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Article · August 2020

DOI: 10.3390/math8091437

Fulga Andreea
Universitatea Transilvania Brasov

Erdal Karapinar
Cankaya University

mathematics

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A Discussion on p-Geraghty Contraction on mw-Quasi-Metric Spaces

Carmen Alegre [1,*], Andreea Fulga [2], Erdal Karapinar [3,4,5,6] and Pedro Tirado [1]

1 Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camí de Vera s/n, 46022 Valencia, Spain; pedtipe@mat.upv.es
2 Department of Mathematics and Computer Science, Transilvania University of Brasov, 500036 Brasov, Romania; afulga@unitbv.ro
3 Institute of Research and Development Duy Tan University, Da Nang 550000, Vietnam; erdalkarapinar@duytan.edu.vn or erdalkarapinar@gmail.com
4 Faculty of Natural Sciences, Duy Tan University, Da Nang 550000, Vietnam
5 Department of Mathematics, Çankaya University, Etimesgut, 06790 Ankara, Turkey
6 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan; karapinar@mail.cmuh.org.tw
* Correspondence: calegre@mat.upv.es

Received: 23 July 2020; Accepted: 24 August 2020; Published: 27 August 2020

Abstract: In this paper we consider a kind of Geraghty contractions by using mw-distances in the setting of complete quasi-metric spaces. We provide fixed point theorems for this type of mappings and illustrate with some examples the results obtained.

Keywords: fixed point; metric space; quasi-metric space

MSC: 46T99; 47H10; 54H25

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1. Introduction and Preliminaries

In 1973, Geraghty proved the following fixed point theorem which generalizes the classical Banach fixed point theorem.

Theorem 1 ([1]). *Let $(\mathcal{X}, d)$ be a complete metric space and consider the set*

$$\mathcal{B} = \{\beta : [0,\infty) \to [0,1) \mid \beta(\varkappa_n) \to 1 \text{ implies } \varkappa_n \to 0\}.$$

The mapping $T : \mathcal{X} \to \mathcal{X}$ has a unique fixed point provided that there exists $\beta \in \mathcal{B}$ such that

$$d(T\varkappa, T\nu) \leq \beta(d(\varkappa,\nu))d(\varkappa,\nu), \quad \text{for all } \varkappa,\nu \in \mathcal{X}.$$

Since then, different authors have proved versions of this theorem in various frameworks (see, for example, [2] and the references quoted therein).

Cho et al. [3] defined the notion of α-Geraghty contraction in the context of a metric space, as follows:

A maping T is called a α-Geraghty contraction if

$$\alpha(\varkappa,\nu)d(T\varkappa,T\nu) \leq \beta(d(\varkappa,\nu)) \max\{d(\varkappa,\nu), d(\nu,T\nu), d(\varkappa,T\varkappa)\}, \quad \text{for all } \varkappa,\nu \in \mathcal{X},$$

where $\alpha : \mathcal{X} \times \mathcal{X} \to [0,\infty)$ is a function with the property

$$\alpha(\varkappa,\nu) \geq 1 \Rightarrow \alpha(T\varkappa, T\nu) \geq 1 \quad \text{for all } \varkappa,\nu \in X.$$

In addition, in [3], the authors provided fixed point theorems for this type of maps in the framework of a complete metric space.

Inspired from the recent results [4,5], we consider a Geraghty type contraction via mw-distances in the context of a complete quasi-metric space.

Before stating our main result, we shall collect fundamental notions and useful results for the sake of completeness. First, we shall state what we understand by the notion of the quasi-metric space. Note that in the literature, "quasi-metric" was used to express several different structure.

Definition 1. *Let X be a nonempty set. A function $q : X \times X \to [0,\infty)$ is a quasi-metric if it satisfies the following axioms for all $\varkappa,\nu,\theta \in X$*

(q_1) *reflexivity, that is*

$$q(\varkappa,\nu) = q(\nu,\varkappa) = 0 \Leftrightarrow \nu = \varkappa$$

(q_2) *the triangle inequality,*

$$q(\varkappa,\nu) \leq q(\varkappa,\theta) + q(\theta,\nu).$$

The pair (X,q) denotes a quasi-metric space.

Remark 1 ([6]). *On a set X, a quasi-metric q induces a topology $\tau(q)$. This topology has as a base the family of open balls $\{B_q(\varkappa,\varepsilon) | \varkappa \in X, \varepsilon > 0\}$, where $B_q(\varkappa,\varepsilon) = \{\nu \in X | q(\varkappa,\nu)) < \varepsilon\}$.*

On a non-empty set X, each quasi-metric q yields a metric by letting

$$q^s(\varkappa,\nu) = \max\{q(\varkappa,\nu), q(\nu,\varkappa)\}.$$

On the other hand, if q is quasi-metric on X, the function q^* defined as $q^*(\varkappa,\theta) = q(\theta,\varkappa)$ for all $\varkappa,\theta \in X$, is also a quasi-metric on X and it is called the conjugate quasi-metric of q.

A sequence $\{\varkappa_n\} \subset X$ converges to $\varkappa \in X$ in the quasi-metric space (X,q) if $\{\varkappa_n\}$ converges to $\varkappa$ with respect to the topology $\tau(q)$, in other words, $\lim_{n\to\infty} q(\varkappa,\varkappa_n) = 0$.

A quasi-metric space (X,q) is called complete ([4,6]) if every Cauchy sequence $\{\varkappa_n\}$ in the metric space (X,q^s) converges with respect to the topology $\tau(q^*)$ (i.e., there exists $\varkappa \in X$ such that $\lim_{n\to\infty} q(\varkappa_n,\varkappa) = 0$).

We shall recall another distance (w-distance, [7,8]) which is defined via a quasi-metric.

Definition 2 ([6]). *A function $p : X \times X \to [0,\infty)$ is a w-distance on a quasi-metric space (X,q) if the following conditions are satisfied:*

(w_1) *$p(\varkappa,\nu) \leq p(\varkappa,\theta) + p(\theta,\nu)$ for any $\varkappa,\nu,\theta \in X$;*
(w_2) *$p(\varkappa,\cdot) : X \to \mathbb{R}_0^+$ is lower semi-continuous on $(X,\tau(q^*))$ for all $\varkappa \in X$;*
(w_3) *for each $\epsilon > 0$ there exists $\delta > 0$ such that if $p(\varkappa,\nu) \leq \delta$ and $p(\varkappa,\theta) \leq \delta$ then $q(\nu,\theta) \leq \epsilon$.*

In the paper [4] the authors remark that a quasi-metric is not necessarily a w-distance and illustrate it by the following example.

Example 1 ([4]). *The function* $q_S(\varkappa,\nu) = \begin{cases} \nu-\varkappa, & \text{if } \nu \geq \varkappa \\ 1, & \text{if } \nu < \varkappa \end{cases}$ *is a quasi-metric on* $\mathbb{R}$ *but is not a* w*-distance because the condition* (w_3) *is not satisfied. Taking* $\delta > 0$ *and* $\epsilon = 1/2$*, for* $\nu = \varkappa + \delta/2$ *and* $\theta = \varkappa + \delta/3$*, we have:*

$$q_S(\varkappa,\nu) = \delta/2 < \delta,\ q_S(\varkappa,\theta) = \delta/3 < \delta, \text{ but } q_S(\theta,\nu) = 1 > 1/2.$$

In what follows, we recall the main tool of this paper, namely, mw-distance:

Definition 3 ([4]). *On a quasi-metric space* (X,q)*, a function* $p : X \times X \to \mathbb{R}_0^+$ *is an* mw*-distance if it satisfies the following conditions:*

(mw_1) $p(\varkappa,\nu) \leq p(\varkappa,\theta) + p(\theta,\nu)$ *for any* $\varkappa,\nu,\theta \in X$;
(mw_2) $p(\varkappa,\cdot) : X \to \mathbb{R}_0^+$ *is lower semi-continuous on* $(X,\tau(q^*))$ *for all* $\varkappa \in X$;
(mw_3) *for each* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $p(\nu,\varkappa) \leq \delta$ *and* $p(\varkappa,\theta) \leq \delta$ *then* $q(\nu,\theta) \leq \epsilon$.

A quasi-metric q on X is an mw-distance on (X,q).
We say that a $m\omega$-distance $p : X \times X \to \mathbb{R}_0^+$ on a quasi-metric space (X,q) is a strong mw-distance (see [4]) if it satisfies the following condition:

(mw_2^*) $p(\cdot,\varkappa) : X \to \mathbb{R}_0^+$ is lower semi-continuous on $(X,\tau(q^*))$ for all $\varkappa \in X$.

Definition 4. *Let* (X,q) *be a quasi-metric space and let* p *be an* mw*-distance. A mapping* $T : X \to X$ *is said to be* p*-lower semi-continuous if the function* $\varkappa \to p(\varkappa,T\varkappa)$ *is lower semi-continuous on the metric space* (X,q^s).

2. Main Results

We start this section by introducing the notion of the p-Geraghty contraction:

Definition 5. *A self-mapping* T *on a quasi-metric space* (X,q) *is a* p*-Geraghty contraction if there exist a strong* mw*-distance* p *on* (X,q) *and a function* $\beta \in \mathcal{B}$ *such that for any* $\varkappa,\nu \in X$

$$p(T\varkappa,T\nu) \leq \beta\left(M_1(\varkappa,\nu)\right) M_1(\varkappa,\nu) \tag{1}$$

and

$$p(T\varkappa,T\nu) \leq \beta\left(M_2(\varkappa,\nu)\right) M_2(\varkappa,\nu) \tag{2}$$

where

$$\begin{aligned} M_1(\varkappa,\nu) &= \max\left\{p(\varkappa,\nu), p(\varkappa,T\varkappa), p(\nu,T\nu)\right\}, \text{ and,} \\ M_2(\varkappa,\nu) &= \max\left\{p(\varkappa,\nu), p(T\varkappa,\varkappa), p(T\nu,\nu)\right\}. \end{aligned}$$

Theorem 2. *Let* (X,q) *be a complete quasi-metric space and* $T : X \to X$ *a* p*-Geraghty contraction. If* T *is a* p*-lower semi-continuous mapping, then* T *has a unique fixed point.*

Proof. Let $\varkappa_0$ be a point on X and consider the sequence $\{\varkappa_n\}$, where $\varkappa_n = T^n\varkappa_0$ for any $n \in \mathbb{N}$. If we can find $n_0 \in \mathbb{N}$ such that $p(\varkappa_{n_0},\varkappa_{n_0+1}) = 0$, replacing $\varkappa$ by $\varkappa_{n_0}$ and ν by $\varkappa_{n_0+1}$ in (1) and since $\beta \in \mathcal{B}$, we have

$$\begin{aligned} p(\varkappa_{n_0+1},\varkappa_{n_0+2}) &= p(T\varkappa_{n_0},T\varkappa_{n_0+1}) \leq \beta\left(M_1(\varkappa_{n_0},\varkappa_{n_0+1})\right) M_1(\varkappa_{n_0},\varkappa_{n_0+1}) \\ &< M_1(\varkappa_{n_0},\varkappa_{n_0+1}) = \max\left\{p(\varkappa_{n_0},\varkappa_{n_0+1}), p(\varkappa_{n_0},T\varkappa_{n_0}), p(\varkappa_{n_0+1},T\varkappa_{n_0+1})\right\} \\ &= p(\varkappa_{n_0+1},\varkappa_{n_0+2}). \end{aligned}$$

Consequently, $p(\varkappa_{n_0+1},\varkappa_{n_0+2}) = 0$. By induction, for every $j \in \mathbb{N}$, we can easily get that $p(\varkappa_{n_0+j},\varkappa_{n_0+j+1}) = 0$ and by (mw_1), $p(\varkappa_n,\varkappa_m) = 0$ for $m > n \geq n_0$.

- Suppose that $p(\varkappa_n, \varkappa_{n+1}) \neq 0$ for $n \in \mathbb{N}$. From (1),

$$p(\varkappa_{n+1}, \varkappa_{n+2}) \quad = p(T\varkappa_n, T\varkappa_{n+1}) \leq \beta\left(M_1(\varkappa_n, \varkappa_{n+1})\right) M_1(\varkappa_n, \varkappa_{n+1}) < M_1(\varkappa_n, \varkappa_{n+1}), \tag{3}$$

where

$$M_1(\varkappa_n, \varkappa_{n+1}) = \max\left\{p(\varkappa_n, \varkappa_{n+1}), p(\varkappa_n, T\varkappa_n), p(\varkappa_{n+1}, T\varkappa_{n+1})\right\} = \max\left\{p(\varkappa_n, \varkappa_{n+1}), p(\varkappa_{n+1}, \varkappa_{n+2})\right\}.$$

As if $M_1(\varkappa_n, \varkappa_{n+1}) = p(\varkappa_{n+1}, \varkappa_{n+2})$, we obtain $p(\varkappa_{n+1}, \varkappa_{n+2}) < p(\varkappa_{n+1}, \varkappa_{n+2})$, a contradiction, we conclude that $p(\varkappa_n, \varkappa_{n+1}) > p(\varkappa_{n+1}, \varkappa_{n+2})$. Thus, the sequence $\{p(\varkappa_n, \varkappa_{n+1})\}$ converges to some $a \geq 0$. If we suppose that $a > 0$, because $\lim_{n\to\infty} M_1(\varkappa_n, \varkappa_{n+1}) = a$, by taking the limit as $n \to \infty$ in (3), we get $\lim_{n\to\infty} \beta(M_1(\varkappa_n, \varkappa_{n+1})) = 1$. Since $\beta \in \mathcal{B}$, we obtain $a = \lim_{n\to\infty} M_1(\varkappa_n, \varkappa_{n+1}) = 0$, which is a contradiction.

Similarly, from (2),

$$\begin{aligned} p(\varkappa_{n+2}, \varkappa_{n+1}) &= p(T\varkappa_{n+1}, T\varkappa_n) \leq \beta\left(M_2(\varkappa_{n+1}, \varkappa_n)\right) M_2(\varkappa_{n+1}, \varkappa_n) < M_2(\varkappa_{n+1}, \varkappa_n) \\ &= \max\left\{p(\varkappa_{n+1}, \varkappa_n), p(T\varkappa_{n+1}, \varkappa_{n+1}), p(T\varkappa_n, \varkappa_n)\right\} = \max\left\{p(\varkappa_{n+1}, \varkappa_n), p(\varkappa_{n+2}, \varkappa_{n+1})\right\}, \end{aligned} \tag{4}$$

and using the same arguments we get that the sequence $\{p(\varkappa_{n+1}, \varkappa_n)\}$ converges to 0.

Consequently,

$$\lim_{n\to\infty} p(\varkappa_n, \varkappa_{n+1}) = 0 = \lim_{n\to\infty} p(\varkappa_{n+1}, \varkappa_n). \tag{5}$$

Note that $\lim_{n\to\infty} p(\varkappa_n, \varkappa_n) = 0$.

- As a next step, we aim to prove that $\{\varkappa_n\}$ is a Cauchy sequence in the metric space (X, q^s).

First, we shall prove that given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(\varkappa_n, \varkappa_m) < \varepsilon$ for all $n > m \geq n_0$.

Assume the contrary. Then, there exists $\varepsilon > 0$ and two sequences of positive integers $\{n(l)\}_{l=1}^{\infty}$ and $\{m(l)\}_{l=1}^{\infty}$ such that $n(l) > m(l) \geq l$, for any $l \in \mathbb{N}$ and

$$p(\varkappa_{n(l)}, \varkappa_{m(l)}) \geq \varepsilon, \; p(\varkappa_{n(l)-1}, \varkappa_{m(l)}) < \varepsilon.$$

Taking into account (mw_1), we have

$$\varepsilon \leq p(\varkappa_{n(l)}, \varkappa_{m(l)}) \leq p(\varkappa_{n(l)}, \varkappa_{n(l)-1}) + p(\varkappa_{n(l)-1}, \varkappa_{m(l)}) < p(\varkappa_{n(l)}, \varkappa_{n(l)-1}) + \varepsilon$$

and from (5) it follows that

$$\lim_{l\to\infty} p(\varkappa_{n(l)}, \varkappa_{m(l)}) = \varepsilon. \tag{6}$$

Again, by (mw_1) we have

$$\begin{aligned} p(\varkappa_{n(l)+1}, \varkappa_{m(l)+1}) &\leq p(\varkappa_{n(l)+1}, \varkappa_{n(l)}) + p(\varkappa_{n(l)}, \varkappa_{m(l)}) + p(\varkappa_{m(l)}, \varkappa_{m(l)+1}), \\ p(\varkappa_{n(l)}, \varkappa_{m(l)}) &\leq p(\varkappa_{n(l)}, \varkappa_{n(l)+1}) + p(\varkappa_{n(l)+1}, \varkappa_{m(l)+1}) + p(\varkappa_{m(l)+1}, \varkappa_{m(l)}), \end{aligned}$$

which means that

$$\begin{aligned} p(\varkappa_{n(l)}, \varkappa_{m(l)}) - p(\varkappa_{n(l)}, \varkappa_{n(l)+1}) - p(\varkappa_{m(l)+1}, \varkappa_{m(l)}) &\leq p(\varkappa_{n(l)+1}, \varkappa_{m(l)+1}) \\ &\leq p(\varkappa_{n(l)+1}, \varkappa_{n(l)}) + p(\varkappa_{n(l)}, \varkappa_{m(l)}) + p(\varkappa_{m(l)}, \varkappa_{m(l)+1}). \end{aligned}$$

Furthermore,

$$\lim_{l\to\infty} p(\varkappa_{n(l)+1}, \varkappa_{m(l)+1}) = \varepsilon. \tag{7}$$

Consequently,

$$\lim_{l\to\infty} M_1(\varkappa_{n(l)}, \varkappa_{m(l)}) = \lim_{l\to\infty} \max\left\{p(\varkappa_{n(l)}, \varkappa_{m(l)}), p(\varkappa_{n(l)}, \varkappa_{n(l)+1}), p(\varkappa_{m(l)}, \varkappa_{m(l)+1})\right\} = \varepsilon$$

and since $\beta \in \mathcal{B}$ it follows that

$$\begin{aligned}\varepsilon &= \lim_{l\to\infty} p(\varkappa_{n(l)+1}, \varkappa_{m(l)+1}) = \lim_{l\to\infty} p(T\varkappa_{n(l)}, T\varkappa_{m(l)}) \\ &\le \lim_{l\to\infty} \beta\left(M_1(\varkappa_{n(l)}, \varkappa_{m(l)})\right) M_1(\varkappa_{n(l)}, \varkappa_{m(l)}) \\ &\le \lim_{l\to\infty} M_1(\varkappa_{n(l)}, \varkappa_{m(l)}) = \varepsilon.\end{aligned}$$

Therefore, $\lim_{l\to\infty} \beta\left(M_1(\varkappa_{n(l)}, \varkappa_{m(l)})\right) = 1$, by which $\varepsilon = \lim_{l\to\infty} M_1(\varkappa_{n(l)}, \varkappa_{m(l)}) = 0$. This is a contradiction. Therefore,

$$\lim_{l\to\infty} p(\varkappa_{n(l)}, \varkappa_{m(l)}) = 0 \text{ for all } n(l) > m(l) \ge l. \tag{8}$$

Then, taking into account (5) and (8), for each $\delta > 0$ there exists $n_1 \in \mathbb{N}$ such that $p(\varkappa_n, \varkappa_{n-1}) \le \frac{\delta}{2}$ and $p(\varkappa_{n-1}, \varkappa_m) \le \frac{\delta}{2}$, $\forall n > m \ge n_1$. Accordingly, since p is a mw-distance, by (mw_3) we get that $q(\varkappa_n, \varkappa_m) < \epsilon$.

Similarly, we prove that given $\varepsilon > 0$ there exists $n_2 \in \mathbb{N}$ such that $q(\varkappa_m, \varkappa_n) < \epsilon$, $\forall m > n \ge n_2$. Therefore,

$$q^s(\varkappa_n, \varkappa_m) < \epsilon$$

for all $n, m \ge max\{n_1, n_2\}$ so that $\{\varkappa_n\}$ is a Cauchy sequence in the metric space $(\mathcal{X}, q^s)$. Then there exists a point $\omega \in \mathcal{X}$ such that $\lim_{n\to\infty} q(\varkappa_n, \omega) = 0$.

- We have to prove now that $\lim_{n\to\infty} p(\varkappa_n, \omega) = 0 = \lim_{n\to\infty} p(\omega, \varkappa_n)$.

Let $\varepsilon > 0$. Since $\lim_{n\to\infty} q(\varkappa_n, \omega) = 0$, we observe by (mw_2) that

$$p(\varkappa_n, \omega) \le p(\varkappa_n, \varkappa_m) + \varepsilon$$

for $n, m \in \mathbb{N}$ sufficiently large. Then, $\lim_{n\to\infty} p(\varkappa_n, \omega) = 0$.

As well, since p is a strong mw-distance, due to (mw_2^*) we have also

$$p(\omega, \varkappa_n) \le p(\varkappa_m, \varkappa_n) + \varepsilon$$

for $n, m \in \mathbb{N}$ sufficiently large. Therefore, $\lim_{n\to\infty} p(\omega, \varkappa_n) = 0$ and by (mw_1), $p(\omega, \omega) = 0$.

- Taking into account that $\lim_{n\to\infty} p(\omega, \varkappa_n) = 0$ and $\lim_{n\to\infty} p(\varkappa_n, \varkappa_n) = 0$, by (mw_3), we obtain that $\lim_{n\to\infty} q(\omega, \varkappa_n) = 0$. Consequently, $\{\varkappa_n\}$ converges to ω in the metric space $(\mathcal{X}, q^*)$. Then, because T is p-lower semi-continuous, we have that given $\varepsilon > 0$ there exists $n' \in \mathbb{N}$ such that

$$p(\omega, T\omega) - p(\varkappa_n, T\varkappa_n) < \varepsilon,$$

for all $n \ge n'$. Therefore, $p(\omega, T\omega) = 0$, and so, by (mw_3), we get $\lim_{n\to\infty} q(\varkappa_n, T\omega) = 0$. On the other hand, by (1), we have

$$\begin{aligned}p(\varkappa_n, T\omega) &= p(T\varkappa_{n-1}, T\omega) \le \beta\left(M_1(\varkappa_{n-1}, \omega)\right) M_1(\varkappa_{n-1}, \omega) < M_1(\varkappa_{n-1}, \omega) \\ &= \max\left\{p(\varkappa_{n-1}, \omega), p(\varkappa_{n-1}, \varkappa_n), p(\omega, T\omega)\right\}.\end{aligned}$$

Hence, since $\lim_{n\to\infty} p(\varkappa_n, \omega) = 0$ and $p(\omega, T\omega) = 0$, by (mw_1) we get

$$\lim_{n\to\infty} p(\varkappa_n, T\omega) = 0. \tag{9}$$

Since $\lim_{n\to\infty} q(\varkappa_n, T\omega) = 0$ and taking into account that p is a strong mw-distance, we have by (mw_2^*) that the function $p(\cdot, \varkappa)$ is lower semi-continuous on $(X, \tau(q^*))$, so that for every $\varepsilon > 0$ there exists $\mu(\varepsilon) \in \mathbb{N}$ such that

$$p(T\omega, \omega) \leq p(\varkappa_n, \omega) + \varepsilon, \quad \text{for all } n \geq \mu(\varepsilon).$$

Since $\lim_{n\to\infty} p(\varkappa_n, \omega) = 0$, we get $p(T\omega, \omega) = 0$ and from (mw_1),

$$p(T\omega, T\omega) = 0,$$

from where, using (mw_3) we obtain $q(T\omega, \omega) = 0 = q(\omega, T\omega)$. Hence $T\omega = \omega$, that is, ω is a fixed point of T.

- Assuming this point is not unique, we can find $\varsigma \in X$ such that $T\omega = \omega \neq \varsigma = T\varsigma$ and from (1) we have

$$\begin{aligned} p(\varsigma, \varsigma) &= p(T\varsigma, T\varsigma) \leq \beta\left(M_1(\varsigma, \varsigma)\right) M_1(\varsigma, \varsigma) \\ &< M_1(\varsigma, \varsigma) = \max\left\{p(\varsigma, \varsigma), p(\varsigma, T\varsigma)\right\} = p(\varsigma, \varsigma). \end{aligned} \tag{10}$$

This is a contradiction. Hence, $p(\varsigma, \varsigma) = 0$. On the other hand,

$$\begin{aligned} p(\omega, \varsigma) &= p(T\omega, T\varsigma) \leq \beta\left(M_1(\omega, \varsigma)\right) M_1(\omega, \varsigma) \\ &< M_1(\omega, \varsigma) = \max\left\{p(\omega, \varsigma), p(\omega, T\omega), p(\varsigma, T\varsigma)\right\} \\ &= \max\left\{p(\omega, \varsigma), p(\omega, \omega), p(\varsigma, \varsigma)\right\} = p(\omega, \varsigma), \end{aligned} \tag{11}$$

which is a contradiction, so that $p(\omega, \varsigma) = 0$. Furthermore, from (mw_3), together with (10) and (11), we obtain that

$$q(\omega, \varsigma) = 0.$$

We prove that $q(\varsigma, \omega) = 0$ in a similar way, hence $\varsigma = \omega$. Then, T has exactly one fixed point. □

In the following, we will show that if the conditions (1) and (2) are modified, the p-lower semi-continuity of T can be replaced by p-lower semi-continuity of T^2 or even eliminated.

Theorem 3. *Let (X, q) be a complete quasi-metric space and $T\colon X \to X$ a mapping such that there exist a strong mw-distance p on (X, q) and a function $\beta \in \mathcal{B}$ such that for any $\varkappa, \nu \in X$*

$$p(T\varkappa, T\nu) \leq \beta\left(m_1(\varkappa, \nu)\right) m_1(\varkappa, \nu) \tag{12}$$

and

$$p(T\varkappa, T\nu) \leq \beta\left(m_2(\varkappa, \nu)\right) m_2(\varkappa, \nu) \tag{13}$$

where

$$m_1(\varkappa, \nu) = \max\left\{p(\varkappa, \nu), p(\nu, T\nu)\right\}, m_2(\varkappa, \nu) = \max\left\{p(\varkappa, \nu), p(T\nu, \nu)\right\}.$$

Suppose also that T^2 is p-lower semi-continuous. Then T has a unique fixed point.

Proof. Let us consider a point $\varkappa_0 \in X$ and as in Theorem 2 we can prove that the sequence $\{\varkappa_n\}$, where $\varkappa_n = T\varkappa_{n-1}$ for any $n \in \mathbb{N}$ is a Cauchy sequence on the metric space (X, q^s). Therefore, there exists $\omega \in X$ such that

$$\lim_{n\to\infty} q(\varkappa_n, \omega) = 0,$$

$$\lim_{n\to\infty} p(\varkappa_n, \omega) = \lim_{n\to\infty} p(\omega, \varkappa_n) = 0 = p(\omega, \omega),$$

and

$$\lim_{n\to\infty} q(\omega, \varkappa_n) = 0.$$

Since T^2 is p-lower semi-continuous, we have that $p(\omega, T^2\omega) = 0$ and taking into account that $\lim_{n\to\infty} p(\varkappa_n, \omega) = 0$ by (mw_3), we get $q(\varkappa_n, T^2\omega) = 0$. Withal, from (mw_2^*) we know that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$p(T^2\omega, \omega) < p(\varkappa_n, \omega) + \varepsilon, \text{ for any } n \geq n_0.$$

Hence, $p(T^2\omega, \omega) = 0$ and thus $p(T^2\omega, T^2\omega) \leq p(T^2\omega, \omega) + p(\omega, T^2\omega) = 0$. Therefore, by (mw_3) $q(T^2\omega, \omega) = 0$ and so $T^2\omega = \omega$.

Finally, we must show that ω is a fixed point for T. Indeed, by (13), and taking into account that $T^2\omega = \omega$, we have

$$\begin{aligned} p(\omega, T\omega) &= p(T^2\omega, T\omega) \leq \beta(m_2(T\omega, \omega))m_2(T\omega, \omega) \\ &< m_2(T\omega, \omega) = \max\{p(T\omega, \omega), p(T\omega, \omega)\} \\ &= p(T\omega, \omega) \end{aligned} \tag{14}$$

and

$$\begin{aligned} p(T\omega, \omega) &= p(T\omega, T^2\omega) \leq \beta(m_2(\omega, T\omega))m_2(\omega, T\omega) \\ &< m_2(\omega, T\omega) = \max\{p(\omega, T\omega), p(T^2\omega, T\omega)\} \\ &= p(\omega, T\omega). \end{aligned} \tag{15}$$

This is a contradiction, so $p(T\omega, \omega) = p(\omega, T\omega) = 0$. Moreover, by (mw_1) we have $p(T\omega, T\omega) = 0$. Therefore, by (mw_3) we get that $q(\omega, T\omega) = q(T\omega, \omega) = 0$, that is, ω is a fixed point of T. The uniqueness of the fixed point is proved as in the previous theorem. □

Theorem 4. *Let (X, q) be a complete quasi-metric space and $T : X \to X$ a mapping such that there exist a strong mw-distance p on (X, q) and a function $\beta \in \mathcal{B}$ such that for any $\varkappa, \nu \in X$*

$$p(T\varkappa, T\nu) \leq \beta(m_1^*(\varkappa, \nu))\, m_1^*(\varkappa, \nu) \tag{16}$$

and

$$p(T\varkappa, T\nu) \leq \beta(m_2^*(\varkappa, \nu))\, m_2^*(\varkappa, \nu) \tag{17}$$

where

$$m_1^*(\varkappa, \nu) = \max\{p(\varkappa, \nu), p(\varkappa, T\varkappa)\}, m_2^*(\varkappa, \nu) = \max\{p(\varkappa, \nu), p(T\varkappa, \varkappa)\}.$$

Then T has a unique fixed point.

Proof. Following the same steps as in the proof of Theorem 2 we obtain that

$$\lim_{n\to\infty} p(\varkappa_n, \varkappa_{n-1}) = \lim_{n\to\infty} p(\varkappa_{n-1}, \varkappa_n) = 0, \text{ for any } n \in \mathbb{N} \tag{18}$$

and there exists $\omega \in X$ such that

$$\lim_{n\to\infty} p(\varkappa_n, \omega) = \lim_{n\to\infty} p(\omega, \varkappa_n) = 0 = p(\omega, \omega). \tag{19}$$

Therefore, by (16), since $\beta \in \mathcal{B}$ we have

$$\begin{aligned} p(\varkappa_n, T\omega) &= p(T\varkappa_{n-1}, T\omega) \leq \beta(m_1^*(\varkappa_{n-1}, \omega))m_1^*(\varkappa_{n-1}, \omega) < m_1^*(\varkappa_{n-1}, \omega) \\ &= \max\{p(\varkappa_{n-1}, \omega), p(\varkappa_{n-1}, \varkappa_n)\} \end{aligned}$$

which together with (18) and (19) gives us that $p(\varkappa_n, T\omega) \to 0$. Moreover, by (mw_3), we obtain $q(\omega, T\omega) = 0$.

Since $p(\omega, \varkappa_n) \to 0$ and $p(\varkappa_n, T\omega) \to 0$, by (mw_1) we have that $p(\omega, T\omega) = 0$. Then,

$$p(T\omega, \varkappa_n) = p(T\omega, T\varkappa_{n-1}) < \max\{p(\omega, \varkappa_{n-1}), p(\omega, T\omega)\} = p(\omega, \varkappa_{n-1}).$$

Therefore, $p(T\omega, \varkappa_n) \to 0$, and from (18) and (mw_3), we obtain that $q(T\omega, \omega) = 0$.
Hence, $q(\omega, T\omega) = q(T\omega, \omega) = 0$, and so $T\omega = \omega$.
Let now $\zeta \in X$ be a point such that $T\zeta = \zeta$. Then by (16)

$$p(\omega, \zeta) = p(T\omega, T\zeta) \leq \beta(m_1^*(\omega, \zeta))m_1^*(\omega, \zeta) < \max\{p(\omega, \zeta), p(\omega, T\omega)\} = p(\omega, \zeta).$$

This is a contradiction, so that $p(\omega, \zeta) = 0$. Since $p(\omega, \omega) = 0$, by (mw_3) we obtain that $q(\omega, \zeta) = 0$. Now we shall prove that $q(\zeta, \omega) = 0$.

$$\begin{aligned} p(\zeta, \zeta) &= p(T\zeta, T\zeta) \leq \beta(m_1^*(\zeta, \zeta))m_1^*(\zeta, \zeta) < m_1^*(\zeta, \zeta) \\ &= \max\{p(\zeta, \zeta), p(\zeta, \zeta)\} = p(\zeta, \zeta) \end{aligned}$$

Then, $p(\zeta, \zeta) = 0$.

$$p(\zeta, \omega) = p(T\zeta, T\omega) \leq \beta(m_1^*(\zeta, \omega))m_1^*(\zeta, \omega) < \max\{p(\zeta, \omega), p(\zeta, T\zeta)\} = p(\zeta, \omega).$$

So that $p(\zeta, \omega) = 0$. Since $p(\zeta, \zeta) = 0$, by (mw_3) we obtain that $q(\zeta, \omega) = 0$. □

Now we give an example where Theorems 3 and 4 can be used but it is not possible to apply Theorem 2.

Example 2. *Let q be the quasi-metric on $\mathbb{R}^+$ defined as $q(\varkappa, \nu) = \max\{\nu - \varkappa, 0\}$ for $\varkappa, \nu \in \mathbb{R}^+$. Then $(\mathbb{R}^+, q)$ is a complete quasi-metric space and $p(\varkappa, \nu) = \nu$ is a strong mw-distance (see Example 11 of [4]). Let us consider the mapping $T : \mathbb{R}^+ \to \mathbb{R}^+$,*

$$T(\varkappa) = \begin{cases} \frac{1}{1+\varkappa}, & \varkappa \geq 1 \\ 0, & \varkappa < 1 \end{cases}$$

and the function $\beta \in \mathcal{B}$, $\beta(t) = \frac{1}{1+t}$. Then,

- *If $\nu < 1$ and $\varkappa \in \mathbb{R}^+$ then $p(T\varkappa, T\nu) = 0$, so that the conditions (16) and (17) hold.*
- *If $\nu \geq 1$ and $\varkappa < 1$, then*

$$m_1^*(\varkappa, \nu) = max\{\nu, 0\} = \nu,$$

$$m_2^*(\varkappa, \nu) = max\{\nu, \varkappa\} = \nu$$

and

$$\beta(m_i^*(\varkappa, \nu))m_i^*(\varkappa, \nu) = \beta(\nu)\nu = \frac{\nu}{1+\nu} \geq \frac{1}{1+\nu} = p(T\varkappa, T\nu).$$

Therefore (16) and (17) are fulfilled.

- *If $\nu \geq 1$, $\varkappa \geq 1$, and $\nu \geq \varkappa$, then*

$$m_1^*(\varkappa, \nu) = max\{\nu, \frac{1}{1+\varkappa}\} = \nu,$$

$$m_2^*(\varkappa, \nu) = max\{\nu, \varkappa\} = \nu$$

and, as before, (16) and (17) hold.

- *If $\nu \geq 1$, $\varkappa \geq 1$, and $\nu \leq \varkappa$, then*

$$m_1^*(\varkappa, \nu) = \nu,$$
$$m_2^*(\varkappa, \nu) = \varkappa$$

and

$$\beta(m_2^*(\varkappa, \nu))m_2^*(\varkappa, \nu) = \beta(\varkappa)\varkappa = \frac{\varkappa}{1+\varkappa} \geq \frac{\nu}{1+\nu} \geq \frac{1}{1+\nu} = p(T\varkappa, T\nu).$$

Then (16) and (17) are also satisfied.

Consequently, from Theorem 4, the mapping T has a unique fixed point.

In this example we can also apply Theorem 3 because (12) and (13) hold and $T^2 = 0$. Nevertheless, it is not possible to apply Theorem 2 because T is not a p-lower semi-continuous mapping. Indeed, the sequence $\{\varkappa_n\}$ where $\varkappa_n = 1 - \frac{1}{n}$ converges to 1 but $p(1, T1) - p(\varkappa_n, T\varkappa_n) = 1/2$, for all $n \in \mathbb{N}$.

The condition that the mw-distance is strong cannot be eliminated in the statement of Theorem 4. The following example shows this.

Example 3. *Let q be the quasi-metric on $\mathbb{N}$ given by $q(\varkappa, \varkappa) = 0$ and $q(\varkappa, \nu) = \frac{1}{\varkappa}$ if $\varkappa \neq \nu$. Clearly, $(\mathbb{N}, q)$ is complete because if $\{\varkappa_n\}$ is a Cauchy sequence in $(\mathbb{N}, q^s)$, then $\{\varkappa_n\}$ converges to $\varkappa$ in (X, q^*) for all $\varkappa \in \mathbb{N}$. Let $p = q$. Then p is an mw-distance which is not strong.*

Indeed, the sequence $\{n\}$ converges to $\varkappa$ in $(\mathbb{N}, q^*)$ for all $\varkappa \in \mathbb{N}$ but

$$p(\varkappa, \nu) - p(n, \nu) = \frac{1}{\varkappa} - \frac{1}{n},$$

i.e., $p(\cdot, \nu)$ is not lower semi-continuous on $(X, \tau(q^*))$.

Let $T : \mathbb{N} \to \mathbb{N}$ given by $T\varkappa = 2\varkappa$ and let $\beta \in \mathcal{B}$, $\beta(t) = \frac{1}{1+t}$. Then

$$m_1^*(\varkappa, \nu) = max\{p(\varkappa, \nu), p(\varkappa, T\varkappa)\} = \frac{1}{\varkappa},$$
$$m_2^*(\varkappa, \nu) = max\{p(\varkappa, \nu), p(T\varkappa, \varkappa)\} = \frac{1}{\varkappa}$$

and

$$\beta(m_i^*(\varkappa, \nu))m_i^*(\varkappa, \nu) = \beta(\frac{1}{\varkappa})\frac{1}{\varkappa} = \frac{1}{1+\varkappa} \geq \frac{1}{2\varkappa} = p(T\varkappa, T\nu).$$

Therefore, (16) and (17) are fulfilled but T has no fixed point.

Author Contributions: Conceptualization, C.A., A.F., E.K. and P.T.; investigation, C.A., A.F., E.K. and P.T.; writing—review and editing, C.A., A.F., E.K. and P.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was partially supported by the Spanish Ministry of Science, Innovation and Universities. Grant number PGC2018-095709-B-C21 and AEI/FEDER, UE funds

Conflicts of Interest: The authors declare no conflict of interest.

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