2 + \frac{1}{2} + 1\right)}$$

$$-2\sum_{n=1}^{\infty} a_n t^{\frac{1}{2}n+\frac{1}{2}}\frac{\Gamma\left(\frac{1}{2}n + 1\right)}{\Gamma\left(\frac{1}{2}n + \frac{1}{2} + 1\right)} + 4\frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 1\right)} + \frac{2t^{\frac{1}{2}+\frac{3}{2}}}{\Gamma\left(\frac{1}{2} + \frac{3}{2} + 1\right)} + \frac{2t^{2+\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + 2 + 1\right)} = 0 \tag{75}$$

Then comparing the coefficients of powers of t, yields the undetermined coefficients $a_n^{'}s$, which will be

$$a_1 = a_2 = a_3 = a_5 = \ldots = 0,\ \ a_4 = 1 \tag{76}$$

Substituting into Eq. (70) to get the zero-order exact solution of Eq. (65) in the form

$$u(t) = 1 + t^2. \tag{77}$$

Example 5.3

Here the case of homogeneous initial value BTE will be solved; for instance, see [29,30], therefore, one finds

$$D^\alpha u(t) + u(t) = 0, \quad \alpha < 1, \tag{78}$$

$$I.C. \quad u(0) = 1, \quad u'(0) = 0. \tag{79}$$

For this objective, set

$$L(u) = D^\alpha u(t), N(u) = u(t), \tag{80}$$

$$U(t) = \sum_{n=0}^{\infty} a_n t^{\alpha n}, \tag{81}$$

Substituting into the Homotopy in Eq. (39) applying the recurrence relation in Eq. (50), it is found that

$$\delta^0 : D^\alpha u_0(t) = D^\alpha\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right], \text{ applying the integral operator } J^\alpha \text{ to get} \tag{82}$$

$$: u_0(t) = \sum_{n=1}^{\infty} a_n t^{\alpha n} + \sum\nolimits_{k=0}^{0} h_k \frac{t^k}{k!} = \sum_{n=1}^{\infty} a_n t^{\alpha n} + 1, \tag{83}$$

$$\delta^1 : D^\alpha u_1(t) + D^\alpha\left[\sum_{n=0}^{\infty} a_n t^{\alpha n}\right] + \sum_{n=1}^{\infty} a_n t^{\alpha n} + 1 = 0, \tag{84}$$

$$\delta^2 : u_2(t) = u_1(t), \tag{85}$$

$$\delta^3 : u_3(t) = u_2(t), \tag{86}$$

Equating $u_1(t) = 0$ in Eq. (84) will result in all u_n's ($n \geq 2, n$ is an integer) will be equal to zero, Therefore, one gets

$$u_1(t) = -\sum_{n=1}^{\infty} a_n t^{\alpha n} - \sum_{n=1}^{\infty} a_n t^{\alpha n+\alpha} \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha n + \alpha + 1)} - \frac{t^\alpha}{\Gamma(\alpha + 1)} = 0. \tag{87}$$

Then compare the coefficients of powers of t to get the undetermined coefficients $a_n^{'}s$, which will be

$$a_1 = \frac{-1}{\Gamma(\alpha + 1)}, \quad a_2 = \frac{1}{\Gamma(2\alpha + 1)}, \tag{88}$$

$$a_3 = \frac{-1}{\Gamma(3\alpha + 1)}, \; a_4 = \frac{1}{\Gamma(4\alpha + 1)}, \ldots \tag{89}$$

Substituting into Eq. (83) to get the zero-order exact solution of Eq. (78) in the form

$$u(t) = 1 + \frac{-1}{\Gamma(\alpha+1)} t^{\alpha} + \frac{1}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{-1}{\Gamma(3\alpha+1)} t^{3\alpha} + \frac{1}{\Gamma(4\alpha+1)} t^{4\alpha} + \ldots \tag{90}$$

$$u(t) = E_{\alpha}(-t)^{\alpha}. \tag{91}$$

The exact solution is obtained, which is the well-know Mittag-Leffler function.

Example 5.4

In this problem, consider the inhomogeneous BTE; for instance, see Sakar et al. [31]

$$D^2 u(t) + D^{\frac{3}{2}} u(t) + u(t) = t^3 + 5t + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}}, t \in (0, 1], \tag{92}$$

$$I.C.\ \ u(0) = 0,\ u^{'}(0) = -1. \tag{93}$$

For this objective, set

$$L(u) = D^{\frac{3}{2}} u(t),\ \ N(u) = D^2 u(t) + u(t), \tag{94}$$

$$U(t) = \sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n}, \tag{95}$$

Substituting into the Homotopy in Eq. (39) applying the recurrence relation in Eq. (50), it is found that

$$\delta^0 : D^{\frac{3}{2}} u_0(t) = D^{\frac{3}{2}} \left[\sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n} \right],\ \ \text{applying the integral operator } J^{\frac{3}{2}} \text{ to get} \tag{96}$$

$$: u_0(t) = \sum_{n=2}^{\infty} a_n t^{\frac{3}{2}n} + \sum\nolimits_{k=0}^{1} h_k \frac{t^k}{k!} = \sum_{n=2}^{\infty} a_n t^{\frac{3}{2}n} - t, \tag{97}$$

$$\delta^1 : D^{\frac{3}{2}} u_1(t) + D^{\frac{3}{2}} \left[\sum_{n=0}^{\infty} a_n t^{\frac{3}{2}n} \right] + u_0^{''}(t) + u_0(t) - t^3 - 5t - \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} = 0, \tag{98}$$

$$\delta^2 : u_2(t) = u_1(t), \tag{99}$$

$$\delta^3 : u_3(t) = u_2(t), \tag{100}$$

$\vdots$

Equating $u_1(t) = 0$ in Eq. (84) will result in all u_n's ($n \geq 2, n$ is an integer) will be equal to zero, Therefore, one gets

$$u_1(t) = -\sum_{n=2}^{\infty} a_n t^{\frac{3}{2}n} - \sum_{n=2}^{\infty} a_n \left(\frac{3n}{2}\right)\left(\frac{3n}{2} - 1\right) \frac{\Gamma\left(\frac{3n}{2} - 1\right)}{\Gamma\left(\frac{3n}{2} + \frac{1}{2}\right)} t^{\frac{3}{2}n + \frac{1}{2}}$$

$$-\sum_{n=2}^{\infty} a_n \frac{\Gamma\left(\frac{3n}{2}+1\right)}{\Gamma\left(\frac{3n}{2}+\frac{5}{2}\right)} t^{\frac{3}{2}n+\frac{3}{2}} + 6\frac{\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)} t^{\frac{5}{2}} + \frac{\Gamma(4)}{\Gamma\left(\frac{11}{2}\right)} t^{\frac{9}{2}} + \frac{8}{\sqrt{\pi}} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(4)} t^3 = 0 \tag{101}$$

Then comparing the coefficients of powers of t to get the undetermined coefficients $a_n^{'}s$, which will be

$$a_2 = 1, \tag{102}$$

$$otherwise, \quad a_n = 0, \tag{103}$$

Substituting into Eq. (83) to get the zero-order exact solution of Eq. (78) in the form

$$u(t) = t^3 - t. \tag{104}$$

6 Conclusion

This article aims to offer an elementary concept of finding an analytical solution to the BTE. Consequently, the work announced to has a successful algorithm to solve BTE. The announced technique manipulated the given problems and gave the exact solution in all cases that appeared in this work with ease. This shows that the technique is powerful, effective, promising, efficient, and reliable in solving linear fractional differential equations or to be specific BTE. The unity of the solution is demonstrated and proved. It is clear from the analysis of the results that this method is characterized by accuracy and ease of implementation as it depends on some of the few steps compared to others.

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