Fractional Differential Equation With a Complex Potential

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Abstract. In this manuscript, we discuss the square-integrable property of a fractional differential equation having a complex-valued potential function and we show that at least one of the linearly independent solutions of the fractional differential equation must be squarely integrable with respect to some function containing the imaginary parts of the spectral parameter and the potential function.

1. Introduction

In 1957 Sims [1] investigated the square-integrable property of the solutions for

$$-f'' + q(t)f = \mu f, \ t \in (a,b), \tag{1}$$

such that μ denote the spectral parameter, q represents a complex-valued function on (a,b) such that $q = q_1 + iq_2$, continuous on (a,b) and has a singularity at a and b. This investigation is a nontrivial generalization of the results of Weyl [2]. In fact, in 1910 Weyl showed that the second-order equation

$$-(p(t)f')' + q(t)f = \mu f, \ t \in [0,\infty), \tag{2}$$

such that p, q denote real-valued functions on $[0,\infty)$, $p > 0$, p^{-1}, q represent localy integrable functions on $[0,\infty)$, has always at least one square-integrable solution on $[0,\infty)$. Moreover, the other independent solution of (2) may be squarely integrable on $[0,\infty)$. The theory including these results is known as limit-point/circle theory and this name is coming from the geometric representation of the fractional equation obtained by the combination of the independent solutions of (2). Sims showed that at a singular point for (1) there may occur one of the following three situations:

    (*i*) a limit-point case but one square-integrable solution,
    (*ii*) a limit-point case and two square-integrable solutions,
    (*iii*) a limit-circle case and two square-integrable solutions.

The second case can not occur when q is real-valued function. Furthermore, Sims gave two examples containing cases (*i*) and (*ii*).

Although the Eq.s (1) and (2) play a central role in quantum-mechanics, modern electromagnetic theory and classical physics, in recent years, the authors have given a special attempt to get some results for the differential equations containing non-integer orders. The results obtained for these non-integer order differential equations are the generalizations of the results obtained for the integer order differential equations. Such generalizations are done by using some special differential expressions and special integrals (for example, see, [3], [4]). One of them is the Riemann-Liouville derivative of order α which is defined as

$$D_{a^+}^\alpha f = \frac{d^m}{dx^m}(I_{a^+}^{m-\alpha} f),$$

where $\mathrm{Re}(\alpha) \in (m-1, m)$ and

$$I_{a^+}^\alpha f = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\sigma)^{\alpha-1} f(\sigma)d\sigma, \ a < x,$$

and the other one is the Caputo derivative of order β which is defined as

$$^C D_{b^-}^\beta f = I_{b^-}^{n-\beta}\left(-\frac{d^n f}{dx^n}\right),$$

where $\mathrm{Re}(\beta) \in (n-1, n)$ and

$$I_{b^-}^\beta f = \frac{1}{\Gamma(\beta)} \int_x^b (\sigma-x)^{\beta-1} f(\sigma)d\sigma, \ x < b.$$

Here Γ represents the Euler gamma function. One of the generalizations of (2) can now be represented by

$$^C D_{b^-}^\beta p(t) D_{a^+}^\alpha f + q(t)f = \mu w(t)f, \tag{3}$$

and some important results for the Eq. (3) (such as symmetry of the equation) and some boundary conditions have been introduced by Klimek and Agrawal in [5] for $\alpha \in (0,1) \cup (1,2)$, $p \neq 0$ for all $t \in [a,b]$, $w > 0$, p, q are real-valued and continuous on $[a,b]$. In [6], the authors showed that the Eq. (3) with $p \equiv w \equiv 1$, q is real-valued continuous function on $[a,b)$ has at least one solution which is squarely integrable on $[a,b)$ and moreover the other independent solution of the same form of (3) may be squarely integrable on the same interval. The results in [6] are the generalization of the results of Weyl for the differential equation with non-integer order and fulfill some gaps on this theory.

Our work will generalize the results of Sims and Uğurlu et. al. for the differential equation with non-integer order having the complex-valued potential function and this will fulfill some gaps on such problems.

2. Differential equation

Below we discuss the equation, namely

$$\tau^\alpha(f) :=^C D_{b^-}^\alpha D_{a^+}^\alpha f + q(t)f = \mu f, \ t \in [a,b). \tag{4}$$

Here μ represents the complex parameter, q is continuous function on each compact subset of $[a,b)$ having a singularity at b, q is a complex-valued function such that $q = q_1 + iq_2$ and $0 < \alpha < 1$.

We define the Wronskian of two functions f_1 and f_2 with the rule

$$W[f_1, f_2] = (I_{a^+}^{1-\alpha} f_1)(D_{a^+}^\alpha f_2) - (D_{a^+}^\alpha f_1)(I_{a^+}^{1-\alpha} f_2).$$

We should note that the Eq. (4) has a unique solution f satisfying the conditions [7]

$$(I_{a^+}^{1-\alpha} f)(l) = l_1, \quad (D_{a^+}^{\alpha} f)(l) = l_2,$$

where l_1 and l_2 are arbitrary complex numbers and $l \in [a, b)$.

Following the same procedure given in [6] for the proofs we may introduce the following Lemmas.

Lemma 2.1. *Let f, g be the solutions of (4).*
(i) If $W[f, g](t_0) = 0$ for $t_0 \in [a, b)$, then f, g are linearly dependent.
(ii) If f, g are linearly dependent solutions of (4), then $W[f, g] \equiv 0$ on $[a, b)$.

Lemma 2.2. *Let f_1, f_2 be the solutions of (4) satisfying*

$$(I_{a^+}^{1-\alpha} f_s)(t_0) = c_{s1}, \quad (D_{a^+}^{\alpha} f_s)(t_0) = c_{s2},$$

where $s = 1, 2$, c_{s1}, c_{s2}, are arbitrary complex numbers and

$$\det \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \neq 0.$$

Then the other solution z of (4) can be represented by f_1 and f_2.

Lemma 2.3. *The set of solutions of (4) obeys a 2 dimensional linear space 2.*

We will consider the ordinary inner product of the functions f and g as follows

$$(f, g) = \int_a^b f \overline{g} dt$$

in the Hilbert space H that contains the functions f satisfying the following

$$\int_a^b |f|^2 dt < \infty.$$

Let D be a subset of H with the functions $f \in H$ such that $D_{a^+}^{\alpha} f$, $^C D_{b^-}^{\alpha} D_{a^+}^{\alpha} f$ are meaningful and $\tau^{\alpha}(f) \in H$. For $f, g \in D$ one obtains the following Lagrange's formula [5]

$$\int_a^b \left[\tau^{\alpha}(f)g - f\tau^{\alpha}(g) \right] dt = W[f, g](b) - W[f, g](a). \tag{5}$$

From (5) we obtain that if $f(t, \mu)$ and $g(t, \mu)$ are the solutions of (4) corresponding to the same value μ then $W[f, g]$ is independent of t and depends only on μ.

For $f, g \in D$ we get

$$(\tau^{\alpha}(f), g) - (f, \tau^{\alpha}(g)) = 2i \int_a^b q_2 f \overline{g} dt + W[f, \overline{g}](b) - W[f, \overline{g}](a). \tag{6}$$

For the further calculations we will use the equation (6).

3. The circle equation

Using Lemma 2.3 we may represent an arbitrary solution of (4) as a combination of two linearly independent solutions of (4).

Let $\xi(t, \mu)$ and $\zeta(t, \mu)$ be the solutions of (4) such that

$$
\begin{aligned}
\left(I_{a^+}^{1-\alpha_1}\xi\right)(a, \mu) &= \sin \delta_1, & \left(I_{a^+}^{1-\alpha_1}\zeta\right)(a, \mu) &= \cos \delta_1, \\
\left(D_{a^+}^{\alpha_1}\xi\right)(a, \mu) &= -\cos \delta_1, & \left(D_{a^+}^{\alpha_1}\zeta\right)(a, \mu) &= \sin \delta_1,
\end{aligned}
$$

where $0 \le \delta_1 < \pi$. Note that $W[\xi, \zeta](a) = 1$ and therefore by Lemma 2.1 they constitute a linearly independent set of solutions.

Let κ be a solution of (4). Then according to Lemma 2.3 one may write $\kappa(t, \mu)$ as

$$
\kappa(t, \mu) = \xi(t, \mu) + m\zeta(t, \mu), \ t \in [a, b),
$$

such that m denotes a constant.

Let consider on the subinterval $[a, c]$ of $[a, b)$ the followings

$$
\cos \gamma_1 \left(I_{a^+}^{1-\alpha_1}f\right)(c) + \sin \gamma_1 \left(D_{a^+}^{\alpha_1}f\right)(c) = 0, \tag{7}
$$

where $0 \le \gamma_1 < \pi$. Since κ is a solution of (4) we obtain from (7) that

$$
(\Lambda + m\Xi) \cos \gamma_1 + (\Theta + m\Phi) \sin \gamma_1 = 0, \tag{8}
$$

where

$$
\Lambda = \left(I_{a^+}^{1-\alpha_1}\xi\right)(c, \mu), \quad \Xi = \left(I_{a^+}^{1-\alpha_1}\zeta\right)(c, \mu), \quad \Theta = \left(D_{a^+}^{\alpha_1}\xi\right)(c, \mu), \quad \Phi = \left(D_{a^+}^{\alpha_1}\zeta\right)(c, \mu).
$$

Since γ_1 is a real number we obtain from (8)

$$
(\overline{\Lambda} + \overline{m}\,\overline{\Xi}) \cos \gamma_1 + (\overline{\Theta} + \overline{m}\,\overline{\Phi}) \sin \gamma_1 = 0. \tag{9}
$$

Therefore (8) and (9) give

$$
\begin{bmatrix} \Lambda + m\Xi & \Theta + m\Phi \\ \overline{\Lambda} + \overline{m}\,\overline{\Xi} & \overline{\Theta} + \overline{m}\,\overline{\Phi} \end{bmatrix} \begin{bmatrix} \cos \gamma_1 \\ \sin \gamma_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10}
$$

Because $\sin \gamma_1$ and $\cos \gamma_1$ can not be zero at the same time for each γ_1, (10) is satisfied if and only if

$$
|m|^2 \left(\Xi\overline{\Phi} - \Phi\overline{\Xi}\right) + m \left(\Xi\overline{\Theta} - \Phi\overline{\Lambda}\right) + \overline{m} \left(\Lambda\overline{\Phi} - \Theta\overline{\Xi}\right) + \Lambda\overline{\Theta} - \Theta\overline{\Lambda} = 0, \tag{11}
$$

where

$$
\begin{aligned}
\Xi\overline{\Phi} - \Phi\overline{\Xi} &= W[\zeta, \overline{\zeta}](c), \\
\Xi\overline{\Theta} - \Phi\overline{\Lambda} &= W[\zeta, \overline{\xi}](c), \\
\Lambda\overline{\Phi} - \Theta\overline{\Xi} &= W[\xi, \overline{\zeta}](c), \\
\Lambda\overline{\Theta} - \Theta\overline{\Lambda} &= W[\xi, \overline{\xi}](c).
\end{aligned}
$$

On the other side the following equation

$$W[\kappa, \overline{\kappa}](c) = 0 \tag{12}$$

is also equal to (11).

Initial conditions given for ζ and (6) imply

$$W[\zeta, \overline{\zeta}](c) = 2i \int_a^b (\mathrm{Im}\mu - q_2) |\zeta|^2 \, dt. \tag{13}$$

Moreover using the definition of κ and (6) we obtain

$$W[\kappa, \overline{\kappa}](c) = 2i \int_a^b (\mathrm{Im}\mu - q_2) |\kappa|^2 \, dt - 2i\mathrm{Im}m. \tag{14}$$

Consequently (11)-(14) implies that

$$\Omega(c) = \frac{W[\kappa, \overline{\kappa}](c)}{W[\zeta, \overline{\zeta}](c)} = 0 \tag{15}$$

denotes a circle equation in the complex $m-$plane under the condition $\mathrm{Im}\mu - q_2 \neq 0$. The expression of its radius becomes

$$r_c = \frac{1}{\left| W[\zeta, \overline{\overline{\zeta}}](c) \right|} \tag{16}$$

and the inside of it is $\Omega(c) < 0$.

Theorem 3.1. *Suppose either* $\mathrm{Im}\mu > 0$ *and* $q_2 \leq 0$ *or* $\mathrm{Im}\mu < 0$ *and* $q_2 \geq 0$. *Then the circles* $\Omega(c) = 0$ *are nested as* $c \to b$.

Proof. Let $\mathrm{Im}\mu > 0$ and $q_2 \leq 0$ and $\Omega(c) = 0$ be a circle in the $m-$plane. Then the points located on the boundary or within the circle $\Omega(c) = 0$ are written as $\Omega(c) \leq 0$ or from (13)-(15) as

$$\frac{\int_a^c (\mathrm{Im}\mu - q_2) |\kappa|^2 \, dt - \mathrm{Im}m}{\int_a^c (\mathrm{Im}\mu - q_2) |\zeta|^2 \, dt} \leq 0. \tag{17}$$

As a result the point m is on or inside $\Omega(c) = 0$ if

$$\int_a^c (\mathrm{Im}\mu - q_2) |\kappa|^2 \, dt \leq \mathrm{Im}m. \tag{18}$$

Now we analyse $\widetilde{c}$ fulfilling $a < \widetilde{c} < c$. Then (18) implies that

$$\int_a^{\widetilde{c}} (\mathrm{Im}\mu - q_2) |\kappa|^2 \, dt < \mathrm{Im}m$$

and therefore m is located inside the circle generated by $\Omega(\widetilde{c}) = 0$.

For $\mathrm{Im}\mu < 0$ and $q_2 \geq 0$ one should reverse the inequality given in (18).

As a result we finished the proof. $\square$

Corollary 3.2. *As $c \to b$ the circles $\Omega(c) = 0$ may converge either to a circle $\Omega(b) = 0$ or a point $\Omega(b) = 0$.*

Theorem 3.3. *Let $\kappa(t, \mu) = \xi(t, \mu) + m\zeta(t, \mu)$ be a solution of (4). Then for $\operatorname{Im}\mu > 0$ and $q_2 \leq 0$ or $\operatorname{Im}\mu < 0$ and $q_2 \geq 0$ the following inequality holds*

$$-\infty < \int_a^b (\operatorname{Im}\mu - q_2) \left| \xi(t, \mu) + m\zeta(t, \mu) \right|^2 dt < \infty.$$

Proof. Firstly we shall consider the case $\operatorname{Im}\mu > 0$ and $q_2 \leq 0$.

We consider m as a point on the $\Omega(b) = 0$. Then for $a < c < b$ m must inside of $\Omega(c) = 0$. Therefore the following inequality must hold

$$\int_a^c (\operatorname{Im}\mu - q_2) \left| \xi + m\zeta \right|^2 dt < \operatorname{Im}m. \tag{19}$$

Taking into account that the right-hand side of the above equation is independent of c we consider $c \to b$ to obtain

$$\int_a^b (\operatorname{Im}\mu - q_2) \left| \xi + m\zeta \right|^2 dt < \operatorname{Im}m$$

and this implies the assertion.

For the second case $\operatorname{Im}\mu < 0$ and $q_2 \geq 0$ we should reverse the inequality given in (19) and a similar argument completes the proof. $\square$

If the circles $\Omega(c) = 0$ converge to a point $\Omega(b) = 0$ as $c \to b$ then the radius of the circles vanish and therefore (13) and (16) imply that the integral

$$\int_a^b (\operatorname{Im}\mu - q_2) \left| \zeta \right|^2 dt$$

is divergent for $\operatorname{Im}\mu > 0$ and $q_2 \leq 0$ or $\operatorname{Im}\mu < 0$ and $q_2 \geq 0$. However, if the limiting circle $\Omega(b) = 0$ is a circle then one has

$$-\infty < \int_a^b (\operatorname{Im}\mu - q_2) \left| \zeta \right|^2 dt < \infty$$

for $\operatorname{Im}\mu > 0$ and $q_2 \leq 0$ or $\operatorname{Im}\mu < 0$ and $q_2 \geq 0$. Consequently for the solutions of (4) we may conclude the following main results of our work.

Theorem 3.4. *Consider the equation (4) with $\operatorname{Im}\mu > 0$, $q_2 \leq 0$ or $\operatorname{Im}\mu < 0$, $q_2 \geq 0$. Then one of the following two situations is possible:*

(i) The limit-circle case: every solution f corresponding to (4) satisfies

$$-\infty < \int_a^b (\operatorname{Im}\mu - q_2) \left| f \right|^2 dt < \infty. \tag{20}$$

(ii) The limit-point case: one of two linearly independent solutions of (4) is not obeying (20).

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