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Research article

Existence and uniqueness of miscible flow equation through porous media with a non singular fractional derivative

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Abstract: In this paper, we discuss the phenomenon of miscible flow with longitudinal dispersion in porous media. This process simultaneously occur because of molecular diffusion and convection. Here, we analyze the governing differential equation involving Caputo–Fabrizio fractional derivative operator having non singular kernel. Fixed point theorem has been used to prove the uniqueness and existence of the solution of governing differential equation. We apply Laplace transform and use technique of iterative method to obtain the solution. Few applications of the main result are discussed by taking different initial conditions to observe the effect on derivatives of different fractional order on the concentration of miscible fluids.

Keywords: Caputo-Fabrizio fractional derivative operator; miscible flow; fixed point theorem; Laplace transform; iterative method
Mathematics Subject Classification: Primary: 34A08; Secondary: 26A33

1. Introduction

Two fluids are defined as miscible if the molecules of the one fluid are free to mix with the molecules of the other fluid. There is no interface between two miscible fluids. A common example of two miscible fluids is water and ethanol. In any proportions it is possible to mix the water and ethanol together to form a single homogeneous phase. When two gases meet, they are always miscible; for example oxygen and nitrogen readily mix in air. Two fluids are defined as immiscible if the two fluids scarcely mix at all at the molecular level, and not at all at the macroscale. The two phases remain distinct and there is a well-defined interface between the two fluids. Common examples of two immiscible fluids are water and most vegetable oils.

The phenomenon of longitudinal dispersion is the process by which miscible fluids in the laminar flow mix in the direction of the flow. The hydrodynamic dispersion is the macroscopic outcome of the actual movements of individual tracer particles throughout the pores and various physical and chemical phenomenon that take place within the pores. This phenomenon simultaneously occurs due to molecular diffusion and convection and has great importance in sea water intrusion into lake and groundwater recharge by polluted water. Immiscible flooding, that is, the oil is displaced by one of the liquid petroleum gas products, Ethane, Propene or Butane. If the reservoir conditions are such that the liquid petroleum gas is in the liquid phase then it is miscible with the oil and theoretically all residual oil can be recovered. This problem has been discussed by several authors from different viewpoints in [10, 14, 16, 17, 24, 25]. Many Important problems in water resources engineering involve the mass-transport of a miscible fluid in a flow. Agarwal et al. [2–5] and Yadav et al. [26] have discussed analytical and numerical solutions of various fractionalized groundwater flow problems. Recently, in [19, 20] authors investigate a new fractional mathematical model involving a non–singular derivative operator to discuss the clinical implications of diabetes and tuberculosis coexistence and dengue fever outbreak based on a system of fractional differential equations.

A fluid is considered to be a continuous material and hence in addition to the velocity of a fluid element, the molecules in this element have random motion. As a result of the random motion, molecules of a certain material in high concentration at one point will spread with time. So the velocity considered here is time dependent. The net molecular motion from a point of higher concentration to one of lower concentration is called molecular diffusion. Fluid flows in nature are usually turbulent, but we have considered the porous medium through which the fluid flows, to be homogeneous and for this reason, in the direction of flow, we assume laminar flow in which miscible fluids mix.

Two fluids, which are contiguous, takes different paths at one moment in any medium. In [15, 23], longitudinal dispersion term is described by the coefficient D for the isotropic media in different connected tubes. The equation of continuity for the mixture is given as [10]

$$\frac{\partial \rho}{\partial \tau} + \nabla.(\rho \bar{v}) = 0, \tag{1.1}$$

where ρ is the density for mixture and $\bar{v}$ is the pore velocity. Without addition or subtraction of dispersive fluids the diffusion equation in homogeneous porous media is given as

$$\frac{\partial c}{\partial \tau} + \nabla.(c\bar{v}) = \nabla.\left[\rho \bar{D} \nabla \left(\frac{c}{\rho}\right)\right], \tag{1.2}$$

where c is the concentration of the fluid A in the host fluid B, $\bar{D}$ is the tensor co-efficient of dispersion with nine components $\bar{D}_{ij}$. For homogeneous porous media at constant temperature, ρ may be considered as a constant and hence Eq (1.2) becomes

$$\frac{\partial c}{\partial \tau} + \nabla.(c\bar{v}) = \nabla.\left[\bar{D} \nabla c\right], \tag{1.3}$$

Along x-axis, $D_{11} = D_L$ and other D_{ij} are zero [22] hence

$$\frac{\partial c(\xi,\tau)}{\partial \tau} = -u\frac{\partial c(\xi,\tau)}{\partial \xi} + D_L \frac{\partial^2 c(\xi,\tau)}{\partial \xi^2}, \tag{1.4}$$

where u is the velocity component along x-axis.

We use Caputo-Fabrizio fractional derivative for the time variable and fractionalize Eq (1.4) as follows:

$$ {}_{0}^{CF}D_{\tau}^{\varsigma}c(\xi,\tau) = -u\frac{\partial c(\xi,\tau)}{\partial \xi} + D_L\frac{\partial^2 c(\xi,\tau)}{\partial \xi^2}, \tag{1.5} $$

subject to the initial condition $c(\xi, 0) = c_0$, $0 < \varsigma < 1$, $\tau > 0$ and $\xi \in (0, L)$.

Caputo fractional derivative given by Caputo and Fabrizio [13] as

$$ {}_{a}^{CF}D_{t}^{\varsigma}f(t) = \frac{B(\varsigma)}{1-\varsigma}\int_a^t f'(\tau)\exp\left(-\varsigma\frac{t-\tau}{1-\varsigma}\right)d\tau, \ a < t < b, \tag{1.6} $$

where $f \in H'(a,b)$ is the Sobolev space, $\varsigma \in (0,1]$ and $B(\varsigma)$ is a normalized function such that $B(0) = B(1) = 1$.

Remark 1.1. *Usually normalized function means that some important property of the mathematical function takes value unity. The general idea, though, is to pick a particular element of a class of functions and reduce it to the "simplest" element in the class, so that you only need to deal with that simpler representative. Normalized function has many applications in science, mathematics and statistics, computer science and technology etc. At few places, the value of $B(\varsigma)$ is taken constant i.e. 1 (see, e.g [13, 18]) and at some places it is defined as $B(\varsigma) = 1 + \varsigma + \frac{\varsigma}{\Gamma(\varsigma)}$, (see, e.g. [9]). In this paper, $B(\varsigma)$ is assumed constant and equal to 1.*

The Caputo-Fabrizio integral of a function $f(t)$ of fractional order ς is given by Nieto and Losada [21] as

$$ {}_{a}^{CF}I_{t}^{\varsigma}f(t) = \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}f(t) + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_a^t f(s)ds, \ t \geq 0, \tag{1.7} $$

where $M(\varsigma) = \frac{2}{2-\varsigma}B(\varsigma)$.

Remark 1.2. *Equation (1.7) can be written as average of the function $f(t)$ and its integral as*

$$ \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} = 1, \tag{1.8} $$

which implies that $M(\varsigma) = \frac{2}{2-\varsigma}$. Nieto and Losada [21] gave Caputo-Fabrizio definition by using Eqs (1.7) and (1.8) as

$$ {}_{a}^{CF}D_{t}^{\varsigma}f(t) = \frac{1}{1-\varsigma}\int_a^t f'(\tau)\exp\left(-\varsigma\frac{t-\tau}{1-\alpha}\right)d\tau. \tag{1.9} $$

Caputo-Fabrizio fractional derivative does not have singularity at $t = \tau$ in its kernel, therefore memory is described better by fractional order with non singular kernels compared to fractional order with singular kernels. Several studies based upon Caputo-Fabrizio fractional derivative operator were made [1, 6–8]. Laplace transform of the Caputo-Fabrizio fractional derivative is given by Caputo and Fabrizio [13] as

$$ \mathcal{L}\left[{}_{0}^{CF}D_{t}^{\varsigma}f(t); s\right] = B(\varsigma)\frac{sF(s) - f(0)}{s + \varsigma(1-s)}, \ 0 < \varsigma \leq 1, \tag{1.10} $$

If exists, The organization of this paper is as follows:
In section 2 we present existence and uniqueness of the solution by using fixed point theorem. In section 3 we derive approximation solution by employing the iterative method. Section 4 deals with applications of obtained iterative formula by taking different initial conditions. In section 5 we observe the effect of the fractional order derivative to the approximate solution and last section 6 contains the conclusion.

2. Existence and uniqueness of the solution

We use fixed point theorem to show the existence of the solution. We transform the Eq (1.5) by applying Caputo-Fabrizio fractional integral (1.7) to obtain

$$c(\xi,\tau)-c(\xi,0)={}_0^{CF}I_\tau^{\varsigma}\left(-u\frac{\partial c(\xi,\tau)}{\partial\xi}+D_L\frac{\partial^2 c(\xi,\tau)}{\partial\xi^2}\right). \tag{2.1}$$

Now applying the definition given by Nieto and Losada [21] on equation (2.1) becomes

$$\begin{aligned}c(\xi,\tau)-c(\xi,0)=&\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\left(-u\frac{\partial c(\xi,\tau)}{\partial\xi}+D_L\frac{\partial^2 c(\xi,\tau)}{\partial\xi^2}\right)\\&+\frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^\tau\left(-u\frac{\partial c(\xi,\tau)}{\partial\xi}+D_L\frac{\partial^2 c(\xi,\tau)}{\partial\xi^2}\right).\end{aligned} \tag{2.2}$$

Theorem 2.1. *$K(\xi,\tau,c)$ satisfy the Lipschitz condition and is contraction if the following inequality holds*

$$0<\left(u\theta_1+D_L\theta_2^2\right)\leq 1, \tag{2.3}$$

where

$$K(\xi,\tau,c)=-u\frac{\partial c(\xi,\tau)}{\partial\xi}+D_L\frac{\partial^2 c(\xi,\tau)}{\partial\xi^2}. \tag{2.4}$$

Proof. Let c and c_1 are two functions then from Eq (2.4) we have

$$\|K(\xi,\tau,c)-K(\xi,\tau,c_1)\|=\left\|-u\frac{\partial\{c(\xi,\tau)-c_1(\xi,\tau)\}}{\partial\xi}+D_L\frac{\partial^2\{c(\xi,\tau)-c_1(\xi,\tau)\})}{\partial\xi^2}\right\|. \tag{2.5}$$

Using the triangular inequality on Eq (2.5), we obtain

$$\|K(\xi,\tau,c)-K(\xi,\tau,c_1)\|\leq u\left\|-\frac{\partial\{c(\xi,\tau)-c_1(\xi,\tau)\}}{\partial\xi}\right\|+D_L\left\|\frac{\partial^2\{c(\xi,\tau)-c_1(\xi,\tau)\})}{\partial\xi^2}\right\|. \tag{2.6}$$

Since the derivatives satisfy the Lipschitz condition, we can find positive parameters θ_1 and θ_2, from Eq (2.6) as

$$u\left\|-\frac{\partial\{c(\xi,\tau)-c_1(\xi,\tau)\}}{\partial\xi}\right\|\leq u\theta_1\|c(\xi,\tau)-c_1(\xi,\tau)\| \tag{2.7}$$

$$D_L \left\| \frac{\partial^2 \{c(\xi,\tau) - c_1(\xi,\tau)\})}{\partial \xi^2} \right\| \leq D_L \theta_2^2 \left\| c(\xi,\tau) - c_1(\xi,\tau) \right\|. \tag{2.8}$$

Using the inequalities (2.7) and (2.8) in Eq (2.6), we obtain

$$\|K(\xi,\tau,c) - K(\xi,\tau,c_1)\| \leq \left(u\theta_1 + D_L\theta_2^2\right) \|c(\xi,\tau) - c_1(\xi,\tau)\|. \tag{2.9}$$

Letting $\left(u\theta_1 + D_L\theta_2^2\right) = R$, we obtain

$$\|K(\xi,\tau,c) - K(\xi,\tau,c_1)\| \leq R\|c(\xi,\tau) - c_1(\xi,\tau)\|. \tag{2.10}$$

Therefore $K(\xi,\tau,c)$ satisfy the Lipschitz condition and if $R \leq 1$, then it is contraction. □

By considering this kernel $K(\xi,\tau,c)$, equation (2.2) reduces as

$$c(\xi,\tau) - c(\xi,0) = \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} K(\xi,\tau,c) + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^\tau K(\xi,y,c)dy. \tag{2.11}$$

Now we consider following recursive formula

$$c_n(\xi,\tau) = \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} K(\xi,\tau,c_{n-1}) + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^\tau K(\xi,y,c_{n-1})dy, \tag{2.12}$$

with initial component $c_0(\xi,\tau) = c(\xi,0)$ and the difference between two consecutive terms of the Eq (2.12) is denoted by $C_n(\xi,\tau)$, is given as

$$\begin{aligned} C_n(\xi,\tau) =& c_n(\xi,\tau) - c_{n-1}(\xi,\tau) \\ =& \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} (K(\xi,\tau,c_{n-1}) - K(\xi,\tau,c_{n-2})) \\ &+ \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^\tau (K(\xi,y,c_{n-1}) - K(\xi,y,c_{n-2}))dy, \end{aligned} \tag{2.13}$$

and also $c_n(\xi,\tau) = \sum_{i=0}^n C_i(\xi,\tau)$. Taking norm on both sides of Eq (2.13), we obtain

$$\begin{aligned} \|C_n(\xi,\tau)\| =& \|c_n(\xi,\tau) - c_{n-1}(\xi,\tau)\| \\ =& \left\| \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} (K(\xi,\tau,c_{n-1}) - K(\xi,\tau,c_{n-2})) \right. \\ &\left. + \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^\tau (K(\xi,y,c_{n-1}) - K(\xi,y,c_{n-2}))dy \right\|. \end{aligned} \tag{2.14}$$

Using the triangular inequality, Eq (2.14) becomes

$$\begin{aligned} \|c_n(\xi,\tau) - c_{n-1}(\xi,\tau)\| \leq& \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)} \|(K(\xi,\tau,c_{n-1}) - K(\xi,\tau,c_{n-2}))\| \\ &+ \frac{2\varsigma}{(2-\varsigma)M(\varsigma)} \int_0^\tau \|(K(\xi,y,c_{n-1}) - K(\xi,y,c_{n-2}))dy\|. \end{aligned} \tag{2.15}$$

Since the kernel $K(\xi,\tau,c)$ satisfies the Lipschitz condition, we obtain

$$\begin{aligned}\|c_n(\xi,\tau)-c_{n-1}(\xi,\tau)\| \leq{}& \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}R\|c_{n-1}(\xi,\tau)-c_{n-2}(\xi,\tau)\| \\ &+\frac{2\varsigma R}{(2-\varsigma)M(\varsigma)}\int_0^\tau \|c_{n-1}(\xi,y)-c_{n-2}(\xi,y)\|dy. \end{aligned} \tag{2.16}$$

Solving further, we obtain

$$\|C_n(\xi,\tau)\| \leq \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}R\|c_{n-1}(\xi,\tau)\| + \frac{2\varsigma R}{(2-\varsigma)M(\varsigma)}\int_0^\tau \|c_{n-1}(\xi,y)\|dy. \tag{2.17}$$

We shall then state the following theorem:

Theorem 2.2. *The fractional miscible flow Eq (1.5) through porous media has exact solution.*

Proof. We have that $c(\xi,\tau)$ is bounded and kernel (1.5) satisfy the Lipschitz condition, then we obtain the following relation using the recursive relation

$$\|C_n(\xi,\tau)\| \leq \|c(\xi,0)\|\left(\frac{2(1-\varsigma)R}{(2-\varsigma)M(\varsigma)}+\frac{2\varsigma R\tau}{(2-\varsigma)M(\varsigma)}\right)^n. \tag{2.18}$$

Now, we prove that (2.18) is a solution, so we assume that

$$c(\xi,\tau)-c(\xi,0)=c_n(\xi,\tau)-A_n(\xi,\tau). \tag{2.19}$$

Therefore, we have

$$\begin{aligned}\|A_n(\xi,\tau)\| ={}& \left\|\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}(K(\xi,\tau,c_n)-K(\xi,\tau,c_{n-1}))\right. \\ &\left.+\frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^\tau (K(\xi,y,c_n)-K(\xi,y,c_{n-1}))dy\right\|, \\ \leq{}& \frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\|(K(\xi,\tau,c_n)-K(\xi,\tau,c_{n-1}))\| \\ &+\frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^\tau \|(K(\xi,y,c_n)-K(\xi,y,c_{n-1}))dy\|, \\ \leq{}& \frac{2(1-\varsigma)R}{(2-\varsigma)M(\varsigma)}\|(c_n-c_{n-1})\| \\ &+\frac{2\varsigma R\tau}{(2-\varsigma)M(\varsigma)}\int_0^\tau \|(c_n-c_{n-1})dy\|. \end{aligned} \tag{2.20}$$

Repeating this process we obtain

$$\|A_n(\xi,\tau)\| \leq \left(\frac{2(1-\varsigma)R}{(2-\varsigma)M(\varsigma)}+\frac{2\varsigma R\tau}{(2-\varsigma)M(\varsigma)}\right)^{n+1} R^{n+1}c_0. \tag{2.21}$$

At $\tau=\tau_0$

$$\|A_n(\xi,\tau)\| \leq \left(\frac{2(1-\varsigma)R}{(2-\varsigma)M(\varsigma)}+\frac{2\varsigma R\tau_0}{(2-\varsigma)M(\varsigma)}\right)^{n+1} R^{n+1}c_0. \tag{2.22}$$

Now taking $\lim_{n\to\infty}\|A_n(\tau)\| \to 0$, hence existence of the solution is proved. □

For the uniqueness of the solution, we let

$$c(\xi,\tau)-c_1(\xi,\tau)=\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}(K(\xi,\tau,c)-K(\xi,\tau,c_1))$$
$$+\frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^\tau (K(\xi,y,c)-K(\xi,y,c_1))dy. \tag{2.23}$$

and then

$$\|c(\xi,\tau)-c_1(\xi,\tau)\|\leq\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}\|(K(\xi,\tau,c)-K(\xi,\tau,c_1))\|$$
$$+\frac{2\varsigma}{(2-\varsigma)M(\varsigma)}\int_0^\tau \|(K(\xi,y,c)-K(\xi,y,c_1))\|dy. \tag{2.24}$$

Now making use of the Lipschitz condition on the kernel, we obtain

$$\|c(\xi,\tau)-c_1(\xi,\tau)\|\leq\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}R\|c(\xi,\tau)-c_1(\xi,\tau)\|$$
$$+\frac{2\varsigma R\tau}{(2-\varsigma)M(\varsigma)}\|c(\xi,\tau)-c_1(\xi,\tau)\|. \tag{2.25}$$

This leads to

$$\|c(\xi,\tau)-c_1(\xi,\tau)\|\left(1-\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}R+\frac{2\varsigma R}{(2-\varsigma)M(\varsigma)}\right)\leq 0. \tag{2.26}$$

Since

$$\left(1-\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}R+\frac{2\varsigma R}{(2-\varsigma)M(\varsigma)}\right)>0 \tag{2.27}$$

If the above condition holds

$$\|c(\xi,\tau)-c_1(\xi,\tau)\|\left(1-\frac{2(1-\varsigma)}{(2-\varsigma)M(\varsigma)}R+\frac{2\varsigma R}{(2-\varsigma)M(\varsigma)}\right)\leq 0. \tag{2.28}$$

implies

$$\|c(\xi,\tau)-c_1(\xi,\tau)\|=0. \tag{2.29}$$

3. Derivation of the approximate solution

In this section, we derive the solution by employing the iterative technique. Assume $B(\varsigma)=1$ in (1.6). Laplace transform technique is used on Eq (1.5) to obtain

$$L\left\{{}_0^{CF}D_t^{\varsigma}c(\xi,\tau)\right\}=L\left(-u\frac{\partial c(\xi,\tau)}{\partial\xi}+D_L\frac{\partial^2 c(\xi,\tau)}{\partial\xi^2}\right)$$
$$\Longrightarrow\ \frac{sc(\xi,s)-c(\xi,0)}{s+\varsigma(1-s)}=L\left(-u\frac{\partial c(\xi,\tau)}{\partial\xi}+D_L\frac{\partial^2 c(\xi,\tau)}{\partial\xi^2}\right). \tag{3.1}$$

Solving further, we obtain

$$c(\xi, s) = \frac{c(\xi, 0)}{s} + \frac{s + \varsigma(1 - s)}{s} L\left(-u\frac{\partial c(\xi, \tau)}{\partial \xi} + D_L \frac{\partial^2 c(\xi, \tau)}{\partial \xi^2}\right). \tag{3.2}$$

Now applying the inverse Laplace transform on Eq (3.2) we obtain

$$c(\xi, \tau) = c(\xi, 0) + L^{-1}\left(\frac{s + \varsigma(1 - s)}{s} L\left(-u\frac{\partial c(\xi, \tau)}{\partial \xi} + D_L \frac{\partial^2 c(\xi, \tau)}{\partial \xi^2}\right)\right). \tag{3.3}$$

Let us assume the following recursive formula

$$c_{n+1}(\xi, \tau) = c_n(\xi, \tau) + L^{-1}\left(\frac{s + \varsigma(1 - s)}{s} L\left(-u\frac{\partial c_n(\xi, \tau)}{\partial \xi} + D_L \frac{\partial^2 c_n(\xi, \tau)}{\partial \xi^2}\right)\right). \tag{3.4}$$

with the initial component $c_0(\xi, \tau) = c(\xi, 0)$. The solution is thus provided by $c(\xi, \tau) = \lim_{n\to\infty} c_n(\xi, \tau)$.

4. Applications

Here, as applications, we have considered the different initial conditions $c(\xi, 0) = c_0(\xi)$ in the Eq (1.5).

1. Consider $c_0(\xi, \tau) = (1 + \xi)^{-n},\ n \in \mathbb{R}^+$, then from Eq (3.4) we have

$$\begin{aligned}
c_1(\xi, \tau) =& c_0(\xi, \tau) + L^{-1}\left(\frac{s + \varsigma(1 - s)}{s} L\left(-u\frac{\partial c_0(\xi, \tau)}{\partial \xi} + D_L \frac{\partial^2 c_0(\xi, \tau)}{\partial \xi^2}\right)\right)\\
=&(1 + \xi)^{-n} + \left(nu(1 + \xi)^{-n-1} + n(n + 1)D_L(1 + \xi)^{-n-2}\right) L^{-1}\left(\frac{s + \varsigma(1 - s)}{s}\right)\\
=&(1 + \xi)^{-n} + \left(nu(1 + \xi)^{-n-1} + n(n + 1)D_L(1 + \xi)^{-n-2}\right)(1 - \varsigma + \varsigma\tau)\\
c_2(\xi, \tau) =& c_1(\xi, \tau) + L^{-1}\left(\frac{s + \varsigma(1 - s)}{s} L\left(-u\frac{\partial c_1(\xi, \tau)}{\partial \xi} + D_L \frac{\partial^2 c_1(\xi, \tau)}{\partial \xi^2}\right)\right)\\
=&(1 + \xi)^{-n} + \left(nu(1 + \xi)^{-n-1} + n(n + 1)D_L(1 + \xi)^{-n-2}\right)(1 - \varsigma + \varsigma\tau)\\
&+ (1 - \xi + \xi\tau)(u^2 n(2n + 1)(1 + \xi)^{-2n-3})\\
&+ (1 - \varsigma)(1 - \varsigma + \varsigma\tau)nuD_L(n + 1)(n + 2)(1 + \xi) + \varsigma(1 + \varsigma)t + \varsigma\frac{\tau^2}{2}.
\end{aligned} \tag{4.1}$$

and so on. By means of these terms, the solution $c(\xi, \tau)$ is given by

$$c(\xi, \tau) = c_0(\xi, \tau) + c_1(\xi, \tau) + c_2(\xi, \tau) + \ldots \tag{4.2}$$

2. Consider $c_0(\xi,\tau) = e^{-\xi}$, then from Eq (3.4) we have

$$
\begin{aligned}
c_1(\xi,\tau) =& c_0(\xi,\tau) + L^{-1}\left(\frac{s+\varsigma(1-s)}{s} L\left(-u\frac{\partial c_0(\xi,\tau)}{\partial \xi} + D_L \frac{\partial^2 c_0(\xi,\tau)}{\partial \xi^2}\right)\right)\\
=& e^{-\xi} + L^{-1}\left(\frac{s+\varsigma(1-s)}{s} L\left(-ue^{-\xi} + D_L e^{-\xi}\right)\right)\\
=& e^{-\xi} - (ue^{-\xi} - D_L e^{-\xi})(1-\varsigma+\varsigma\tau)\\
c_2(\xi,\tau) =& c_1(\xi,\tau) + L^{-1}\left(\frac{s+\varsigma(1-s)}{s} L\left(-u\frac{\partial c_1(\xi,\tau)}{\partial \xi} + D_L \frac{\partial^2 c_1(\xi,\tau)}{\partial \xi^2}\right)\right)\\
=& e^{-\xi} - (ue^{-\xi} - D_L e^{-\xi})(1-\varsigma+\varsigma\tau) + L^{-1}\left(\frac{s+\varsigma(1-s)}{s}\right.\\
& \left. L\left(ue^{-\xi} + ue^{-\xi}(u+D_L)(1-\varsigma+\varsigma\tau) + D_L e^{-\xi} + D_L e^{-\xi}(u+D_L)(1-\varsigma+\varsigma\tau)\right)\right)\\
=& e^{-\xi} - (ue^{-\xi} - e^{-\xi} D_L)(1-\varsigma+\varsigma\tau) + e^{-\xi}(1-\varsigma)(u+D_L)(1+u-u\varsigma+D_L-\varsigma D_L)\\
& + \varsigma(1-\varsigma)(u+D_L)^2 e^{-\xi}\tau + e^{-\xi}\varsigma\tau(u+D_L)(1+u-u\varsigma+D_L-D_L\varsigma)\\
& + \frac{1}{2}\varsigma^2 e^{-\xi}\tau^2(u+D_L)^2,
\end{aligned}
\tag{4.3}
$$

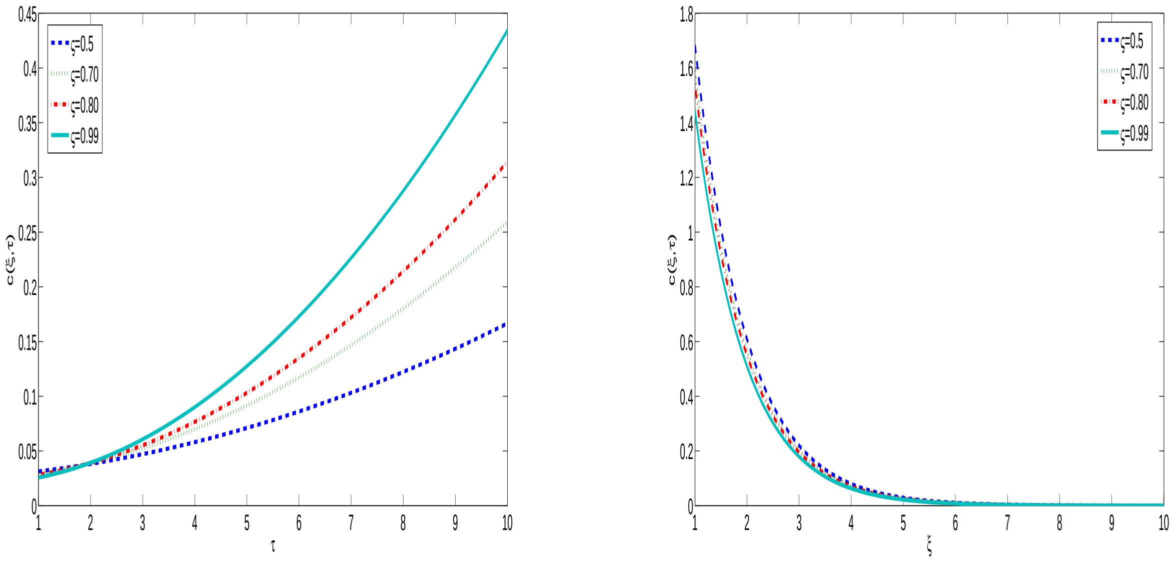
and so on. By means of these terms, the solution $c(\xi,\tau)$ is given by

$$
c(\xi,\tau) = c_0(\xi,\tau) + c_1(\xi,\tau) + c_2(\xi,\tau)\ldots \tag{4.4}
$$

5. Results discussions

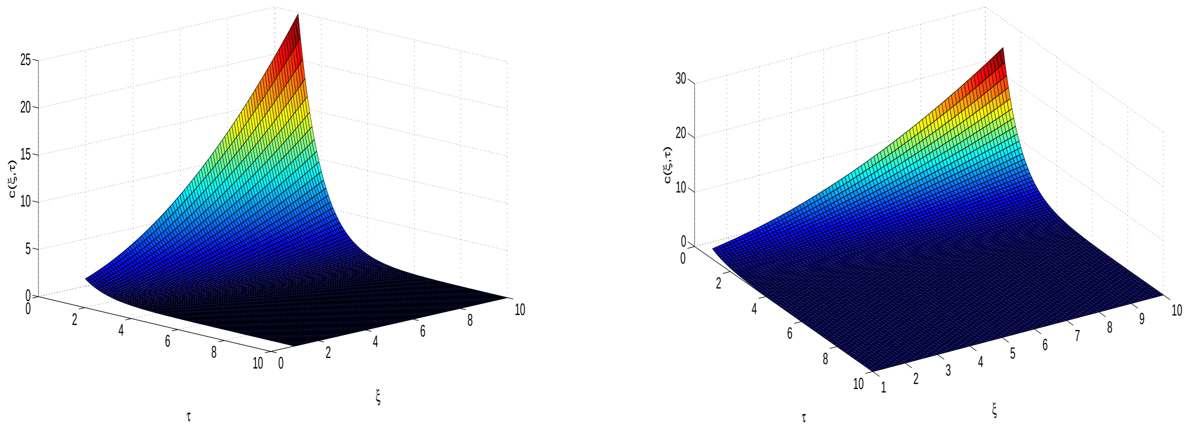
To obtain the effect of the fractional order derivatives to the approximation solution (3.4) of the governing differential Eq (1.5), we plotted graphs for various values of fractional order $\varsigma = 0.50,\ 0.70,\ 0.80$ and 0.99. We observe the behavior of the miscible flow with longitudinal dispersion with respect to time and space variables in Figure 1 (a) and (b) respectively and also plotted a 3-dimensional graph in Figure 2. The initial condition considered here is $c(\xi,0) = e^{-\xi}$ for fixed values of $u = 0.5$ and $D_L = 0.6$.

The developed mathematical formulation is helpful in predicting the possible concentration of a given dissolved substance in homogeneous, isotropic porous media and vary exponentially with time in Figure 1 (a). In Figure 1 (b) the concentration is decreasing with respect to ξ for $\tau > 0$. The obtained concentration of dispersing element in Eq (3.4) can be helpful to the study of salinity intrusion in groundwater and to making the predictions of the possible contamination of groundwater flows.



(a) **(b)**

Figure 1. Concentration profiles for $\varsigma = 0.5,\ 0.7,\ 0.8$ and 0.99 with respect to time and space variable respectively for the fixed values of $u = 0.5,\ D_L = 0.6$.



(a) **(b)**

Figure 2. Concentration profiles for $\varsigma = 0.99$ with respect to time and space variable respectively in different directions for the fixed values of $u = 0.5,\ D_L = 0.6$.

6. Conclusions

- In this paper, the phenomenon of the longitudinal dispersion in the flow of two miscible fluids through porous media is investigated by applying Caputo-Fabrizio fractional derivative.
- Existence and uniqueness of the solution is proved by using fixed point theorem. Solution is obtained by using iterative method and some numerical simulations are performed.

- The results described in Figures shows the behaviours of the concentration with respect to time and space variables for different order and fixed values of the parameters.
- The obtained results of dispersing element can be used to analyzing the salinity intrusion in groundwater and to making the predictions of contamination in groundwater.
- The recent features of the Caputo–Fabrizio fractional derivative operator provide more realistic models, which help us to adjust better the dynamical behaviours of the real–world phenomena as discussed in [11, 12].

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Conflict of interest

The authors declare no conflict of interest in this paper.

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