Research Article

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Some new extensions for fractional integral operator having exponential in the kernel and their applications in physical systems

https://doi.org/10.1515/phys-2020-0114
received April 02, 2020; accepted June 04, 2020

Abstract: The key purpose of this study is to suggest a new fractional extension of Hermite–Hadamard, Hermite–Hadamard–Fejér and Pachpatte-type inequalities for harmonically convex functions with exponential in the kernel. Taking into account the new operator, we derived some generalizations that capture novel results under investigation with the aid of the fractional operators. We presented, in general, two different techniques that can be used to solve some new generalizations of increasing functions with the assumption of convexity by employing more general fractional integral operators having exponential in the kernel have yielded intriguing results. The results achieved by the use of the suggested scheme unfold that the used computational outcomes are very accurate, flexible, effective and simple to perform to examine the future research in circuit theory and complex waveforms.

Keywords: convex function, harmonically convex functions, Hermite-Hadamard inequality, Hermite-Hadamard-Fejér inequality, Pachpatte-type inequality

1 Introduction and preliminaries

The Hermite–Hadamard inequality is a well-known, paramount and extensively used inequality in the applied literature of mathematical inequalities [1–17]. This inequality is of pivotal significance, because of other classical inequalities, such as Hardy, Opial, Lynger, Ostrowski, Minkowski, Hölder, Ky-Fan, Beckenbach–Dresher, Levinson, arithmetic-geometric, Young, Olsen and Gagliardo–Nirenberg inequalities, but the most distinguished inequality is the Hermite–Hadamard-type inequality [18,19], which is stated as:

$$\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right) \le \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(z)\mathrm{d}z \le \frac{\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)}{2}. \quad (1.1)$$

Inequality (1.1) and its generalizations, refinements, extensions and converses have many applications in different fields of science, for example, electrical engineering, mathematical statistics, financial economics, information theory, guessing and coding [20–23]. Convexity has played a crucial role in the advancement of different areas of science and technology. Due to its robustness, convex functions and convex sets have been generalized and extended in various areas. It has been proved that a function is convex, if and only if, it satisfies an integral inequality (1.1). In the present scenario, we propose an innovative class of functional variants for harmonically convex functions and several other generalizations for the convexity theory as novel fractional operators with the exponential kernel are new and effectively applicable.

In [21], Fejér contemplated the important generalizations that are the weighted generalization of the Hermite–Hadamard inequality.

Let $\mathcal{I} \subseteq \mathbb{R}$ and a function $\mathcal{P} : \mathcal{I} \to \mathbb{R}$ be a convex function. Then, the inequalities

$$\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)\int_{\eta_1}^{\eta_2}\mathcal{Q}(z)\mathrm{d}z \le \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(z)\mathcal{Q}(z)\mathrm{d}z \le \frac{\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)}{2}\int_{\eta_1}^{\eta_2}\mathcal{Q}(z)\mathrm{d}z \quad (1.2)$$

hold, where $\mathcal{Q} : \mathcal{I} \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{\eta_1+\eta_2}{2}$.

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In [24], Pachpatte presented two novel versions of Hermite–Hadamard variants for products of convex functions as follows.

Let $\mathcal{P}, Q : \mathcal{I} \to \mathbb{R}$ be two non-negative and convex functions, then

$$2\mathcal{P}\left(\frac{\eta_1+\eta_2}{2}\right)Q\left(\frac{\eta_1+\eta_2}{2}\right) \le \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(z)Q(z)\mathrm{d}z + \frac{\mathcal{P}(\eta_1)Q(\eta_1)+\mathcal{P}(\eta_2)Q(\eta_2)}{6} + \frac{\mathcal{P}(\eta_1)Q(\eta_2)+\mathcal{P}(\eta_2)Q(\eta_1)}{3}$$

and

$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\mathcal{P}(z)Q(z)\mathrm{d}z \le \frac{\mathcal{P}(\eta_1)Q(\eta_1)+\mathcal{P}(\eta_2)Q(\eta_2)}{6} + \frac{\mathcal{P}(\eta_1)Q(\eta_2)+\mathcal{P}(\eta_2)Q(\eta_1)}{3}.$$

Iscan [25] gave the concept of harmonically convex functions.

Definition 1.1. [25] Let a real interval $\mathcal{K} \subset \mathbb{R}\backslash\{0\}$ and a function $\mathcal{P} : \mathcal{K} \to \mathbb{R}$ is said to be harmonically convex, if

$$\mathcal{P}\left(\frac{xy}{\zeta x+(1-\zeta)y}\right) \le \zeta\mathcal{P}(y)+(1-\zeta)\mathcal{P}(x) \tag{1.3}$$

for all $x, y \in \mathcal{K}$ and $\zeta \in [0, 1]$. If inequality (1.4) holds in the reversed direction, then $\mathcal{P}$ is called the harmonically concave function.

It is worth mentioning that the Jensen harmonic convexity has applications in the electrical circuit theory and other branches of sciences. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocal of the individual resistance value and then considering the reciprocal of their total. For example, if s_1 and s_2 are the resistance of two parallel resistors, then the total resistance

$$S = \frac{1}{\frac{1}{s_1}+\frac{1}{s_2}} = \frac{s_1 s_2}{s_1+s_2},$$

which is half of the harmonic mean. The "conductivity effective mass" of a semiconductor is also defined as the harmonic mean of the effective masses along with the three crystallographic directions. Also, harmonically convex functions have unwanted higher frequencies that superimposed on the fundamental waveform creating a distorted wave pattern [26].

Definition 1.2. [25] A function $\mathcal{P} : \mathcal{K} \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2\eta_1\eta_2}{\eta_1+\eta_2}$, if

$$Q(r) = Q\left(\frac{1}{\frac{1}{\eta_1}+\frac{1}{\eta_2}-z}\right) \tag{1.4}$$

holds for all $z \in \mathcal{I}$.

A few decades ago, classical calculus has been revolutionized by tremendous innovations. The study of differentiation and integration to a fractional order has caught importance and popularity among researchers compared to classical differentiation and integration. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property. For instance, various applications and comprehensive strategy of the fractional calculus are addressed in the works of Baleanu et al. [27], Miller and Ross [28] and Kilbas et al. [29]. A good review of different fractional operators can be found in ref. [22,23,27,30–33]. It has been proved that differential equations with fractional order process more accurately than integer-order differential equations do, and fractional arrangers provide excellent performance of the description of hereditary attributes than integer-order arrangers. Applications can be found in complex viscoelastic media, electrical spectroscopy, porous media, cosmology, environmental science, medicine (the modeling of infectious diseases), signal and image processing, materials and many others.

Moreover, fractional integral inequalities have several applications in scientific areas that can be found in the existing literature, see ref. [22,23,34–43]. The uses of variants in applied sciences are generally studied and now it is a profoundly appealing research-oriented area where the researchers also investigate the existence and uniqueness of the solutions of fractional differential equations. Adil Khan et al. [1] derived the Hermite–Hadamard inequality for s-convex functions. Rashid et al. [44] contemplated weighted generalizations of Hermite–Hadamard inequalities for extended generalized Mittag–Leffler functions as fractional operators.

Following the aforementioned trend, we use the fractional integral operator for the integrable functions to establish Hermite–Hadamard, Hermite–Hadamard–Fejér and Pachpatte-type integral inequalities for harmonically convex functions. Additionally, several other generalizations by a more general fractional integral operator having exponential in the kernel are deliberated. Our consequences are more fascinating and effectively applicable than the existing ones. Finally, a complete agreement is achieved between the proposed method and inequalities for convexity to manifest about the performance and applicability of the more general operator.

Here, we recall some concerned definitions from the existing literature.

We now demonstrate some essential ideas associated with the fractional integral, which is mainly due to Ahmed et al. [2].

Definition 1.3. [2] Let $\mathcal{P} \in L_1([\eta_1, \eta_2])$. The fractional integrals $\mathcal{J}^{\gamma}_{\eta_1}$ and $\mathcal{J}^{\gamma}_{\eta_2}$ of order $\gamma > 0$ are stated as:

$$\mathcal{J}^{\gamma}_{\eta_1}\mathcal{P}(r) = \frac{1}{\gamma}\int_{\eta_1}^{r} e^{-\frac{1-\gamma}{\gamma}(r-\delta)}\mathcal{P}(\delta)\mathrm{d}\delta, \quad r > \delta \tag{1.5}$$

and

$$\mathcal{J}^{\gamma}_{\eta_1}\mathcal{P}(r) = \frac{1}{\gamma}\int_{r}^{\eta_2} e^{-\frac{1-\gamma}{\gamma}(\delta-r)}\mathcal{P}(\delta)\mathrm{d}\delta, \quad r < \delta. \tag{1.6}$$

Furthermore, we introduced the more general concept of the fractional integral operator having exponential in the kernel as follows.

Definition 1.4. Let $\mathcal{P} : \mathcal{I} \to \mathbb{R}, (0 < \eta_1 < \eta_2)$ be a function such that $\mathcal{P}$ be a positive and integrable, also Ψ be a differentiable and strictly increasing on (η_1, η_2). Then, the fractional integral operators $\mathcal{J}^{\gamma,\Psi}_{\eta_1}$ and $\mathcal{J}^{\gamma,\Psi}_{\eta_2}$ of order $\gamma > 0$ are stated as:

$$\mathcal{J}^{\gamma,\Psi}_{\eta_1}\mathcal{P}(r) = \frac{1}{\gamma}\int_{\eta_1}^{r} e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\delta))}\Psi'(\delta)\mathcal{P}(\delta)\mathrm{d}\delta, \quad r > \delta \tag{1.7}$$

and

$$\mathcal{J}^{\gamma,\Psi}_{\eta_1}\mathcal{P}(r) = \frac{1}{\gamma}\int_{r}^{\eta_2} e^{-\frac{1-\gamma}{\gamma}(\Psi(\delta)-\Psi(r))}\Psi'(\delta)\mathcal{P}(\delta)\mathrm{d}\delta, \quad r < \delta. \tag{1.8}$$

Next, we define the one-sided definition of a more general fractional integral operator having exponential in their kernel as follows.

Definition 1.5. Let $\mathcal{P} : \mathcal{I} \to \mathbb{R}, (0 < \eta_1 < \eta_2)$ be a function such that $\mathcal{P}$ be a positive and integrable, also Ψ be a differentiable and strictly increasing on (η_1, η_2). Then, the one-sided fractional integral operator $\mathcal{J}^{\gamma,\Psi}_{\eta_1}$ is stated as:

$$\mathcal{J}^{\gamma,\Psi}_{0^+,r}\mathcal{P}(r) = \frac{1}{\gamma}\int_{0}^{r} e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\delta))}\Psi'(\delta)\mathcal{P}(\delta)\mathrm{d}\delta, \quad r > \delta. \tag{1.9}$$

Throughout Sections 2 to 4, we set $\vartheta = \frac{1-\gamma}{\gamma}\left(\frac{\eta_2-\eta_1}{\eta_2\eta_1}\right)$.

2 Hermite–Hadamard-type inequality for Harmonic convex functions using fractional integral having exponential in the kernel

In this section, we derive the Hermite–Hadamard inequality for harmonically convex functions in the frame of a new fractional integral operator as follows.

Theorem 2.1. *For $\gamma > 0$ and let there is a positive function $\mathcal{P} : \mathcal{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ with $\eta_2 > \eta_1$ and $\mathcal{P} \in L_1([\eta_1, \eta_2])$. If $\mathcal{P}$ is a harmonically convex function on $\mathcal{I}$, then*

$$\begin{aligned}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) &\le \frac{1-\gamma}{2(1-e^{-\vartheta})}\left[\mathcal{J}^{\gamma}_{\frac{1}{\eta_1}^-}(\mathcal{P}\circ Q)\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\frac{1}{\eta_2}^+}(\mathcal{P}\circ Q)\left(\frac{1}{\eta_1}\right)\right] \\ &\le \frac{\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)}{2},\end{aligned} \tag{2.1}$$

where $Q(r) = \frac{1}{r}$, $r \in \left[\frac{1}{\eta_2}, \frac{1}{\eta_1}\right]$.

Proof. By utilizing harmonically convexity of $\mathcal{P}$ on $\mathcal{I}$, we have for every $z_1, z_2 \in \mathcal{I}$ having $\zeta = \frac{1}{2}$,

$$\mathcal{P}\left(\frac{2z_1z_2}{z_1+z_2}\right) \le \frac{\mathcal{P}(z_1)+\mathcal{P}(z_2)}{2}, \tag{2.2}$$

choosing $z_1 = \frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}$, $z_2 = \frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}$, takes the form:

$$\begin{aligned}2\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) &\le \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right) \\ &\quad + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right).\end{aligned} \tag{2.3}$$

Conducting product on both sides of (2.7) by $e^{-\vartheta\zeta}$, then integrating with respect to ζ from 0 to 1, we get

$$\begin{aligned}\frac{2(1-e^{-\vartheta})}{\vartheta}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) &\le \left[\int_0^1 e^{-\vartheta\zeta}\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)d\zeta + \int_0^1 e^{-\vartheta\zeta}\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)d\zeta\right]\\ &\le \left(\frac{\eta_1\eta_2}{\eta_2-\eta_1}\right)\left[\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{\left(-\frac{ab(1-\gamma)}{\gamma(\eta_2-\eta_1)}\left(z-\frac{1}{\eta_2}\right)\right)}\mathcal{P}\left(\frac{1}{r}\right)dr + \int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{\left(-\frac{ab(1-\gamma)}{\gamma(\eta_2-\eta_1)}\left(\frac{1}{\eta_1}-z\right)\right)}\mathcal{P}\left(\frac{1}{r}\right)dr\right]\\ &= \gamma\left(\frac{\eta_1\eta_2}{\eta_2-\eta_1}\right)\left[\mathcal{J}^{\gamma}_{\frac{1}{\eta_1}-}(\mathcal{P}\circ Q)\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\frac{1}{\eta_2}+}(\mathcal{P}\circ Q)\left(\frac{1}{\eta_1}\right)\right]\end{aligned}$$

and established the first inequality.

For the proof of the second inequality in (2.1), we first note that if $\mathcal{P}$ is a harmonically convex function, then, for $\zeta \in [0,1]$, it yields

$$\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right) \le \zeta\mathcal{P}(\eta_1)+(1-\zeta)\mathcal{P}(\eta_2)$$

and

$$\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right) \le \zeta\mathcal{P}(\eta_2)+(1-\zeta)\mathcal{P}(\eta_1).$$

By adding the above inequalities, we have

$$\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)+\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right) \le \mathcal{P}(\eta_1)+\mathcal{P}(\eta_2). \tag{2.4}$$

Then, multiplying on both sides of (2.4) by $e^{-\vartheta\zeta}$ and integrating the inequality with respect to ζ from 0 to 1, one obtains

$$\int_0^1 e^{-\vartheta\zeta}\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)d\zeta + \int_0^1 e^{-\vartheta\zeta}\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)d\zeta \le [\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)]\int_0^1 e^{-\vartheta\zeta}d\zeta.$$

As a result, we have

$$\gamma\left(\frac{\eta_1\eta_2}{\eta_2-\eta_1}\right)\left[\mathcal{I}^{\gamma}_{\frac{1}{\eta_1}-}(\mathcal{P}\circ Q)\left(\frac{1}{\eta_2}\right)+\mathcal{I}^{\gamma}_{\frac{1}{\eta_2}+}(\mathcal{P}\circ Q)\left(\frac{1}{\eta_1}\right)\right] \le \frac{1-e^{-\vartheta}}{\vartheta}[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)].$$

The proof is completed. □

Remark. In the limiting case, when $\gamma \to 1$, observe that

$$\lim_{\gamma\to 1}\frac{1-\gamma}{2\left(1-e^{-\frac{1-\gamma}{\gamma}\left(\frac{\eta_2-\eta_1}{\eta_1\eta_2}\right)}\right)} = \frac{\eta_1\eta_2}{2(\eta_2-\eta_1)},$$

which is proposed by Iscan in [34].

3 Hermite–Hadamard–Fejér-type inequality for harmonically convex functions

In order to prove our main result, we need the following lemma which will help us in proving the Hermite–Hadamard–Fejér-type inequality.

Lemma 3.1. *For $\gamma > 0$ and let there is a function $Q : \mathcal{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ integrable and harmonically symmetric with respect to $\frac{2\eta_1\eta_2}{\eta_1+\eta_2}$, then*

$$\begin{aligned}\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)+}(Q\circ\mathcal{U})\left(\frac{1}{\eta_1}\right) &= \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)-}(Q\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\\ &= \frac{1}{2}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)+}(Q\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)-}(Q\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\right],\end{aligned} \tag{3.1}$$

where $\gamma > 0$ and $\mathcal{U}(z) = \frac{1}{z}$, $z \in \left[\frac{1}{\eta_2},\frac{1}{\eta_1}\right]$.

Proof. By the given assumption and substituting $\zeta = \frac{1}{\eta_1}+\frac{1}{\eta_2}-z$ in the following integral and performing some computation we get

$$\begin{aligned}\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)+}(Q\circ\mathcal{U})\left(\frac{1}{\eta_1}\right) &= \frac{1}{\gamma}\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{-\vartheta\left(\frac{1}{\eta_1}-\zeta\right)}Q\left(\frac{1}{\zeta}\right)d\zeta\\ &= -\frac{1}{\gamma}\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{-\vartheta\left(\zeta-\frac{1}{\eta_2}\right)}Q\left(\frac{1}{\frac{1}{\eta_1}+\frac{1}{\eta_2}-z}\right)dz\\ &= \frac{1}{\gamma}\int_{\frac{1}{\eta_1}}^{\frac{1}{\eta_2}} e^{-\vartheta\left(\zeta-\frac{1}{\eta_2}\right)}Q\left(\frac{1}{z}\right)dz\\ &= \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)-}(Q\circ\mathcal{U})\left(\frac{1}{\eta_2}\right),\end{aligned} \tag{3.2}$$

the required result. □

Theorem 3.2. *For $\gamma > 0$ and let there is a harmonically convex function $\mathcal{P} : \mathcal{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that $\mathcal{P} \in L_1([\eta_1, \eta_2])$ with $\eta_2 > \eta_1$ and $\eta_1, \eta_2 \in \mathcal{I}$. Also, if there is non-negative, integrable and harmonically symmetric with respect to $\frac{2\eta_1\eta_2}{\eta_1+\eta_2}$, then*

$$\begin{aligned}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)&\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right]\\ &\leq\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right]\\ &\leq\frac{\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)}{2}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right],\end{aligned}\tag{3.3}$$

where $\mathcal{U}(z) = \frac{1}{z}$, $\quad z \in \left[\frac{1}{\eta_2}, \frac{1}{\eta_1}\right]$.

Proof. Since $\mathcal{P}$ is a harmonically convex function on $\mathcal{I}$, we have inequality (2.4) for all $\zeta \in [0, 1]$. Multiplying on both sides of (2.4) by $e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)$, and then integrating the inequality with respect to ζ from 0 to 1, we obtain

$$\begin{aligned}2\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)&\int_0^1 e^{-\vartheta\zeta}\mathcal{U}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)d\zeta\\ &\leq\int_0^1 e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)d\zeta\\ &+\int_0^1 e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)d\zeta.\end{aligned}\tag{3.4}$$

Setting $z = \frac{\zeta\eta_2+(1-\zeta)\eta_1}{\eta_1\eta_2}$ and utilizing that $\mathcal{Q}$ is harmonically symmetric, we have

$$\begin{aligned}\frac{2\eta_1\eta_2}{\eta_2-\eta_1}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)&\int_{\frac{1}{\eta_2}}^{\frac{2\eta_1\eta_2}{\eta_1+\eta_2}} e^{\left(\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(z-\frac{1}{\eta_2}\right)\right)}\mathcal{Q}\left(\frac{1}{\frac{1}{\eta_1}+\frac{1}{\eta_2}-z}\right)dr\\ &\leq\frac{\eta_1\eta_2}{\eta_2-\eta_1}\left[\int_{\frac{1}{\eta_2}}^{\frac{2\eta_1\eta_2}{\eta_1+\eta_2}} e^{\left(\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(z-\frac{1}{\eta_2}\right)\right)}\mathcal{Q}\left(\frac{1}{z}\right)\mathcal{P}\left(\frac{1}{\frac{1}{\eta_1}+\frac{1}{\eta_2}-z}\right)dz\right.\\ &\left.+\int_{\frac{1}{\eta_2}}^{\frac{2\eta_1\eta_2}{\eta_1+\eta_2}} e^{\left(\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(z-\frac{1}{\eta_2}\right)\right)}\mathcal{Q}\left(\frac{1}{z}\right)\mathcal{P}\left(\frac{1}{z}\right)dz\right].\end{aligned}\tag{3.5}$$

It follows that

$$\begin{aligned}\frac{2\eta_1\eta_2\gamma}{\eta_2-\eta_1}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)&\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\\ &\leq\gamma\frac{\eta_1\eta_2}{\eta_2-\eta_1}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\right.\\ &\left.+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right].\end{aligned}\tag{3.6}$$

Applying Lemma 3.1 on the left hand side of (3.6), we have

$$\begin{aligned}\frac{\eta_1\eta_2}{\eta_2-\eta_1}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)&\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\right.\\ &\left.+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right]\\ &\leq\frac{\eta_1\eta_2}{\eta_2-\eta_1}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\right.\\ &\left.+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right].\end{aligned}\tag{3.7}$$

For the proof of the second inequality in (3.3), multiplying on both sides of (2.4) by $e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)$ and integrating the inequality with respect to ζ from 0 to 1, we obtain

$$\begin{aligned}&\int_0^1 e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)d\zeta\\ &+\int_0^1 e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)d\zeta\\ &\leq[\mathcal{P}(\eta_1)+\mathcal{P}(\eta_2)]\int_0^1 e^{-\vartheta\zeta}\mathcal{Q}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)d\zeta.\end{aligned}\tag{3.8}$$

Again, setting $z = \frac{\zeta\eta_2+(1-\zeta)\eta_1}{\eta_1\eta_2}$ and after simple calculations, we conclude the second inequality (3.3). ☐

Remark 3.1. In Theorem 3.2:
(1) If one takes $\gamma \to 1$, then one has Theorem 4 in [35].
(2) If one takes $\mathcal{Q}(r) = 1$ and $\gamma \to 1$, then one has Theorem 2 in [35].

4 Pachpatte-type inequalities for harmonically convex functions

Theorem 4.1. *For $\gamma > 0$ and let there are two harmonically convex functions $\mathcal{P}, \mathcal{W} : \mathcal{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ such that $\mathcal{P}, \mathcal{W} \in L_1([\eta_1, \eta_2])$ with $\eta_2 > \eta_1$ and $\eta_1, \eta_2 \in \mathcal{I}$, then*

$$\begin{aligned}\frac{\gamma\eta_1\eta_2}{\eta_2-\eta_1}&\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}\mathcal{P}\left(\frac{1}{\eta_2}\right)\mathcal{W}\left(\frac{1}{\eta_2}\right)+\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}\mathcal{P}\left(\frac{1}{\eta_1}\right)\mathcal{W}\left(\frac{1}{\eta_1}\right)\right]\\ &\leq\Upsilon_1(\eta_1,\eta_2)\frac{\vartheta^2-2\vartheta+4-(\vartheta^2+2\vartheta+4)e^{-\vartheta}}{2\vartheta^3}\\ &+\Upsilon_2(\eta_1,\eta_2)\frac{\vartheta-2+e^{-\vartheta}(\vartheta+2)}{\vartheta^3}\end{aligned}\tag{4.1}$$

and

$$2\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) \le \frac{1-\gamma}{2(1-e^{-\vartheta})}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}\mathcal{P}\left(\frac{1}{\eta_2}\right)\mathcal{W}\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}\mathcal{P}\left(\frac{1}{\eta_1}\right)\mathcal{W}\left(\frac{1}{\eta_1}\right)\right] + \frac{\vartheta^2-2\vartheta+4-e^{-\vartheta}(\vartheta^2+2\vartheta+4)}{2\vartheta^2(1-e^{-\vartheta})}\Upsilon_2(\eta_1,\eta_2) + \frac{\vartheta-2+(\vartheta+2)e^{-\vartheta}}{\vartheta^2(1-e^{-\vartheta})}\Upsilon_1(\eta_1,\eta_2), \tag{4.2}$$

where

$$\Upsilon_1(\eta_1,\eta_2) = [\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_2)] \tag{4.3}$$

and

$$\Upsilon_2(\eta_1,\eta_2) = [\mathcal{P}(\eta_1)\mathcal{W}(\eta_2) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_1)]. \tag{4.4}$$

Proof. Since $\mathcal{P}$ and $\mathcal{W}$ are harmonically convex functions on $\mathcal{I}$, then, for $\zeta \in [0,1]$,

$$\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right) \le \zeta^2\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + (1-\zeta)^2\mathcal{P}(\eta_2)\mathcal{W}(\eta_2) + \zeta(1-\zeta)[\mathcal{P}(\eta_1)\mathcal{W}(\eta_2) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_1)] \tag{4.5}$$

and

$$\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right) \le \zeta^2\mathcal{P}(\eta_2)\mathcal{W}(\eta_2) + (1-\zeta)^2\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + \zeta(1-\zeta)[\mathcal{P}(\eta_1)\mathcal{W}(\eta_2) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_1)]. \tag{4.6}$$

Adding (4.5) and (4.6), we have

$$\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right) + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right) \le (2\zeta^2-2\zeta+1)[\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_2)] + 2\zeta(1-\zeta)[\mathcal{P}(\eta_1)\mathcal{W}(\eta_2) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_1)]. \tag{4.7}$$

Multiplying on both sides of (4.7) by $e^{-\vartheta\zeta}$ and integrating the inequality with respect to ζ from 0 to 1, we have

$$\int_0^1 e^{-\vartheta\zeta}\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathrm{d}\zeta + \int_0^1 e^{-\vartheta\zeta}\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathrm{d}\zeta \le [\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_2)]\int_0^1 e^{-\vartheta\zeta}(2\zeta^2-2\zeta+1)\mathrm{d}\zeta + 2[\mathcal{P}(\eta_1)\mathcal{W}(\eta_2) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_1)]\int_0^1 e^{-\vartheta\zeta}(2\zeta^2-2\zeta+1) \times \zeta(1-\zeta)\mathrm{d}\zeta$$

Consequently, we get

$$\frac{\gamma ab}{\eta_2-\eta_1}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}\mathcal{P}\left(\frac{1}{\eta_2}\right)\mathcal{W}\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}\mathcal{P}\left(\frac{1}{\eta_1}\right)\mathcal{W}\left(\frac{1}{\eta_1}\right)\right] \le [\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_2)]\frac{(1-e^{-\vartheta})(\vartheta^2+2\vartheta+4)}{2\vartheta^3} + [\mathcal{P}(\eta_1)\mathcal{W}(\eta_2) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_1)]\frac{\vartheta-2+e^{-\vartheta}(\vartheta+2)}{\vartheta^3} = \Upsilon_1(\eta_1,\eta_2)\frac{\vartheta^2-2\vartheta+4-e^{-\vartheta}(\vartheta^2+2\vartheta+4)}{2\vartheta^3} + \Upsilon_2(\eta_1,\eta_2)\frac{\vartheta-2+e^{-\vartheta}(\vartheta+2)}{\vartheta^3},$$

which completes the proof of (4.8).

Furthermore, we prove inequality (4.2). By utilizing the harmonically convexity of the functions $\mathcal{P}$ and $\mathcal{W}$ on $\mathcal{I}$, we have for all $z_1, z_2 \in \mathcal{I}$

$$\mathcal{P}\left(\frac{2z_1z_2}{z_1+z_2}\right) \le \frac{\mathcal{P}(z_1)+\mathcal{P}(z_2)}{2} \tag{4.8}$$

and

$$\mathcal{W}\left(\frac{2z_1z_2}{z_1+z_2}\right) \le \frac{\mathcal{W}(z_1)+\mathcal{W}(z_2)}{2}. \tag{4.9}$$

Substituting $z_1 = \frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}$ and $z_2 = \frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}$, we have

$$4\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)\mathcal{W}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right) \le \left[\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right) + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right) + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right) + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\right]$$

$$= \left[\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2+(1-\zeta)\eta_1}\right) + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1+(1-\zeta)\eta_2}\right)\right] + (\zeta^2+(1-\zeta)^2)[\mathcal{P}(\eta_2)\mathcal{W}(\eta_1) + \mathcal{P}(\eta_1)\mathcal{W}(\eta_2)] + 2\zeta(1-\zeta)[\mathcal{P}(\eta_1)\mathcal{W}(\eta_1) + \mathcal{P}(\eta_2)\mathcal{W}(\eta_2)]$$

$$
\begin{aligned}
&= \left[\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2 + (1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2 + (1-\zeta)\eta_1}\right)\right.\\
&\quad \left. + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1 + (1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1 + (1-\zeta)\eta_2}\right)\right]\\
&\quad + (2\zeta^2 - 2\zeta + 1)\Upsilon_2(\eta_1, \eta_2) + 2\zeta(1-\zeta)\Upsilon_1(\eta_1, \eta_2).
\end{aligned} \tag{4.10}
$$

Multiplying on both sides of (4.11) by $e^{-\vartheta\zeta}$ and then integrating the resulting inequality with respect to $\zeta \in [0, 1]$, we have

$$
\begin{aligned}
&\frac{4(1-e^{-\vartheta})}{\vartheta}\mathcal{P}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)\mathcal{W}\left(\frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right)\\
&\le \int_0^1 e^{-\vartheta\zeta}\left[\mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_2 + (1-\zeta)\eta_1}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_2 + (1-\zeta)\eta_1}\right)\right.\\
&\quad \left. + \mathcal{P}\left(\frac{\eta_1\eta_2}{\zeta\eta_1 + (1-\zeta)\eta_2}\right)\mathcal{W}\left(\frac{\eta_1\eta_2}{\zeta\eta_1 + (1-\zeta)\eta_2}\right)\right]\mathrm{d}\zeta\\
&\quad + \Upsilon_2(\eta_1, \eta_2)\int_0^1 e^{-\vartheta\zeta}(2\zeta^2 - 2\zeta + 1)\mathrm{d}\zeta\\
&\quad + 2\Upsilon_1(\eta_1, \eta_2)\int_0^1 \zeta(1-\zeta)\mathrm{d}\zeta\\
&= \frac{\gamma\eta_1\eta_2}{\eta_2 - \eta_1}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}\mathcal{P}\left(\frac{1}{\eta_2}\right)\mathcal{W}\left(\frac{1}{\eta_2}\right)\right.\\
&\quad \left. + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}\mathcal{P}\left(\frac{1}{\eta_1}\right)\mathcal{W}\left(\frac{1}{\eta_1}\right)\right]\\
&\quad + \frac{\vartheta^2 - 2\vartheta + 4 - (\vartheta^2 + 2\vartheta + 4)e^{-\vartheta}}{\vartheta^3}\Upsilon_2(\eta_1, \eta_2)\\
&\quad + \frac{\vartheta - 2 + (\vartheta + 2)e^{-\vartheta}}{\vartheta^3}\Upsilon_1(\eta_1, \eta_2),
\end{aligned} \tag{4.11}
$$

after suitable rearrangements, we get the desired inequality 4.2. □

Lemma 4.2. *For $\gamma > 0$ and let there is a differentiable function $\mathcal{P} : \mathcal{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ on $\mathcal{I}^\circ$ such that $\mathcal{P}' \in L_1([\eta_1, \eta_2])$ with $\eta_2 > \eta_1$ and $\eta_1, \eta_2 \in \mathcal{I}$. Also, if there is a non-negative, integrable and harmonically symmetric with respect to $\frac{2\eta_1\eta_2}{\eta_1+\eta_2}$, then*

$$
\begin{aligned}
&\frac{\mathcal{P}(\eta_1) + \mathcal{P}(\eta_2)}{2}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right)\right.\\
&\quad \left. + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right]\\
&\le \left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right]\\
&= \frac{1}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\right)\right.\\
&\quad \left.\times(\mathcal{P}\circ\mathcal{U})'(\zeta)\mathrm{d}\zeta\right]\\
&\quad - \frac{1}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\zeta}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\right)\right.\\
&\quad \left.\times(\mathcal{P}\circ\mathcal{U})'(\zeta)\mathrm{d}\zeta\right],
\end{aligned} \tag{4.12}
$$

where $\mathcal{U}(z) = \frac{1}{z}$, $\quad z \in \left[\frac{1}{\eta_2}, \frac{1}{\eta_1}\right]$.

Proof. Consider

$$
\begin{aligned}
I &= \frac{1}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\right)\right.\\
&\quad \left.\times(\mathcal{P}\circ\mathcal{U})'(\zeta)\mathrm{d}\zeta\right]\\
&\quad - \frac{1}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\zeta}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\right)\right.\\
&\quad \left.\times(\mathcal{P}\circ\mathcal{U})'(\zeta)\right]\mathrm{d}\zeta = I_1 - I_2.
\end{aligned} \tag{4.13}
$$

Now

$$
\begin{aligned}
I_1 &= \frac{1}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\right)(\mathcal{P}\circ\mathcal{U})'(\zeta)\mathrm{d}\zeta\right]\\
&= \frac{1}{\gamma}\left(\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\right)(\mathcal{P}\circ\mathcal{U})(\zeta)\Bigg|_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\\
&\quad - \int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\zeta\right)}(\mathcal{Q}\circ\mathcal{U})(\zeta)(\mathcal{P}\circ\mathcal{U})(\zeta)\mathrm{d}\delta\\
&= \frac{1}{\gamma}(\mathcal{P}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\zeta\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\mathrm{d}\delta\\
&\quad - \int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\zeta\right)}(\mathcal{Q}\circ\mathcal{U})(\zeta)\mathrm{d}\zeta\\
&= \left[\mathcal{P}(\eta_1)\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right) - \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}\mathcal{Q}\circ\mathcal{U})\left(\frac{1}{\eta_1}\right)\right],
\end{aligned}
$$

taking into account Lemma 3.1, we have

$$I_1 = \frac{\mathcal{P}(\eta_1)}{2}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(Q \circ \mathcal{U})\left(\frac{1}{\eta_1}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(Q \circ \mathcal{U})\left(\frac{1}{\eta_2}\right)\right] - \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}Q \circ \mathcal{U})\left(\frac{1}{\eta_1}\right).$$

Analogously,

$$I_2 = \frac{1}{\gamma}\left[\int_{\zeta}^{\frac{1}{\eta_2}}\left(\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right)(\mathcal{P} \circ \mathcal{U})'(\zeta)\mathrm{d}\zeta\right]$$
$$= \frac{\mathcal{P}(\eta_2)}{2}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(Q \circ \mathcal{U})\left(\frac{1}{\eta_1}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(Q \circ \mathcal{U})\left(\frac{1}{\eta_2}\right)\right] - \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{P}Q \circ \mathcal{U})\left(\frac{1}{\eta_2}\right).$$

Substituting I_1 and I_2 in (4.19), then we get the desired identity (4.12). □

For the sake of simplicity, we symbolize

$$\Lambda^{\delta}_{\frac{1}{\eta_1},\frac{1}{\eta_2}}(\mathcal{P}, Q, \mathcal{U};\vartheta) = \frac{\mathcal{P}(\eta_1) + \mathcal{P}(\eta_2)}{2}\left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(Q \circ \mathcal{U})\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(Q \circ \mathcal{U})\left(\frac{1}{\eta_1}\right)\right] - \left[\mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_1}\right)^-}(\mathcal{P}Q \circ \mathcal{U})\left(\frac{1}{\eta_2}\right) + \mathcal{J}^{\gamma}_{\left(\frac{1}{\eta_2}\right)^+}(\mathcal{P}Q \circ \mathcal{U})\left(\frac{1}{\eta_1}\right)\right].$$

Theorem 4.3. *For $\gamma > 0$ and let there is a differentiable function $\mathcal{P} : \mathcal{I} \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ on $\mathcal{I}^\circ$ (the interior of $\mathcal{I}$) such that $\mathcal{P}' \in L_1([\eta_1, \eta_2])$ with $\eta_2 > \eta_1$ and $\eta_1, \eta_2 \in \mathcal{I}$. Also, there is a continuous and harmonically function $Q : \mathcal{I} \to \mathbb{R}$ symmetric with respect to $\frac{2\eta_1\eta_2}{\eta_1+\eta_2}$. If $|\mathcal{P}'|$ is harmonically convex on $\mathcal{I}$, then*

$$\left|\Lambda^{\delta}_{\frac{1}{\eta_1},\frac{1}{\eta_2}}(\mathcal{P}, Q, \mathcal{U};\vartheta)\right| \le \frac{\|Q\|_\infty\eta_1\eta_2(\eta_2 - \eta_1)}{\gamma}[\mathcal{A}_1|\mathcal{P}'(\eta_1)| + \mathcal{A}_2|\mathcal{P}'(\eta_2)|], \tag{4.14}$$

where $\|Q\|_\infty = \sup_{\eta\in[\eta_1,\eta_2]} |Q(\zeta)|$,

$$\mathcal{A}_1 := \left(\int_0^{\frac{1}{2}} \frac{e^{-\vartheta(1-\kappa)} - e^{-\vartheta\kappa}}{(\kappa\eta_2 + (1-\kappa)\eta_1)^2}\kappa\mathrm{d}\kappa + \int_{\frac{1}{2}}^1 \frac{e^{-\vartheta\kappa} - e^{-\vartheta(1-\kappa)}}{(\kappa\eta_2 + (1-\kappa)\eta_1)^2}\kappa\mathrm{d}\kappa\right)$$

and

$$\mathcal{A}_2 := \left(\int_0^{\frac{1}{2}} \frac{e^{-\vartheta(1-\kappa)} - e^{-\vartheta\kappa}}{(\kappa\eta_2 + (1-\kappa)\eta_1)^2}(1-\kappa)\mathrm{d}\kappa + \int_{\frac{1}{2}}^1 \frac{e^{-\vartheta\kappa} - e^{-\vartheta(1-\kappa)}}{(\kappa\eta_2 + (1-\kappa)\eta_1)^2}(1-\kappa)\mathrm{d}\kappa\right).$$

Proof. Using Lemma 4.2, we have

$$\left|\Lambda^{\delta}_{\frac{1}{\eta_1},\frac{1}{\eta_2}}(\mathcal{P}, Q, \mathcal{U};\vartheta)\right| \le \frac{1}{\gamma}\int_{\frac{1}{\eta_2}}^{\frac{1}{\eta_1}}\left|\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta. - \int_{\zeta}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right|(\mathcal{P} \circ \mathcal{U})'(\zeta)\mathrm{d}\zeta. \tag{4.15}$$

Utilizing harmonically symmetric property of Q with respect to $\frac{2\eta_1\eta_2}{\eta_1+\eta_2}$ and establishing the relation for the right hand side

$$\left|\int_{\frac{1}{\eta_2}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta - \int_{\zeta}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right|$$
$$= \left|\int_{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}^{\frac{1}{\eta_1}} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta - \int_{\frac{1}{\eta_1}}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right|$$
$$= \left|\int_{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right| \tag{4.16}$$
$$\le \begin{cases} \left|\int_{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right|, & \zeta \in \left[\frac{1}{\eta_2}, \frac{2\eta_1\eta_2}{\eta_1+\eta_2}\right], \\ \left|\int_{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}^{\zeta} e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(Q \circ \mathcal{U})(\delta)\mathrm{d}\delta\right|, & \zeta \in \left[\frac{2\eta_1\eta_2}{\eta_1+\eta_2}, \frac{1}{\eta_1}\right]. \end{cases}$$

Using (4.19) and (4.16), we have

$$
\begin{aligned}
&\left|\Lambda^{\delta}_{\frac{1}{\eta_1},\frac{1}{\eta_2}}(\mathcal{P},\mathcal{Q},\mathcal{U};\vartheta)\right| \\
&\leq \frac{1}{\gamma}\int_{\frac{1}{\eta_2}}^{\frac{\eta_1+\eta_2}{2\eta_1\eta_2}}\left(\int_{\zeta}^{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}\left|e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\right|\mathrm{d}\delta\right) \\
&\quad\times|(\mathcal{P}\circ\mathcal{U})'(\zeta)|\mathrm{d}\zeta \\
&\quad+\frac{1}{\gamma}\int_{\frac{\eta_1+\eta_2}{2\eta_1\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}^{\zeta}\left|e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}(\mathcal{Q}\circ\mathcal{U})(\delta)\right|\mathrm{d}\delta\right) \\
&\quad\times|(\mathcal{P}\circ\mathcal{U})'(\zeta)|\mathrm{d}\zeta \\
&\leq \frac{\|\mathcal{Q}\|_{\infty}}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{\eta_1+\eta_2}{2\eta_1\eta_2}}\left(\int_{\zeta}^{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}\mathrm{d}\delta\right)\frac{1}{\zeta^2}\left|\mathcal{P}'\left(\frac{1}{\zeta}\right)\right|\mathrm{d}\zeta\right. \\
&\quad\left.+\int_{\frac{\eta_1+\eta_2}{2\eta_1\eta_2}}^{\frac{1}{\eta_1}}\left(\int_{\frac{1}{\eta_1}+\frac{1}{\eta_2}-\zeta}^{\zeta}e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}\mathrm{d}\delta\right)\frac{1}{\zeta^2}\left|\mathcal{P}'\left(\frac{1}{\zeta}\right)\right|\mathrm{d}\zeta\right] \\
&\quad\times\frac{\|\mathcal{Q}\|_{\infty}}{\gamma}\left[\int_{\frac{1}{\eta_2}}^{\frac{\eta_1+\eta_2}{2\eta_1\eta_2}}e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}\right. \\
&\quad-e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}\frac{1}{\zeta^2}\left|\mathcal{P}'\left(\frac{1}{\zeta}\right)\right|\mathrm{d}\zeta \\
&\quad\left.+\int_{\frac{\eta_1+\eta_2}{2\eta_1\eta_2}}^{\frac{1}{\eta_1}}\left[e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\delta-\frac{1}{\eta_2}\right)}-e^{\frac{-\vartheta\eta_1\eta_2}{\eta_2-\eta_1}\left(\frac{1}{\eta_1}-\delta\right)}\right]\frac{1}{\zeta^2}\left|\mathcal{P}'\left(\frac{1}{\zeta}\right)\right|\mathrm{d}\zeta\right].
\end{aligned}
\tag{4.17}
$$

Substituting $\zeta=\frac{\kappa\eta_2+(1-\kappa)\eta_1}{\eta_1\eta_2}$ in (4.17), we have

$$
\begin{aligned}
&\left|\Lambda^{\delta}_{\frac{1}{\eta_1},\frac{1}{\eta_2}}(\mathcal{P},\mathcal{Q},\mathcal{U};\vartheta)\right| \\
&\leq\frac{\|\mathcal{Q}\|_{\infty}\eta_1\eta_2(\eta_2-\eta_1)}{\gamma} \\
&\quad\times\left[\int_0^{\frac{1}{2}}\left[\frac{e^{-\vartheta(1-\kappa)}-e^{-\vartheta\kappa}}{(\kappa\eta_2+(1-\kappa)\eta_1)^2}\right]\left|\mathcal{P}'\left(\frac{\eta_1\eta_2}{\kappa\eta_2+(1-\kappa)\eta_1}\right)\right|\mathrm{d}\kappa\right] \\
&\quad+\left[\int_{\frac{1}{2}}^{1}\left[\frac{e^{-\vartheta\kappa}-e^{-\vartheta(1-\kappa)}}{(\kappa\eta_2+(1-\kappa)\eta_1)^2}\right]\left|\mathcal{P}'\left(\frac{\eta_1\eta_2}{\kappa\eta_2+(1-\kappa)\eta_1}\right)\right|\mathrm{d}\kappa.\right]
\end{aligned}
\tag{4.18}
$$

Using harmonically convexity of $|\mathcal{P}'|$ on $\mathcal{I}$, we have

$$
\begin{aligned}
&\left|\Lambda^{\delta}_{\frac{1}{\eta_1},\frac{1}{\eta_2}}(\mathcal{P},\mathcal{Q},\mathcal{U};\vartheta)\right| \\
&\leq\frac{\|\mathcal{Q}\|_{\infty}\eta_1\eta_2(\eta_2-\eta_1)}{\gamma} \\
&\quad\times\left[\left(\int_0^{\frac{1}{2}}\frac{e^{-\vartheta(1-\kappa)}-e^{-\vartheta\kappa}}{(\kappa\eta_2+(1-\kappa)\eta_1)^2}\kappa\mathrm{d}\kappa\right.\right. \\
&\quad\left.+\int_{\frac{1}{2}}^{1}\frac{e^{-\vartheta\kappa}-e^{-\vartheta(1-\kappa)}}{(\kappa\eta_2+(1-\kappa)\eta_1)^2}\kappa\mathrm{d}\kappa\right)|\mathcal{P}'(\eta_1)| \\
&\quad+\left(\int_0^{\frac{1}{2}}\frac{e^{-\vartheta(1-\kappa)}-e^{-\vartheta\kappa}}{(\kappa b+(1-\kappa)a)^2}(1-\kappa)\mathrm{d}\kappa\right. \\
&\quad\left.\left.+\int_{\frac{1}{2}}^{1}\frac{e^{-\vartheta\kappa}-e^{-\vartheta(1-\kappa)}}{(\kappa\eta_2+(1-\kappa)\eta_1)^2}(1-\kappa)\mathrm{d}\kappa\right)|\mathcal{P}'(\eta_2)|\right] \\
&=\frac{\|\mathcal{Q}\|_{\infty}\eta_1\eta_2(\eta_2-\eta_1)}{\gamma}[\mathcal{A}_1|\mathcal{P}'(\eta_1)| \\
&\quad+\mathcal{A}_2|\mathcal{P}'(\eta_2)|].
\end{aligned}
\tag{4.19}
$$

This completes the proof. □

5 Some new generalizations for convex functions via fractional integral having exponential in the kernel

Throughout this article, we assume that $\Psi(\zeta)$ is an increasing, positive and monotone function defined on $[0,\infty)$ such that $\Psi(0)=0$, and $\Psi'(\zeta)$ is continuous on $[0,\infty)$.

Theorem 5.1. *For $\gamma>0$ and let there are two positive functions $\mathcal{P}$ and $\mathcal{U}$ with $\mathcal{P}\leq\mathcal{U}$ defined on $[0,\infty)$. Moreover, there is an increasing function $\mathcal{P}$ and a decreasing function $\frac{\mathcal{P}}{\mathcal{U}}$ defined on $[0,\infty)$, then for a convex function Φ with $\Phi(0)=0$, the fractional integral operator defined in (1.9) satisfies the inequality*

$$
\frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\mathcal{P}(r)]}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\mathcal{U}(r)]}\geq\frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\Phi(\mathcal{P}(r))]}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\Phi(\mathcal{U}(r))]}.
\tag{5.1}
$$

Proof. By the given hypothesis, the function $\frac{\Phi(r)}{r}$ is increasing. As $\mathcal{P}$ is increasing, so is the function $\frac{\Phi(r)}{r}$. Obviously, the function $\frac{\mathcal{P}(r)}{\mathcal{U}(r)}$ is decreasing. Thus, for all $\zeta, \lambda \in [0, \infty)$, we have

$$\left(\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)} - \frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\right)\left(\frac{\mathcal{P}(\lambda)}{\mathcal{U}(\lambda)} - \frac{\mathcal{P}(\zeta)}{\mathcal{U}(\zeta)}\right) \geq 0. \tag{5.2}$$

It follows that

$$\begin{aligned}&\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\frac{\mathcal{P}(\lambda)}{\mathcal{U}(\lambda)} + \frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\frac{\mathcal{P}(\zeta)}{\mathcal{U}(\zeta)} - \frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\frac{\mathcal{P}(\lambda)}{\mathcal{U}(\lambda)}\\ &- \frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\frac{\mathcal{P}(\zeta)}{\mathcal{U}(\zeta)} \geq 0.\end{aligned} \tag{5.3}$$

Multiplying (5.3) by $\mathcal{U}(\zeta)\mathcal{U}(\lambda)$, we have

$$\begin{aligned}&\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{P}(\lambda)\mathcal{U}(\zeta) + \frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\zeta)\mathcal{U}(\lambda)\\ &- \frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)\\ &- \frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{P}(\zeta)\mathcal{U}(\lambda) \geq 0.\end{aligned} \tag{5.4}$$

Multiplying (5.4) by $\frac{1}{\gamma}e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)$, which is positive because $\zeta \in (0, r), r > 0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}&\frac{1}{\gamma}\int_0^r e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)\mathrm{d}\zeta\\ &+ \frac{1}{\gamma}\int_0^r e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\zeta)\mathcal{U}(\lambda)\mathrm{d}\zeta\\ &- \frac{1}{\gamma}\int_0^r e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)\mathrm{d}\zeta\\ &- \frac{1}{\gamma}\int_0^r e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{P}(\zeta)\mathcal{U}(\lambda)\mathrm{d}\zeta \geq 0.\end{aligned}$$

This follows that

$$\begin{aligned}&\mathcal{P}(\lambda)\mathcal{J}_{0^+,r}^{\gamma,\Psi}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right)\\ &+\left(\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{U}(\lambda)\right)\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{P}(r))\\ &-\left(\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\lambda)\right)\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{U}(r))\\ &-\mathcal{U}(\lambda)\mathcal{J}_{0^+,r}^{\gamma,\Psi}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{P}(r)\right) \geq 0.\end{aligned} \tag{5.5}$$

Again, multiplying (5.5) by $\frac{1}{\gamma}e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\Psi(\lambda))}\Psi'(\lambda)$, which is positive because $\lambda \in (0, r), r > 0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}&\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{P}(r))\mathcal{J}_{0^+,r}^{\gamma,\Psi}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right)\\ &+\mathcal{J}_{0^+,r}^{\gamma,\Psi}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right)\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{P}(r))\\ &\geq \mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{U}(r))\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\Phi(\mathcal{P}(r))\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{U}(r)).\end{aligned} \tag{5.6}$$

It follows that

$$\frac{\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{P}(r))}{\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\mathcal{U}(r))} \geq \frac{\mathcal{J}_{0^+,r}^{\gamma,\Psi}(\Phi(\mathcal{P}(r)))}{\mathcal{J}_{0^+,r}^{\gamma,\Psi}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right)}. \tag{5.7}$$

Now, since $\mathcal{P} \leq \mathcal{U}$ on $[0, \infty)$ and $\frac{\Phi(r)}{r}$ is an increasing function, for $\zeta, \lambda \in [0, z)$, we have

$$\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)} \leq \frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}. \tag{5.8}$$

Multiplying both sides of (5.8) by $\frac{1}{\gamma}e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)$, which is positive because $\zeta \in (0, r), r > 0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}&\frac{1}{\gamma}\int_0^\zeta e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{U}(\zeta)\mathrm{d}\zeta\\ &\leq \frac{1}{\gamma}\int_0^\zeta e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}\mathcal{U}(\zeta)\mathrm{d}\zeta.\end{aligned}$$

By utilizing (1.9), it follows that

$$\mathcal{J}_{0^+,r}^{\gamma,\Psi}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right) \leq \mathcal{J}_{0^+,r}^{\gamma,\Psi}((\Phi(\mathcal{U}(r))). \tag{5.9}$$

Hence from (5.7) and (5.9), we get (5.1). ☐

Theorem 5.2. *For $\gamma > 0$ and let there are two positive functions $\mathcal{P}$ and $\mathcal{U}$ with $\mathcal{P} \leq \mathcal{U}$ defined on $[0, \infty)$. Moreover, there is an increasing function $\mathcal{P}$ and a decreasing function $\frac{\mathcal{P}}{\mathcal{U}}$ defined on $[0, \infty)$, then for a convex function Φ with $\Phi(0) = 0$, the fractional integral operator defined in* (1.9) *satisfies the inequality*

$$\frac{\mathcal{J}_{0^+,r}^{\gamma,\Psi}[\mathcal{P}(r)]\mathcal{J}_{0^+,r}^{\beta,\Psi}[\Phi(\mathcal{U}(r))] + \mathcal{J}_{0^+,r}^{\beta,\Psi}[\mathcal{P}(r)]\mathcal{J}_{0^+,r}^{\gamma,\Psi}[\Phi(\mathcal{U}(r))]}{\mathcal{J}_{0^+,r}^{\gamma,\Psi}[\mathcal{U}(r)]\mathcal{J}_{0^+,r}^{\beta,\Psi}[\Phi(\mathcal{P}(r))] + \mathcal{J}_{0^+,r}^{\beta,\Psi}[\mathcal{U}(r)]\mathcal{J}_{0^+,r}^{\gamma,\Psi}[\Phi(\mathcal{P}(r))]} \geq 1. \tag{5.10}$$

Proof. By the given hypothesis, the function $\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}$ is increasing. As $\mathcal{P}$ is increasing, so is the function $\frac{\Phi(r)}{r}$. Obviously, the function $\frac{\mathcal{P}(r)}{\mathcal{U}(r)}$ is decreasing for all

$\zeta, \lambda \in [0, r)$. Multiplying $\frac{1}{\beta}e^{-\frac{1-\beta}{\beta}(\Psi(r)-\Psi(\lambda))}\Psi'(\lambda)$, which is positive because $\lambda \in (0, r)$, $\lambda > 0$ and integrating the resulting identity from 0 to r, we get

$$\begin{aligned}&\mathcal{J}^{\beta,\Psi}_{0^+,r}(\mathcal{P}(r))\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right)\\&\quad+\mathcal{J}^{\beta,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right)\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{P}(r))\\&\geq\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{U}(r))\mathcal{J}^{\beta,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{P}(r)\right)\mathcal{J}^{\beta,\Psi}_{0^+,r}(\mathcal{U}(r)).\end{aligned}\tag{5.11}$$

Now, since $\mathcal{P} \leq \mathcal{U}$ on $[0, \infty)$ and $\frac{\Phi(r)}{r}$ is an increasing function, for $\zeta, \lambda \in [0, r)$, we have

$$\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)} \leq \frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}.\tag{5.12}$$

Multiplying (5.12) by $\frac{1}{\gamma}e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\mathcal{U}(\zeta)$, which is positive because $\zeta \in (0, r)$, $r > 0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}&\frac{1}{\gamma}\int_0^r e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{U}(\zeta)\mathrm{d}\zeta\\&\leq\frac{1}{\gamma}\int_0^r e^{-\frac{1-\gamma}{\gamma}(\Psi(r)-\psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}\mathcal{U}(\zeta)\mathrm{d}\zeta.\end{aligned}\tag{5.13}$$

By utilizing (1.9), it follows that

$$\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\right) \leq \mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{U}(r))).\tag{5.14}$$

Hence from (5.8), (5.11) and (5.14), we get the required result. □

Remark. If ones take $\beta = \gamma$, then Theorem 5.2 will become Theorem 5.1.

Theorem 5.3. *For $\gamma > 0$ and let there are three positive functions $\mathcal{P}, \mathcal{U}$ and $\mathcal{Q}$ defined on $[0, \infty)$ with $\mathcal{P} \leq \mathcal{U}$ defined on $[0,\infty)$. Moreover, there are increasing functions $\mathcal{P}, \mathcal{Q}$ and a decreasing function $\frac{\mathcal{P}}{\mathcal{U}}$ defined on $[0, \infty)$, then for a convex function Φ with $\Phi(0) = 0$, the fractional integral operator defined in (1.9) satisfies the inequality*

$$\frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\mathcal{P}(r)]}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\mathcal{U}(r)]} \geq \frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\Phi(\mathcal{P}(r))\mathcal{Q}(r)]}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\Phi(\mathcal{U}(r))\mathcal{Q}(r)]}.\tag{5.15}$$

Proof. Since $\mathcal{P} \leq \mathcal{U}$ on $[0, \infty)$ and $\frac{\Phi(r)}{r}$ is increasing, for $\zeta, \lambda \in [0, r)$, we have

$$\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)} \leq \frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}.\tag{5.16}$$

Multiplying (5.16) by $\frac{1}{\gamma}e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\mathcal{U}(\zeta)\mathcal{Q}(\zeta)$, which is positive because $\zeta \in (0, r)$, $r > 0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}&\frac{1}{\gamma}\int_0^r e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{U}(\zeta)\mathcal{Q}(\zeta)\mathrm{d}\zeta\\&\leq\frac{1}{\gamma}\int_0^r e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}\mathcal{U}(\zeta)\mathcal{Q}(\zeta)\mathrm{d}\zeta.\end{aligned}\tag{5.17}$$

By utilizing (1.9), it follows that

$$\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right) \leq \mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{U}(r))\mathcal{Q}(r)).\tag{5.18}$$

Also, since the function Φ is convex and such that $\Phi(0) = 0$, $\frac{\Phi(\zeta)}{\zeta}$ is increasing. Since $\mathcal{P}$ is increasing, so is $\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}$. Clearly, the function $\frac{\mathcal{P}(\zeta)}{\mathcal{U}(\zeta)}$ is decreasing for all $\zeta, \lambda \in [0, r)$, $r > 0$. Thus,

$$\begin{aligned}&\left(\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{Q}(\zeta)-\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{Q}(\lambda)\right)(\mathcal{P}(\lambda)\mathcal{U}(\zeta)\\&\quad-\mathcal{P}(\zeta)\mathcal{U}(\lambda)) \geq 0.\end{aligned}\tag{5.19}$$

It follows that

$$\begin{aligned}&\frac{\Phi(\mathcal{P}(\zeta))\mathcal{Q}(\zeta)}{\mathcal{P}(\zeta)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)-\frac{\Phi(\mathcal{P}(\lambda))\mathcal{Q}(\lambda)}{\mathcal{P}(\lambda)}\mathcal{P}(\zeta)\mathcal{U}(\lambda)\\&-\frac{\Phi(\mathcal{P}(\lambda))\mathcal{Q}(\lambda)}{\mathcal{P}(\lambda)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)\\&-\frac{\Phi(\mathcal{P}(\zeta))\mathcal{Q}(\zeta)}{\mathcal{P}(\zeta)}\mathcal{P}(\lambda)\mathcal{U}(\zeta) \geq 0.\end{aligned}\tag{5.20}$$

Multiplying (5.20) by $\frac{1}{\gamma}e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)$, which is positive because $\zeta \in (0, r)$, $r > 0$ and integrating the resulting identity from 0 to ζ, we have

$$\begin{aligned}&\frac{1}{\gamma}\int_0^r e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)\mathcal{Q}(\zeta)\mathrm{d}\zeta\\&+\frac{1}{\gamma}\int_0^r e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\zeta)\mathcal{U}(\lambda)\mathcal{Q}(\lambda)\mathrm{d}\zeta\\&-\frac{1}{\gamma}\int_0^r e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{P}(\lambda)\mathcal{U}(\lambda)\mathcal{Q}(\zeta)\mathrm{d}\zeta.\\&-\frac{1}{\gamma}\int_0^r e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\mathcal{P}(\lambda)\mathcal{U}(\zeta)\mathcal{Q}(\lambda)\mathrm{d}\zeta \geq 0\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathcal{P}(\lambda)\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\\
&+\left(\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{U}(\lambda)\mathcal{Q}(\lambda)\right)\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{P}(r))\\
&-\left(\frac{\Phi(\mathcal{P}(\lambda))}{\mathcal{P}(\lambda)}\mathcal{U}(\lambda)\mathcal{Q}(\lambda)\right)\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{U}(r))\\
&-\mathcal{U}(\lambda)\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\geq 0.
\end{aligned}\tag{5.21}$$

Again, multiplying (5.21) by $\frac{1}{\gamma}e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\lambda))}\Psi'(\lambda)$, which is positive because $\lambda\in(0,z),\lambda>0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}
&\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{P}(r))\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\\
&+\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{P}(r))\\
&\geq\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{U}(r))\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{P}(r))\mathcal{Q}(r))\\
&+\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{P}(r))\mathcal{Q}(r))\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{U}(r)).
\end{aligned}\tag{5.22}$$

It follows that

$$\frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{P}(r))}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{U}(r))}\geq\frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{P}(r)))\mathcal{Q}(r)}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)}.\tag{5.23}$$

Hence from (5.22) and (5.23), we get the required result. □

Theorem 5.4. *For $\gamma>0$ and let there are three positive functions $\mathcal{P},\mathcal{U}$ and $\mathcal{Q}$ defined on $[0,\infty)$ with $\mathcal{P}\leq\mathcal{U}$ defined on $[0,\infty)$. Moreover, there are increasing functions $\mathcal{P},\mathcal{Q}$ and a decreasing function $\frac{\mathcal{P}}{\mathcal{U}}$ defined on $[0,\infty)$, then for a convex function Φ with $\Phi(0)=0$, the fractional integral operator defined in (1.9) satisfies the inequality*

$$\begin{aligned}
&\frac{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\mathcal{P}(r)]\mathcal{J}^{\beta}_{0^+,r}[\Phi(\mathcal{P}(r))\mathcal{Q}(r)]+\mathcal{J}^{\beta}_{0^+,r}[\mathcal{P}(r)]\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\Phi(\mathcal{P}(r))\mathcal{Q}(r)]}{\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\mathcal{U}(r)]\mathcal{J}^{\beta}_{0^+,r}[\Phi(\mathcal{P}(r))\mathcal{Q}(r)]+\mathcal{J}^{\beta}_{0^+,r}[\mathcal{U}(r)]\mathcal{J}^{\gamma,\Psi}_{0^+,r}[\Phi(\mathcal{P}(r))\mathcal{Q}(r)]}\\
&\geq 1.
\end{aligned}\tag{5.24}$$

Proof. Multiplying both sides of (5.24) by $\frac{1}{\beta}e^{-\frac{\beta-1}{\beta}(\Psi(r)-\Psi(\lambda))}\Psi'(\lambda)$, which is positive because $\lambda\in(0,r),\lambda>0$ and integrating the inequality from 0 to r, yields

$$\begin{aligned}
&\mathcal{J}^{\beta,\Psi}_{0^+,r}(\mathcal{P}(r))\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\\
&+\mathcal{J}^{\beta,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{P}(r))\\
&\geq\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\mathcal{U}(r))\mathcal{J}^{\beta,\Psi}_{0^+,r}(\Phi(\mathcal{P}(r))\mathcal{Q}(r))\\
&+\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{P}(r))\mathcal{Q}(r))\mathcal{J}^{\beta,\Psi}_{0^+,r}(\mathcal{U}(r)).
\end{aligned}\tag{5.25}$$

Since $\mathcal{P}\leq\mathcal{U}$ on $[0,\infty)$ and $\frac{\Phi(r)}{r}$ is increasing, for $\zeta,\lambda\in[0,r)$, we have

$$\frac{\Phi(\mathcal{P}(\zeta))}{\mathcal{P}(\zeta)}\leq\frac{\Phi(\mathcal{U}(\zeta))}{\mathcal{U}(\zeta)}.\tag{5.26}$$

Multiplying both sides of (5.26) by $\frac{1}{\gamma}e^{-\frac{\gamma-1}{\gamma}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\mathcal{U}(\zeta)\mathcal{Q}(\zeta)$, $\zeta\in(0,r),\zeta>0$ and integrating the resulting identity from 0 to ζ, we have

$$\mathcal{J}^{\gamma,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\leq\mathcal{J}^{\gamma,\Psi}_{0^+,r}(\Phi(\mathcal{U}(r))\mathcal{Q}(r)).\tag{5.27}$$

Similarly, multiplying both sides of (5.26) by $\frac{1}{\beta}e^{-\frac{\beta-1}{\beta}(\Psi(r)-\Psi(\zeta))}\Psi'(\zeta)\mathcal{U}(\zeta)\mathcal{Q}(\zeta)$, $\zeta\in(0,r),\zeta>0$ and integrating the inequality from 0 to r, yields

$$\mathcal{J}^{\beta,\Psi}_{0^+,r}\left(\frac{\Phi(\mathcal{P}(r))}{\mathcal{P}(r)}\mathcal{U}(r)\mathcal{Q}(r)\right)\leq\mathcal{J}^{\beta,\Psi}_{0^+,r}(\Phi(\mathcal{U}(r))\mathcal{Q}(r)).\tag{5.28}$$

Hence, we get the required result. □

Remark. If ones take $\beta=\gamma$, then Theorem 5.4 will become Theorem 5.3.

6 Conclusions

In this work, we have fruitfully applied the fractional integral operators with an exponential kernel to derive the Hermite–Hadamard, Hermite–Hadamard–Fejér and Pachpatte-type integral inequalities involving the fractional integral operator essentially using the functions having the harmonically convexity property. The key procedure of the new adaption in extended form with an exponential kernel to the more general fractional integral operator is helpful in deriving several generalizations for the convexity theory. Finally, the present investigation illuminates the effectiveness of the considered operator. We presented two different schemes and show that the results of the proposed method are in excellent agreement with the results of the Riemann–Liouville fractional integral operator which approves the validity of the derived outcomes. From the obtained results, it can be noted that both the featured techniques are reliable and efficient to handle the different nonlinear problems appearing in science and engineering. We conclude that the results derived in this article are general in character and give some contributions to circuit theory and complex waveforms. Such a potential connection needs further investigation.

Acknowledgements: The authors would like to express their sincere thanks to the support of the National Natural Science Foundation of China.

Availability of supporting data: Not applicable.

Competing interests: The authors declare that they have no competing interests.

Funding: This work was supported by the National Natural Science Foundation of China (Grant No. 61673169).

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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