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Research Article

A k-Dimensional System of Fractional Finite Difference Equations

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We investigate the existence of solutions for a k-dimensional system of fractional finite difference equations by using the Kranoselskii's fixed point theorem. We present an example in order to illustrate our results.

1. Introduction

The fractional calculus revealed during the last decade its huge potential applications in many branches of science and engineering (see, e.g., [1–9]). A new and promising direction within fractional calculus is the discrete fractional calculus (see [6, 7, 10–14]). The advantages of this type of calculus are that it treats better phenomena with memory effect (see [10, 11, 14]). We recall that some researchers have been investigating discrete fractional calculus for special equations via very definite boundary conditions (see, e.g., [12, 13, 15–24] and the references therein). Many researchers could focus on this field by considering natural potential of fractional finite difference equations. In this paper, we investigate the existence of solutions for k-dimensional system of fractional finite difference equations:

$$
\begin{gathered}
\Delta^{\nu_1} y_1(t) + f_1\left(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)\right) = 0, \\
\Delta^{\nu_2} y_2(t) + f_2\left(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)\right) = 0, \\
\vdots \\
\Delta^{\nu_k} y_k(t) + f_k\left(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)\right) = 0, \\
y_1(\nu_1-2) = \Delta y_1(\nu_1+b) = 0, \\
y_2(\nu_2-2) = \Delta y_2(\nu_2+b) = 0, \\
\vdots \\
y_k(\nu_k-2) = \Delta y_k(\nu_k+b) = 0,
\end{gathered}
\tag{1}
$$

where $b \in \mathbb{N}_0$, $1 < \nu_i \le 2$, and $f_i : \mathbb{R}^k \to \mathbb{R}$ are continuous functions for $i = 1, 2, \ldots, k$. One-dimensional version of the problem has been studied by Goodrich [18]. Also, Pan et al. studied two-dimensional version of the problem [24]. We show that the problem (1) is equivalent to a summation equation and by using Krasnoselskii's fixed point theorem we investigate solutions of the problem. In this way, we present an example to illustrate our result.

2. Preliminaries

It is known that the finite fractional difference theory is important in many branches of science and engineering (see, e.g., [13, 16, 18, 19, 21, 25, 26] and the references therein). The Gamma function is defined by $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$ for the complex numbers z in which the real part of z is positive (see [8]). Note that the domain of the Gamma function is $\mathbb{R} \setminus \{0, -1, -2, -3, \ldots\}$ (see [8]). Now, we recall $t^{\underline{\nu}} := \Gamma(t+1)/\Gamma(t+1-\nu)$ for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined (see [16]). If $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$ (see [16]). We recall that $\Delta^\beta t^{\underline{\mu}} = \Gamma(\mu+1)t^{\underline{\mu-\beta}}/\Gamma(\mu-\beta+1)$ (see [16]). One can verify that $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu+1)$ and $t^{\underline{\nu+1}}/t^{\underline{\nu}} = t-\nu$.

In this paper, we use the standard notations $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \ldots, b\}$ for all real numbers a and b whenever $b-a$ is a natural number. Let $\nu > 0$ with $m-1 < \nu < m$ for some natural number m. Then, the νth fractional sum of f based at a is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a}^{t-\nu} (t-\sigma(k))^{\underline{\nu-1}} f(k) \tag{2}$$

for all $t \in \mathbb{N}_{a+\nu}$, where $\sigma(k) = k+1$ is the forward jump operator (see [16]). Similarly, we define $\Delta_a^\nu f(t) = (1/\Gamma(-\nu)) \sum_{k=a}^{t+\nu} (t-\sigma(k))^{\underline{-\nu-1}} f(k)$ for all $t \in \mathbb{N}_{a+N-\nu}$. Note that the domain of $\Delta_a^r f(t)$ is $\mathbb{N}_{a+N-r}$ for $r > 0$ and $\mathbb{N}_{a-r}$ for $r < 0$ (see [16]). Also, for the natural number $\nu = m$, we have to recall the formula

$$\Delta_a^\nu f(t) = \Delta^m f(t) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(t+m-i). \tag{3}$$

We define the trivial sum $\Delta_a^0 f(t) = f(t)$ for all $t \in \mathbb{N}_a$.

Lemma 1 (see [13]). *Let $h : \mathbb{N}_a \to \mathbb{R}$ be a mapping and m a natural number. Then, the general solution of the equation $\Delta_{a+\nu-m}^\nu y(t) = h(t)$ is given by $y(t) = \sum_{i=1}^m -C_i(t-a)^{\underline{\nu-i}} + \Delta_a^{-\nu}h(t)$ for all $t \in \mathbb{N}_{a+\nu-m}$, where $C_1, \ldots, C_m$ are arbitrary constants.*

Let $h : \mathbb{N}_{\nu-m} \times \mathbb{R} \to \mathbb{R}$ be a mapping and m a natural number. By using a similar proof, one can check that the general solution of the equation $\Delta_{\nu-m}^\nu y(t) = h(t+\nu-m+1, y(t+\nu-m+1))$ is given by

$$y(t) = \sum_{i=1}^{m} -C_i t^{\underline{\nu-i}} + \Delta^{-\nu} h(t+\nu-m+1, y(t+\nu-m+1)) \tag{4}$$

for all $t \in \mathbb{N}_{\nu-m}$. In particular, the general solution has the following representation:

$$y(t) = \sum_{i=1}^{m} -C_i t^{\underline{\nu-i}} + \frac{1}{\Gamma(\nu)} \times \sum_{s=0}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} h(s+\nu-m+1, y(s+\nu-m+1)) \tag{5}$$

for all $t \in \mathbb{N}_{\nu-m}$. By considering the details, note that $\sum_{k=a}^b (t-\sigma(k))^{\underline{-\nu-1}} f(k) = 0$ whenever $b < a$. Also for $\nu, \mu > 0$ with $m-1 < \nu \le m$ and $n-1 < \mu \le n$, the domain of the operator Δ is given by $\mathcal{D}\{\Delta_a^{-\nu} f\} = \mathbb{N}_{a+\nu}$, $\mathcal{D}\{\Delta_a^\nu f\} = \mathbb{N}_{a+m-\nu}$, $\mathcal{D}\{\Delta_{a+n-\mu}^{-\nu}\Delta_a^\mu f\} = \mathbb{N}_{a+n+\nu-\mu}$, $\mathcal{D}\{\Delta_{a+\mu}^{\nu}\Delta_a^{-\mu} f\} = \mathbb{N}_{a+\mu+m-\nu}$, $\mathcal{D}\{\Delta_{a+n-\mu}^{\nu}\Delta_a^{\mu} f\} = \mathbb{N}_{a+n-\mu+m-\nu}$, and $\mathcal{D}\{\Delta_{a+\mu}^{-\nu}\Delta_a^{-\mu} f\} = \mathbb{N}_{a+\nu+\mu}$ (for more details see [13, 21, 22]). One can find next result about composing a difference with a sum in [12].

Lemma 2. *Let $f : \mathbb{N}_a \to \mathbb{R}$ be a map, $k \in \mathbb{N}_0$, and $\mu > k$ with $n-1 < \mu \le n$. Then $\Delta^k \Delta_a^{-\mu} f(t) = \Delta_a^{k-\mu} f(t)$ for all $t \in \mathbb{N}_{a+\mu}$ and $\Delta^k \Delta_a^\mu f(t) = \Delta_a^{k+\mu} f(t)$ for all $t \in \mathbb{N}_{a+n-\mu}$.*

By using Lemma 1 and last lemma for $k = 1$, we get

$$\Delta y(t) = \sum_{i=1}^{m} -C_i (\nu-i) t^{\underline{\nu-i-1}} + \frac{1}{\Gamma(\nu-1)} \times \sum_{s=0}^{t-\nu+1} (t-\sigma(s))^{\underline{\nu-2}} h(t+\nu-m+1, y(t+\nu-m+1)). \tag{6}$$

We are going to use this in our main results. A nonempty, closed subset $P \neq \{0\}$ of a topological vector space E is called a cone whenever $P \cap (-P) = \{0\}$ and $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b (for more details and examples see [27] and references therein).

Lemma 3 (see [28]). *Let X be a Banach space and K a cone in X. Assume that B_1 and B_2 are open subsets of X such that $0 \in B_1$ and $\overline{B_1} \subseteq B_2$. Suppose that $T : K \cap (\overline{B_2} \setminus B_1) \to K$ is a completely continuous operator. If either $\|Ty\| \le \|y\|$ for all $y \in K \cap \partial B_1$ and $\|Ty\| \ge \|y\|$ for all $y \in K \cap \partial B_2$ or $\|Ty\| \ge \|y\|$ for all $y \in K \cap \partial B_1$ and $\|Ty\| \le \|y\|$ for all $y \in K \cap \partial B_2$, then T has at least one fixed point in $K \cap (\overline{B_2} \setminus B_1)$.*

3. Main Result

In this section we provide the main results. For next result, consider the problem (1).

Lemma 4. *The fractional finite difference equation*

$$\Delta^{\nu_i} y_i(t) + f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)) = 0 \tag{$*$}$$

via the boundary conditions $y_i(\nu_i - 2) = \Delta y_i(\nu_i + b) = 0$ *has a solution* y_{i0} *if and only if* y_{i0} *is a solution of the summation equation* $y_i(t) = \sum_{s=0}^{b+1} G_i(t,s) f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1))$, *where the Green function* $G_i(t,s)$ *is given by*

$$G_i(t,s) = \frac{1}{\Gamma(\nu_i)} \times \begin{cases} \dfrac{t^{\underline{\nu_i-1}}}{(\nu_i+b)^{\underline{\nu_i-2}}}(\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} - (t-\sigma(s))^{\underline{\nu_i-1}} \\ \quad s \le t-\nu \le b+1, \\ \dfrac{t^{\underline{\nu_i-1}}}{(\nu_i+b)^{\underline{\nu_i-2}}}(\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} \\ \quad t-\nu+1 \le s \le b+1 \end{cases} \tag{7}$$

for all $s \in \mathbb{N}_0^{b+1}$. *Here,* $i \in \{1,2,\ldots,k\}$ *and* $(*)$ *is one of equations of the system.*

Proof. Let $h_i(t+\nu_i-1) := f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1))$ and let y_{i0} be a solution of the fractional finite difference equation $\Delta^{\nu_i} y_i(t) + f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)) = 0$. By using Lemma 1, we get $y_{i0}(t) = C_1 t^{\underline{\nu_i-1}} + C_2 t^{\underline{\nu_i-2}} - (1/\Gamma(\nu_i)) \sum_{s=0}^{t-\nu_i} (t-\sigma(s))^{\underline{\nu_i-1}} h_i(s+\nu_i-1)$. By using the boundary condition $y_{i0}(\nu_i-2) = 0$, we obtain

$$0 = C_1(\nu_i-2)^{\underline{\nu_i-1}} + C_2(\nu_i-2)^{\underline{\nu_i-2}} - \frac{1}{\Gamma(\nu_i)} \sum_{s=0}^{-2} (\nu_i-2-\sigma(s))^{\underline{\nu_i-1}} h_i(s+\nu_i-1) \tag{8}$$

and so $C_2 = 0$. Now by using the boundary condition $\Delta y_{i0}(\nu_i+b) = 0$, we get

$$0 = C_1(\nu_i-1)(\nu_i+b)^{\underline{\nu_i-2}} - \frac{1}{\Gamma(\nu_i-1)} \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1). \tag{9}$$

Hence, $C_1 = (1/((\nu_i+b)^{\underline{\nu_i-2}}\Gamma(\nu_i))) \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1)$ and so

$$\begin{aligned} y_{i0}(t) &= \frac{t^{\underline{\nu_i-1}}}{(\nu_i+b)^{\underline{\nu_i-2}}\Gamma(\nu_i)} \times \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1) \\ &\quad - \frac{1}{\Gamma(\nu_i)} \sum_{s=0}^{t-\nu_i} (t-\sigma(s))^{\underline{\nu_i-1}} h_i(s+\nu_i-1) \\ &= \sum_{s=0}^{b+1} G_i(t,s) f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)). \end{aligned} \tag{10}$$

Now, let y_{i0} be a solution of the fractional sum equation

$$y_i(t) = \sum_{s=0}^{b+1} G_i(t,s) f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)). \tag{11}$$

Then, $y_{i0}(t) = (t^{\underline{\nu_i-1}}/(\nu_i+b)^{\underline{\nu_i-2}}\Gamma(\nu_i)) \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1) - (1/\Gamma(\nu_i)) \sum_{s=0}^{t-\nu_i} (t-\sigma(s))^{\underline{\nu_i-1}} h_i(s+\nu_i-1)$. Since $(\nu_i-2)^{\underline{\nu_i-1}} = 0$, we get $y_{i0}(\nu_i-2) = 0$. Also,

$$\begin{aligned} \Delta y_{i0}(t) &= \frac{(\nu_i-1)t^{\underline{\nu_i-2}}}{(\nu_i+b)^{\underline{\nu_i-2}}\Gamma(\nu_i)} \times \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1) \\ &\quad - \frac{1}{\Gamma(\nu_i-1)} \sum_{s=0}^{t-\nu_i+1} (t-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1). \end{aligned} \tag{12}$$

Hence, we get

$$\begin{aligned} \Delta y_{i0}(\nu_i+b) &= \frac{(\nu_i-1)(\nu_i+b)^{\underline{\nu_i-2}}}{(\nu_i+b)^{\underline{\nu_i-2}}\Gamma(\nu_i)} \times \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1) \\ &\quad - \frac{1}{\Gamma(\nu_i-1)} \sum_{s=0}^{b+1} ((\nu_i+b)-\sigma(s))^{\underline{\nu_i-2}} \times h_i(s+\nu_i-1) = 0. \end{aligned} \tag{13}$$

Moreover, $\Delta^{\nu_i} y_{i0}(t) = (\Delta^{\nu_i} t^{\underline{\nu_i-1}}/(\nu_i+b)^{\underline{\nu_i-2}}\Gamma(\nu_i)) \sum_{s=0}^{b+1} (\nu_i+b-\sigma(s))^{\underline{\nu_i-2}} h_i(s+\nu_i-1) - \Delta^{\nu_i}\Delta^{-\nu_i} h_i(s+\nu_i-1)$. Since $\Delta^{\nu_i} t^{\underline{\nu_i-1}} = \Gamma(\nu_i) t^{\underline{\nu_i-1-\nu_i}}/\Gamma(\nu_i-\nu_i) = 0$ and $\Delta^{\nu_i}\Delta^{-\nu_i} h_i(s+\nu_i-1) = h_i(s+\nu_i-1)$, we get

$$\Delta^{\nu_i} y_{i0}(t) + f_i(y_1(t+\nu_1-1), y_2(t+\nu_2-1), \ldots, y_k(t+\nu_k-1)) = 0. \tag{14}$$

This completes the proof. □

Hereafter, for simplicity we use the notations $I_i := \mathbb{N}_{\nu_i-1}^{\nu_i+b+1}$ and $J_i := [((\nu_i+b)/4), ((3(\nu_i+b))/4)]$ for all $i = 1, 2, \ldots, k$.

Lemma 5 (see [18]). *The Green function* (7) *satisfies* $G_i(t,s) \geq 0$ *for all* $t \in I_i$ *and* $s \in \mathbb{N}_0^{b+1}$ *and* $\max_{t\in I_i} G_i(t,s) = G_i(s+\nu_i-1,s)$ *for all* $s \in \mathbb{N}_0^{b}$ *and there exist* $\lambda_i \in (0,1)$ *such that*

$$\min_{t\in J_i} G_i(t,s) \geq \lambda_i \max_{t\in I_i} G_i(t,s) = \lambda_i G_i(s+\nu_i-1,s) \tag{15}$$

for all $s \in \mathbb{N}_0^{b+1}$.

Goodrich showed that $\lambda_i = \min\{\gamma_1^i, \gamma_2^i\}$ (see [18]), where $\gamma_1^i = ((b+\nu_i)/4)^{\underline{\nu_i-1}}/(b+\nu_i)^{\underline{\nu_i-1}}$ and

$$\gamma_2^i = \frac{1}{(3(b+\nu_i)/4)^{\underline{\nu_i-1}}} \times \left[\left(\frac{3(b+\nu_i)}{4}\right)^{\underline{\nu_i-1}} - \frac{(b+1)(3(b+\nu_i)/4-1)^{\underline{\nu_i-1}}\Gamma(\nu_i+b+1)}{\Gamma(b+3)(\nu_i+b-1)^{\underline{\nu_i-1}}}\right]. \tag{16}$$

Note that γ_2^i can be written in the simple form $\gamma_2^i = (\nu_i+2)/3(b+2)$, because

$$\begin{aligned}\gamma_2^i &= \frac{1}{(3(b+\nu_i)/4)^{\underline{\nu_i-1}}} \times \left[\left(\frac{3(b+\nu_i)}{4}\right)^{\underline{\nu_i-1}} - \frac{(b+1)(3(b+\nu_i)/4-1)^{\underline{\nu_i-1}}\Gamma(\nu_i+b+1)}{\Gamma(b+3)(\nu_i+b-1)^{\underline{\nu_i-1}}}\right] \\ &= 1 - \frac{(b+1)(3(b+\nu_i)/4-1)^{\underline{\nu_i-1}}\Gamma(\nu_i+b+1)}{(3(b+\nu_i)/4)^{\underline{\nu_i-1}}\Gamma(b+3)(\nu_i+b-1)^{\underline{\nu_i-1}}} \\ &= 1 - \frac{(b+1)(3(b+\nu_i)/4-\nu_i+1)\Gamma(\nu_i+b+1)}{(3(b+\nu_i)/4)\Gamma(b+3)(\Gamma(\nu_i+b)/\Gamma(b+1))} \\ &= 1 - \frac{(b+1)(3(b+\nu_i)/4-\nu_i+1)(\nu_i+b)\Gamma(\nu_i+b)}{(3(b+\nu_i)/4)(b+1)(b+2)\Gamma(b+1)(\Gamma(\nu_i+b)/\Gamma(b+1))} \\ &= \frac{\nu_i+2}{3(b+2)}.\end{aligned} \tag{$**$}$$

Note that $(**)$ hold because $(a-1)^{\underline{b}}/a^{\underline{b}} = (a-b)/a$. Suppose that $\mathscr{A}_i$ is the Banach space of the maps $u : \mathbb{N}_{\nu_i-2}^{\nu_i+b} \to \mathbb{R}$ via the usual maximum norm $\|u\| = \max\{|u(t)| : t \in \mathbb{N}_{\nu_i-2}^{\nu_i+b}\}$. Consider the space $\mathscr{X} = \mathscr{A}_1 \times \mathscr{A}_2 \times \cdots \times \mathscr{A}_k$ via the norm $\|(y_1,y_2,\ldots,y_k)\|_{\mathscr{X}} = \|y_1\| + \|y_2\| + \cdots + \|y_k\|$. It is clear that $(\mathscr{X}, \|\cdot\|_{\mathscr{X}})$ is a Banach space (see [29]). Now, define the map $T : \mathscr{X} \to \mathscr{X}$ by

$$T(y_1,y_2,\ldots,y_k)(t_1,t_2,\ldots,t_k) = \begin{pmatrix} T_1(y_1,y_2,\ldots,y_k)(t_1) \\ T_2(y_1,y_2,\ldots,y_k)(t_2) \\ \vdots \\ T_k(y_1,y_2,\ldots,y_k)(t_k) \end{pmatrix}, \tag{17}$$

where $T_i(y_1,y_2,\ldots,y_k)(t) = \sum_{s=0}^{b+1} G_i(t,s) f_i(y_1(s+\nu_1-1), y_2(s+\nu_2-1),\ldots,y_k(s+\nu_k-1))$ for $i=1,2,\ldots,k$. Also, consider the cone $\mathscr{K}$ defined by

$$\begin{aligned}\mathscr{K} = \{&(y_1,y_2,\ldots,y_k) \in \mathscr{X} : y_i \geq 0, \\ &\min_{(t_1,t_2,\ldots,t_k)\in J_1\times J_2\times\cdots\times J_k} [y_1(t_1) + y_2(t_2) + \cdots + y_k(t_k)] \\ &\geq \lambda \|(y_1,y_2,\ldots,y_k)\|_{\mathscr{X}}\},\end{aligned} \tag{18}$$

where $\lambda = \min_{1\leq i\leq k}\lambda_i$. First, for the operator T we show that $T(\mathscr{K}) \subseteq \mathscr{K}$ whenever the functions f_i are nonnegative for $i=1,2,\ldots,k$. Let $(y_1,y_2,\ldots,y_k) \in \mathscr{K}$. Then, we have

$$\begin{aligned}&\min_{(t_1,t_2,\ldots,t_k)\in J_1\times J_2\times\cdots\times J_k} \sum_{n=1}^{k} T_n(y_1,y_2,\ldots,y_k)(t_n) \\ &\geq \sum_{n=1}^{k} \min_{t_n\in J_n} T_n(y_1,y_2,\ldots,y_k)(t_n) \\ &= \sum_{n=1}^{k} \min_{t_n\in J_n} \sum_{s=0}^{b+1} G_n(t_n,s) f_n\begin{pmatrix} y_1(s+\nu_1-1) \\ y_2(s+\nu_2-1) \\ \vdots \\ y_k(s+\nu_k-1)\end{pmatrix} \\ &\geq \sum_{n=1}^{k} \lambda_n \max_{t_n\in I_n} \sum_{s=0}^{b+1} G_n(t_n,s) f_n\begin{pmatrix} y_1(s+\nu_1-1) \\ y_2(s+\nu_2-1) \\ \vdots \\ y_k(s+\nu_k-1)\end{pmatrix} \\ &= \sum_{n=1}^{k} \lambda_n \|T_n(y_1,y_2,\ldots,y_k)\| \\ &\geq \lambda \sum_{n=1}^{k} \|T_n(y_1,y_2,\ldots,y_k)\| \\ &= \lambda \|T(y_1,y_2,\ldots,y_k)\|_{\mathscr{X}},\end{aligned} \tag{19}$$

where $\lambda = \min_{1\leq n\leq k}\lambda_n$. Hence, $T(y_1,y_2,\ldots,y_k) \in \mathscr{K}$ and so $T(\mathscr{K}) \subseteq \mathscr{K}$. For providing our main result, we use similar conditions which have been given by Goodrich in [18] and Henderson et al. in [30].

Theorem 6. *Suppose that $f_i \in C([0,\infty)^k)$ for all $i = 1,2,\ldots,k$:*

$$\lim_{(y_1,y_2,\ldots,y_k)\to(0^+,0^+,\ldots,0^+)} \frac{f_i(y_1,y_2,\ldots,y_k)}{y_1+y_2+\cdots+y_k} = f_i^*,$$
$$\lim_{(y_1,y_2,\ldots,y_k)\to(+\infty,+\infty,\ldots,+\infty)} \frac{f_i(y_1,y_2,\ldots,y_k)}{y_1+y_2+\cdots+y_k} = f_i^{**} \tag{20}$$

such that $\sum_{s=0}^{b+1} G_i(s+\nu_i-1,s)(f_i^*+\epsilon) \le 1/k$ *and* $\sum_{s=0}^{b+1} \lambda G_i(s+\nu_i-1,s)(f_i^{**}-\epsilon) \ge 1/k$ *for some*

$$0 < \epsilon < \min\{f_i^{**} : 1 \le i \le k\}, \tag{21}$$

where G_i is the Green function (7) and $\lambda = \min_{1\le i\le k}\lambda_i$. Then the k-dimensional system of fractional finite difference equations (1) has at least one solution.

Proof. Consider the operator $T : \mathcal{K} \to \mathcal{K}$ defined by (17) and the cone $\mathcal{K}$. It is clear that T is completely continuous because it is a summation operator on a finite set. Choose $\delta_1 > 0$ such that

$$f_i(y_1,y_2,\ldots,y_k) \le (f_i^*+\epsilon)(y_1+y_2+\cdots+y_k) \tag{22}$$

for all $\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} < \delta_1$. Put $\mathcal{B}_1 = \{(y_1,y_2,\ldots,y_k) \in \mathcal{X} : \|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} < \delta_1\}$. Then, $0 \in \mathcal{B}_1$ and $\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} = \delta_1$ for all $(y_1,y_2,\ldots,y_k) \in \mathcal{K}\cap\partial\mathcal{B}_1$. Also, we have

$$\begin{aligned}
&\|T_i(y_1,y_2,\ldots,y_k)\| \\
&= \max_{t_i\in I_i}\sum_{s=0}^{b+1} G_i(t_i,s) f_i(y_1(s+\nu_1-1), \\
&\qquad y_2(s+\nu_2-1),\ldots,y_k(s+\nu_k-1)) \\
&\le \sum_{s=0}^{b+1} G_i(s+\nu-1,s)(f_i^*+\epsilon)(y_1+y_2+\cdots+y_k) \\
&\le \|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}\sum_{s=0}^{b+1} G_i(s+\nu-1,s)(f_i^*+\epsilon) \\
&\le \frac{1}{k}\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}
\end{aligned} \tag{23}$$

for all $(y_1,y_2,\ldots,y_k) \in \mathcal{K}\cap\partial\mathcal{B}_1$. Hence,

$$\begin{aligned}
\|T(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} &= \sum_{i=1}^{k}\|T_i(y_1,y_2,\ldots,y_k)\| \\
&\le k\times\frac{1}{k}\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} \\
&= \|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}
\end{aligned} \tag{24}$$

for all $(y_1,y_2,\ldots,y_k) \in \mathcal{K}\cap\partial\mathcal{B}_1$. Now, choose $\beta \in \mathbb{R}$ such that $\beta > \delta_1$ and

$$f_i(y_1,y_2,\ldots,y_k) \ge (f_i^{**}-\epsilon)(y_1+y_2+\cdots+y_k) \tag{25}$$

for all $\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} \ge \beta$. Also, choose δ_2 such that $(1/k)\beta \le \delta_2 \le \lambda\beta\min_{1\le i\le k}\sum_{s=0}^{b+1} G_i(s+\nu-1,s)(f_i^{**}-\epsilon)$. Now, put $\mathcal{B}_2 = \{(y_1,y_2,\ldots,y_k) \in \mathcal{X} : \|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} < k\delta_2\}$. Then, $\overline{\mathcal{B}_1} \subseteq \mathcal{B}_2$ and

$$\begin{aligned}
&y_1(t_1)+y_2(t_2)+\cdots+y_k(t_k) \\
&\ge \min_{(t_1,t_2,\cdots,t_k)\in J_1\times J_2\times\cdots\times J_k}[y_1(t_1)+y_2(t_2) \\
&\qquad +\cdots+y_k(t_k)] \\
&\ge \lambda\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}
\end{aligned} \tag{26}$$

for all $(y_1,y_2,\ldots,y_k) \in \mathcal{K}\cap\partial\mathcal{B}_2$. Thus, by using (25) we get

$$\begin{aligned}
&\|T_i(y_1,y_2,\ldots,y_k)\| \\
&= \max_{t_i\in I_i}\sum_{s=0}^{b+1} G_i(t_i,s) f_i(y_1(s+\nu_1-1), y_2(s+\nu_2-1),\ldots, \\
&\qquad y_k(s+\nu_k-1)) \\
&\ge \sum_{s=0}^{b+1} G_i(s+\nu-1,s)(f_i^{**}-\epsilon)(y_1+y_2+\cdots+y_k) \\
&\ge \lambda\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}\sum_{s=0}^{b+1} G_i(s+\nu-1,s)(f_i^{**}-\epsilon) \\
&\ge \frac{1}{k}\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}
\end{aligned} \tag{27}$$

for all $(y_1,y_2,\ldots,y_k) \in \mathcal{K}\cap\partial\mathcal{B}_2$. Hence,

$$\begin{aligned}
\|T(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} &= \sum_{i=1}^{k}\|T_i(y_1,y_2,\ldots,y_k)\| \\
&\ge k\times\frac{1}{k}\|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}} \\
&= \|(y_1,y_2,\ldots,y_k)\|_{\mathcal{X}}
\end{aligned} \tag{28}$$

for all $(y_1,y_2,\ldots,y_k) \in \mathcal{K}\cap\partial\mathcal{B}_2$. By using Lemma 3, T has at least one fixed point $(y_{10},y_{20},\ldots,y_{k0})$ in $\mathcal{K}\cap(\overline{\mathcal{B}_2}\setminus\mathcal{B}_1)$ and so by using Lemma 4, the k-dimensional system of fractional finite difference equations (1) has at least one solution. □

4. Example

Here, we provide an example to illustrate our last result.

Example 1. Consider the 5-dimensional fractional finite difference equation system:

$$\begin{aligned}
&\Delta^{1.2} y_1(t) + f_1(y_1(t+0.2), y_2(t+0.4), y_3(t+0.5), \\
&\qquad y_4(t+0.6), y_5(t+0.8)) = 0, \\
&\Delta^{1.4} y_2(t) + f_2(y_1(t+0.2), y_2(t+0.4), y_3(t+0.5), \\
&\qquad y_4(t+0.6), y_5(t+0.8)) = 0, \\
&\Delta^{1.5} y_2(t) + f_3(y_1(t+0.2), y_2(t+0.4), y_3(t+0.5), \\
&\qquad y_4(t+0.6), y_5(t+0.8)) = 0, \\
&\Delta^{1.6} y_2(t) + f_4(y_1(t+0.2), y_2(t+0.4), y_3(t+0.5), \\
&\qquad y_4(t+0.6), y_5(t+0.8)) = 0, \\
&\Delta^{1.8} y_2(t) + f_5(y_1(t+0.2), y_2(t+0.4), y_3(t+0.5), \\
&\qquad y_4(t+0.6), y_5(t+0.8)) = 0, \\
&y_1(-0.8) = \Delta y_1(9.2) = 0, \\
&y_2(-0.6) = \Delta y_2(9.4) = 0, \\
&y_3(-0.5) = \Delta y_3(9.5) = 0, \\
&y_4(-0.4) = \Delta y_4(9.6) = 0, \\
&y_5(-0.2) = \Delta y_5(9.8) = 0.
\end{aligned} \tag{29}$$

We show that the problem has at least one solution, where

$$\begin{aligned}
&f_1(y_1, y_2, y_3, y_4, y_5) \\
&\quad = \frac{(y_1 + y_2 + \cos y_3)(y_1 + y_2 + y_3 + y_4 + y_5)}{y_1 + y_2 + 1000}, \\
&f_2(y_1, y_2, y_3, y_4, y_5) \\
&\quad = 3e^{-10/(y_1+y_2+y_3+1)}(y_1 + y_2 + y_3 + y_4 + y_5), \\
&f_3(y_1, y_2, y_3, y_4, y_5) \\
&\quad = (y_1 + y_2 + y_3 + y_4 + y_5)\begin{cases} 5y_1 + \dfrac{1}{1000} & y_1 < 1, \\ 2.001 + \dfrac{3}{y_1} & y_1 \geq 1, \end{cases} \\
&f_4(y_1, y_2, y_3, y_4, y_5) \\
&\quad = \left(\frac{3y_3 - \sin y_5}{2y_3 + 1} + \frac{1}{1000}\right)(y_1 + y_2 + y_3 + y_4 + y_5), \\
&f_5(y_1, y_2, y_3, y_4, y_5) \\
&\quad = (y_1 + y_2 + y_3 + y_4 + y_5)\begin{cases} e^{-8\sin(y_2)/y_2} & y_2 > 0, \\ e^{-8} & y_2 = 0. \end{cases}
\end{aligned} \tag{30}$$

Let $\nu_1 = 1.2$, $\nu_2 = 1.4$, $\nu_3 = 1.5$, $\nu_4 = 1.6$, $\nu_5 = 1.8$, $b = 8$, and $k = 5$. Thus, the system (29) is a special case of the system (1). It is easy to check that $f_i \in C([0,\infty)^5)$ for $i = 1, 2, 3, 4, 5$. Put $\gamma_2^i = (\nu_i + 2)/3(8 + 2)$,

$$\gamma_1^i = \frac{((8 + \nu_i)/4)^{\underline{\nu_i - 1}}}{(8 + \nu_i)^{\underline{\nu_i - 1}}} = \frac{\Gamma(3 + \nu_i/4) \times \Gamma(10)}{\Gamma(4 - 3\nu_i/4) \times \Gamma(9 + \nu_i)}, \tag{31}$$

and $\lambda_i = \min\{\gamma_1^i, \gamma_2^i\}$ for $i = 1, 2, 3, 4, 5$. Then, by a calculation we get $\lambda_1 = 0.1066$, $\lambda_2 = 0.1133$, $\lambda_3 = 0.1166$, $\lambda_4 = 0.1200$, and $\lambda_5 = 0.1266$. Thus, $\lambda = \min\{\lambda_i : i = 1, 2, 3, 4, 5\} = 0.1066$. On the other hand by calculation of some limits, one can get that $f_1^* = 10^{-3}$, $f_1^{**} = 1$, $f_2^* = 3e^{-10}$, $f_2^{**} = 3$, $f_3^* = 10^{-3}$, $f_3^{**} = 2.001$, $f_4^* = 10^{-3}$, $f_4^{**} = 1.501$, $f_5^* = e^{-8}$, and $f_5^{**} = 1$. Moreover, we have

$$\begin{aligned}
&\sum_{s=0}^{b+1} G_1(s + \nu_1 - 1, s) \\
&\quad = \sum_{s=0}^{9} G_1(s + 0.2, s) \\
&\quad = \sum_{s=0}^{9} \frac{(s + 0.2)^{\underline{0.2}}}{9.2^{\underline{-0.8}}}(9.2 - \sigma(s))^{\underline{-0.8}} \\
&\quad = \sum_{s=0}^{9} \frac{\Gamma(s + 1.2)\Gamma(11)\Gamma(9.2 - s)}{\Gamma(s + 1)\Gamma(10.2)\Gamma(10 - s)} \\
&\quad \geq \frac{\Gamma(11)}{\Gamma(10.2)} \sum_{s=0}^{9} \frac{\Gamma(9.2 - s)}{\Gamma(10 - s)} \geq 6 \times 6 = 36, \\
&\sum_{s=0}^{b+1} G_1(s + \nu_1 - 1, s) \\
&\quad = \sum_{s=0}^{9} \frac{\Gamma(s + 1.2)\Gamma(11)\Gamma(9.2 - s)}{\Gamma(s + 1)\Gamma(10.2)\Gamma(10 - s)} \\
&\quad \leq \frac{\Gamma(11)}{\Gamma(10.2)} \sum_{s=0}^{9} \frac{\Gamma(s + 1.2)}{\Gamma(s + 1)} \leq 6 \times 13 = 78.
\end{aligned} \tag{32}$$

Similarly, we obtain

$$\sum_{s=0}^{b+1} G_2(s + \nu_2 - 1, s) \geq 6 \times 6 = 36,$$

$$\sum_{s=0}^{b+1} G_2(s + \nu_2 - 1, s) \leq 6 \times 19 = 114,$$

$$\begin{aligned}
&\sum_{s=0}^{b+1} G_3\left(s+\nu_3-1, s\right) \geq 6 \times 6=36, \\
&\sum_{s=0}^{b+1} G_3\left(s+\nu_3-1, s\right) \leq 6 \times 22=132, \\
&\sum_{s=0}^{b+1} G_4\left(s+\nu_4-1, s\right) \geq 6 \times 6=36, \\
&\sum_{s=0}^{b+1} G_4\left(s+\nu_4-1, s\right) \leq 6 \times 27=162, \\
&\sum_{s=0}^{b+1} G_5\left(s+\nu_5-1, s\right) \geq 6 \times 7=42, \\
&\sum_{s=0}^{b+1} G_5\left(s+\nu_5-1, s\right) \leq 6 \times 38=228.
\end{aligned} \tag{33}$$

Now, let $\epsilon = 0.0001$. Then, $0 < \epsilon < \min\{f_i^{**} : i = 1, 2, 3, 4, 5\}$ and we have

$$\begin{aligned}
&\sum_{s=0}^{b+1} \lambda G_1\left(s+\nu_1-1, s\right)\left(f_1^{**}-\epsilon\right) \\
&\quad \geq 0.1066 \times 36 \times (1-0.0001)=3.8372 \geq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} G_1\left(s+\nu_1-1, s\right)\left(f_1^{*}+\epsilon\right) \\
&\quad \leq 78 \times\left(10^{-3}+0.0001\right)=0.0858 \leq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} \lambda G_2\left(s+\nu_2-1, s\right)\left(f_2^{**}-\epsilon\right) \\
&\quad \geq 0.1066 \times 36 \times (3-0.0001)=11.5124 \geq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} G_2\left(s+\nu_2-1, s\right)\left(f_2^{*}+\epsilon\right) \\
&\quad \leq 114 \times\left(3e^{-10}+0.0001\right)=0.02692 \leq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} \lambda G_3\left(s+\nu_3-1, s\right)\left(f_3^{**}-\epsilon\right) \\
&\quad \geq 0.1066 \times 36 \times (2.001-0.0001)=7.6786 \geq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} G_3\left(s+\nu_3-1, s\right)\left(f_3^{*}+\epsilon\right) \\
&\quad \leq 132 \times\left(10^{-3}+0.0001\right)=0.1452 \leq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} \lambda G_4\left(s+\nu_4-1, s\right)\left(f_4^{**}-\epsilon\right) \\
&\quad \geq 0.1066 \times 36 \times (1.501-0.0001)=5.7598 \geq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} G_4\left(s+\nu_4-1, s\right)\left(f_4^{*}+\epsilon\right) \\
&\quad \leq 162 \times\left(10^{-3}+0.0001\right)=0.1782 \leq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} \lambda G_5\left(s+\nu_5-1, s\right)\left(f_5^{**}-\epsilon\right) \\
&\quad \geq 0.1066 \times 37 \times (1-0.0001)=3.9438 \geq \frac{1}{5}, \\
&\sum_{s=0}^{b+1} G_5\left(s+\nu_5-1, s\right)\left(f_5^{*}+\epsilon\right) \\
&\quad \leq 228 \times\left(e^{-8}+0.0001\right)=0.0992 \leq \frac{1}{5}.
\end{aligned} \tag{34}$$

Thus by using Theorem 6, the 5-dimensional system of fractional finite difference equations (29) has at least one solution.

5. Conclusions

In this paper, based on main idea of Goodrich we review the existence of solutions for a k-dimensional system of fractional finite difference equations. In fact we are going to extend the work of Goodrich in a sense. We give an example to illustrate our last result.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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