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A discussion on a generalized Geraghty multi-valued mappings and applications

Hojjat Afshari[1], Maryam Atapour[1] and Erdal Karapınar[2,3,4,5*]

*Correspondence:
erdalkarapinar@duytan.edu.vn;
erdalkarapinar@yahoo.com
[2]Institute of Research and
Development, Duy Tan University,
550000 Da Nang, Vietnam
[3]Faculty of Natural Sciences, Duy
Tan University, 550000 Da Nang,
Vietnam
Full list of author information is
available at the end of the article

Abstract

This research intends to investigate the existence results for both coincidence points and common fixed point of generalized Geraghty multi-valued mappings endowed with a directed graph. The proven results are supported by an example. We also consider fractional integral equations as an application.

Keywords: Directed graph; Generalized Geraghty multi-valued mappings; Coincidence point; Fractional integral equation

1 Introduction and preliminaries

The metric fixed point theory, which has wide common application potential in distinct quantitative sciences, develops itself in parallel with the developments in applied areas [8, 14–16]. One of the examples of this trend was given by Echenique [12] who considered metric fixed point theory in the framework of graphs. Later, Espinola and Kirk [13] applied some well-known fixed point results and techniques to the graph theory. On the other hand, in 2008, Jachymski [17] proposed a new contraction, the concept graph contraction, in order to generalize the distinguished Banach contraction principle.

In this paper, we shall consider generalized Geraghty multi-valued mappings in the context of complete metric spaces endowed with a graph. After investigating the existence and uniqueness of a fixed point for such mappings, we shall consider a fractional integral equation [1–7, 9, 10, 18–21, 23–25, 27] and we solve this equation via our obtained results.

Throughout the paper, we presume that all considered sets and subsets are non-empty. The pair (M, d) denotes a metric space and (M^*, d) represent complete metric space. Let Δ be the diagonal of $M \times M$. Assume that Γ is a directed graph with vertex set $V(\Gamma)$ which is coincided with M and edge set $E(\Gamma)$ which contains Δ. Suppose there is not any parallel edges in G. We shall say that $g : M \to M$ is a Banach G-contraction if:

(i) $(x, y) \in E(\Gamma)$ implies $(g(x), g(y)) \in E(\Gamma)$, for all $x, y \in M$,

(ii) $\exists 0 < \alpha < 1$ such that, for $x, y \in M$,

$$(x, y) \in E(\Gamma) \quad \Rightarrow \quad d\big(g(x), g(y)\big) \le \alpha d(x, y).$$

Assume that (M, d) be a metric space and $P_{b,cl}(M)$ be the family of bounded and closed sets in M. For $x \in M$ and $M_1, M_2 \in P_{b,cl}(M)$,

$$D(x, M_1) = \inf_{a \in M_1} d(x, a),$$

$$D(M_1, M_2) = \sup_{a \in M_1} D(M_1, M_2).$$

Define $H : P_{b,cl}(M) \times P_{b,cl}(M) \to [0, \infty)$ with

$$H(M_1, M_2) = \max\left\{ \sup_{x \in M_1} d(x, M_2), \sup_{y \in M_2} d(y, M_2) \right\},$$

for $M_1, M_2 \in P_{b,cl}(M)$. H is the famous Hausdorff metric.

Lemma 1.1 ([11]) *Let (M, d) be a metric space. For $M_1, M_2 \in P_{b,cl}(M)$ and $x, y \in M$, the following relations hold*:
  (1) $D(x, M_2) \leq d(x, b), b \in M_2,$
  (2) $D(x, M_2) \leq H(M_1, M_2),$
  (3) $D(x, M_1) \leq d(x, y) + D(y, M_1),$
  (4) *for all $\omega \in M_1$, there exists some $\varphi \in M_2$ such that $d(\omega, \varphi) \leq qH(M_1, M_2)$, where*
      $q > 1$.

Definition 1.2 ([28]) For a set M, let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with $V(\Gamma) = M$. Then:
  (i) a mapping $T : M \to P_{b,cl}(M)$ is said to be graph preserving if it preserves the edges, i.e.,

   $$\text{if } (x, y) \in E(\Gamma), \text{then } (u, v) \in E(\Gamma) \text{ for all } u \in Tx \text{ and } v \in Ty,$$

  (ii) mappings $S, T : M \to P_{b,cl}(M)$ are said to be mixed graph preserving respect to h_1, h_2 if the preserve the edges, i.e., for $x, y \in M$, if $(h_1(x), h_2(y)) \in E(\Gamma)$, then $(x, \tau) \in E(\Gamma)$ for all $x \in Tx$ and $\tau \in Sy$ and

   $$\text{if } \big(h_2(x), h_1(y)\big) \in E(\Gamma), \text{then } (b, r) \in E(\Gamma) \text{ for all } b \in Sx \text{ and } r \in Ty.$$

Consider the class

$$\Psi := \big\{ \psi : [0, \infty) \to [0, \infty), \text{increasing, continuous, and } \psi(ct) \leq c\psi(t),$$
$$\text{for all } c > 1, \text{with } \psi(0) = 0 \big\}.$$

Let $\mathcal{F}$ denote the family of all functions $\beta : [0, \infty) \to [0, 1)$.

2 Main results

We shall start this section by introducing the notion of the generalized Geraghty-type G-multi-valued mapping.

Definition 2.1 On a metric space (M, d), let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph with vertex set $V(\Gamma) = M$ and the set $E(\Gamma)$ of its edges such that $E(\Gamma) \supseteq \Delta$. For $h_1, h_2 : M \to M$ and

$S, T : M \to P_{b,cl}(M)$; S, T are called a generalized h_1, h_2-Geraghty-type G-multi-valued mapping provided that

(i) S, T are mixed graph preserving respect to h_1, h_2;

(ii) for $x, y \in M$ with $(h_1(x), h_2(y)) \in E(\Gamma)$, there exists $L \geq 0$ such that, for

$$P\big(h_1(x), h_2(y)\big) = \max\bigg\{ d\big(h_1(x), h_2(y)\big), D\big(h_1(x), Tx\big), D\big(h_2(y), Sy\big),$$

$$\frac{D(h_1(x), Tx) + D(h_2(y), Sy)}{2} \bigg\} \quad \text{and} \tag{1}$$

$$Q\big(h_1(x), h_2(y)\big) = \min\big\{ D\big(h_1(x), Sy\big), D\big(h_2(y), Tx\big) \big\}, \tag{2}$$

we have

$$\psi\big(H(Sx, Ty)\big) \leq \gamma\big(\psi\big(P\big(h_1(x), h_2(y)\big)\big)\big)\psi\big(P\big(h_1(x), h_2(y)\big)\big)$$

$$+ L\phi\big(Q\big(h_1(x), h_2(y)\big)\big), \tag{3}$$

and if $(h_2(x), h_1(y)) \in E(\Gamma)$ then

$$\psi\big(H(Sx, Ty)\big) \leq \gamma\big(\psi\big(P\big(h_2(x), h_1(y)\big)\big)\big)\psi\big(P\big(h_2(x), h_1(y)\big)\big)$$

$$+ L\phi\big(Q\big(h_2(x), h_1(y)\big)\big), \tag{4}$$

where $\gamma \in \mathcal{F}$ and $\psi, \phi \in \psi$.

Theorem 2.2 *Let (M, d) be a complete metric space endowed with a graph $\Gamma = (V(\Gamma), E(\Gamma))$, $h_1, h_2 : M \to M$ are surjective and $S, T : M \to P_{b,cl}(M)$ generalized h_1, h_2-Geraghty-type G-multi-valued mapping in (M, d). Suppose*

(i) *$\exists\, x_0 \in M$ such that $(h_1(x_0), u) \in E(\Gamma)$ for some $u \in Tx_0$,*

(ii) *if $(h_1(x), h_2(y)) \in E(\Gamma)$, then $(e, f) \in E(\Gamma)$ for all $e \in Tx$ and $f \in Sy$ and if $(h_2(x), h_1(y)) \in E(\Gamma)$, then $(w, r) \in E(\Gamma)$ for all $w \in Sx$ and $r \in Ty$,*

(iii) *for $\{x_n\}_{n \in \mathbb{N}}$ in M, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(\Gamma)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ such that $(x_{n_k}, x) \in E(\Gamma)$ for $n_k \in \mathbb{N}$.*

Then there exist $u, v \in M$ such that $h_1(u) \in Tu$ or $h_2(v) \in Sv$.

Proof Regarding that h_2 is surjective, one can find $x_1 \in M$ such that $h_2(x_1) \in Tx_0$ and $(h_1(x_0), h_2(x_1)) \in E(\Gamma)$. Let $q = \dfrac{1}{\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}}$. Then $q > 1$, so

$$0 < D\big(h_2(x_1), Sx_1\big) \leq H(Tx_0, Sx_1) < qH(Tx_0, Sx_1).$$

By Lemma 1.1 and (ii), h_1 is surjective, which implies that there exists $x_2 \in M$ with $h_1(x_2) \in Sx_1$ and $(h_2(x_1), h_1(x_2)) \in E(\Gamma)$, hence

$$\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big) < \psi\big(qH(Tx_0, Sx_1)\big) \leq q\psi\big(H(Tx_0, Sx_1)\big)$$

$$\leq q\gamma\big(\psi\big(P\big(h_1(x_0), h_2(x_1)\big)\big)\big)\psi\big(P\big(h_1(x_0), h_2(x_1)\big)\big)$$

$$+ qL\phi\big(Q\big(h_1(x_0), h_2(x_1)\big)\big), \tag{5}$$

where

$$P\big(h_1(x_0), h_2(x_1)\big) = \max\bigg\{d\big(h_1(x_0), h_2(x_1)\big), D\big(h_1(x_0), Tx_0\big), D\big(h_2(x_1), Sx_1\big),$$

$$\frac{D(h_1(x_0), Tx_0) + D(h_2(x_1), Sx_1)}{2}\bigg\}$$

$$= \max\big\{d\big(h_1(x_0), h_2(x_1)\big), d\big(h_2(x_1), h_1(x_2)\big)\big\} \tag{6}$$

and

$$Q\big(h_1(x_0), h_2(x_1)\big) = \min\big\{D\big(h_1(x_0), Sx_0\big), D\big(h_2(x_1), Tx_0\big)\big\}$$

$$\leq \min\big\{d\big(h_1(x_0), h_2(x_1)\big), d\big(h_2(x_1), h_2(x_1)\big)\big\} = 0. \tag{7}$$

If we have

$$\max\big\{d\big(h_1(x_0), h_2(x_1)\big), d\big(h_2(x_1), h_1(x_2)\big)\big\} = d\big(h_2(x_1), h_1(x_2)\big),$$

then by (5) we get

$$\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big) < \psi\big(qH(Tx_0, Sx_1)\big) \leq q\psi\big(H(Tx_0, Sx_1)\big)$$

$$\leq q\gamma\big(\psi\big(P\big(h_1(x_0), h_2(x_1)\big)\big)\big)\psi\big(P\big(h_1(x_0), h_2(x_1)\big)\big)$$

$$+ qL\phi\big(Q\big(h_1(x_0), h_2(x_1)\big)\big)$$

$$= q\gamma\big(\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big)\big)\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big)$$

$$= \sqrt{\gamma\big(\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big)\big)}\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big)$$

$$< \psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big),$$

a contradiction.

Hence, we obtain $\max\{d(h_1(x_0), h_2(x_1)), d(h_2(x_1), h_1(x_2))\} = d(h_1(x_0), h_2(x_1))$ and so by (5)

$$\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big) \leq \sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big). \tag{8}$$

Keeping $\psi \in \psi$, in mind together with $\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))} < 1$, we get

$$\psi\left(\frac{1}{\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}}d\big(h_2(x_1), h_1(x_2)\big)\right)$$

$$\leq \frac{1}{\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}}\psi\big(d\big(h_2(x_1), h_1(x_2)\big)\big)$$

$$< \psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big). \tag{9}$$

Since ψ is increasing, we have

$$d\big(h_2(x_1), h_1(x_2)\big) \leq \sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}d\big(h_1(x_0), h_2(x_1)\big).$$

Recall that $h_1(x_2) \in Sx_1$ and $h_2(x_1) \in Tx_0$. Choose

$$q_1 = \frac{\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}\psi(d(h_1(x_0), h_2(x_1)))}{\psi(d(h_2(x_1), h_1(x_2)))}.$$

By (8), we have $q_1 > 1$. If $h_1(x_2) \in Tx_2$, then x_2 forms a coincidence point for h_1 and T. We presume that $h_1(x_2) \notin Tx_2$. We get

$$0 < \psi\big(d\big(h_1(x_2), Tx_2\big)\big) \leq \psi\big(H(Sx_1, Tx_2)\big) < q_1\psi\big(H(Sx_1, Tx_2)\big).$$

Hence, there exists $h_2(x_3) \in Tx_2$ such that $(h_1(x_2), h_2(x_3)) \in E(\Gamma)$ and

$$\begin{aligned}
\psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big) &< q_1\psi\big(H(Sx_1, Tx_2)\big) \\
&\leq q_1\gamma\big(\psi\big(P\big(h_1(x_1), h_2(x_2)\big)\big)\big)\psi\big(P\big(h_1(x_1), h_2(x_2)\big)\big) \\
&\quad + q_1 L\phi\big(Q\big(h_1(x_1), h_2(x_2)\big)\big).
\end{aligned}$$

Similarly, $P(h_1(x_1), h_2(x_2)) \leq d(h_1(x_1), h_2(x_2))$ and $Q(h_1(x_1), h_2(x_2)) = 0$. By (8) and a property of (γ), we have

$$\begin{aligned}
\psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big) & \\
&\leq \sqrt{\gamma\big(\psi\big(d\big(h_1(x_1), h_2(x_2)\big)\big)\big)}\psi\big(d\big(h_1(x_1), h_2(x_2)\big)\big) \\
&\leq \sqrt{\gamma\big(\psi\big(d\big(h_1(x_1), h_2(x_2)\big)\big)\big)}\sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big). \quad (10)
\end{aligned}$$

By (8) and $\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))} < 1$, we have

$$\psi\big(d\big(h_1(x_1), h_2(x_2)\big)\big) \leq \psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big).$$

The function γ is increasing, by (10), we obtain

$$\psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big) \leq \big(\sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}\big)^2 \psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big). \quad (11)$$

Again, by (9),

$$d\big(h_1(x_2), h_2(x_3)\big) \leq \big(\sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}\big)^2 d\big(h_1(x_0), h_2(x_1)\big)$$

manifestly, $h_1(x_2) \neq h_2(x_3)$. Take

$$q_2 = \frac{(\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))})^2 \psi(d(h_1(x_0), h_2(x_1)))}{\psi(d(h_1(x_2), h_2(x_3)))}.$$

Then $q_2 > 1$. If $h_2(x_3) \in Sx_3$, then x_3 is a coincidence point of h_2 and S. Assume that $h_2(x_3) \notin Sx_3$. Then

$$0 < \psi\big(d\big(h_2(x_3), Sx_3\big)\big) \leq \psi\big(H(Tx_2, Sx_3)\big) < q_2\psi\big(H(Tx_2, Sx_3)\big).$$

Thus there exists $h_1(x_4) \in Sx_3$ such that $(h_2(x_3), h_1(x_4)) \in E(\Gamma)$ and

$$
\begin{aligned}
\psi\big(d\big(h_2(x_3), h_1(x_4)\big)\big) &< q_2 \psi\big(H(Tx_2, Sx_3)\big) \\
&\leq q_2 \gamma\big(\psi\big(P\big(h_1(x_2), h_2(x_3)\big)\big)\big) \psi\big(P\big(h_1(x_2), h_2(x_3)\big)\big) \\
&\quad + q_2 L \phi\big(Q\big(h_1(x_2), h_2(x_3)\big)\big).
\end{aligned}
\tag{12}
$$

Similarly, $P(h_1(x_2), h_2(x_3)) \leq d(h_1(x_2), h_2(x_3))$ and $Q(h_1(x_2), h_2(x_3)) = 0$.
  So, by (12),

$$
\begin{aligned}
&\psi\big(d\big(h_2(x_3), h_1(x_4)\big)\big) \\
&\leq \sqrt{\gamma\big(\psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big)\big)} \psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big) \\
&\leq \sqrt{\gamma\big(\psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big)\big)} \Big(\sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}\Big)^2 \\
&\quad \times \psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big).
\end{aligned}
\tag{13}
$$

By (11) and $\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}^2 < 1$, we have

$$
\psi\big(d\big(h_1(x_2), h_2(x_3)\big)\big) \leq \psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big).
$$

Again, γ is increasing, so using (13),

$$
d\big(h_2(x_3), h_1(x_4)\big) \leq \Big(\sqrt{\gamma\big(\psi\big(d\big(h_1(x_0), h_2(x_1)\big)\big)\big)}\Big)^3 d\big(h_1(x_0), h_2(x_1)\big).
$$

It is clear that $h_2(x_3) \neq h_1(x_2)$. Put

$$
q_3 = \frac{\big(\sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}\big)^3 \psi(d(h_1(x_0), h_2(x_1)))}{\psi(d(h_1(x_2), h_2(x_3)))}.
$$

Then $q_3 > 1$. Continuing this process, we construct a sequence $\{h(x_n)\}$ in M such that

$$
h_1(x_{2n}) \in Sx_{2n-1}, \qquad h_2(x_{2n-1}) \in T(x_{2n-2})
$$
$$
\big(h_1(x_{2n-2}), h_1(x_{2n-1})\big), \big(h_1(x_{2n-1}), h_1(x_{2n})\big) \in E(\Gamma).
$$

Define the sequence $\{h(x_n)\}$ as follows:

$$
h(x_n) = \begin{cases} h_1(x_n), & n \text{ is even,} \\ h_2(x_n), & n \text{ is odd.} \end{cases}
$$

Let $t = \sqrt{\gamma(\psi(d(h_1(x_0), h_2(x_1))))}$, then $0 < t < 1$. We have

$$
d\big(h(x_n), h(x_{n+1})\big) \leq t^n d\big(h_1(x_0), h_2(x_1)\big).
$$

Since $0 < t < 1$, we have

$$
\sum_{n=0}^{\infty}\big(d\big(h(x_n), h(x_{n+1})\big)\big) \leq d\big(h_1(x_0), h_2(x_1)\big) \sum_{n=0}^{\infty} t^n < \infty.
$$

Thus, the sequence $\{h(x_n)\}$ is Cauchy in (M, d). Since M is complete, the sequence $h(x_n)$ converge to point w for some $w \in M$. Let $u, v \in M$ with $h_1(u) = w = h_2(v)$. By (iii), there exists $\{h(x_{n_k})\}$ such that $(h(x_{n_k}), h_1(u)) \in E(\Gamma)$ for any $n \in \mathbb{N}$. We assert that $h_1(u) \in Tu$ or $h_2(v) \in Sv$.

Let $\mathcal{A} = \{n_k | n_k \text{ is even}\}$ and $\mathcal{B} = \{n_k | n_k \text{ is odd}\}$. Then obviously $\mathcal{A} \cup \mathcal{B}$ is infinite and so at least $\mathcal{A}$ or $\mathcal{B}$ must be infinite. In the case that $\mathcal{A}$ is infinite, for each $h_2(x_{n_k+1})$, $n_k \in \mathcal{A}$, we have

$$
\begin{aligned}
D\big(h_2(v), S(v)\big) &\leq d\big(h_2(v), h_2(x_{n_k+1})\big) + D\big(h_2(x_{n_k+1}), S(v)\big) \\
&\leq d\big(h_2(v), h_2(x_{n_k+1})\big) + H\big(T(x_{n_k}), S(v)\big) \\
&\leq d\big(h_2(v), h_2(x_{n_k+1})\big) + \gamma\big(\psi\big(P\big(h_1(x_{n_k}), h_2(v)\big)\big)\big)\psi\big(P\big(h_1(x_{n_k}), h_2(v)\big)\big) \\
&\quad + L\phi\big(Q\big(h_1(x_{n_k}), h_2(v)\big)\big) \\
&= d\big(h_2(v), h_2(x_{n_k+1})\big) + \gamma\big(\psi\big(d\big(h_1(x_{n_k}), h_1(v)\big)\big)\big)\psi\big(d\big(h_1(x_{n_k}), h_2(v)\big)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
P\big(h_1(x_{n_k}), h_2(v)\big) = \max\Big\{ &d\big(h_1(x_{n_k}), h_2(v)\big), D\big(h_1(x_{n_k}), Tx_{n_k}\big), D\big(h_2(v), Sv\big), \\
&\frac{D(h_1(x_{n_k}), Tx_{n_k}) + D(h_2(v), Sv)}{2} \Big\} \quad \text{and}
\end{aligned} \tag{14}
$$

$$
Q\big(h_1(x_{n_k}), h_2(v)\big) = \min\big\{D\big(h_1(x_{n_k}), Sv\big), D\big(h_2(v), Tx_{n_k}\big)\big\}.
$$

With $n \to \infty$, we get $Q(h_1(x_{n_k}), h_2(v)) = 0$ and with regard to (14) we obtain

$$
\psi\big(D\big(h_2(v), S(v)\big)\big) \leq \gamma\big(\psi\big(h_2(v), S(v)\big)\big)\psi\big(D\big(h_2(v), S(v)\big)\big) < \psi\big(D\big(h_2(v), S(v)\big)\big),
$$

which is a contradiction, unless $D(h_2(v), S(v)) = 0$. Since Tv is closed, thus $h_2(v) \in S(v)$. Similarly, we can prove that $h_1(v) \in T(v)$ when $\mathcal{B}$ is infinite. This completes the proof. We notice also that if both $\mathcal{A}, \mathcal{B}$ are infinite, then $h_1(v) \in T(v)$ and $h_2(v) \in S(v)$. $\qquad\square$

Example 2.3 Let $M = [0, 1]$ and d be the standard metric on M. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a directed graph with $V(\Gamma) = M$ and

$$
E(\Gamma) = \left\{(x, x), \left(0, \frac{1}{4}\right), \left(\frac{1}{4}, 0\right), \left(0, \frac{1}{16}\right), \left(\frac{1}{16}, 0\right), \left(\frac{1}{4}, \frac{1}{16}\right), \left(\frac{1}{16}, \frac{1}{4}\right) : x \in M\right\}.
$$

Let $T : M \to P_{b,cl}(M)$ be defined by

$$
Tx = \begin{cases} \{\frac{1}{4}\} & \text{if } x = 1, \\ \{0, \frac{1}{4}\} & \text{if } x \in (0, 1) - \{\frac{1}{4}, \frac{1}{2}\}, \\ \{\frac{1}{16}\} & \text{if } x \in \{0, \frac{1}{4}, \frac{1}{2}\}, \end{cases}
$$

$$
Sx = \begin{cases} \{\frac{1}{16}\} & \text{if } x = 0, \frac{1}{16}, \frac{1}{4}, 1, \\ \{0, \frac{1}{16}\} & \text{if } x \in (0, 1) - \{\frac{1}{16}, \frac{1}{4}\}. \end{cases}
$$

Let $h_1, h_2 : M \to M$ be defined by $h_1(x) = x^2$, $h_2(x) = x$. Consider $\psi(t) = t$ and $\gamma(t) = \frac{t+1}{t+2}$. Then it is evident that S, T are mixed Geraghty-type G-multi-valued respect to h_1, h_2. Note that (i), (ii) and (iii) of Theorem 2.2 hold. Besides, if $(h(x), h(y)) \in E(\Gamma)$, then $H(Th(x), Th(y)) = 0$. Hence, for all $x, y \in M$ we have $(h(x), h(y)) \in E(\Gamma)$, ergo

$$\psi\big(H(Sx, Ty)\big) \leq \gamma\big(\psi\big(d\big(h_1(x), h_2(y)\big)\big)\big)\psi\big(d\big(h_1(x), h_2(y)\big)\big),$$

for $x, y \in M$ with $(h_1(x), h_2(y)) \in E(\Gamma)$.

If $x, y \in M$ with $(h_2(x), h_1(y)) \in E(\Gamma)$, then

$$\psi\big(H(Sx, Ty)\big) \leq \gamma\big(\psi\big(d\big(h_2(x), h_1(y)\big)\big)\big)\psi\big(d\big(h_2(x), h_1(y)\big)\big).$$

By Theorem 2.2, there exist $u, v \in M$ such that $h_1(u) \in Tu$ or $h_2(v) \in Sv$. In this example, $u = \frac{1}{4}$ or $u = \frac{1}{16}$.

If in Theorem 2.2, we set $h_1 = h_2 = h$, then we get the following corollary.

Corollary 2.4 *Let (M, d) be a complete metric space with the directed graph Γ, $h : M \to M$ is surjective map and $S, T : M \to P_{b,cl}(M)$ be h-graph preserving with*

$$\psi\big(H(Sx, Ty)\big) \leq \gamma\big(\psi\big(d\big(h(x), h(y)\big)\big)\big)\psi\big(d\big(h(x), h(y)\big)\big),$$

for all $x, y \in M$ with $(h(x), h(y)) \in E(\Gamma)$. Suppose
  (i) *there exists $x_0 \in M$ such that $(h(x_0), u) \in E(\Gamma)$ for some $u \in Tx_0$;*
  (ii) *for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in M, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(\Gamma)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ such that $(x_{n_k}, x) \in E(\Gamma)$ for $n_k \in \mathbb{N}$.*
Then there exist $u, v \in M$ such that $h(u) \in Tu$ or $h(v) \in Sv$.

Definition 2.5 Let (M, d) be a metric space endowed with a partial order $\leq$. For each $M_1, M_2 \in M$, $M_1 \preceq M_2$ if $\omega_1 \leq \omega_2$ for any $\omega_1 \in M_1$, $\omega_2 \in M_2$, $h : M \to M$ a surjective map, and $T : M \to P_{b,cl}(M)$. T is said to be h-increasing if for any $x, y \in M$, $h(x) \leq h(y)$ implies $Tx \preceq Ty$.

Theorem 2.6 *Let (M, d) be a complete metric space with partially order $\leq$, $h : M \to M$ be a surjective map and $T : M \to P_{b,cl}(M)$ be a multi-valued mapping. Suppose that*
  (i) *T is h-increasing;*
  (ii) *there exist $x_0 \in M$ and $u \in Tx_0$ such that $h(x_0) \leq u$;*
  (iii) *for each sequence x_k such that $h(x_k) \leq h(x_{k+1})$, $k \in \mathbb{N}$ and $h(x_k)$ converges to $h(x)$ for some $x \in M$, then $h(x_k) \leq h(x)$;*
  (iv) *for $x, y \in M$ with $h(x) \leq h(y)$, we have*

$$\psi\big(H(Tx, Ty)\big) \leq \gamma\big(\psi\big(d\big(h(x), h(y)\big)\big)\big)\psi\big(d\big(h(x), h(y)\big)\big),$$

where $\gamma \in \mathcal{F}$ and $\psi, \phi \in \psi$.
Then there is $u \in M$ so that $h(u) \in Tu$. In addition, if h is injective, then there is a unique $u \in M$ such that $h(u) \in Tu$.

Proof Let $\Gamma = (V(\Gamma), E(\Gamma))$, be a graph with $V(\Gamma) = M$ and

$$E(\Gamma) = \big\{(x, y) | x \le y\big\},$$

let $(h(x), h(y)) \in E(\Gamma)$, then $h(x) \le h(y)$ and by (i), $Tx \preceq Ty$. For each $u \in Tx$, $v \in Ty$, we have $u \le v$, thus $(u, v) \in E(\Gamma)$. That is, T is h-graph preserving. By (ii), there exist $x_0 \in M$ and $u \in Tx_0$ such that $h(x_0) \le u$. So $(h(x), u) \in E(\Gamma)$ and hence the property (i) in Corollary 2.4 is satisfied. Moreover, we obtain the property (ii) of Corollary 2.4 from the assumption (iii). Set $S = T$, then the S, T are h-graph preserving mappings and fulfill

$$\psi\big(H(Tx, Ty)\big) \le \gamma\big(\psi\big(d\big(h(x), h(y)\big)\big)\big)\psi\big(P\big(h(x), h(y)\big)\big),$$

for all $x, y \in M$ with $(h(x), h(y)) \in E(\Gamma)$. By Corollary 2.4 we get $h(u) \in Tu$ for some $u \in M$.

Now, in addition, we suppose that h is injective. Let $u, v \in M$ be such that $h(u) \in Tu$ and $h(v) \in Tv$. Suppose, on the contrary, that $h(u) \ne h(v)$. We assume, without loss of generality, that $h(u) < h(v)$. Since $h(u) \in Tu$ and $h(v) \in Tv$, it yields $D(h(u), Tu) = D(h(v), Tv)$ and hence

$$\psi\big(d\big(h(u), h(v)\big)\big) \le \psi\big(H(Tu, Tv)\big) \le \gamma\big(\psi\big(d\big(h(u), h(v)\big)\big)\big)\psi\big(d\big(h(u), h(v)\big)\big),$$
$$< \psi\big(d\big(h(u), h(v)\big)\big).$$

This leads to a contradiction. Thus $h(u) = h(v)$. Since h is injective, we have $u = v$.   □

Corollary 2.7 *Let (M, d) be a complete metric space endowed with a graph $\Gamma = (V(\Gamma), E(\Gamma))$, $S, T : M \to M$ be generalized mappings such that*

$$\psi\big(d(Sx, Ty)\big) \le \gamma\big(\psi\big(d(x, y)\big)\big)\psi\big(d(x, y)\big), \tag{15}$$

mapping in (M, d). Suppose that
  (i) *$\exists x_0 \in M$ such that $(x_0, u) \in E(\Gamma)$ for some $u \in Tx_0$,*
  (ii) *if $(x, y) \in E(\Gamma)$, then $(e, f) \in E(\Gamma)$ for all $e \in Tx$ and $f \in Sy$ and if $(x, y) \in E(\Gamma)$, then $(w, r) \in E(\Gamma)$ for all $w \in Sx$ and $r \in Ty$,*
  (iii) *for $\{x_n\}_{n \in \mathbb{N}}$ in M, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(\Gamma)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k \in \mathbb{N}}$ such that $(x_{n_k}, x) \in E(\Gamma)$ for $n_k \in \mathbb{N}$.*
Then there exist $u, v \in M$ such that $u \in Tu$ or $v \in Sv$.

3 Application
In this section, we apply our theorem for a solution of the following integral system:

$$\begin{cases} x(t) = \int_a^b K(t, s)f(t, s, x(s), y(s))\, ds + h(t), \\ y(t) = \int_a^b K(t, s)f(t, s, y(s), x(s))\, ds + h(t), \quad t \in [a, b], \end{cases} \tag{16}$$

where $X := C([a, b], \mathbb{R})$ with $\|x\|_\infty = \sup_{t \in [a, b]} |x(t)|$, for $x \in C([a, b], \mathbb{R})$.

Let G be a graph, defined by $V(\Gamma) = X$, and

$$E(\Gamma) = \big\{(x, y) \in X \times X : x(t) \le y(t) \text{ or } y(t) \le x(t) \text{ for } t \in [a, b]\big\}.$$

Let d be the metric induced by the norm. It follows that (X, d) is a complete metric space endowed with a directed graph Γ. Define mappings $S, T : X \to X$ by

$$
\begin{cases}
Sx(t) = \int_a^b K(t,s)f(t,s,x(s),y(s))\,ds + h(t), \\
Ty(t) = \int_a^b K(t,s)f(t,s,y(s),x(s))\,ds + h(t),
\end{cases}
$$

where $x, y \in C([a,b], \mathbb{R})$.

Theorem 3.1 *Consider Eq. (16) and suppose that*

1. *$f : [a,b] \times [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function and there exists $M > 0$ such that $\max_{t \in [a,b]} \int_a^b K(t,s)\,ds < M$.*

2.

$$
\left| f\big(t,s,y(s),x(s)\big) - f\big(t,s,x(s),y(s)\big) \right| \le \frac{1}{\alpha} \frac{|y(s) - x(s)|^2}{1 + |y(s) - x(s)|}.
$$

3. *There exists a function $x_0 \in X$ such that*

$$
x_0(t) \le \int_a^b K(t,s)f\big(t,s,x_0(s),Tx_0(s)\big)\,ds + h(t), \quad t \in [a,b].
$$

4. *The inequality $x(s) \le y(s)$ implies $Tx(s) \le Sy(s)$ and $Sx(s) \le Ty(s)$.*

5. *For $\{x_n\}_{n\in\mathbb{N}}$ in M, if $x_n \to x$ and $x_n(t) \le x_{n+1}(t)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k\in\mathbb{N}}$ such that $x_{n_k}(t) \le x(t)$ for $n_k \in \mathbb{N}$, $t \in [a,b]$.*

Then there exist $u, v \in X$ such that $u(t) = \int_a^b K(t,s)f(t,s,u(s),v(s))\,ds + h(t)$ or $v(t) = \int_a^b K(t,s)f(t,s,v(s),u(s))\,ds + h(t)$.

Proof Let $x, y \in X$, using (2), we get

$$
\begin{aligned}
\left| Sx(t) - Ty(t) \right| &= \left| \int_a^b K(t,s)f\big(t,s,x(s),y(s)\big)\,ds - \int_a^b K(t,s)f\big(t,s,y(s),x(s)\big)\,ds \right| \\
&\le \sup_{t \in [a,b]} \left| f\big(t,s,x(s),y(s)\big) - f\big(t,s,y(s),x(s)\big) \right| M \\
&\le \frac{1}{M} \frac{|x(s) - y(s)|^2}{1 + |x(s) - y(s)|} M = \frac{|x(s) - y(s)|^2}{1 + |x(s) - y(s)|} \\
&= \gamma\big(|x(s) - y(s)|\big)|x(s) - y(s)|,
\end{aligned}
$$

where $\gamma(t) = \frac{t}{t+1}$. With setting $\psi(t) = t$, the condition 15 in Corollary 2.7 holds. With considering (3) and (4) from assumption of theorem and by definition of the graph Γ we deduce the conditions (i) and (ii) of in Corollary 2.7 hold. Also by the condition (5) from assumption of theorem, the condition (iii) of Corollary 2.7 holds. As a result, we have

$$
u(t) = \int_a^b K(t,s)f\big(t,s,u(s),v(s)\big)\,ds + h(t), \quad \text{or}
$$

$$
v(t) = \int_a^b K(t,s)f\big(t,s,v(s),u(s)\big)\,ds + h(t). \qquad \square
$$

Now we use the results of our findings to solve fractional differential equations.

Definition 3.2 ([22, 26]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$I_{0^+}^\alpha f(\varsigma) = \frac{1}{\Gamma(\alpha)} \int_0^\varsigma (\varsigma - \eta)^{\alpha-1} f(\eta) \, d\eta,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 3.3 ([22, 26]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$D_{0^+}^\alpha f(\varsigma) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int_0^\varsigma (\varsigma - \eta)^{n-\alpha-1} f(\eta) \, d\eta,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

In this paper we discuss the local existence and uniqueness of positive solutions for the following coupled system of fractional boundary value problem subject to integral boundary conditions:

$$\begin{cases} D_{0^+}^\alpha u(\varsigma) + f(\varsigma, u(\varsigma), v(\varsigma)) = 0, \\ D_{0^+}^\alpha v(\varsigma) + f(\varsigma, v(\varsigma), u(\varsigma)) = 0, \quad 0 < \varsigma < 1, \\ u(0) = 0, \qquad u(1) = \int_0^1 \phi(\varsigma) u(\varsigma) \, d\varsigma, \\ v(0) = 0, \qquad v(1) = \int_0^1 \psi(\varsigma) v(\varsigma) \, d\varsigma, \end{cases} \tag{17}$$

where $1 < \alpha \le 2$, $\phi, \psi \in L^1[0, 1]$ are nonnegative and $f \in C([0, 1] \times [0, \infty), [0, \infty))$ and D is the standard Riemann–Liouville fractional derivative. The functions $\phi(\varsigma)$, $\psi(\varsigma)$ satisfy the following conditions:

$$(Q) \quad \phi, \psi : [0, 1] \to [0, +\infty) \quad \text{with } \phi, \psi \in L^1[0, 1]$$

and

$$\sigma_1 := \int_0^1 \phi(\varsigma) \varsigma^{\alpha-1} \, d\varsigma;$$

$$\sigma_2 := \int_0^1 \varsigma^{\alpha-1}(1 - \varsigma) \phi(\varsigma) \, d\varsigma, \quad \varsigma \in (0, 1).$$

Lemma 3.4 ([29]) *If $\int_0^1 \phi(\varsigma) \varsigma^{\alpha-1} \, d\varsigma \neq 1$, then for any $\sigma \in C[0, 1]$, the unique solution of the following boundary value problem*:

$$\begin{cases} D_{0^+}^\alpha u(\varsigma) + \sigma(\varsigma) = 0, \quad 0 < \varsigma < 1, \\ u(0) = 0, \qquad u(1) = \int_0^1 \phi(\varsigma) u(\varsigma) \, d\varsigma, \end{cases}$$

is given by

$$u(\varsigma) = \int_0^1 G_{1\alpha}(\varsigma, \eta) \sigma(\eta) \, d\eta,$$

where

$$G_\alpha(\varsigma, \eta) = G_{1\alpha}(\varsigma, \eta) + G_{2\alpha}(\varsigma, \eta),$$

$$G_{1\alpha}(\varsigma, \eta) = \frac{1}{\Gamma(\alpha)} \begin{cases} \varsigma^{\alpha-1}(1-\eta)^{\alpha-1} - (\eta - \varsigma)^{\alpha-1}, \\ \varsigma^{\alpha-1}(1-\eta)^{\alpha-1}, \end{cases} \tag{18}$$

$$G_{2\alpha}(\varsigma, \eta) = \frac{\varsigma^{\alpha-1}}{1 - \int_0^1 \phi(\varsigma)\varsigma^{\alpha-1}\,d\varsigma} \int_0^1 \phi(\varsigma)G_{1\alpha}(\varsigma, \eta)\,d\varsigma.$$

Then $G_\alpha(\varsigma, \eta)$ is a Green's function.

Lemma 3.5 ([29]) *Let $\alpha \in (1, 2]$. Assume that (Q) holds. Then the functions $G_{1\alpha}(\varsigma, \eta)$ have the following property*:

$$\frac{(\alpha-1)\sigma_2\eta(1-\eta)^{\alpha-1}\varsigma^{\alpha-1}}{(1-\sigma_1)\Gamma(\alpha)} \le G_\alpha \le \frac{(1-\eta)^{\alpha-1}\varsigma^{\alpha-1}}{\Gamma(\alpha)(1-\sigma_1)}.$$

Lemma 3.6 ([29]) *Assume that (Q) holds and $f(y, x, y)$ continuous, then $(u, v) \in X \times X$ is a solution of the system (17) if and only if it is a solution of the integral equations*

$$\begin{cases} u(y) = \int_0^1 G_\alpha(y, x)f(x, u(x), v(x))\,dx, \\ v(y) = \int_0^1 G_\alpha(y, x)f(x, v(x), u(x))\,dx. \end{cases}$$

Define mappings $S, T : X \to X$ by

$$\begin{cases} Su(y) = \int_0^1 G_\alpha(y, x)f(x, u(x), v(x))\,dx, \\ Tv(y) = \int_0^1 G_\alpha(y, x)f(x, v(x), u(x))\,dx. \end{cases}$$

Theorem 3.7 *Consider Eq. (17) and suppose that*
1. *$f : [0, 1] \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ is continuous function such that*

$$\left| f(t, s, y(s), x(s)) - f(t, s, x(s), y(s)) \right| \le \frac{(1-s)^{\alpha-1}\varsigma^{\alpha-1}}{\Gamma(\alpha)} \frac{|y(s) - x(s)|^2}{1 + |y(s) - x(s)|}.$$

2. *There exists a function $x_0 \in X$ such that*

$$x_0(t) \le \int_a^b K(t, s)f(t, s, x_0(s), Tx_0(s))\,ds, \quad t \in [a, b].$$

3. *The inequality $x(s) \le y(s)$ implies $Tx(s) \le Sy(s)$ and $Sx(s) \le Ty(s)$.*
4. *For $\{x_n\}_{n\in\mathbb{N}}$ in M, if $x_n \to x$ and $x_n(t) \le x_{n+1}(t)$ for $n \in \mathbb{N}$, then there is a subsequence $\{x_{n_k}\}_{n_k\in\mathbb{N}}$ such that $x_{n_k}(t) \le x(t)$ for $n_k \in \mathbb{N}$, $t \in [a, b]$.*

Then there exist $u, v \in X$ such that

$$u(t) = \int_a^b G_\alpha(t, s)f(t, s, u(s), v(s))\,ds \quad or$$

$$v(t) = \int_a^b G_\alpha(t, s)f(t, s, v(s), u(s))\,ds.$$

Proof Let $x, y \in X$, using (2), we get

$$
\begin{aligned}
\left| Sx(t) - Ty(t) \right| &= \left| \int_a^b G_\alpha(t,s) f\big(t,s,x(s),y(s)\big)\,ds - \int_a^b G_\alpha(t,s) f\big(t,s,y(s),x(s)\big)\,ds \right| \\
&\leq \sup_{t \in [a,b]} \left| f\big(t,s,x(s),y(s)\big) - f\big(t,s,y(s),x(s)\big) \right| \frac{\Gamma(\alpha)}{(1-s)^{\alpha-1}\varsigma^{\alpha-1}} \\
&\leq \frac{(1-s)^{\alpha-1}\varsigma^{\alpha-1}}{\Gamma(\alpha)} \frac{|x(s)-y(s)|^2}{1+|x(s)-y(s)|} \frac{\Gamma(\alpha)}{(1-s)^{\alpha-1}\varsigma^{\alpha-1}} = \frac{|x(s)-y(s)|^2}{1+|x(s)-y(s)|} \\
&= \gamma\big(\left|x(s)-y(s)\right|\big)\left|x(s)-y(s)\right|,
\end{aligned}
$$

where $\gamma(t) = \frac{t}{t+1}$. With setting $\psi(t) = t$, the condition (15) in Corollary 2.7 holds. With considering (3) and (4) from the assumption of the theorem and by definition of the graph Γ we deduce the conditions (i) and (ii) in Corollary (2.7) hold. Also by the condition (5) from assumption of theorem, the condition (iii) of Corollary 2.7 holds. As a result, we have

$$
u(t) = \int_a^b G_\alpha(t,s) f\big(t,s,u(s),v(s)\big)\,ds \quad \text{or}
$$

$$
v(t) = \int_a^b G_\alpha(t,s) f\big(t,s,v(s),u(s)\big)\,ds.
$$

Author details
[1]Department of Mathematics, Faculty of Basic Science, University of Bonab, Bonab, Iran. [2]Institute of Research and Development, Duy Tan University, 550000 Da Nang, Vietnam. [3]Faculty of Natural Sciences, Duy Tan University, 550000 Da Nang, Vietnam. [4]Department of Mathematics, Cankaya University, Mimar Sinan Caddesi, 06790 Ankara, Turkey. [5]Department of Medical Research, China Medical University, Hsueh-Shih Road, Taichung, 40402, Taiwan.

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