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A class of fractal Hilbert-type inequalities obtained via Cantor-type spherical coordinates

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We present a class of higher dimensional Hilbert-type inequalities on a fractal set $(\mathbb{R}_+^{\alpha n})^k$. The crucial step in establishing our results are higher dimensional spherical coordinates on a fractal space. Further, we impose the corresponding conditions under which the constants appearing in the established Hilbert-type inequalities are the best possible. As an application, our results are compared with the previous results known from the literature.

KEYWORDS

Cantor-type spherical coordinates, fractal set, local fractional integral, the best possible constant, the Hilbert inequality

MSC CLASSIFICATION

26D15; 26D10

1 | INTRODUCTION AND PRELIMINARIES

Let p and q be a pair of nonnegative conjugate parameters, that is, $1/p + 1/q = 1$, $p > 1$. The celebrated Hilbert inequality in its integral form (see, e.g., Hardy et al.[1]) states that

$$\int_{\mathbb{R}_+^2} \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin\frac{\pi}{p}} \left(\int_{\mathbb{R}_+} f^p(x)dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+} g^q(y)dy \right)^{\frac{1}{q}} \tag{1}$$

holds for every pair of nonnegative integrable functions $f, g : \mathbb{R}_+ \to \mathbb{R}$. In addition, the constant $\pi/\sin\frac{\pi}{p}$ is the best possible in inequality (1). This means that it cannot be replaced by a smaller positive constant so that the inequality remains valid.

Although classical, inequality (1) is still a research topic by many authors. In the last few decades, Hilbert-type inequalities have been extensively studied by numerous mathematicians. A whole series of generalizations included inequalities with more general kernels, weight functions, and integration domains, as well as improvements of the basic Hilbert inequality (1). For more details about Hilbert-type inequalities, the reader is referred to monographs[1-3] and the references therein.

The basic task of the local fractional calculus is to handle diverse nondifferentiable problems appearing in complex systems of the real-world phenomena. For example, the nondifferentiability occurring in engineering and science has been modeled by the local fractional ordinary or partial differential equations. On the other hand, the local fractional calculus also plays an important role in pure mathematics. Namely, in the last few years, a whole variety of classical inequalities have been extended to hold on fractal spaces (see previous studies[4-11] and the references therein).

For the reader's convenience, we give now basic definitions and properties of the local fractional derivative and integral developed in Yang[12] (see also Yang[13]).

Let $\mathbb{R}^\alpha$, $0<\alpha\leq 1$, be an α-type fractal set of real numbers. Then, $\mathbb{R}^\alpha$ is a field with addition and multiplication defined by $x^\alpha + y^\alpha = (x+y)^\alpha$ and $x^\alpha \cdot y^\alpha = (xy)^\alpha$, $x^\alpha, y^\alpha \in \mathbb{R}^\alpha$. Clearly, the additive identity is 0^α, while multiplicative identity is 1^α.

A nondifferentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$ is said to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that $|x-x_0|<\delta$ implies that $|f(x)-f(x_0)|<\varepsilon^\alpha$. The set of local fractional continuous functions on interval I is denoted by $C_\alpha(I)$. The local fractional derivative of f of order α at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Gamma(1+\alpha)(f(x)-f(x_0))}{(x-x_0)^\alpha},$$

where Γ is a usual Gamma function defined by $\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$, $a>0$. If $f^{(\alpha)}(x)$ is well-defined for every $x\in I$, we say that f belongs to $D_\alpha(I)$.

The local fractional integral is defined as follows. Let $f\in C_\alpha[a,b]$ and let $P = \{t_0, t_1, \dots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a,b]$ such that $a=t_0<t_1<\dots<t_{N-1}<t_N=b$. Moreover, let $\Delta t_j = t_{j+1}-t_j$, $j = 0,\dots,N-1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$. In this setting, the local fractional integral of f on the interval $[a,b]$ of order α (denoted by ${}_aI_b^{(\alpha)}f(x)$) is defined by

$${}_aI_b^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha.$$

The above definition implies that ${}_aI_b^\alpha f(x) = 0$ if $a=b$ and ${}_aI_b^\alpha f(x) = -{}_bI_a^\alpha f(x)$ if $a<b$. The Newton–Leibnitz formula on fractal space asserts that if $f=g^{(\alpha)} \in C_\alpha[a,b]$, then ${}_aI_b^\alpha f(x) = g(b)-g(a)$. In particular, if $f(x)=x^{k\alpha}$, $k\in\mathbb{R}$, then

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha} - a^{(k+1)\alpha}). \tag{2}$$

A change of variables theorem in the above-described setting asserts that if g is a differentiable transformation and $(f\circ g)\in C_\alpha[g(a), g(b)]$, then holds the relation

$${}_aI_b^\alpha(f\circ g)(s)[g'(s)]^\alpha = {}_{g(a)}I_{g(b)}^\alpha f(x). \tag{3}$$

Clearly, if $\alpha = 1$, then the local fractional calculus reduces to classical real calculus. For more details about the above presented concept of fractional differentiability and integrability, the reader is referred to Yang[12] and the references therein.

The main goal of this paper is a study of multidimensional fractal Hilbert-type inequalities. Two-dimensional fractal Hilbert-type inequalities have been extensively studied in the aforementioned papers.[5,7-9] Recently, Krnić and Vuković[6] provided a unified approach to k-dimensional fractal Hilbert-type inequalities. In particular, they proved that if $\sum_{i=1}^{k}\frac{1}{p_i} = 1$, $p_i > 1$, $f_i \in C_\alpha(\mathbb{R}_+)$, and $s \leq \min_{1\leq i\leq k}\left\{k + \frac{1}{2}p_i\right\}$, then holds the inequality

$$\begin{aligned}&\frac{1}{\Gamma^k(1+\alpha)}\int_{\mathbb{R}_+^k}\frac{\prod_{i=1}^k f_i(x_i)}{\left(\sum_{j=1}^k x_j^\alpha\right)^s}(dx_1)^\alpha(dx_2)^\alpha\dots(dx_k)^\alpha\\ &\leq \frac{\prod_{i=1}^k \Gamma_\alpha\left(\frac{s-k+p_i}{p_i}\right)}{\Gamma_\alpha(s)}\prod_{i=1}^k\left[\frac{1}{\Gamma(1+\alpha)}\int_{\mathbb{R}_+} x_i^{\alpha(k-1)-\alpha s} f_i^{p_i}(x_i)(dx_i)^\alpha\right]^{\frac{1}{p_i}},\end{aligned} \tag{4}$$

where $\Gamma_\alpha(\cdot)$, $0<\alpha\leq 1$, is a local fractal Gamma function defined by

$$\Gamma_\alpha(x) = \frac{1}{\Gamma(1+\alpha)}\int_{\mathbb{R}_+} E_\alpha(-t^\alpha)t^{\alpha(x-1)}(dt)^\alpha, \tag{5}$$

and $E_\alpha(y)=\sum_{k=0}^{\infty}\frac{y^k}{\Gamma(1+\alpha k)}$ is the Mittag–Leffler function. Moreover, it has been showed that the constant appearing on the right-hand side of (4) is the best possible. For more details about fractal Gamma function defined by (5), the reader is referred to Jumarie.[14]

Our main goal is to extend inequality (4) to hold on the fractal space $(\mathbb{R}_+^{\alpha n})^k$. To do this, we first need to introduce higher dimensional spherical coordinates on a fractal space.

2 | CANTOR-TYPE *n*-DIMENSIONAL SPHERICAL COORDINATES

In this section, we introduce higher dimensional spherical coordinates on a fractal space, which will be necessary in deriving multidimensional Hilbert-type inequalities. An initial step in this direction are fractal versions of trigonometric functions. The sine function defined on the fractal set $\mathbb{R}^\alpha$ is given by

$$\sin_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)},\ x\in\mathbb{R},$$

while the fractal cosine function is defined by

$$\cos_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(1+2k\alpha)},\ x\in\mathbb{R}.$$

The above functions are related by a basic identity $\sin_\alpha^2 x+\cos_\alpha^2 x=1$, while their local fractional derivatives are given by

$$\frac{d^\alpha}{dx^\alpha}(\sin_\alpha(x^\alpha))=\cos_\alpha(x^\alpha),\ \frac{d^\alpha}{dx^\alpha}(\cos_\alpha(x^\alpha))=-\sin_\alpha(x^\alpha).$$

For more details about fractal trigonometric functions, the reader is referred to monographs[12,15] and the references therein.

Now, the n-dimensional spherical coordinates on the fractal set $\mathbb{R}^\alpha$, usually referred to as the Cantor-type n-dimensional spherical coordinates, are defined by

$$\begin{aligned}
x_1^\alpha &= r^\alpha\cos_\alpha(\varphi_1^\alpha),\\
x_2^\alpha &= r^\alpha\sin_\alpha(\varphi_1^\alpha)\cos_\alpha(\varphi_2^\alpha),\\
x_3^\alpha &= r^\alpha\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\cos_\alpha(\varphi_3^\alpha),\\
&\dots\\
x_{n-1}^\alpha &= r^\alpha\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\ \dots\ \sin_\alpha(\varphi_{n-2}^\alpha)\cos_\alpha(\varphi_{n-1}^\alpha),\\
x_n^\alpha &= r^\alpha\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\ \dots\ \sin_\alpha(\varphi_{n-2}^\alpha)\sin_\alpha(\varphi_{n-1}^\alpha),
\end{aligned}\tag{6}$$

where $0\le r\le R$, $0\le\varphi_1,\varphi_2,\ \dots\ ,\varphi_{n-2}\le\pi$, and $0\le\varphi_{n-1}\le 2\pi$. It should be noted here that the above transformation (6) is an extension of classical n-dimensional spherical coordinates to a fractal space. Therefore, similarly to a classical setting, using the basic identity $\sin_\alpha^2 x+\cos_\alpha^2 x=1$, it follows that

$$x_1^{2\alpha}+x_2^{2\alpha}+\ \dots\ +x_n^{2\alpha}=r^{2\alpha}.\tag{7}$$

Lemma 1. *The local fractional Jacobian of transformation (6) is*

$$J_n=\Gamma(1+\alpha)r^{\alpha(n-1)}\sin_\alpha^{n-2}(\varphi_1^\alpha)\sin_\alpha^{n-3}(\varphi_2^\alpha)\ \dots\ \sin_\alpha^2(\varphi_{n-3}^\alpha)\sin_\alpha(\varphi_{n-2}^\alpha),\tag{8}$$

for $n\ge 3$.

Proof. We prove our assertion by a mathematical induction. If $n = 3$, we consider the transformation

$$\begin{aligned} x_1^\alpha &= F(r,\varphi_1,\varphi_2) = r^\alpha \cos_\alpha(\varphi_1^\alpha),\\ x_2^\alpha &= G(r,\varphi_1,\varphi_2) = r^\alpha \sin_\alpha(\varphi_1^\alpha)\cos_\alpha(\varphi_2^\alpha),\\ x_3^\alpha &= H(r,\varphi_1,\varphi_2) = r^\alpha \sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\cos_\alpha(\varphi_3^\alpha). \end{aligned}$$

The corresponding Jacobian is equal to

$$\begin{aligned} J_3 &= \det\begin{bmatrix} F_r^{(\alpha)} & F_{\varphi_1}^{(\alpha)} & F_{\varphi_2}^{(\alpha)}\\ G_r^{(\alpha)} & G_{\varphi_1}^{(\alpha)} & G_{\varphi_2}^{(\alpha)}\\ H_r^{(\alpha)} & H_{\varphi_1}^{(\alpha)} & H_{\varphi_2}^{(\alpha)} \end{bmatrix}\\ &= \det\begin{bmatrix} \Gamma(1+\alpha)\cos_\alpha(\varphi_1^\alpha) & -r^\alpha\sin_\alpha(\varphi_1^\alpha) & 0\\ \Gamma(1+\alpha)\sin_\alpha(\varphi_1^\alpha)\cos_\alpha(\varphi_2^\alpha) & r^\alpha\cos_\alpha(\varphi_1^\alpha)\cos_\alpha(\varphi_2^\alpha) & -r^\alpha\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\\ \Gamma(1+\alpha)\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha) & r^\alpha\cos_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha) & r^\alpha\sin_\alpha(\varphi_1^\alpha)\cos_\alpha(\varphi_2^\alpha) \end{bmatrix}\\ &= \Gamma(1+\alpha)r^{2\alpha}\sin_\alpha(\varphi_1^\alpha), \end{aligned}$$

where we have used basic properties of fractal trigonometric functions, as well as the formula $d^\alpha/dx^\alpha(x^{k\alpha}/\Gamma(1+k\alpha)) = x^{(k-1)\alpha}/\Gamma(1+(k-1)\alpha)$. Hence, formula (8) holds for $n = 3$.

Now, suppose that (8) holds for $n-1$. Then, the Jacobian of the corresponding n-dimensional transformation can be rewritten as

$$J_n = \det\left[\begin{array}{ccc:c} & & & 0\\ & & & 0\\ & \cos_\alpha(\varphi_{n-1}^\alpha)J_{n-1} & & \vdots\\ & & & 0\\ & & & -r^\alpha s(0)\\ \hdashline \Gamma(1+\alpha)s(0) & r^\alpha s(1) & \dots & r^\alpha s(n-1) \end{array}\right],$$

where $s(i) = \sin_\alpha(\varphi_1^\alpha)\,\dots\,\sin_\alpha(\varphi_{i-1}^\alpha)\cos_\alpha(\varphi_i^\alpha)\sin_\alpha(\varphi_{i+1}^\alpha)\,\dots\,\sin_\alpha(\varphi_{n-1}^\alpha)$, $1 \le i \le n-1$, and $s(0) = \sin_\alpha(\varphi_1^\alpha)\,\dots\,\sin_\alpha(\varphi_{n-1}^\alpha)$. Finally, expanding the previous determinant along the last column and using the induction hypothesis, we obtain

$$\begin{aligned} J_n &= (-1)^{(n-1)+n}(-r^\alpha)\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\,\dots\,\sin_\alpha(\varphi_{n-1}^\alpha)(\sin_\alpha(\varphi_{n-1}^\alpha)J_{n-1})\\ &\quad + (-1)^{n+n}r^\alpha\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\,\dots\,\sin_\alpha(\varphi_{n-2}^\alpha)\cos_\alpha(\varphi_{n-1}^\alpha)(\cos_\alpha(\varphi_{n-1}^\alpha)J_{n-1})\\ &= r^\alpha\sin_\alpha(\varphi_1^\alpha)\sin_\alpha(\varphi_2^\alpha)\,\dots\,\sin_\alpha(\varphi_{n-2}^\alpha)J_{n-1}(\sin_\alpha^2(\varphi_{n-1}^\alpha) + \cos_\alpha^2(\varphi_{n-1}^\alpha))\\ &= \Gamma(1+\alpha)r^{\alpha(n-1)}\sin_\alpha^{n-2}(\varphi_1^\alpha)\sin_\alpha^{n-3}(\varphi_2^\alpha)\,\dots\,\sin_\alpha^2(\varphi_{n-3}^\alpha)\sin_\alpha(\varphi_{n-2}^\alpha), \end{aligned}$$

as claimed. □

By virtue of Cantor-type n-dimensional spherical coordinates, we will establish some particular Hilbert-type inequalities on the fractal space $(\mathbb{R}_+^{\alpha n})^k$. In order to summarize our further discussion, we use the following abbreviations. For $\mathbf{x_i} = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$, $i = 1, 2, \dots, k$, we define

$$|\mathbf{x_i^\alpha}| = \sqrt{x_{i1}^{2\alpha} + x_{i2}^{2\alpha} + \dots + x_{in}^{2\alpha}}$$

and

$$\mathbf{X} = (\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_k}),\ (d\mathbf{X})^\alpha = \prod_{i=1}^{k}(d\mathbf{x_i})^\alpha,\ (\hat{d}^i\mathbf{X})^\alpha = \prod_{j=1, j\neq i}^{k}(d\mathbf{x_j})^\alpha.$$

In particular, if $n = 1$, we have that

$$X = (x_1, x_2, \ldots, x_k),\ (dX)^\alpha = \prod_{i=1}^{k}(dx_i)^\alpha,\ (\hat{d}^i X)^\alpha = \prod_{j=1, j\neq i}^{k}(dx_j)^\alpha.$$

In order to pass from Cartesian to spherical coordinates, it is necessary to use the change of variables theorem for fractional integrals, which refers to a nondifferentiable transformation. More precisely, if D, $S^{(\beta)}$ are fractal surfaces, $f \in C_\alpha(D)$, $g_i \in C_\alpha(S^{(\beta)})$, and $x_i = g_i(U)$, $i = 1, 2, \ldots, k$, is a nondifferentiable variable transformation, then holds

$$\frac{1}{\Gamma^k(1+\alpha)}\int_D f(X)(dX)^\alpha = \frac{1}{\Gamma^k(1+\alpha)}\int_{S^{(\beta)}} f(g_1(U), \ldots, g_k(U))|J|(dU)^\alpha, \tag{9}$$

where J denotes the local fractional Jacobian of the corresponding transformation. For more details about the above transformation, the reader is referred to Yang et al.[15]

Remark 1. To illustrate the change of variable formula given by (9), we consider a unit sphere and ball. Let S^{n-1} denotes a unit sphere in $\mathbb{R}^n$, while S_α^{n-1} stands for the corresponding sphere in $\mathbb{R}^{\alpha n}$. Then, passing to n-dimensional spherical coordinates and using formula (9), we obtain that its area $|S_\alpha^{n-1}|$ is given by

$$\begin{aligned}|S_\alpha^{n-1}| &= \frac{1}{\Gamma^n(1+\alpha)}\int_{S^{n-1}}(dX)^\alpha \\ &= \frac{1}{\Gamma^{n-1}(1+\alpha)}\int_0^\pi \sin_\alpha^{n-2}(\varphi_1^\alpha)(d\varphi_1)^\alpha \ldots \int_0^\pi \sin_\alpha(\varphi_{n-2}^\alpha)(d\varphi_{n-2})^\alpha \\ &\times \int_0^{2\pi}(d\varphi_{n-1})^\alpha.\end{aligned}$$

In particular, if $n = 3$, then

$$\begin{aligned}|S_\alpha^2| &= \frac{1}{\Gamma^3(1+\alpha)}\int_{S^2} dS^2 = \frac{1}{\Gamma^2(1+\alpha)}\int_0^\pi \sin_\alpha(\varphi_1^\alpha)(d\varphi_1)^\alpha \int_0^{2\pi}(d\varphi_2)^\alpha \\ &= \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)}(1-\cos_\alpha(\pi^\alpha)).\end{aligned} \tag{10}$$

Let V^n be a unit ball in $\mathbb{R}^n$ and V_α^n the corresponding ball in $\mathbb{R}^{\alpha n}$. Then, its volume is given by the formula

$$\begin{aligned}|V_\alpha^n| &= \frac{1}{\Gamma^n(1+\alpha)}\int_{V^n}(dX)^\alpha \\ &= \frac{1}{\Gamma^{n-1}(1+\alpha)}\int_0^\pi \sin_\alpha^{n-2}(\varphi_1^\alpha)(d\varphi_1)^\alpha \ldots \int_0^\pi \sin_\alpha(\varphi_{n-2}^\alpha)(d\varphi_{n-2})^\alpha \\ &\times \int_0^{2\pi}(d\varphi_{n-1}^\alpha)\int_0^1 r^{\alpha(n-1)}(dr)^\alpha \\ &= |S_\alpha^{n-1}|\int_0^1 r^{\alpha(n-1)}(dr)^\alpha = \frac{\Gamma(1+\alpha)\Gamma(1+(n-1)\alpha)}{\Gamma(1+n\alpha)}|S_\alpha^{n-1}|.\end{aligned} \tag{11}$$

It should be noted here that if $\alpha = 1$, then (11) reduces to the well-known identity

$$|V^n| = \frac{\Gamma(2)|S^{n-1}|\Gamma(n)}{\Gamma(n+1)} = \frac{|S^{n-1}|}{n}.$$

If $n = 3$, then relations (10) and (11) provide an explicit formula for a volume of unit ball V_α^3:

$$|V_\alpha^3| = \frac{1}{\Gamma^3(1+\alpha)} \int_{V^3} dV^3 = \frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} |S_\alpha^2| = \frac{(2\pi)^\alpha\ \Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(1-\cos_\alpha(\pi^\alpha)).$$

Clearly, if the fractal dimension is equal to 1, then we obtain the well-known formulas for the area of a unit sphere in $\mathbb{R}^3$ and the volume of a unit ball in $\mathbb{R}^3$, that is, $|S^2| = 4\pi$ and $|V^3| = \frac{4\pi}{3}$.

The n-dimensional sphere S_α^{n-1} will appear naturally in the forthcoming results. The following integral formula will be crucial in establishing the Hilbert-type inequality on $(\mathbb{R}_+^{\alpha n})^k$.

Lemma 2. *Let* $\sum_{i=1}^{k} \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \dots, k$. *If* $\varepsilon > 0$, $s > 0$ *and* $\tilde{A}_i \in \mathbb{R}$, $i = 1, 2, \dots, k$, *then holds the identity*

$$\frac{1}{\Gamma^{kn}(1+\alpha)} \int_{|\boldsymbol{x}_1^\alpha|\geq 1,\dots,|\boldsymbol{x}_k^\alpha|\geq 1} \frac{\prod_{i=1}^k |\boldsymbol{x}_i^\alpha|^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{\left(\sum_{j=1}^k |\boldsymbol{x}_j^\alpha|\right)^s} (d\boldsymbol{X})^\alpha = |S_\alpha^{n-1}|^k \int_{(1,\infty)^k} \frac{\prod_{i=1}^k t_i^{\alpha(n-1)+\alpha\tilde{A}_i - \frac{\alpha\varepsilon}{p_i}}}{\left(\sum_{j=1}^k t_j^\alpha\right)^s} (dT)^\alpha. \tag{12}$$

Proof. Denote by I the left-hand side of (12). Then, applying the Fubini theorem (see, e.g., Yang[12]), we have that

$$I = \frac{1}{\Gamma^n(1+\alpha)} \int_{|\mathbf{x}_1^\alpha|\geq 1} |\mathbf{x}_1^\alpha|^{\tilde{A}_1 - \frac{\varepsilon}{p_1}} \times \left(\frac{1}{\Gamma^{(k-1)n}(1+\alpha)} \int_{|\mathbf{x}_2^\alpha|\geq 1,\dots,|\mathbf{x}_k^\alpha|\geq 1} \frac{\prod_{i=2}^k |\mathbf{x}_i^\alpha|^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{\left(\sum_{j=1}^k |\mathbf{x}_j^\alpha|\right)^s} (\hat{d}^1\mathbf{X})^\alpha \right) (d\mathbf{x}_1)^\alpha.$$

Moreover, applying the n-dimensional spherical coordinates given by (6), we have

$$\begin{aligned}
x_{11}^\alpha &= t_1^\alpha \cos_\alpha(\varphi_{11}^\alpha),\\
x_{12}^\alpha &= t_1^\alpha \sin_\alpha(\varphi_{11}^\alpha)\cos_\alpha(\varphi_{12}^\alpha),\\
x_{13}^\alpha &= t_1^\alpha \sin_\alpha(\varphi_{11}^\alpha)\sin_\alpha(\varphi_{12}^\alpha)\cos_\alpha(\varphi_{13}^\alpha),\\
&\dots\\
x_{1n-1}^\alpha &= t_1^\alpha \sin_\alpha(\varphi_{11}^\alpha)\sin_\alpha(\varphi_{12}^\alpha)\ \dots\ \sin_\alpha(\varphi_{1n-2}^\alpha)\cos_\alpha(\varphi_{1n-1}^\alpha),\\
x_{1n}^\alpha &= t_1^\alpha \sin_\alpha(\varphi_{11}^\alpha)\sin_\alpha(\varphi_{12}^\alpha)\ \dots\ \sin_\alpha(\varphi_{1n-2}^\alpha)\sin_\alpha(\varphi_{1n-1}^\alpha),
\end{aligned}$$

where $1 \leq t_1 \leq \infty$, $0 \leq \varphi_{11}, \varphi_{12}, \dots, \varphi_{1n-2} \leq \pi$, and $0 \leq \varphi_{1n-1} \leq 2\pi$. The local fractional Jacobian of the above transformation is equal to

$$\Gamma(1+\alpha)t_1^{\alpha(n-1)} \sin_\alpha^{n-2}(\varphi_{11}^\alpha) \sin_\alpha^{n-3}(\varphi_{12}^\alpha)\ \dots\ \sin_\alpha^2(\varphi_{1n-3}^\alpha)\sin_\alpha(\varphi_{1n-2}^\alpha).$$

Now, since $|\mathbf{x}_1^\alpha| = t_1^\alpha$, applying the above spherical coordinates, we arrive at the expression

$$I = \frac{1}{\Gamma^{n-1}(1+\alpha)} \int_0^\pi \sin_\alpha^{n-2}(\varphi_{11}^\alpha)(d\varphi_{11})^\alpha \dots \int_0^\pi \sin_\alpha(\varphi_{1n-2}^\alpha)(d\varphi_{1n-2})^\alpha \int_0^{2\pi} (d\varphi_{1n-1})^\alpha$$
$$\times \int_1^\infty t_1^{\alpha\tilde{A}_1 - \alpha\frac{\varepsilon}{p_1} + \alpha(n-1)}$$
$$\times \left(\frac{1}{\Gamma^{(k-1)n}(1+\alpha)} \int_{|\mathbf{x}_2^\alpha|\geq 1, \dots, |\mathbf{x}_k^\alpha|\geq 1} \frac{\prod_{i=2}^k |\mathbf{x}_\mathbf{i}^\alpha|^{\tilde{A}_i - \frac{\varepsilon}{p_i}}}{\left(t_1^\alpha + \sum_{j=2}^k |\mathbf{x}_\mathbf{j}^\alpha|\right)^s} (\hat{d}^1\mathbf{X})^\alpha \right) (dt_1)^\alpha.$$

Finally, repeating the above procedure and using the formula for the area $|S_\alpha^{n-1}|$ (see Remark 1), we obtain identity (12), as claimed. □

3 | A CLASS OF HIGHER DIMENSIONAL HILBERT-TYPE INEQUALITIES

Now, our aim is to generalize the Hilbert-type inequality (4) to hold on a fractal space $(\mathbb{R}_+^{\alpha n})^k$, where $0 < \alpha \leq 1$ and $n \in \mathbb{N}$, $n > 1$. More precisely, we deal with a class of Hilbert-type inequalities involving the homogeneous kernel

$$K(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k) = \left(\sum_{j=1}^k |\mathbf{x}_i^\alpha| \right)^{-s}, \tag{13}$$

where $s > 0$, $0 < \alpha \leq 1$, and $\mathbf{x}_i \in \mathbb{R}_+^n$, $i = 1, 2, \dots, k$.

The first step in deriving Hilbert-type inequalities is the famous Hölder inequality. A multidimensional fractal version of the Hölder inequality (see Krnić and Vuković[6]) asserts that if $\sum_{i=1}^k \frac{1}{p_i} = 1$, $p_i > 1$, and $F_i \in C_\alpha(\Omega^k)$, $i = 1, 2, \dots, k$, where Ω is a fractal space, then holds the inequality

$$\frac{1}{\Gamma^k(1+\alpha)} \int_{\Omega^k} \prod_{i=1}^k F_i(X)(dX)^\alpha \leq \prod_{i=1}^k \left(\frac{1}{\Gamma^k(1+\alpha)} \int_{\Omega^k} F_i^{p_i}(X)(dX)^\alpha \right)^{\frac{1}{p_i}}. \tag{14}$$

Another important step in establishing our results will be Cantor-type spherical coordinates developed in the previous section. In this setting, we will establish a class of inequalities with constants expressed in terms of a fractal Gamma function defined by relation (5). In addition, we will exploit the following integral formula:

$$\frac{1}{\Gamma^k(1+\alpha)} \int_{\mathbb{R}_+^k} \frac{\prod_{i=1}^k x_i^{\alpha(r_i-1)}}{\left(1^\alpha + \sum_{i=1}^k x_i^\alpha\right)^{\sum_{i=1}^{k+1} r_i}} (dX)^\alpha = \frac{\prod_{i=1}^{k+1} \Gamma_\alpha(r_i)}{\Gamma_\alpha\left(\sum_{i=1}^{k+1} r_i\right)}, \; r_i > 0, \tag{15}$$

which has been proved in Krnić and Vuković.[6]

Now, we are able to derive the Hilbert-type inequality involving kernel (13), on a fractal space $(\mathbb{R}_+^{\alpha n})^k$.

Theorem 1. *Let* $\sum_{i=1}^{k}\frac{1}{p_i}=1$, $p_i>1$, $i=1,2,\dots,k$, *and let* A_{ij}, $i,j=1,2,\dots,k$, *be real parameters such that* $\sum_{i=1}^{k}A_{ij}=0$, *for* $j=1,2,\dots,k$. *If* $f_i\in C_\alpha(\mathbb{R}_+^n)$, $i=1,2,\dots,k$, *are nonnegative functions, then holds the inequality*

$$\begin{aligned}&\frac{1}{\Gamma^{kn}(1+\alpha)}\int_{(\mathbb{R}_+^n)^k}\frac{\prod_{i=1}^{k}f_i(\boldsymbol{x_i})}{\left(\sum_{j=1}^{k}|\boldsymbol{x_j^\alpha}|\right)^s}(d\mathbf{X})^\alpha\\&\leq M\prod_{i=1}^{k}\left[\frac{1}{\Gamma^n(1+\alpha)}\int_{\mathbb{R}_+^n}|\boldsymbol{x_i^\alpha}|^{(k-1)n-s+p_i\beta_i}f_i^{p_i}(\boldsymbol{x_i})(d\boldsymbol{x_i})^\alpha\right]^{\frac{1}{p_i}},\end{aligned}\tag{16}$$

where $\beta_i=\sum_{j=1}^{k}A_{ij}$, $i=1,2,\dots,k$, *and*

$$\begin{aligned}M=&\frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}\Gamma_\alpha(s)}\\&\times\prod_{i,j=1,i\neq j}^{k}\Gamma_\alpha(n+p_iA_{ij})^{\frac{1}{p_i}}\prod_{i=1}^{k}\Gamma_\alpha(s-(k-1)n-p_i\beta_i+p_iA_{ii})^{\frac{1}{p_i}}.\end{aligned}\tag{17}$$

Proof. Since $\sum_{i=1}^{k}A_{ij}=0$, for $j=1,2,\dots,k$, the left-hand side of inequality (16) can be transformed in the following way:

$$\begin{aligned}I:=&\frac{1}{\Gamma^{kn}(1+\alpha)}\int_{(\mathbb{R}_+^n)^k}\frac{\prod_{i=1}^{n}f_i(\mathbf{x_i})}{\left(\sum_{j=1}^{k}|\mathbf{x_j^\alpha}|\right)^s}(d\mathbf{X})^\alpha\\=&\frac{1}{\Gamma^{kn}(1+\alpha)}\int_{(\mathbb{R}_+^n)^k}\frac{1}{\left(\sum_{j=1}^{k}|\mathbf{x_j^\alpha}|\right)^s}\prod_{i=1}^{k}(f_i(\mathbf{x_i})\prod_{j=1}^{k}|\mathbf{x_i^\alpha}|^{A_{ij}})(d\mathbf{X})^\alpha.\end{aligned}$$

Now, applying the Hölder inequality (14) and the Fubini theorem (see, e.g., Yang[12]), we have that

$$I\leq\prod_{i=1}^{k}\left(\frac{1}{\Gamma^n(1+\alpha)}\int_{\mathbb{R}_+^n}|\mathbf{x_i^\alpha}|^{p_iA_{ii}}(f_i\omega_i)^{p_i}(\mathbf{x_i})(d\mathbf{x_i})^\alpha\right)^{\frac{1}{p_i}},\tag{18}$$

where

$$\omega_i^{p_i}(\mathbf{x_i})=\frac{1}{\Gamma^{(k-1)n}(1+\alpha)}\int_{(\mathbb{R}_+^n)^{k-1}}\frac{\prod_{j=1,j\neq i}^{k}|\mathbf{x_j^\alpha}|^{p_iA_{ij}}}{\left(\sum_{j=1}^{k}|\mathbf{x_j^\alpha}|\right)^s}(\hat{d}^i\mathbf{X})^\alpha.$$

Without loss of generality, it suffices to estimate the weight $\omega_1(\mathbf{x_1})$. Namely, passing to n-dimensional spherical coordinates (see also the proof of Lemma 2), it follows that

$$\omega_1^{p_1}(\mathbf{x_1})=\frac{|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}}\int_{\mathbb{R}_+^{k-1}}\frac{\prod_{j=2}^{k}t_j^{\alpha(n-1)+\alpha p_1A_{1j}}}{\left(|\mathbf{x_1^\alpha}|+\sum_{j=2}^{k}t_j^\alpha\right)^s}(\hat{d}^1T)^\alpha.$$

Now, by a suitable change of variables $u_j = t_j/|\mathbf{x_1}|$, $(dt_j)^\alpha = |\mathbf{x_1^\alpha}|(du_j)^\alpha$, $j = 2, 3, \dots, k$, the previous formula reduces to

$$\omega_1^{p_1}(\mathbf{x_1}) = \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}}|\mathbf{x_1^\alpha}|^{(k-1)n-s+p_1(\beta_1-A_{11})} \times \frac{1}{\Gamma^{k-1}(1+\alpha)}\int_{\mathbb{R}_+^{k-1}} \frac{\prod_{j=2}^k u_j^{\alpha(n-1)+\alpha p_1 A_{1j}}}{\left(1^\alpha + \sum_{j=2}^k u_j^\alpha\right)^s}(\hat{d}^1 U)^\alpha, \tag{19}$$

where $\beta_i = \sum_{j=1}^{k} A_{ij}$, for $i = 1, 2, \dots, k$. Moreover, applying the integral formula (15), we obtain

$$\omega_1^{p_1}(\mathbf{x_1}) = \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}\Gamma_\alpha(s)}\Gamma_\alpha(s-(k-1)n-p_1(\beta_1-A_{11})) \times \prod_{j=2}^k \Gamma_\alpha(n+p_1A_{1j})|\mathbf{x_1^\alpha}|^{(k-1)n-s+p_1(\beta_1-A_{11})},$$

and similarly,

$$\omega_i^{p_i}(\mathbf{x_i}) = \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}\Gamma_\alpha(s)}\Gamma_\alpha(s-(k-1)n-p_i(\beta_i-A_{ii})) \times \prod_{j=1, j\neq i}^k \Gamma_\alpha(n+p_iA_{ij})|\mathbf{x_i^\alpha}|^{(k-1)n-s+p_i(\beta_i-A_{ii})}, \tag{20}$$

for $i = 2, 3, \dots, k$. Finally, combining relations (18)–(20), we obtain (16), as claimed. □

We have already mentioned that the constant $\pi/\sin\frac{\pi}{p}$ appearing in inequality (1) is the best possible. The natural question that now arises is whether it is possible to find the appropriate conditions under which the constant M, given by (17), is the best possible in inequality (16). To do this, we will first simplify the form of M, so that the product of fractional Gamma functions does not include exponents. Therefore, we set the following conditions on the parameters A_{ij}:

$$n + p_jA_{ji} = s - (k-1)n - p_i(\beta_i - A_{ii}), \ \ j \neq i, \ \ i, j \in \{1, 2, \dots, k\}. \tag{21}$$

So if (21) holds, then the constant M defined by (17) reduces to

$$M^* = \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}\Gamma_\alpha(s)}\prod_{i=1}^k \Gamma_\alpha(n+\tilde{A}_i), \tag{22}$$

where the parameters $\tilde{A}_i$, $i = 1, 2, \dots, k$, are defined by

$$\tilde{A}_i = p_jA_{ji}, \ \ i \neq j. \tag{23}$$

Moreover, it is easy to see that the parameters $\tilde{A}_i$ satisfy relation $\sum_{i=1}^k \tilde{A}_i = s - kn$, so in the above-described setting, inequality (16) reduces to

$$\frac{1}{\Gamma^{kn}(1+\alpha)}\int_{(\mathbb{R}_+^n)^k} \frac{\prod_{i=1}^k f_i(\mathbf{x_i})}{\left(\sum_{j=1}^k |\mathbf{x_j^\alpha}|\right)^s}(d\mathbf{X})^\alpha \leq M^* \prod_{i=1}^k \left[\frac{1}{\Gamma^n(1+\alpha)}\int_{\mathbb{R}_+^n} |\mathbf{x_i^\alpha}|^{-n-p_i\tilde{A}_i} f_i^{p_i}(\mathbf{x_i})(d\mathbf{x_i})^\alpha\right]^{\frac{1}{p_i}}. \tag{24}$$

Now, our goal is to prove that M^* is the best possible constant in inequality (24).

Theorem 2. *Let* $\sum_{i=1}^{k}\frac{1}{p_i}=1$, $p_i>1$, $i=1,2,\dots,k$, *and let* A_{ij}, $i,j=1,2,\dots,k$, *be real parameters such that* $\sum_{i=1}^{k}A_{ij}=0$, *for* $j=1,2,\dots,k$. *Further, suppose that (21) holds and let* $\tilde{A}_i\le\frac{1}{2}$, $i=1,2,\dots,k$, *where the parameters* $\tilde{A}_i$ *are defined by (23). Then, the constant* M^* *is the best possible in (24).*

Proof. Suppose contrary that M^* is not the best possible constant in (24). This means that there exists a positive real number $M_1<M^*$ such that (24) still holds when M^* is replaced by M_1.

Now, consider functions $\tilde{f}_i:\mathbb{R}_+^n\mapsto\mathbb{R}^\alpha$ defined by

$$\tilde{f}_i(\mathbf{x_i})=\begin{cases}0^\alpha, & |\mathbf{x_i^\alpha}|<1\\ |\mathbf{x_i^\alpha}|^{\tilde{A}_i-\frac{\varepsilon}{p_i}}, & |\mathbf{x_i^\alpha}|\ge 1\end{cases},\quad i=1,\dots,k,$$

where $\varepsilon>0$ is small enough. Our intention is to substitute these functions in inequality (24), where M^* is replaced by M_1.

Utilizing the n-dimensional spherical coordinates, we have that the right-hand side of (24) reduces to

$$\begin{aligned}&M_1\prod_{i=1}^{k}\left[\frac{1}{\Gamma^n(1+\alpha)}\int_{|\mathbf{x_i^\alpha}|\ge1}|\mathbf{x_i^\alpha}|^{-n-\varepsilon}(d\mathbf{x_i})^\alpha\right]^{\frac{1}{p_i}}\\ &=\frac{M_1|S_\alpha^{n-1}|}{2^{\alpha n}}\int_1^\infty t^{-\alpha n-\alpha\varepsilon}t^{\alpha n-\alpha}(dt)^\alpha=\frac{M_1|S_\alpha^{n-1}|}{2^{\alpha n}\varepsilon^\alpha}.\end{aligned}\tag{25}$$

On the other hand, let J stand for the left-hand side of inequality (24) for the above choice of functions $\tilde{f}_i$. Then, applying Lemma 2 and the Fubini theorem, we obtain

$$\begin{aligned}J&=\frac{1}{\Gamma^{kn}(1+\alpha)}\int_{|\mathbf{x_1^\alpha}|\ge1,\dots,|\mathbf{x_k^\alpha}|\ge1}\frac{\prod_{i=1}^{k}|\mathbf{x_i^\alpha}|^{\tilde{A}_i-\frac{\varepsilon}{p_i}}}{\left(\sum_{j=1}^{k}|\mathbf{x_j^\alpha}|\right)^s}(d\mathbf{X})^\alpha\\ &=\frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^k}{2^{\alpha kn}}\int_1^\infty t_1^{\alpha(n-1)+\alpha\tilde{A}_1-\frac{\alpha\varepsilon}{p_1}}\\ &\times\left(\frac{1}{\Gamma^{k-1}(1+\alpha)}\int_{(1,\infty)^{k-1}}\frac{\prod_{i=2}^{k}t_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(\sum_{j=1}^{k}t_j^\alpha\right)^s}(\hat{d}^1T)^\alpha\right)(dt_1)^\alpha.\end{aligned}$$

Furthermore, using the change of variables $u_i=\frac{t_i}{t_1}$, $(dt_i)^\alpha=t_1^\alpha(du_i)^\alpha$, $i=2,3,\dots,k$, we arrive at the relation

$$\begin{aligned}J&=\frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^k}{2^{\alpha kn}}\int_1^\infty t_1^{-\alpha\varepsilon-\alpha}\\ &\times\left(\frac{1}{\Gamma^{k-1}(1+\alpha)}\int_{(1/t_1,\infty)^{k-1}}\frac{\prod_{i=2}^{k}u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^{k}u_i^\alpha\right)^s}(\hat{d}^1U)^\alpha\right)(dt_1)^\alpha.\end{aligned}$$

Now, it is easy to check the validity of the inequality

$$
\begin{aligned}
J \geq{}& \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^k}{2^{\alpha k n}} \int_1^\infty t_1^{-\alpha\varepsilon-\alpha} \\
&\times \left(\frac{1}{\Gamma^{k-1}(1+\alpha)} \int_{(0,\infty)^{k-1}} \frac{\prod_{i=2}^k u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^k u_i^\alpha\right)^s} (\hat{d}^1 U)^\alpha \right) (dt_1)^\alpha \\
&- \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^k}{2^{\alpha k n}} \int_1^\infty t_1^{-\alpha\varepsilon-\alpha} \\
&\times \left(\frac{1}{\Gamma^{k-1}(1+\alpha)} \sum_{j=2}^k \int_{\mathbb{D}_j} \frac{\prod_{i=2}^k u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^k u_i^\alpha\right)^s} (\hat{d}^1 U)^\alpha \right) (dt_1)^\alpha,
\end{aligned}
$$

where $\mathbb{D}_j = \{(u_2, u_3, \dots, u_k); 0 < u_j \leq \frac{1}{t_1}, 0 < u_l < \infty, l \neq j\}$, $j = 2, 3, \dots, k$. In addition, due to integral formula (15), the above inequality can be transformed in the following way:

$$
\begin{aligned}
J \geq{}& \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^k}{\varepsilon^\alpha 2^{\alpha k n}\Gamma_\alpha(s)} \Gamma_\alpha\left(n+\tilde{A}_1+\varepsilon-\frac{\varepsilon}{p_1}\right) \prod_{i=2}^k \Gamma_\alpha\left(n+\tilde{A}_i-\frac{\varepsilon}{p_i}\right) \\
&- \frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^k}{2^{\alpha k n}} \int_1^\infty t_1^{-\alpha\varepsilon-\alpha} \\
&\times \left(\frac{1}{\Gamma^{k-1}(1+\alpha)} \sum_{j=2}^k \int_{\mathbb{D}_j} \frac{\prod_{i=2}^k u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^k u_i^\alpha\right)^s} (\hat{d}^1 U)^\alpha \right) (dt_1)^\alpha.
\end{aligned}
\tag{26}
$$

Clearly, without loss of generality, it suffices to estimate the integral

$$
\frac{1}{\Gamma^{k-1}(1+\alpha)} \int_{\mathbb{D}_2} \frac{\prod_{i=2}^k u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^k u_i^\alpha\right)^s} (\hat{d}^1 U)^\alpha.
$$

More precisely, we have that

$$
\begin{aligned}
&\frac{1}{\Gamma^{k-1}(1+\alpha)} \int_{\mathbb{D}_2} \frac{\prod_{i=2}^k u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^k u_i^\alpha\right)^s} (\hat{d}^1 U)^\alpha \\
&\leq \frac{1}{\Gamma^{k-2}(1+\alpha)} \int_{(0,\infty)^{k-2}} \frac{\prod_{i=3}^k u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=3}^k u_i^\alpha\right)^s} (du_3)^\alpha (du_4)^\alpha \dots (du_k)^\alpha \\
&\quad \times \frac{1}{\Gamma(1+\alpha)} \int_0^{1/t_1} u_2^{\alpha(n-1)+\alpha\tilde{A}_2-\frac{\alpha\varepsilon}{p_2}} (du_2)^\alpha.
\end{aligned}
$$

Furthermore, since $\tilde{A}_i \leq \frac{1}{2}$, $i = 1, 2, \ldots, k$, taking into account a change of variable formula (3) with $y = u_2^{n-\frac{\varepsilon}{p_2}+\frac{1}{2}}$, as well as the integral formula (15), we obtain the following estimate:

$$\frac{1}{\Gamma^{k-1}(1+\alpha)}\int_{\mathbb{D}_2}\frac{\prod_{i=2}^{k}u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^{k}u_i^\alpha\right)^s}(\hat{d}^1U)^\alpha$$
$$\leq \frac{\Gamma_\alpha\left(2n+\tilde{A}_1+\tilde{A}_2+\varepsilon-\frac{\varepsilon}{p_1}-\frac{\varepsilon}{p_2}\right)}{\Gamma_\alpha(s)\left(n-\frac{\varepsilon}{p_2}+\frac{1}{2}\right)^\alpha}\prod_{i=3}^{k}\Gamma_\alpha\left(n+\tilde{A}_i-\frac{\varepsilon}{p_i}\right)t_1^{\frac{\alpha\varepsilon}{p_2}-\alpha n-\frac{\alpha}{2}}.$$

Consequently, it follows that

$$\frac{1}{\Gamma^{k-1}(1+\alpha)}\int_{\mathbb{D}_i}\frac{\prod_{i=2}^{k}u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^{k}u_i^\alpha\right)^s}(\hat{d}^1U)^\alpha \leq t_1^{\frac{\alpha\varepsilon}{p_2}-\alpha n-\frac{\alpha}{2}}O_i(1),$$

for $i = 1, 2, \ldots, k$, when $\varepsilon \to 0$. Now, taking into account the above inequalities and making use of transformation formula (3) with $y = t_1^{-n-\frac{1}{2}-\varepsilon\left(1-\frac{1}{p_2}\right)}$, we arrive at the following relation:

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_1^\infty t_1^{-\alpha\varepsilon-\alpha}\\
&\times\left(\frac{1}{\Gamma^{k-1}(1+\alpha)}\sum_{i=2}^{k}\int_{\mathbb{D}_i}\frac{\prod_{i=2}^{k}u_i^{\alpha(n-1)+\alpha\tilde{A}_i-\frac{\alpha\varepsilon}{p_i}}}{\left(1^\alpha+\sum_{i=2}^{k}u_i^\alpha\right)^s}(\hat{d}^1U)^\alpha\right)(dt_1)^\alpha\\
&\leq\frac{1}{\Gamma(1+\alpha)}\int_1^\infty t_1^{-\alpha\varepsilon-\alpha}t_1^{\frac{\alpha\varepsilon}{p_2}-\alpha n-\frac{\alpha}{2}}\sum_{i=2}^{k}O_i(1)(dt_1)^\alpha\\
&=\frac{O(1)}{\Gamma(1+\alpha)}\int_1^\infty t_1^{-\frac{3\alpha}{2}-\alpha n-\frac{\alpha\varepsilon}{q_2}}(dt_1)^\alpha = O(1),\ \text{when } \varepsilon\to 0^+.
\end{aligned}\tag{27}$$

Finally, taking into account (25)–(27), respectively, it follows that $M^* \leq M_1$ when ε tends to 0, which is opposite to our initial assumption $M_1 < M^*$. Hence, M^* is the best possible constant in (24). □

We will now consider a special case of Theorem 1 which represents extension of the corresponding result from our recent paper.[6] Namely, consider the parameters A_{ij}, $i, j = 1, 2, \ldots k$, defined by

$$A_{ii} = \frac{(nk-s)(p_i-1)}{p_i^2},\ A_{ij} = \frac{s-nk}{p_ip_j},\ i \neq j,\ i, j = 1, 2, \ldots, k. \tag{28}$$

Then, we have that

$$\sum_{i=1}^{k}A_{ij} = \sum_{i\neq j}\frac{s-nk}{p_ip_j}+\frac{(nk-s)(p_j-1)}{p_j^2} = \frac{s-nk}{p_j}\left(\sum_{i=1}^{k}\frac{1}{p_i}-1\right) = 0,$$

for $j = 1, 2, \ldots, k$. In addition, due to the symmetry, we also have that $\beta_i = \sum_{j=1}^{k}A_{ij} = 0$. Therefore, the above parameters, defined by (28), satisfy conditions given by (21). In order to obtain the inequality with a best possible constant, we need to take into account the set of conditions $\tilde{A}_i \leq \frac{1}{2}$, $i = 1, 2, \ldots, n$, which is in this setting equivalent to $s \leq \min_{1\leq i\leq k}\{nk+\frac{1}{2}p_i\}$. Thus, Theorems 1 and 2 provide the following consequence.

Corollary 1. *Let* $\sum_{i=1}^{k}\frac{1}{p_i}=1$, $p_i>1$, $i=1,2,\dots,k$. *If* $s\le \min_{1\le i\le k}\{nk+\frac{1}{2}p_i\}$, *then holds the inequality*

$$\begin{aligned}&\frac{1}{\Gamma^{kn}(1+\alpha)}\int_{(\mathbb{R}_+^n)^k}\frac{\prod_{i=1}^k f_i(\boldsymbol{x_i})}{\left(\sum_{j=2}^k|\boldsymbol{x_j^\alpha}|\right)^s}(d\boldsymbol{X})^\alpha\\ &\le L^*\prod_{i=1}^k\left[\frac{1}{\Gamma^n(1+\alpha)}\int_{\mathbb{R}_+^n}|\boldsymbol{x_i^\alpha}|^{n(k-1)-s}f_i^{p_i}(\boldsymbol{x_i})(d\boldsymbol{x_i})^\alpha\right]^{\frac{1}{p_i}},\end{aligned} \tag{29}$$

where the constant

$$L^*=\frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{n-1}|^{k-1}}{2^{\alpha(k-1)n}\Gamma_\alpha(s)}\prod_{i=1}^k\Gamma_\alpha\left(\frac{s-nk+np_i}{p_i}\right)$$

is the best possible.

Remark 2. It should be noted here that Corollary 1 is a higher dimensional extension of inequality (4) stated in the Introduction. Namely, if $n=1$, then inequality (29) reduces to

$$\begin{aligned}&\frac{1}{\Gamma^{k}(1+\alpha)}\int_{(\mathbb{R}_+)^k}\frac{\prod_{i=1}^k f_i(x_i)}{\left(\sum_{j=2}^k x_j^\alpha\right)^s}(dX)^\alpha\\ &\le L\prod_{i=1}^k\left[\frac{1}{\Gamma(1+\alpha)}\int_{\mathbb{R}_+}x_i^{\alpha(k-1)-\alpha s}f_i^{p_i}(x_i)(dx_i)^\alpha\right]^{\frac{1}{p_i}},\end{aligned}$$

where the constant

$$L=\frac{\Gamma^{k-1}(1+\alpha)|S_\alpha^{0}|^{k-1}}{2^{\alpha(k-1)}\Gamma_\alpha(s)}\prod_{i=1}^k\Gamma_\alpha\left(\frac{s-k+p_i}{p_i}\right) \tag{30}$$

is the best possible. It remains to show that the constant L coincides with the constant appearing on the right-hand side of inequality (4). To see this, note that

$$|V_\alpha^1|=\frac{\Gamma(1+\alpha)|S_\alpha^0|\Gamma(1)}{\Gamma(1+\alpha)}=|S_\alpha^0|,$$

by (11). On the other hand, the volume of unit ball V_α^1 is equal to

$$\begin{aligned}|V_\alpha^1|&=\frac{1}{\Gamma(1+\alpha)}\int_{V^1}dV^1=\frac{1}{\Gamma(1+\alpha)}\int_{-1}^{1}(dx)^\alpha\\ &=\frac{1^\alpha-(-1)^\alpha}{\Gamma(1+\alpha)}=\frac{2^\alpha}{\Gamma(1+\alpha)}.\end{aligned}$$

Hence, substituting the last relation in (30), we obtain $L=\frac{1}{\Gamma_\alpha(s)}\prod_{i=1}^k\Gamma_\alpha\left(\frac{s-k+p_i}{p_i}\right)$, as claimed.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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