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Ghulam Mustafa
The Islamia University of Bahawalpur

Aamir Shahzad
Riphah International University

Faheem Khan
University of Sargodha

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A New Computational Approach to Estimate the Subdivision Depth of *n*-Ary Subdivision Scheme

GHULAM MUSTAFA[1], AAMIR SHAHZAD[2], FAHEEM KHAN[2], DUMITRU BALEANU[3,4,5], AND YUMING CHU[6,7]

[1]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur 63100, Pakistan
[2]Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan
[3]Department of Mathematical Sciences, Cankaya University, 06530 Ankara, Turkey
[4]Institute of Space Sciences, 077125 Magurele-Bucharest, Romania
[5]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[6]Department of Mathematics, Huzhou University, Huzhou 313000, China
[7]Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science and Technology, Changsha 410114, China

Corresponding author: Yuming Chu (chuyuming2005@126.com)

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ABSTRACT The n-ary subdivision scheme has traditionally been designed to generate smooth curve and surface from control polygon. In this paper, we propose a new subdivision depth computation technique for n-ary subdivision scheme. The existing techniques do not ensure the computation of subdivision depth unless some strong condition is assumed on the mask of the scheme. But our technique relaxes the effect of strong condition assumed on the mask of the scheme by increasing the number of convolution steps. Consequently, a more precise subdivision depth technique for a given error tolerance is presented in this paper.

INDEX TERMS Curves and surfaces, n-ary subdivision schemes, convolution, error, distance, subdivision depth.

I. INTRODUCTION

The n-ary subdivision scheme (nASS) is defined as the set of n-rules with respect to a sequence of control polygons. The nASS takes a polygon as an input and produces a refined polygon by applying n-rules on each edge of a coarse polygon. The repeated applications of these n-rules on the polygons produce a sequence of refined polygons. The sequence of refined polygons converges to a smooth shape. Let $X^0, X^1, \ldots, X^k, \ldots$ be a sequence of polygons and X^∞ be a limiting shape then the distance (also called error) between polygon and limiting shape approaches to zero as k approaches to ∞. In literature, there are different types of nASS but few of them are listed in [1]–[9]. Most of the researchers have discussed the some of well known properties of nASS such as smoothness/continuity, Hölder regularity, approximation order and support of the scheme. But few work has been done on error and subdivision depths of nASS. Let the designer has error tolerance ϵ and the polygon is divided k-times. If the distance between refined polygon at kth level and the limiting shape is less than ϵ then k is called the subdivision depth of the limiting shape with respect to ϵ. In other words, subdivision depth tells us the number of subdivision steps needed to meet the designer error tolerance.

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TABLE 1. The convolution constants of 6-point binary scheme for curve.

T_1	T_2	T_3	T_4
0.742000	0.442177	0.285892	0.183215

A first attempt to find the distance between polygon and the limiting shape is done in [10] for binary subdivision schemes. Its generalization for ternary and quaternary schemes was done in [11], [12]. Its further generalization for n-ary scheme was done in [13]. In this reference, the technique to compute the subdivision depth has also been introduced. The distance of a subdivision surface to its control polyhedron has been computed in [14]. Generally, the above techniques do not ensure the computations of subdivision depth unless some strong condition is assumed on the mask of the schemes. The condition for curve case is $\delta_1 < 1$ while for surface case is $\delta_2 < 1$, where δ_1 and δ_2 are defined in ([13], Equations (5) and (6)).

TABLE 2. Subdivision depths of 6-point binary scheme for curve.

T_{c_0} / ϵ	0.0266	0.0031	0.0003	$4.35e{-}5$	$5.12e{-}6$	$6.02{-}7$	$7.09e{-}8$
T_1	11	18	26	33	40	47	54
T_2	3	6	8	11	14	16	19
T_3	2	4	5	7	9	10	12
T_4	1	3	4	5	6	8	9

TABLE 3. The convolution constants of 6-point binary scheme for surface.

Y_1Z_1	Y_2Z_2	Y_3Z_3	Y_4Z_4
0.897800	0.452181	0.232054	0.117641

The generalizations of the work of [10], [11] is done by [15]–[17] by using the convolution technique.

The error bounds of Doo-Sabin surfaces have been computed by Huawei *et al.* [18] in 2002. The different versions of the error bounds and subdivision depths of Catmull-Clark surfaces have been presented by [19]–[21]. The error bounds and subdivision depths of Loop subdivision surfaces have been computed by [22], [23]. But all these techniques have not been extended for the computation of error bounds and subdivision depths for n-ary, $n > 2$ schemes yet.

In this paper, we attempt to find the generalized version of the work done in [13] by using the convolution technique. Our technique can relax the effect of strong condition assumed on the mask of the schemes by increasing the number of convolution steps. Using the proposed technique, very less number of iterations (subdivision depths) are required to reach the user given error tolerance. So, this method reduces the burden of computational cost.

Rest of the paper is arranged as: In Section 2, we find the subdivision depth of the n-ary schemes for the generation of curves. Section 3 is devoted for the generalization of the work presented in Section 2 for surface case. The applications of our computational technique are given in Section 4. Section 5 is for conclusion.

II. SUBDIVISION DEPTH OF *n*-ARY SCHEME FOR CURVE

As is usually the case in subdivision depth papers, we will first describe our subdivision depth technique in a curve setting and then generalize it for a surface.

A. PRELIMINARY RESULTS FOR CURVE

Let $X^k = \{x_i^k; i \in \mathbb{Z}\}$ be a control polygon with points in $\mathbb{R}^N$, where $N \geq 2$ and k be an integer (non-negative), which indicates the number of iterations (subdivision level). A generalized n-ary subdivision scheme for curve is described as

$$x_{ni+\alpha}^{k+1} = \sum_{j=0}^{N-1} a_{\alpha,j} x_{i+j}^k, \quad \alpha = 0, 1, \ldots, n-1, \tag{1}$$

with

$$\sum_{j=0}^{N-1} a_{\alpha,j} = 1, \quad \alpha = 0, 1, \ldots, n-1. \tag{2}$$

By [13], we have

$$\begin{cases} b_{\beta,j} = \sum\limits_{h=0}^{j} \left(a_{\beta,h} - a_{\beta+1,h}\right), & \beta = 0, 1, \ldots, n-2, \\ b_{n-1,j} = a_{0,j} - \sum\limits_{\beta=0}^{n-2} b_{\beta,j}, & \end{cases}$$

such that

$$\sum_{j=0}^{N-1} |b_{\beta,j}| < 1, \quad \beta = 0, 1, \ldots, n-2, \text{ and } \sum_{j=0}^{N-1} |b_{n-1,j}| < 1.$$

We introduce the coefficients for $u = 0, 1, \ldots, N-1$, such that

$$t_{nu+\gamma} = b_{\gamma,u}, \quad \text{where } \gamma = 0, 1, 2, \ldots, n-1. \tag{3}$$

In the field of science, mathematics and engineering the convolution product has been used. It is a process that can be used for various branches of signal processing, edge detection and data smoothing. Here in the following section, we define some important results regarding the convolution of n-ary subdivision scheme for curve.

B. ONE DIMENSIONAL CONVOLUTION REFORMULATION

Let $(x_l)_{l\geq 0}$ be a limited length vector and $(t_l)_{l\in N} = (t_l)_{l=0}^{nN-1}$, with $t_l = 0$ if $l \geq nN$. The one time convolution product of $x = (x_l)$ and $t = (t_l)$ of n-ary subdivision scheme for curve is given by

$$(x^{(0)} \star t)_f = \sum_{l=0}^{\lfloor f/n \rfloor} x_l t_{f-nl}, \tag{4}$$

TABLE 4. Subdivision depths of 6-point binary scheme for surface.

$Y_{c_0}Z_{c_0}$ / ϵ	0.0313	0.0018	0.0001	$6.44e{-}6$	$3.80e{-}7$	$2.24e{-}8$
Y_1Z_1	47	73	99	126	152	178
Y_2Z_2	4	8	11	15	18	22
Y_3Z_3	2	4	6	8	10	12
Y_4Z_4	1	3	4	5	7	8

TABLE 5. The convolution constants of 4-point ternary scheme for curve.

T_1	T_2	T_3	T_4
0.441358	0.176393	0.067478	0.027136

where $\lfloor . \rfloor$ denotes the integer part. Similarly, we get the reformulation for c_0th convolutions

$$((\ldots(((x^{(0)} \star t)^{(0)}) \star t)^{(0)} \star \ldots \star t)^{(0)} \star t)_f = \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} x_l E_{l,f}^{c_0}, \tag{5}$$

with

$$\begin{cases} E_{l,f}^1 = t_{f-nl}, \\ E_{l,f}^{c_0} = \sum\limits_{e=nl}^{\lfloor f/n^{c_0-1} \rfloor} E_{l,e}^1 E_{e,f}^{c_0-1}, \qquad c_0 \geq 2. \end{cases} \tag{6}$$

By (5), we get

$$\|((\ldots(((x^{(0)} \star t)^{(0)}) \star t)^{(0)} \star \ldots \star t)^{(0)} \star t)\|_\infty \leq \|x\|_\infty \sup_f \left\{ \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} |E_{l,f}^{c_0}| \right\}. \tag{7}$$

Lemma 1: The term $E_{l,f}^{c_0}$ given in the right hand side of inequality (7) has the following relation for n-ary subdivision scheme

$$E_{l,f}^{c_0} = E_{l+1,f+n^{c_0}}^{c_0}. \tag{8}$$

Proof: Here, we begin the induction process over c_0.

- Case $c_0 = 1$

$$E_{l,f}^1 = t_{f-nl} = t_{f+n-n(l+1)} = E_{l+1,f+n}^1, \tag{9}$$

similarly

$$E_{l+1,f}^1 = t_{f-n(l+1)} = E_{l,f-n}^1. \tag{10}$$

We suppose that for an integer M, it is true for $c_0 = M$, so

$$E_{l,f}^M = E_{l+1,f+n^M}^M. \tag{11}$$

Now, we will prove for

- Case $c_0 = M + 1$

Consider

$$E_{l,f}^{M+1} = \sum_{d=nl}^{\lfloor f/n^M \rfloor} E_{l,d}^1 E_{d,f}^M.$$

By using (11), we acquire

$$E_{l,f}^{M+1} = \sum_{d=nl}^{\lfloor f/n^M \rfloor} E_{l,d}^1 E_{d+1,f+n^M}^M.$$

Now, replace d by $d - n$, then

$$E_{l,f}^{M+1} = \sum_{d=n(l+1)}^{\lfloor f/n^M + n \rfloor} E_{l,d-n}^1 E_{d-(n-1),f+n^M}^M.$$

Using (10) and (11), we have

$$E_{l,f}^{M+1} = E_{l+1,f+n^{M+1}}^{M+1}.$$

This completes the proof. □

Similarly, in the following lemma, we can deduce an another relation for the same term $E_{l,f}^{c_0}$.

Lemma 2: The term $E_{l,f}^{c_0}$ has the following relation for n-ary subdivision scheme

$$E_{l,f}^{c_0} = E_{l-1,f-n^{c_0}}^{c_0}. \tag{12}$$

Proof: Here, we start the induction process over c_0.

- Case $c_0 = 1$

$$E_{l,f}^1 = t_{f-nl} = t_{f-n-n(l-1)} = E_{l-1,f-n}^1, \tag{13}$$

similarly

$$E_{l-1,f}^1 = t_{f-n(l-1)} = E_{l,f+n}^1. \tag{14}$$

We suppose that it is true for $c_0 = M$, that is

$$E_{l,f}^M = E_{l-1,f-n^M}^M. \tag{15}$$

Now, we will prove for

- Case $c_0 = M + 1$

Consider

$$E_{l,f}^{M+1} = \sum_{d=nl}^{\lfloor f/n^M \rfloor} E_{l,d}^1 E_{d,f}^M.$$

By using (15), we acquire

$$E_{l,f}^{M+1} = \sum_{d=nl}^{\lfloor f/n^M \rfloor} E_{l,d}^1 E_{d-1,f-n^M}^M.$$

Now, replace d by $d + n$, then

$$E_{l,f}^{M+1} = \sum_{d=n(l-1)}^{\lfloor f/n^M - n \rfloor} E_{l,d+n}^1 E_{d+(n-1),f-n^M}^M.$$

Using (14) and (15), we have

$$E_{l,f}^{M+1} = E_{l-1,f-n^{M+1}}^{M+1}.$$

This completes the proof. □

TABLE 6. Subdivision depths of 4-point ternary scheme for curve.

T_{c_0} / ϵ	$1.26e{-}3$	$1.35e{-}5$	$1.45e{-}7$	$1.56e{-}9$	$1.67e{-}11$	$1.79e{-}13$	$1.92e{-}15$
T_1	6	12	17	23	28	34	40
T_2	3	5	8	11	13	16	18
T_3	2	3	5	7	8	10	12
T_4	1	3	4	5	6	8	9

TABLE 7. The convolution constants of 4-point ternary scheme for surface.

Y_1Z_1	Y_2Z_2	Y_3Z_3	Y_4Z_4
0.505401	0.233098	0.096089	0.040774

Corollary 3: *The term* $\sup_f\left\{\sum_{l=0}^{\lfloor f/n^{c_0}\rfloor}|E_{l,f}^{c_0}|\right\}$ *presented in the right hand side of the inequality (7) has the following alternate form*

$$T_{c_0}=\sup_f\left\{\sum_{l=0}^{\lfloor f/n^{c_0}\rfloor}|E_{l,f}^{c_0}|\right\}=\sup_{f\in\Sigma(c_0,N)}\left\{\sum_{l=0}^{\lfloor f/n^{c_0}\rfloor}|E_{l,f}^{c_0}|\right\}. \quad (16)$$

Proof: Assume that $t = \{t_0, t_1, \ldots, t_{nN-1}\}$, with $N \in \mathbb{N}$ and $\Omega(c_0, N) = (n^{c_0} - (n-1))(nN-1)$. Then for $f > \Omega(c_0, N)$ and by using (6), we acquire

$$E_{0,f}^{c_0} = 0. \quad (17)$$

If $f > \Omega(c_0, N) + wn^{c_0}$ then by Lemma 1, we get

$$E_{w,f}^{c_0} = 0. \quad (18)$$

Now by using (17) and (18), we get (16). □

C. SUBDIVISION DEPTH OF THE SCHEME FOR CURVE

Now firstly, we present some results for computing the distance between two consecutive polygons. Secondly, we compute the distance between kth level polygon and limiting curve. Then we describe an important theorem regarding the subdivision depth.

Theorem 4: *Let X^k and X^{k+1} be two consecutive polygons obtained from the subdivision scheme* (1) *then the distance between these polygons is*

$$\|X^{k+1} - X^k\|_\infty \le \sigma\tau(T_{c_0})^k, \quad (19)$$

where $T_{c_0}, c_0 \ge 1$ defined in (16), $\tau = \max_i\left\|\triangle x_i^0\right\|$, and

$$\sigma = \max\left\{\left|\sum_{j=0}^{N-2}\tilde{a}_{\alpha,j}\right|, \alpha = 0, 1, \ldots, n-1\right\},$$

where

$$\begin{cases}\tilde{a}_{\alpha,0} = \sum_{h=1}^{N-1} a_{\alpha,h} - \frac{\alpha}{n},\\ \tilde{a}_{\alpha,j} = \sum_{h=j+1}^{N-1} a_{\alpha,h}, \quad j \ge 1, \quad \alpha = 0, 1, \ldots, n-1.\end{cases}$$

Proof: Similar to the proof given in [13]. □

Theorem 5: *Let X^k and X^∞ be kth level polygon and limiting curve respectively obtained from the subdivision scheme* (1) *then the distance between them is*

$$\left\|X^\infty - X^k\right\|_\infty \le \sigma\tau\left(\frac{(T_{c_0})^k}{1-T_{c_0}}\right), \quad (20)$$

where $c_0 \ge 1$, such that $T_{c_0} < 1$.

Proof: Similar to the proof given in [13]. □

Theorem 6: *Let k be the subdivision depth and θ^k be the distance between X^k and X^∞. For arbitrary $\epsilon > 0$, if*

$$k \ge \log_{T_{c_0}}\left(\frac{\epsilon(1-T_{c_0})}{\sigma\tau}\right), \quad (21)$$

then $\theta^k \le \epsilon$.

Proof: Since by (20)

$$\theta^k = \left\|X^\infty - X^k\right\|_\infty \le \sigma\tau\left(\frac{(T_{c_0})^k}{1-T_{c_0}}\right),$$

therefore to attain given error tolerance $\epsilon > 0$, consider

$$\sigma\tau\left(\frac{(T_{c_0})^k}{1-T_{c_0}}\right) \le \epsilon,$$

which implies

$$\frac{\sigma\tau}{\epsilon(1-T_{c_0})} \le (T_{c_0}^{-1})^k.$$

Now taking logarithm, we have

$$k \ge \frac{\log\left(\frac{\sigma\tau}{\epsilon(1-T_{c_0})}\right)}{\log T_{c_0}^{-1}} = \frac{\log\left(\frac{\sigma\tau}{\epsilon(1-T_{c_0})}\right)}{-\log T_{c_0}}$$
$$= -\log_{T_{c_0}}\left(\frac{\sigma\tau}{\epsilon(1-T_{c_0})}\right) = \log_{T_{c_0}}\left(\frac{\sigma\tau}{\epsilon(1-T_{c_0})}\right)^{-1},$$

which implies

$$k \ge \log_{T_{c_0}}\left(\frac{\epsilon(1-T_{c_0})}{\sigma\tau}\right),$$

then $\theta^k \le \epsilon$. This completes the proof. □

III. SUBDIVISION DEPTH OF *n*-ARY SCHEME FOR SURFACE

The surface case is the generalization of the curve case: We perform two dimensional convolution followed by the computation of distance between polygons to compute the subdivision depth.

A. PRELIMINARY RESULTS FOR SURFACE

Let $X^k = \{x_{i,j}^k; i, j \in \mathbb{Z}\}$ be a polygon at kth level with points in R^N, where $N \ge 2$. A tensor product of n-ary subdivision scheme (1) is described as

$$x_{ni+\alpha,nj+\beta}^{k+1} = \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} a_{\alpha,r}a_{\beta,s}x_{i+r,j+s}^k,$$
$$\alpha, \beta = 0, 1, \ldots, n-1, \quad (22)$$

where $a_{\alpha,r}$ and $a_{\beta,s}$ satisfies (2).

TABLE 8. Subdivision depths of 4-point ternary scheme for surface.

$Y_{c_0}Z_{c_0}$ / ϵ	$6.59e-3$	$1.11e-4$	$1.18e-6$	$3.21e-8$	$5.45e-10$	$9.24e-12$	$1.56e-13$
Y_1Z_1	7	13	19	25	31	37	43
Y_2Z_2	3	6	9	11	14	17	20
Y_3Z_3	2	4	5	7	9	10	12
Y_4Z_4	1	3	4	5	6	8	9

TABLE 9. The convolution constants of 4-point quaternary scheme for curve.

T_1	T_2	T_3	T_4
0.250000	0.062500	0.015625	0.003906

We introduce the coefficients for $u, v = 0, 1, \ldots, N-1$ such that

$$\begin{cases} y_{nu+\gamma} = a_{\gamma,N-u-1}, \\ z_{nv+\gamma} = b_{\gamma,N-v-1} \qquad \gamma = 0, 1, \ldots, n-1. \end{cases} \tag{23}$$

B. TWO DIMENSIONAL CONVOLUTION REFORMULATION

Let a vector $x = x_{l,m}$ has limited length and $(y_l) = (y_l)_{l=0}^{nN-1}$, $(z_m) = (z_m)_{m=0}^{nN-1}$ with $y_l = z_m = 0$ if $l, m \geq nN$. The convolution product of $x = (x_{l,m})$, $y = (y_l)$ and $z = (z_m)$ for n-ary tensor product subdivision scheme for surface is given by

$$x_{f,g}^{c_0} = \left(x^{c_0-1;0} \star yz\right)_{f,g} = \sum_{l=0}^{\lfloor f/n \rfloor} \sum_{m=0}^{\lfloor g/n \rfloor} x_{l,m}^{c_0-1} y_{f-nl} z_{g-nm}. \tag{24}$$

Similarly, we acquire the reformulation for c_0-th convolutions

$$\begin{aligned} x_{f,g}^{c_0} &= (\ldots(((x^{c_0-1;0} \star yz) \star yz) \star \ldots \star yz) \star yz)_{f,g} \\ &= \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} \sum_{m=0}^{\lfloor g/n^{c_0} \rfloor} x_{l,m}^{0} E_{l,f}^{c_0,y} E_{m,g}^{c_0,z}, \end{aligned} \tag{25}$$

with

$$\begin{cases} E_{l,f}^{c_0,y} = \displaystyle\sum_{p=nl}^{\lfloor f/n^{c_0-1} \rfloor} E_{l,p}^{c_0-1,y} E_{p,f}^{c_0-1,y}, \\ E_{m,g}^{c_0,z} = \displaystyle\sum_{q=nm}^{\lfloor g/n^{c_0-1} \rfloor} E_{m,q}^{c_0-1,z} E_{q,g}^{c_0-1,z}. \end{cases} \tag{26}$$

From (25), we have

$$\max_{f,g} |x_{f,g}^{c_0}| \leq \max_{f,g} \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} \sum_{m=0}^{\lfloor g/n^{c_0} \rfloor} |E_{l,f}^{c_0,y}||E_{m,g}^{c_0,z}| \max_{l,m} |x_{l,m}^{0}| \tag{27}$$

and

$$\begin{aligned} &\max_{f,g} \left\{ \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} \sum_{m=0}^{\lfloor g/n^{c_0} \rfloor} |E_{l,f}^{c_0,y}||E_{m,g}^{c_0,z}| \right\} \\ &\quad = \max_{f,g \in \Sigma(c_0,N)} \left\{ \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} \sum_{m=0}^{\lfloor g/n^{c_0} \rfloor} |E_{l,f}^{c_0,y}||E_{m,g}^{c_0,z}| \right\}. \end{aligned} \tag{28}$$

Also

$$Y_{c_0} = \max_{f \in \Sigma(c_0,N)} \left\{ \sum_{l=0}^{\lfloor f/n^{c_0} \rfloor} |E_{l,f}^{c_0,y}| \right\} \tag{29}$$

and

$$Z_{c_0} = \max_{g \in \Sigma(c_0,N)} \left\{ \sum_{m=0}^{\lfloor g/n^{c_0} \rfloor} |E_{m,g}^{c_0,z}| \right\}. \tag{30}$$

C. SUBDIVISION DEPTH OF THE SCHEME FOR SURFACE

In this section, we first estimate the distance between two successive polygon X^k and X^{k+1} obtained from (22) then we estimate the distance between polygon X^k and the limiting surface X^∞. After that, we present the subdivision Depth of the scheme for surface.

Theorem 7: Let X^k and X^{k+1} be two consecutive polygons obtained from the subdivision scheme (22) then the distance between these polygons is

$$\|X^{k+1} - X^k\|_\infty \leq (Y_{c_0} Z_{c_0})^k \sum_{h=1}^{3} (\xi_h)(\eta_{\alpha,\beta}^h), \tag{31}$$

where Y_{c_0} and Z_{c_0} for $c_0 \geq 1$ are defined in (29) − (30) and $\eta_{\alpha,\beta}^h$, ξ_h, $\alpha, \beta = 0, 1, \ldots, n-1$ are defined as

$$\begin{aligned} \eta_{\alpha,\beta}^1 &= \left| a_{\beta,0} \sum_{h=1}^{N-1} a_{\alpha,h} - \frac{\alpha(n-\beta)}{n^2} \right| + \left| a_{\beta,0} \sum_{s=1}^{N-2} \tilde{a}_{\alpha,s} \right|, \\ \eta_{\alpha,\beta}^2 &= \left| \sum_{h=1}^{N-1} a_{\beta,h} - \frac{\beta}{n} \right| + \left| \sum_{r=0}^{N-1} a_{\alpha,r} \sum_{s=1}^{N-2} \tilde{a}_{\beta,s} \right|, \\ \eta_{\alpha,\beta}^3 &= \left| \sum_{h=1}^{N-1} a_{\alpha,h} \sum_{h=1}^{N-1} a_{\beta,h} - \frac{\alpha\beta}{n^2} \right| + \left| \sum_{h=1}^{N-1} a_{\beta,h} \sum_{s=1}^{N-2} \tilde{a}_{\alpha,s} \right|, \\ \Delta_{i,j,1}^k &= x_{i+1,j}^k - x_{i,j}^k, \quad \Delta_{i,j,2}^k = x_{i,j+1}^k - x_{i,j}^k, \\ \Delta_{i,j,3}^k &= x_{i+1,j+1}^k - x_{i,j+1}^k, \\ \xi_h &= \max_{i,j} \|\Delta_{i,j,h}^0\|, \quad h = 1, 2, 3. \end{aligned}$$

Proof: Similar to the proof given in [13]. □

Theorem 8: Let X^k and X^∞ be kth level polygon and limiting surface respectively obtained from the subdivision scheme (22) then the distance between them is

$$\|X^\infty - X^k\|_\infty \leq \nu \left(\frac{(Y_{c_0} Z_{c_0})^k}{1 - Y_{c_0} Z_{c_0}} \right), \tag{32}$$

where $c_0 \geq 1$, such that $Y_{c_0} Z_{c_0} < 1$ and ν is defined as

$$\nu = \max_{\alpha,\beta} \left\{ \sum_{h=1}^{3} (\xi_h)(\eta_{\alpha,\beta}^h), \alpha, \beta = 0, 1, \ldots, n-1 \right\}.$$

Proof: Similar to the proof given in [13]. □

TABLE 10. Subdivision depths of 4-point quaternary scheme for curve.

T_{c_0} / ϵ	$4.41e{-4}$	$1.72e-6$	$6.73e{-9}$	$2.63e{-11}$	$1.03e{-13}$	$4.01e{-16}$	$1.57e{-18}$
T_1	4	8	12	16	20	24	28
T_2	2	4	6	8	10	12	14
T_3	1	3	4	5	7	8	9
T_4	1	2	3	4	5	6	7

TABLE 11. The convolution constants of 4-point quaternary scheme for surface.

Y_1Z_1	Y_2Z_2	Y_3Z_3	Y_4Z_4
0.264140	0.065694	0.016672	0.004209

Theorem 9: Let k be the subdivision depth and ϑ^k be the distance between X^k and X^∞. For arbitrary $\epsilon > 0$, if

$$k \geq \log_{(Y_{c_0}Z_{c_0})}\left(\frac{\epsilon(1-Y_{c_0}Z_{c_0})}{\nu}\right), \tag{33}$$

then $\vartheta^k \leq \epsilon$.

Proof: Since by (32)

$$\vartheta^k = \|X^\infty - X^k\|_\infty \leq \nu\left(\frac{(Y_{c_0}Z_{c_0})^k}{1-Y_{c_0}Z_{c_0}}\right).$$

To obtain given tolerance $\epsilon > 0$, consider

$$\nu\left(\frac{(Y_{c_0}Z_{c_0})^k}{1-Y_{c_0}Z_{c_0}}\right) \leq \epsilon,$$

which implies

$$\left(\frac{\nu}{\epsilon(1-Y_{c_0}Z_{c_0})}\right) \leq ((Y_{c_0}Z_{c_0})^{-1})^k.$$

Now taking logarithm, we have

$$k \geq \frac{\log\left(\frac{\nu}{\epsilon(1-Y_{c_0}Z_{c_0})}\right)}{\log(Y_{c_0}Z_{c_0})^{-1}} = \frac{\log\left(\frac{\nu}{\epsilon(1-Y_{c_0}Z_{c_0})}\right)}{-\log(Y_{c_0}Z_{c_0})}$$
$$= -\log_{(Y_{c_0}Z_{c_0})}\left(\frac{\nu}{\epsilon(1-Y_{c_0}Z_{c_0})}\right),$$

which implies

$$k \geq \log_{(Y_{c_0}Z_{c_0})}\left(\frac{\nu}{\epsilon(1-Y_{c_0}Y_{c_0})}\right)^{-1},$$

which further implies

$$k \geq \log_{(Y_{c_0}Z_{c_0})}\left(\frac{\epsilon(1-Y_{c_0}Z_{c_0})}{\nu}\right),$$

then $\vartheta^k \leq \epsilon$. This completes the proof. □

IV. NUMERICAL APPLICATIONS

In this section, the subdivision depths of some well known n-ary approximating as well as interpolating subdivision schemes are presented. The subdivision depths are presented both in tabular and graphical forms.

Example 10: If we take $n = 1$ and $\nu = 0$ in ([5], Theorem 1), we get the following coefficients/mask involved in the affine combinations of the 6-point binary interpolating scheme.

$$(a_{0,0}, a_{0,1}, a_{0,2}, a_{0,3}, a_{0,4}, a_{0,5}) = (0, 0, 1, 0, 0, 0),$$
$$(a_{1,0}, a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}) = \left(w, -\frac{1}{16} - 3w, \frac{9}{16} + 2w, \frac{9}{16} + 2w, -\frac{1}{16} - 3w, w\right). \tag{34}$$

- *Curve case: The convolution constants T_{c_0} for $c_0 \geq 1$ of the scheme (34) are presented in Table 1. In Table 2, subdivision depths are shown and their graphical view is shown in Figure 1(a).*
- *Surface case: The convolution constants $Y_{c_0}Z_{c_0}$ for $c_0 \geq 1$ of the tensor product of the scheme (34) are presented in Table 3. In Table 4, subdivision depths are shown and their graphical view is shown in Figure 1(b).*

It has been observed that the subdivision depth decreases with the increase of convolution steps. That is we need less number of iterations to get the required result by increasing the number of convolution steps.

Example 11: If we take $w = -\frac{35}{24}$ in ([6], Equation 9), we get 4-point ternary approximating scheme with following coefficients.

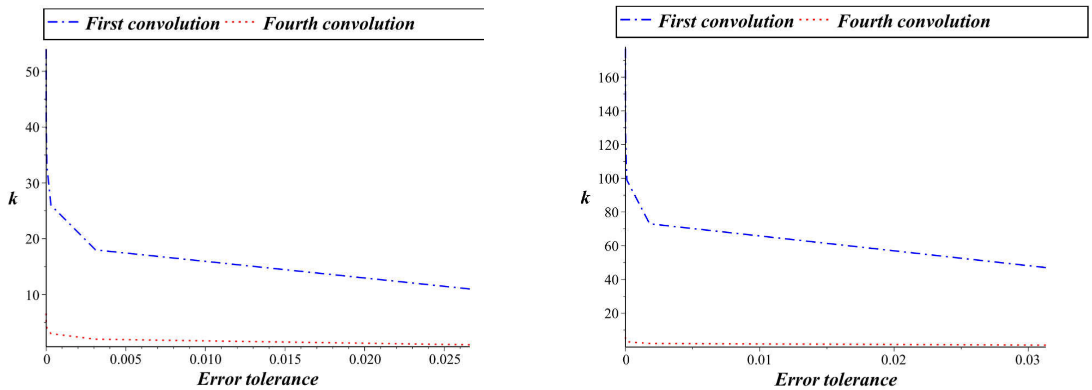
$$(a_{0,0}, a_{0,1}, a_{0,2}, a_{0,3}) = \left(-\frac{55}{1296}, \frac{385}{432}, \frac{77}{432}, -\frac{35}{1296}\right),$$
$$(a_{1,0}, a_{1,1}, a_{1,2}, a_{1,3}) = \left(-\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, -\frac{1}{16}\right),$$
$$(a_{2,0}, a_{2,1}, a_{2,2}, a_{2,3}) = \left(-\frac{35}{1296}, \frac{77}{432}, \frac{385}{432}, -\frac{55}{1296}\right). \tag{35}$$

- *Curve case: The convolution constants T_{c_0} of the scheme (35) are gathered in Table 5. In Table 6, subdivision depths are shown and their graphical view is shown in Figure 2(a).*
- *Surface case: The convolution constants $Y_{c_0}Z_{c_0}$ of the tensor product of the scheme (35) are presented in Table 7. In Table 8, subdivision depths are shown and their graphical view is shown in Figure 2(b).*

Example 12: If we set $n = 4$, $b = 0$ and the free parameter as $a_4 = \frac{7}{32} - \frac{7}{64}\mu$, $a_5 = \frac{15}{128} - \frac{5}{64}\mu$, $a_6 = \frac{7}{128} - \frac{3}{64}\mu$, $a_7 = \frac{1}{64} - \frac{1}{64}\mu$ in ([3], Equations (4.9)-(4.10)), we get the following coefficients of the 4-point quaternary approximating

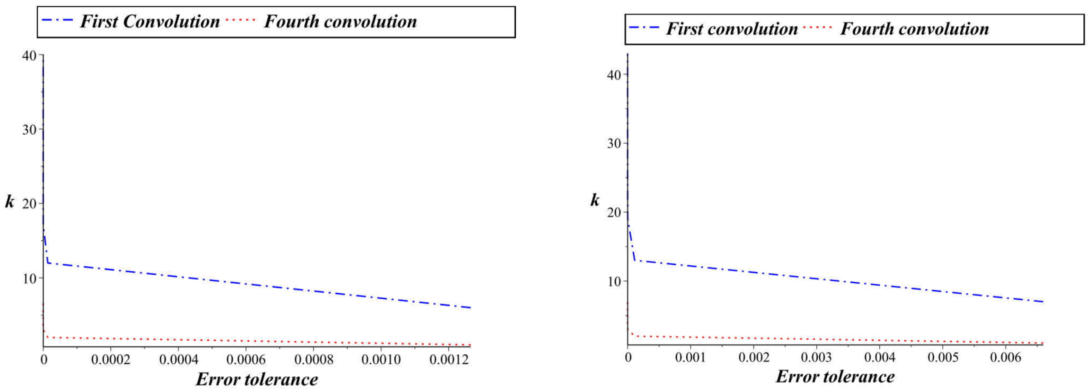
TABLE 12. Subdivision depths of 4-point quaternary scheme for surface.

$Y_{c_0}Z_{c_0}$ / ϵ	$1.51e{-3}$	$6.36e{-6}$	$2.68e{-8}$	$1.13e{-10}$	$4.74e{-13}$	$1.99e{-15}$	$8.4e{-18}$
Y_1Z_1	4	8	13	17	21	25	29
Y_2Z_2	2	4	6	8	10	12	14
Y_3Z_3	1	3	4	5	7	8	9
Y_4Z_4	1	2	3	4	5	6	7



(a) Curve case (b) Surface case

FIGURE 1. The subdivision depths of 6-point binary scheme for curve and surface at first and fourth convolution steps with respect to the user-specified error tolerance.



(a) Curve case (b) Surface case

FIGURE 2. The subdivision depths of 4-point ternary scheme for curve and surface at first and fourth convolution steps with respect to the user-specified error tolerance.

TABLE 13. The convolution constants of 3-point quinary scheme for curve.

T_1	T_2	T_3	T_4
0.680000	0.308800	0.128576	0.051796

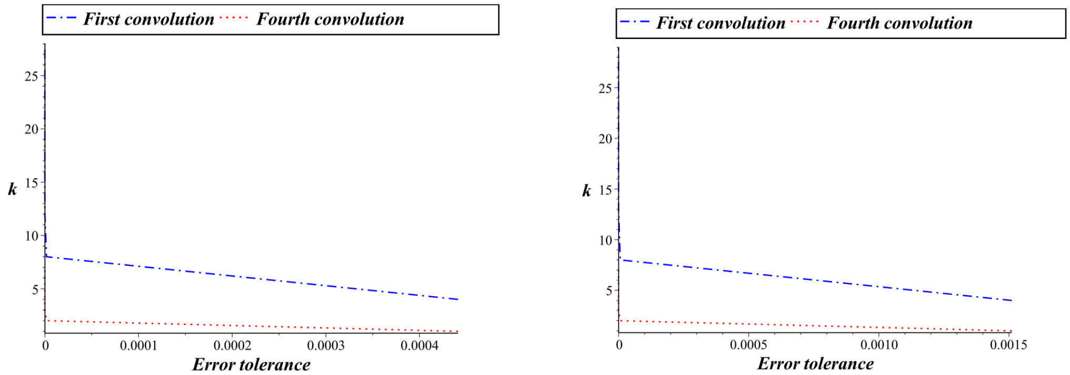
scheme

$$(a_{0,0}, a_{0,1}, a_{0,2}, a_{0,3}) = \left(\frac{7}{32} - \frac{7}{64}\mu, \frac{29}{64} + \frac{13}{64}\mu, \frac{5}{16} - \frac{5}{64}\mu, \frac{1}{64} - \frac{1}{64}\mu\right),$$
$$(a_{1,0}, a_{1,1}, a_{1,2}, a_{1,3}) = \left(\frac{15}{128} - \frac{5}{64}\mu, \frac{57}{128} + \frac{7}{64}\mu, \frac{49}{128} + \frac{1}{64}\mu, \frac{7}{128} - \frac{3}{64}\mu\right),$$
$$(a_{2,0}, a_{2,1}, a_{2,2}, a_{2,3}) = \left(\frac{7}{128} - \frac{3}{64}\mu, \frac{49}{128} + \frac{1}{64}\mu, \frac{57}{128} + \frac{7}{64}\mu, \frac{15}{128} - \frac{5}{64}\mu\right),$$
$$(a_{3,0}, a_{3,1}, a_{3,2}, a_{3,3}) = \left(\frac{1}{64} - \frac{1}{64}\mu, \frac{5}{16} - \frac{5}{64}\mu, \frac{29}{64} + \frac{13}{64}\mu, \frac{7}{32} - \frac{7}{64}\mu\right). \quad (36)$$

- *Curve case: The convolution constants of the scheme (36) are gathered in Table 9. In Table 10, subdivision depths are shown and their graphical view is shown in Figure 3(a).*
- *Surface case: The convolution constants of the tensor product of the scheme (36) are presented in Table 11.*

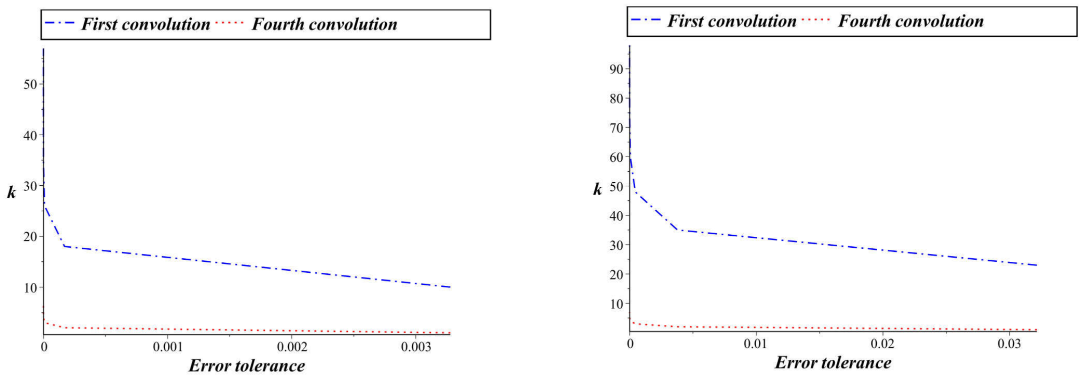
TABLE 14. Subdivision depths of 3-point quinary scheme for curve.

T_{c_0} / ϵ	$3.27e{-3}$	$h1.69e{-4}$	$8.79e{-6}$	$4.55e{-7}$	$2.35e{-8}$	$1.22e{-9}$	$6.32e{-11}$
T_1	10	18	26	34	41	49	57
T_2	3	5	8	10	13	15	18
T_3	1	3	4	6	7	9	10
T_4	1	2	3	4	5	6	7



(a) Curve case (b) Surface case

FIGURE 3. **The subdivision depths of 4-point quaternary scheme for curve and surface at first and fourth convolution steps with respect to the user-specified error tolerance.**



(a) Curve case (b) Surface case

FIGURE 4. **The subdivision depths of 3-point quinary scheme for curve and surface at first and fourth convolution steps with respect to the user-specified error tolerance.**

TABLE 15. **The convolution constants of 3-point quinary scheme for surface.**

Y_1Z_1	Y_2Z_2	Y_3Z_3	Y_4Z_4
0.843200	0.467894	0.238464	0.117563

In Table 12, subdivision depths are shown and their graphical view is shown in Figure 3(b).

Example 13: If we put $n = 5$ and $b = 0$ in ([4], Equations (3.10)-(3.14)), we get the coefficients of 3-point quinary interpolating scheme

$$(a_{0,0}, a_{0,1}, a_{0,2}) = \left(\frac{7}{25}, \frac{21}{25}, -\frac{3}{25}\right),$$
$$(a_{1,0}, a_{1,1}, a_{1,2}) = \left(\frac{3}{25}, \frac{24}{25}, -\frac{2}{25}\right),$$
$$(a_{2,0}, a_{2,1}, a_{2,2}) = (0, 1, 0),$$
$$(a_{3,0}, a_{3,1}, a_{3,2}) = \left(-\frac{2}{25}, \frac{24}{25}, \frac{3}{25}\right),$$
$$(a_{4,0}, a_{4,1}, a_{4,2}) = \left(-\frac{3}{25}, \frac{21}{25}, \frac{7}{25}\right). \quad (37)$$

- *Curve case: The convolution constants of the scheme (37) are presented in Table 13. In Table 14, subdivision depths are shown and their graphical view is shown in Figure 4(a).*
- *Surface case: The convolution constants of the tensor product of the scheme (37) are presented in Table 15. In Table 16, subdivision depths are shown and their graphical view is shown in Figure 4(b).*

V. CONCLUSION AND FUTURE WORK

The main purpose of this research was to provide an optimal technique to compute the subdivision depth. In other

TABLE 16. **Subdivision depths of 3-point quinary scheme for surface.**

$Y_{c_0}Z_{c_0}$ / ϵ	$3.21e{-2}$	h$3.77e{-3}$	$4.44e{-4}$	$5.22e{-5}$	$6.14e{-6}$	$7.21e{-7}$	$8.48{-8}$
Y_1Z_1	23	35	48	60	73	85	98
Y_2Z_2	3	6	9	12	15	18	20
Y_3Z_3	2	3	5	6	8	9	11
Y_4Z_4	1	2	3	4	5	6	7

word, the aim was to predict the number of subdivision steps required to get an error-tolerant shape. In this paper, we have presented the technique to compute the depth of n-ary subdivision scheme. The advantage of this technique over the existing technique is that the strong condition imposed on the mask/coefficients of the scheme can be knocked out by increasing the number of convolution steps. We have also presented the subdivision depths of binary, ternary, quaternary and quinary schemes in this paper. These examples sentence that the proposed technique is valid and applicable for the computation of depth. The authors are looking, as a future work, to extend the computational technique of the subdivision depth of n-ary subdivision scheme for the generation of the shapes in higher dimensions.

AVAILABILITY OF DATA AND MATERIAL

"Data sharing not applicable to this article as no datasets were generated or analysed during the current study."

COMPETING INTERESTS

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GHULAM MUSTAFA received the Ph.D. degree in mathematics from the University of Science and Technology of China (USTC), China, in 2004, sponsored by Cultural Exchange Scholarship of China, and the Ph.D. degree from Durham University, U.K., sponsored by Association of Commonwealth Universities U.K., in 2011. He also visited the School of Mathematical Sciences, USTC, in 2010, to do collaborative research under Pakistan Program for Collaborative Research-Foreign Visit of Local Faculty Member. From 2013 to 2014, he was a Visiting Fellow with USTC sponsored by Chinese Academy of Sciences. He is currently a Full Professor of mathematics and the Chairman Department of Mathematics with The Islamia University of Bahawalpur, Pakistan. He has completed two research projects sponsored by Higher Education Commission (HEC) Pakistan under National Research Program for Universities (NRPU). He has published more than 100 research articles. His research interests include geometric modeling, subdivision schemes, applied approximation theory, and solution of differential equations. He also serves as a reviewer of journals.

AAMIR SHAHZAD received the M.Sc. and M.Phil. degrees in mathematics from the University of Sargodha, Sargodha, Pakistan, where he is currently pursuing the Ph.D. degree with the Department of Mathematics.

FAHEEM KHAN received the M.Sc. degree in mathematics from Bahauddin Zakariya University, Multan, Pakistan, and the Ph.D. degree from The Islamia University of Bahawalpur, Pakistan. He was a Research Associate with the University of Birmingham, U.K. His research interests include numerical computing, numerical solution of ODEs, and CAGD.

DUMITRU BALEANU received the B.Sc. degree in physics from the University of Craiova, Romania, in 1988, the M.Sc. degree from the University of Bucharest, Romania, in 1989, and the Ph.D. degree from the Institute of Atomic Physics, Romania, in 1996. He is currently a Professor with the Institute of Space Sciences, Romania. Since 2000, he has been a Visiting Staff Member with Cankaya University, Turkey. He published more than 1000 articles in journals indexed in SCI. He is a co-editor of five books published by Springer. He is the coauthor of three books published by Elsevier and World Scientific. He is an editorial board member of more than six ISI journals. He is on the 2015 Highly Cited Researcher list in mathematics.

YUMING CHU was born in Huzhou, Zhejiang, China, in June 1966. He received the B.S. degree from the Hangzhou Normal University, Hangzhou, China, in 1988, and the M.S. and Ph.D. degrees from Hunan University, Changsha, China, in 1991 and 1994, respectively. He worked as an Assistant Professor, from 1994 to 1996, and as an Associate Professor, from 1997 to 2002, with the Department of Mathematics, Hunan Normal University, Changsha. Since 2002, he has been a Professor and the Dean of the Department of Mathematics, Huzhou University, Huzhou. His current research interests include special functions, functional analysis, numerical analysis, operator theory, ordinary differential equations, partial differential equations, inequalities theory and applications, and robust filtering and control.