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Research article

A Razumikhin approach to stability and synchronization criteria for fractional order time delayed gene regulatory networks

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Abstract: This manuscript is concerned with the stability and synchronization for fractional-order delayed gene regulatory networks (FODGRNs) via Razumikhin approach. First of all, the existence of FODGRNs are established by using homeomorphism theory, 2-norm based on the algebraic method and Cauchy Schwartz inequality. The uniqueness of this work among the existing stability results are, the global Mittag-Leffler stability of FODGRNs is explored based on the fractional-order Lyapunov Razumikhin approach. In the meanwhile, two different controllers such as linear feedback and adaptive feedback control, are designed respectively. With the assistance of fractional Razumikhin theorem and our designed controllers, we have established the global Mittag-Leffler synchronization and adaptive synchronization for addressing master-slave systems. Finally, three numerical cases are given to justify the applicability of our stability and synchronization results.

Keywords: gene regulatory networks; fractional-order; existence and stability; synchronization; linear feedback control; adaptive feedback control
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1. Introduction

Fractional calculus, which is the general development of old calculus, has a remarkable ability to examine the world around us through the areas of both applied and pure mathematics, respectively. It has also gained considerable attention from the genetics, physics, chemistry, and computer research communities. The best way to make discoveries in mathematics is by adding some new theories to evaluate the current results. Frequently the results may fail. But, often it may pave a way to a new field of research works. Similarly, the fractional-order calculus is a perfect answer to the nonsensical question "what if the derivative order is non-integer?" by Leibnitz during the late sixteenth century. One of the most important properties of the differential fractional equation is its ability to track a motion of an object continuously and instantly of non-local nature. Besides, it contains more memory of the systems. The fractional-order models are easier to understand the complexity of the dynamic system with greater precision contrast to the integer-order differential models. Due to the memory properties, several researchers are integrating the memory properties into nonlinear dynamical systems, and lot of important results about fractional order nonlinear dynamical systems have been reported in recent literature, see Ref [3, 11, 12].

In an organism, gene expression is regulated by RNA, DNA, protein, and tiny molecules. Gene Regulatory Network (GRNs) defined the interconnections between these two. GRNs are viewed as complex networks. Each gene is regarded as a node, and the regulatory link between these genes is known as a relation between the nodes. To unleash the cure for deadly diseases such as cancer and AIDS, a greater understanding of the complex networks of GRNs is essential. Time delays in both biological and artificial neural networks are unavoidable because of the limited speed of information processing. Generally, there are typically two types of gene regulatory networks, such as the boolean model and the continuous model. The continuous model is commonly used for the study of GRNs, and several important results about GRNs with time delays had been well documented, see Ref [1, 16, 19].

On the other hand, stability theory is the flexible branch of science and engineering that deals with the behavioral effect for linear and nonlinear systems of dynamic structures. The investigation on various stability problems of time-delayed GRNs (TDGRNs) are accounted [20, 24, 26, 36]. Paper by Luo et al., has analyzed the existence and Lagrange stability of TDGRNs in Lyapunov's sense based on novel algebraic method and stability theory [20]. In [24], the authors demonstrated the delay-dependent finite time stability issues of TDGRNs with impulses based on the LMI approach and Lyapunov stability theory. In [26], the problem of the stability criterion of TDGRNs with impulsive effects was analyzed. By employing the LMI techniques, convex combination approach, and Lyapunov-Krasovskii functional, the sufficient conditions to assure the global asymptotic stability analysis of the proposed TDGRNs model. In [36], the authors researched the global exponential delay-dependent stability criterion of TDGRNs under distributed delays based on LMI techniques and Lyapunov-Krasovskii functional approach.

Nowadays, the synchronization of dynamic systems is advancing as a dominant research field and has drawn a great deal of interest from researchers of diverse field. Its application found in many fields like secure communication, image, and signaling process. Many types of synchronization results are available in recent works including Mittag-Leffler, asymptotic, quasi and pinning synchronization, and so forth [7–10, 17, 18, 30, 33]. GRN synchronization is essential for knowing the synergic behavior between the more than one gene networks through the connections of gene signals and their products. The advantages of researching GRN synchronization are acquiring knowledge about a gene's internal processes even at cellular levels. Some significant results about synchronization for time-delayed GRNs (TDGRNs) had been studied in recent years. For example, by means of observer-based non-fragile and linear feedback control, LMI techniques, and Lyapunov-Krasovskii method, Ali et al. demonstrate some sufficient criteria for the global asymptotic synchronization issues of TDGRNs under uncertainty [2]. By exploiting finite-time control techniques, robust analysis, and theory of finite-time stability, Jiang et al. investigate stochastic synchronization in finite time analysis for TDGRNs under parameter uncertainties [14]. Depending on the pinning control strategy, some famous inequality approaches, matrix theory, and event-triggered condition, Yue et al.analyze the cluster synchronization analysis for GRNs with coupling terms [35]. The research works in fractional order gene regulatory with delay arguments has been undergone exciting development in recent years, and some meaningful scientific results had been obtained. By using a hybrid control approach, the authors experimented with the bifurcation analysis for FODGRNs [15]. By utilizing the principle of Banach contraction mapping and absolute Lyapunov functional with 1-norm, the authors exhibited the several stability criteria of fractional order GRNs [25]. Depending on the principle of Banach contraction, Lyapunov functional with 1-norm, linear feedback, and adaptive feedback techniques, the sufficient criteria to ensure the finite time delay-independent synchronization problem of considered FODGRNs via Razumikhin approach [23]. By employing the diffusion and stability theory, the authors have demonstrated with the issues of local stability and instability criteria for bifurcation diffusion FODGRNs [29]. Unfortunately, there is no work done on the existence, stability, and synchronization for FODGRNs via Razumikhin approach and quadratic Lyapunov approach, this situation motivates further discussion for global Mittag-Leffler stability and adaptive synchronization of FODGRNs. The essential theme of this manuscript lies in the following aspects:

1) By means of homeomorphism theory and Cauchy Schwartz inequality, a sufficient condition is presented to ascertain the existence and uniqueness of the equilibrium point for FODGRNs.
2) Based on fractional Lyapunov method, fractional-order Razumikhin theorem, and some traditional inequality techniques, a sufficient condition is established for global Mittag-Leffler stability of the proposed networks.
3) According to feedback control technique, two kinds feedback controllers are designed to guarantee the synchronization of a class of master-slave fractional order time delayed gene regulatory networks. One is linear feedback control, which is better and simpler to execute over the other controls. Another one is adaptive feedback control, which is designed to prevent the high feedback gains and it is regarded as the more versatile one. Since, it can adjust the coupling weights by itself.
4) The proposed results in this paper are still true for global exponential stability and synchronization of integer-order GRNs with time delay effects, and these results do not discuss in the previous works of literature.

The scheme of this paper is as planned out as follows. We present the key concepts about the calculus of fractional order, essential lemmas, and the system description in Section 2. In Section 3 and Section 4, we present the main results of this manuscript. In Section 5, we include the numerical results and its simulations. Lastly, we draw some conclusions in Section 6.

Notations: The required notations are displayed as follows: $\mathbb{R}^m$ refers to the space of m-dimensional space. A set of all $m \times m$ real matrix is described by $\mathbb{R}^{m\times m}$. $sign(\cdot)$ indicate the signum function. $\mathcal{E}_{\lambda,\mu}$ and $\mathcal{E}_{\lambda,1}$ refers to the two parameter and one parameter Mittag-Leffler functions, respectively. For any matrix $B = (b_{pq})_{m\times m}$, $|B| = (|b_{pq}|)_{m\times m}$. The greatest and smallest eigenvalues of matrix B is represented by Φ_M and Φ_m, respectively. The symmetric term in a matrix is displayed by ✠. The operator norm of a matrix B is denoted by $\|B\| = \sqrt{\Phi_M\big(B^T B\big)}$. $\Gamma(\cdot)$ is the gamma function. $\mathbb{C}([-\tau, 0], \mathbb{R}^m)$ indicate the group of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^m$, where time lag $\tau > 0$ and the signum function is referred by $sign(\cdot)$.

2. Basic tools and research problem

This section comprises of the rudimentary fractional-order definitions, lemmas which are further employed in the subsequent section.

2.1. Basic tools of Caputo-fractional operator

Definition 2.1 [22] The $\lambda - th$ fractional order for integral function $\ell(t)$ is denoted as:

$$ {}_0^C D_t^{-\lambda}\ell(t) = \frac{1}{\Gamma(\lambda)} \int_{t_0}^{t} (t-\theta)^{\lambda-1}\ell(\theta)\,\mathrm{d}\theta. $$

Definition 2.2 [22] The $\lambda - th$ Caputo type fractional order for a function $\ell(t)$ is denoted as:

$$ {}_0^C D_t^{\lambda}\ell(t) = \frac{1}{\Gamma(m-\lambda)} \int_{t_0}^{t} \frac{\ell^{(m)}(\theta)}{(t-\theta)^{\lambda-m+1}}\,\mathrm{d}\theta, $$

where $t \geq t_0$ and $m - 1 < \lambda < m \in \mathbb{Z}^+$.

Lemma 2.3 [13] For $0 < \lambda < 1$, $\ell(t) \in \mathbb{R}^m$ be a continuously vector valued differentiable function, then for any $t \geq t_0$

$$ {}_0^C D_t^{\lambda}\{\ell^T(t)\mathcal{X}\ell(t)\} \leq 2\ell^T(t)\mathcal{X}\{D_t^{\lambda}\ell(t)\}, $$

where $\mathcal{X} \in \mathbb{R}^{m\times m}$ is a positive definite symmetric matrix.

Lemma 2.4 [22] If $\ell(t) \in C^m([0, +\infty), \mathbb{R})$, then

$$ {}_0^C D_t^{-\lambda}\Big({}_0^C D_t^{\lambda}\ell(t)\Big) = \ell(t) - \sum_{x=0}^{m-1} \frac{t^x}{x!}\ell^{(x)}(t_0). $$

where $m - 1 < \lambda < m,\ (m \in \mathbb{Z}^+,\ m \geq 1)$.

If $0 < \lambda < 1$, then ${}_0^C D_t^{-\lambda}\Big({}_0^C D_t^{\lambda}\ell(t)\Big) = \ell(t) - \ell(t_0)$.

Lemma 2.5 [31] For $0 < \lambda < 1$, if $\ell(t)$ is continuously derivable function on $[0, +\infty)$, then there exist a constants $\vartheta_1 > 0$ and $\vartheta_2 > 0$ such that

$$ {}_0^C D_t^\lambda \ell(t) \leq -\vartheta_1 \ell(t) + \vartheta_2, \ t \geq t_0, $$

then

$$ \ell(t) \leq \ell(t_0)\mathcal{E}_\lambda\big(-\vartheta_1(t-t_0)^\lambda\big) + \vartheta_2 t^\lambda \mathcal{E}_{\lambda,\lambda+1}\big(-\vartheta_1(t-t_0)^\lambda\big), \ t \geq t_0. $$

Lemma 2.6 [34] For $0 < \lambda < 1$, a nondecreasing derivable function $\ell(t)$ is defined on positive, then there exist a constant $\vartheta > 0$ such that

$$ {}_0^C D_t^\lambda [\ell(t) - \vartheta]^2 \leq 2[\ell(t) - \vartheta] {}_0^C D_t^\lambda \ell(t). $$

Lemma 2.7 [37] For $0 < \lambda < 1$, $\ell(t)$ be a continuously vector valued differentiable function, then

$$ {}_0^C D_t^\lambda |\ell(t)| \leq sgn(\ell(t)) {}_0^C D_t^\lambda \ell(t). $$

2.2. *Research problem*

We consider a class of Caputo-sense FODGRNs in this manuscript as follows:

$$ \begin{cases} {}_0^C D_t^\lambda g_p(t) = -a_p g_p(t) + \sum_{q=1}^m b_{pq} f_q\big(h_q(t-\sigma_1)\big) + F_p, \\ {}_0^C D_t^\lambda h_p(t) = -c_p h_p(t) + d_p g_p(t-\sigma_2), \end{cases} \tag{2.1} $$

where $p = 1, 2, .., m$, $0 < \lambda < 1$ signifies the fractional order, $g_p(t) \in \mathbb{R}^m$ and $h_p(t) \in \mathbb{R}^m$ indicate the concentrations of mRNA and protein of pth node at time t, respectively. a_p and c_p are degradation velocities of mRNA and protein molecule, respectively. Moreover, d_p represents the translation rate. The time lags are denoted as $\sigma_1 > 0$ and $\sigma_2 > 0$. b_{pq} is coupling matrix. Besides, the functions $f_q(\cdot)$ represents the nonlinear protein feedback regulation, which are commonly indicated in the Hill form as

$$ f_q(\nu) = \frac{\left(\frac{\nu}{\varsigma_q}\right)^{\mathcal{H}_q}}{\left[1 + \left(\frac{\nu}{\varsigma_q}\right)^{\mathcal{H}_q}\right]}, $$

where $\mathcal{H}_q$ is the Hill coefficients and α_y signifies non-negative constants. The coupling matrix of the network $B = (b_{pq})_{m\times m} \in \mathbb{R}^{m\times m}$ are represents as follows:

$$ b_{pq} = \begin{cases} -\varphi_{pq}, & q \quad \text{is a repressor of gene} \quad p \\ 0, & q \quad \text{does not regulate gene} \quad p \\ \varphi_{pq}, & q \quad \text{is a initiator of gene} \quad p. \end{cases} $$

Presently, we define G_p as $G_p = \sum_{y\in\hat{G}} b_{pq}$, where $\hat{G}$ indicate the set of all repressor of gene p. It's significant to mention that Caputo's definition was the most celebrated definition due to its properties such as derivative of constant is zero. Further, the Cauchy problem defined in the sense of Caputo's definition has an interpretation of integer-order initial values. Therefore, the initial values combined with FODGRNs (2.1) in the sense of Caputo type can be described as:

$$g_p(t) = \omega_p(t),\ h_p(t) = \varpi_p(t),\ t \in [-\tau = \max\{\sigma_1, \sigma_2\}, 0],$$

where $\omega_p(t),\ \varpi_p(t) \in \mathbb{C}([-\tau, 0], \mathbb{R}^m)$ and its norm is defined by

$$\|\varpi\| = \sum_{p=1}^{m} \sup_{-\sigma_1 \le \theta \le 0} \left\{ \left| \varpi_p(\theta) \right| \right\},\ \|\omega\| = \sum_{p=1}^{m} \sup_{-\sigma_2 \le \theta \le 0} \left\{ \left| \omega_p(\theta) \right| \right\}.$$

The vector form of FODGRNs (2.1) is given as

$$\begin{cases} {}_0^C D_t^\lambda g(t) = -Ag(t) + Bf(h(t-\sigma_1)) + F \\ {}_0^C D_t^\lambda h(t) = -Ch(t) + Dg(t-\sigma_2), \end{cases} \tag{2.2}$$

where $g(t) = (g_1(t), ..., g_m(t))^T$, $h(t) = (h_1(t), .., h_m(t))^T$, $A = diag\{a_1, ..., a_m\}$, $B = (b_{pq})_{m\times m}$, $C = diag\{c_1, ..., c_m\}$, $D = diag\{d_1, ..., d_m\}$, $f(h(t)) = (f_1(h_1(t), ..., f_m(h_m(t))^T$ and $F = (F_1, ..., F_m)^T$.

In the development of main results, the following Assumption and Lemma's are important.

Assumption 1. The feedback function $f_q(\cdot)$ is monotonically increasing, it is fulfilled that

$$0 \le \frac{f_q(v) - f_q(w)}{v - w} \le \beta_q,\ q = 1, 2, \ldots, m,$$

for all $v, w \in \mathbb{R}$ with $v \neq w$.

Lemma 2.8 [5] A uniformly continuous function $\ell(t)$ is defined on positive interval and $\int_{t_0}^{t} \ell(\theta)d\theta$ exists and is bounded, then $\lim_{t\to+\infty} \ell(t) = 0$.

Lemma 2.9 [6] For any $v, w \in \mathbb{R}^m$ and $\mathcal{R} \in \mathbb{R}^{m\times m}$ is a positive definite matrix, then

$$v^T w \le \frac{1}{2} v^T \mathcal{R} v + \frac{1}{2} w^T \mathcal{R}^{-1} w.$$

Lemma 2.10 [21] If a continuous map $\Upsilon : \mathbb{R}^m \to \mathbb{R}^m$ holds the following conditions:

(1). $\Upsilon(g)$ is injective on $\mathbb{R}^m$, that is $\Upsilon(g) \neq \Upsilon(h)\ \forall\ g \neq h$.

(2). $\|\Upsilon(g)\| \to +\infty$ as $\|g\| \to +\infty$.

Then, $\Upsilon(g)$ is homeomorphism of $\mathbb{R}^m$.

3. Existence and global Mittag-Leffler stability

In this part, we will derive the existence and stability of FODGRNs (2.1) by using the following Definitions.

Definition 3.1 The vectors $g^\star = \left(g_1^\star, ..., g_m^\star\right)^T$ and $h^\star = \left(h_1^\star, ..., h_m^\star\right)^T$ is an equilibrium points of FODGRNs (2.1), if and only if

$$\begin{cases} -a_p g_p^\star + \sum_{q=1}^m b_{pq} f_q(h_q^\star) + F_p = 0, \\ -c_p h_p^\star + d_p g_p^\star = 0, \ p = 1, 2, ..m. \end{cases}$$

Definition 3.2 The equilibrium point of FODGRNs (2.1) is said to be global Mittag-Leffler stable, if there exists two positive constants $\zeta_1 > 0$ and $\zeta_2 > 0$ such that for any solution $(g(t) - g^\star,\ h(t) - h^\star)$ of FODGRNs (2.1) with initial conditions $(\omega(t) - g^\star, \varpi(t) - h^\star)$ such that

$$\sum_{p=1}^m \left(g_p(t) - g_p^\star\right)^2 + \sum_{p=1}^m \left(h_p(t) - h_p^\star\right)^2 \leq \left\{\mathcal{N}(\omega(t) - g^\star, \varpi(t) - h^\star)\mathcal{E}_{\lambda,1}(\zeta_1(t - t_0)^\lambda)\right\}^{\zeta_2},$$

for $t \geq t_0$, where t_0 is starting time, $\mathcal{N}(0, 0) = 0$, $\mathcal{N}(\omega, \varpi) \geq 0$, and $\mathcal{N}(\omega, \varpi)$ refers to locally Lipschitz with respect to $\varpi \in \mathbb{C}([-\sigma_1, 0], \mathbb{R}^m)$ and $\omega \in \mathbb{C}([-\sigma_2, 0], \mathbb{R}^m)$.

Theorem 3.3 Under Assumption 1, the existence of equilibrium point $(g^\star, h^\star)$ of FODGRNs (2.1) is unique, where $g^\star = \left(g_1^\star, ..., g_m^\star\right)^T$ and $h^\star = \left(h_1^\star, ..., h_m^\star\right)^T$ if there exist positive constants $\alpha_p > 0$ for $p = 1, 2, ..m$ such that the condition is established:

$$\xi_p = \frac{2a_p c_p}{d_p} - \sum_{q=1}^m |b_{pq}|\beta_q - \sum_{q=1}^m \frac{\alpha_q}{\alpha_p}|b_{qp}|\beta_p > 0, \ p = 1, 2, .., m. \tag{3.1}$$

Proof. According to Definition 3.1, it easy to obtain

$$g_p^\star = \frac{c_p}{d_p} h_p^\star, \ p = 1, 2, .., m,$$

which prove that if the existence of equilibrium point $(g_p^\star, h_p^\star)$ of FODGRNs (2.1) is unique, so we only to establish the existence of unique equilibrium point $h_p^\star$.

Define $\Upsilon(h) = \left(\Upsilon_1(h), ..., \Upsilon_m(h)\right)^T$, $h = \left(h_1, ..., h_m\right)^T \in \mathbb{R}^m$, where

$$\Upsilon_p(h_p) = -\frac{a_p c_p}{d_p} h_p^\star + \sum_{q=1}^m b_{pq} f_q(h_q^\star) + F_p, \ p = 1, 2, .., m. \tag{3.2}$$

In the following, we will demonstrate that $\Upsilon(h)$ is homeomorphism of $\mathbb{R}^m$ onto itself based on Lemma 2.10. That is (i). If $h_p \neq k_p$, then $\Upsilon_p(h_p) \neq \Upsilon_p(k_p)$, $p = 1, 2, .., m$. (ii). $\|\Upsilon(h)\| \to +\infty$ as $\|h\| \to +\infty$.

Firstly, we show (i). If there exist $h_p,\ k_p \in \mathbb{R}^m$ holds $h_p \neq k_p$ such that $\Upsilon_p(h_p) \neq \Upsilon_p(k_p)$ for $p = 1, 2, .., m$. Then,

$$\frac{a_p c_p}{d_p}(h_p) - \frac{a_p c_p}{d_p}(k_p) = \sum_{q=1}^{m} b_{pq}[f_q(h_q) - f_q(k_q)]$$

According to Assumption 1, it follows that

$$\frac{a_p c_p}{d_p}|h_p - k_p| \leq \sum_{q=1}^{m} |b_{pq}|\beta_q|h_q - k_q|. \tag{3.3}$$

From (3.3), we have

$$\begin{aligned}
\sum_{p=1}^{m} 2\alpha_p \frac{a_p c_p}{d_p}|h_p - k_p|^2 &= \sum_{p=1}^{m} 2\alpha_p|h_p - k_p|\Big(\frac{a_p c_p}{d_p}|h_p - k_p|\Big) \\
&\leq \sum_{p=1}^{m} 2\alpha_p|h_p - k_p|\Big(\sum_{q=1}^{m} |b_{pq}|\beta_q|h_q - k_q|\Big) \\
&\leq \sum_{p=1}^{m}\sum_{q=1}^{m} 2\alpha_p|b_{pq}|\beta_q|h_p - k_p||h_q - k_q| \\
&\leq \sum_{p=1}^{m}\sum_{q=1}^{m} \alpha_p|b_{pq}|\beta_q\Big(|h_p - k_p|^2 + |h_q - k_q|^2\Big) \\
&= \sum_{p=1}^{m}\sum_{q=1}^{m} \Big[\alpha_p|b_{pq}|\beta_q + \alpha_q|b_{qp}|\beta_p\Big]|h_p - k_p|^2.
\end{aligned} \tag{3.4}$$

Together with (3.1) and (3.4), we sustain

$$\sum_{p=1}^{m} \xi_p|h_p - k_p|^2 \leq 0. \tag{3.5}$$

Based on inequality (3.2) and (3.5), it follows that $|h_p - k_p| = 0$, which leads to a contradiction with our assumption.

Next, we show (ii). From (3.2), we get

$$\begin{aligned}
&\textstyle\sum_{p=1}^{m} 2\alpha_p\Big(\Upsilon_p(h_p) - \Upsilon_p(0)\Big)sign(h_p)|h_p| \\
&\leq -\sum_{p=1}^{m} 2\alpha_p\frac{a_p c_p}{d_p}|h_p|^2 + \sum_{p=1}^{m}\sum_{q=1}^{m} 2\alpha_p|b_{pq}|\beta_q|h_q||h_p| \\
&\leq \sum_{p=1}^{m}\Big[-\frac{2a_p c_p}{d_p} + \sum_{q=1}^{m}|b_{pq}|\beta_q + \sum_{q=1}^{m}\frac{\alpha_q}{\alpha_p}|b_{qp}|\beta_p\Big]|h_p|^2 \\
&\leq -\xi^{\min}||h||^2,
\end{aligned}$$

where $\xi^{\min} = \min_{1\le p\le m}\{\xi_p\}$. Then, we have

$$\begin{aligned}||h||^2 &\le -\frac{2}{\xi^{\min}}\sum_{p=1}^{m}\alpha_p\Big(\Upsilon_p(h_p)-\Upsilon_p(0)\Big)sign(h_p)|h_p|\\ &\le \frac{2\alpha^M}{\xi^{\min}}\sum_{p=1}^{m}\Big|\Upsilon_p(h_p)-\Upsilon_p(0)\Big||h_p|, \end{aligned}\tag{3.6}$$

$\alpha^M = \max_{1\le p\le m}\{\alpha_p\}$. By means of famous Cauchy-Schwartz inequality and the above inequality (3.6), we obtain

$$||h||^2 \le \frac{2\alpha^M}{\xi^{\min}}\Bigg(\sum_{p=1}^{m}\Big|\Upsilon_p(h_p)-\Upsilon_p(0)\Big|^2\Bigg)^{\frac{1}{2}}\Bigg(\sum_{p=1}^{m}|h_p|^2\Bigg)^{\frac{1}{2}},$$

which implies to

$$\begin{aligned}||h|| &\le \frac{2\alpha^M}{\xi^{\min}}\Big\|\Upsilon(h)-\Upsilon(0)\Big\|\\ &\le \frac{2\alpha^M}{\xi^{\min}}\Big(\Big\|\Upsilon(h\Big\|+\Big\|\Upsilon(0)\Big\|\Big).\end{aligned}$$

Based on above discussions, we see that $||\Upsilon(h)|| \to +\infty$ as $||h|| \to +\infty$. In view of Lemma 2.10, $\Upsilon(h)$ is homeomorphism on $\mathbb{R}^m$, which indicates, the existence of equilibrium point $(g_p^\star, h_p^\star)$ of FODGRNs (2.1) is unique, and the proof of Theorem 3.3 is ended.

Remark 3.4 There exist other methods to obtain the existence of equilibrium, such as Schauder's fixed point theorem, Banach fixed point theorem, Browner's fixed point theorem and Krasnoselskii fixed point theorem, and Homotopy invariance theorem. In [23, 25], based on the theory of fractional calculus, the contraction mapping principle and the norm-1 properties, the existence and uniqueness of the equilibrium point of the fractional order genetic regulatory networks is discussed. Different from above mentioned references [23, 25], we have discussed the existence and uniqueness by homeomorphism theory.

Transform $(g_p^\star, h_p^\star)$ of FODGRNs (2.1) to origin via the transformation $v_p(t) = g_p(t) - g_p^\star$ and $w_p(t) = h_p(t) - h_p^\star$ for $p = 1, 2, .., m$. Then, the FODGRNs error system is:

$$\begin{cases} {}_0^C D_t^\lambda v_p(t) = -a_p v_p(t) + \sum_{q=1}^{m} b_{pq}\tilde{f}_q(w_q(t-\sigma_1)) \\ {}_0^C D_t^\lambda w_p(t) = -c_p w_p(t) + d_p v_p(t-\sigma_2), \end{cases}\tag{3.7}$$

for $p = 1, 2, .., m$ and $\tilde{f}_q(w_q(t-\sigma_1)) = \Big(f_q(w_q(t-\sigma_1)+h_q^\star) - h_q^\star\Big)$. The vector form of FODGRNs (3.7) is given as

$$\begin{cases} {}_0^C D_t^\lambda v(t) = -Av(t) + B\tilde{f}(w(t-\sigma_1)) \\ {}_0^C D_t^\lambda w(t) = -Cw(t) + Dv(t-\sigma_2), \end{cases}\tag{3.8}$$

where $v(t) = (v_1(t), ..., v_m(t))^T$, $w(t) = (w_1(t), .., w_m(t))^T$, $\tilde{f}(w(t)) = (\tilde{f}_1(w_1(t), ..., \tilde{f}_m(w_m(t))^T$.

Theorem 3.5 Under Assumption 1, the existence of equilibrium point $(g^\star, h^\star)$ of FODGRNs (2.1) is globally Mittag-Leffler stable if there exist two positive diagonal matrices $\mathcal{X} \in \mathbb{R}^{m\times m}$ and $\mathcal{Y} \in \mathbb{R}^{m\times m}$ such that

$$\Phi_M(\mathcal{X})\bigg(-2\|A\| + \|B\|\|S\|\frac{\Phi_M(\mathcal{X}) + \Phi_m(\mathcal{Y})}{\Phi_m(\mathcal{Y})} + \frac{\Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})}\|D\|\bigg) < 0, \tag{3.9}$$

$$\Phi_M(\mathcal{Y})\bigg(-2\|C\| + \|D\|\frac{\Phi_m(\mathcal{X}) + \Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})} + \frac{\Phi_M(\mathcal{X})}{\Phi_m(\mathcal{Y})}\|B\|\|S\|\bigg) < 0, \tag{3.10}$$

where $S = diag\{\beta_1, ..., \beta_m\}$.

Proof. According to conditions (3.9) and (3.10) from Theorem 3.5 that there exist two positive scalars $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\Phi_M(\mathcal{X})\bigg(-2\|A\| + \|B\|\|S\|\frac{\Phi_M(\mathcal{X}) + \Phi_m(\mathcal{Y})}{\Phi_m(\mathcal{Y})} + \frac{\Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})}\|D\|\bigg) < -\gamma_1, \tag{3.11}$$

$$\Phi_M(\mathcal{Y})\bigg(-2\|C\| + \|D\|\frac{\Phi_m(\mathcal{X}) + \Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})} + \frac{\Phi_M(\mathcal{X})}{\Phi_m(\mathcal{Y})}\|B\|\|S\|\bigg) < -\gamma_2. \tag{3.12}$$

Consider the following subsequent Lyapunov-Razumikhin functional:

$$H(v(t), w(t)) \;=\; v^T(t)\mathcal{X}v(t) + w^T(t)\mathcal{Y}w(t) \tag{3.13}$$

Noting that

$$\Phi_m(\mathcal{X})\|v(t)\|^2 + \Phi_m(\mathcal{Y})\|w(t)\|^2 \le H(v(t), w(t)) \le \Phi_M(\mathcal{X})\|v(t)\|^2 + \Phi_M(\mathcal{Y})\|w(t)\|^2. \tag{3.14}$$

Then, based on Lemma 2.3, and the Caputo-derivative of $H(v(t), w(t))$ with respect to FODGRNs error system (3.7), we have

$${}_0^C D_t^\lambda H(v(t), w(t)) \;\le\; 2\Phi_M(\mathcal{X})\|v(t)\|\, {}_0^C D_t^\lambda\, \|v(t)\| + 2\Phi_M(\mathcal{Y})\|w(t)\|\, {}_0^C D_t^\lambda\, \|w(t)\|$$

From Assumption 1 and Lemma 2.7, we sustain

$$\begin{aligned}
{}_0^C D_t^\lambda H(v(t), w(t)) \;\le\;& 2\Phi_M(\mathcal{X})\|v(t)\| sign(v(t))\, {}_0^C D_t^\lambda\, \{v(t)\} \\
&+2\Phi_M(\mathcal{Y})\|w(t)\| sign(w(t))\, {}_0^C D_t^\lambda\, \{w(t)\} \\
\le\;& 2\Phi_M(\mathcal{X})\|v(t)\| sign(v(t)) \Big\{-Av(t) + BSw(t-\sigma_1)\Big\} \\
&+2\Phi_M(\mathcal{Y})\|w(t)\| sign(w(t)) \Big\{-Cw(t) + Dv(t-\sigma_2)\Big\} \\
\le\;& -2\Phi_M(\mathcal{X})\|A\|\|v(t)\|^2 - 2\Phi_M(\mathcal{Y})\|C\|\|w(t)\|^2 \\
&+2\Phi_M(\mathcal{X})\|B\|\|S\|\|v(t)\|\|w(t-\sigma_1)\| \\
&+2\Phi_M(\mathcal{Y})\|D\|\|w(t)\|\|v(t-\sigma_2)\| \\
\le\;& -2\Phi_M(\mathcal{X})\|A\|\|v(t)\|^2 - 2\Phi_M(\mathcal{Y})\|C\|\|w(t)\|^2
\end{aligned}$$

$$+2\Phi_M(\mathcal{X})||B||||S||\sup_{t-\sigma_1\le\theta\le t}\Big\{||v(t)||||w(\theta)||\Big\}$$
$$+2\Phi_M(\mathcal{Y})||D||\sup_{t-\sigma_2\le\theta\le t}\Big\{||w(t)||||v(\theta)||\Big\}.$$

By using famous inequality $2|\mu_1||\mu_2| \le \mu_1^2 + \mu_2^2$, for $\sup_{t-\sigma_1\le\theta\le t}\Big\{||v(t)||||w(\theta)||\Big\}$ and $\sup_{t-\sigma_2\le\theta\le t}\Big\{||w(t)||||v(\theta)||\Big\}$, we sustain

$$\begin{aligned} {}_0^C D_t^\lambda H(v(t),w(t)) &\le -2\Phi_M(\mathcal{X})||A||||v(t)||^2 - 2\Phi_M(\mathcal{Y})||C||||w(t)||^2 \\ &+\Phi_M(\mathcal{X})||B||||S||\Bigg[||v(t)||^2 + \sup_{t-\sigma_1\le\theta\le t}||w(\theta)||^2\Bigg] \\ &+\Phi_M(\mathcal{Y})||D||\Bigg[\sup_{t-\sigma_2\le\theta\le t}||v(\theta)||^2 + ||w(t)||^2\Bigg]. \end{aligned} \tag{3.15}$$

For any function $v(t)$ and $w(t)$ that hold the following Razumikhin criteria, see Ref [27, 32]

$$H(v(\theta),w(\theta)) \le H(v(t),w(t)),\ t-\tau\le\theta\le t,$$

we get

$$\begin{aligned} \Phi_m(\mathcal{X})||v(\theta)||^2 + \Phi_m(\mathcal{Y})||w(\theta)||^2 &\le v^T(\theta)\mathcal{X}v(\theta) + w^T(\theta)\mathcal{Y}w(\theta) \\ &\le v^T(t)\mathcal{X}v(t) + w^T(t)\mathcal{Y}w(t) \\ &\le \Phi_M(\mathcal{X})||v(t)||^2 + \Phi_M(\mathcal{Y})||w(t)||^2, \end{aligned}$$

and hence

$$||v(\theta)||^2 \le \frac{\Phi_M(\mathcal{X})||v(t)||^2 + \Phi_M(\mathcal{Y})||w(t)||^2}{\Phi_m(\mathcal{X})} \tag{3.16}$$

$$||w(\theta)||^2 \le \frac{\Phi_M(\mathcal{X})||v(t)||^2 + \Phi_M(\mathcal{Y})||w(t)||^2}{\Phi_m(\mathcal{Y})} \tag{3.17}$$

for $t-\tau\le\theta\le t$. From (3.15)-(3.17), one can obtain

$$\begin{aligned} {}_0^C D_t^\lambda H(v(t),w(t)) &\le -2\Phi_M(\mathcal{X})||A||||v(t)||^2 - 2\Phi_M(\mathcal{Y})||C||||w(t)||^2 \\ &+\Phi_M(\mathcal{X})||B||||S||\Bigg[||v(t)||^2 + \frac{\Phi_M(\mathcal{X})||v(t)||^2 + \Phi_M(\mathcal{Y})||w(t)||^2}{\Phi_m(\mathcal{Y})}\Bigg] \\ &+\Phi_M(\mathcal{Y})||D||\Bigg[\frac{\Phi_M(\mathcal{X})||v(t)||^2 + \Phi_M(\mathcal{Y})||w(t)||^2}{\Phi_m(\mathcal{X})} + ||w(t)||^2\Bigg] \\ &\le \Phi_M(\mathcal{X})\Bigg(-2||A|| + ||B||||S||\frac{\Phi_M(\mathcal{X}) + \Phi_m(\mathcal{Y})}{\Phi_m(\mathcal{Y})} + \frac{\Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})}||D||\Bigg) \\ &\times||v(t)||^2 + \Phi_M(\mathcal{Y})\Bigg(-2||C|| + ||D||\frac{\Phi_m(\mathcal{X}) + \Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})} \\ &+\frac{\Phi_M(\mathcal{X})}{\Phi_m(\mathcal{Y})}||B||||S||\Bigg)||w(t)||^2 \end{aligned}$$

$$\begin{aligned} &\leq -\gamma_1\|v(t)\|^2 - \gamma_2\|w(t)\|^2 \\ &\leq -\gamma^m\Big(\|v(t)\|^2 + \|w(t)\|^2\Big) \end{aligned} \tag{3.18}$$

where $\gamma^m = \min\{\gamma_1, \gamma_2\}$. According to inequality (3.14), we get

$$ {}_0^C D_t^\lambda H(v(t), w(t)) \leq -\frac{\gamma^m}{\delta^M}\Big(\|v(t)\|^2 + \|w(t)\|^2\Big) \tag{3.19}$$

where $\delta^M = \max\{\Phi_M(\mathcal{X}), \Phi_M(\mathcal{Y})\}$. According to inequality (3.19) and based on Lemma 2.5, we have

$$\begin{aligned} H(v(t), w(t)) \leq & \sup_{-\tau\leq\theta\leq 0}\Big\{H(\omega(\theta) - g^\star, \varpi(\theta) - h^\star)\Big\} \\ & \times\mathcal{E}_{\lambda,1}\Big(-\frac{\gamma^m}{\delta^M}(t-t_0)^\lambda\Big)\forall\ t \in [t_0, +\infty). \end{aligned}$$

Then, by using inequality (3.14), we get

$$\begin{aligned} \delta^m\Big(\|v(t)\|^2 + \|w(t)\|^2\Big) &\leq H(v(t), w(t)) \\ &\leq \sup_{-\tau\leq\theta\leq 0}\Big\{H(\omega(\theta) - g^\star, \varpi(\theta) - h^\star)\Big\}\mathcal{E}_{\lambda,1}\Big(-\frac{\gamma^m}{\delta^M}(t-t_0)^\lambda\Big) \\ &\leq \delta^M\Bigg[\sum_{p=1}^m \sup_{-\sigma_2\leq\theta\leq 0}\Big(\omega_p(\theta) - g_p^\star\Big)^2 + \sum_{p=1}^m \sup_{-\sigma_1\leq\theta\leq 0}\Big(\varpi_p(\theta) - h_p^\star\Big)^2\Bigg] \\ &\quad \times\mathcal{E}_{\lambda,1}\Big(-\frac{\gamma^m}{\delta^M}(t-t_0)^\lambda\Big) \end{aligned}$$

where $\delta^m = \min\{\Phi_m(\mathcal{X}), \Phi_m(\mathcal{Y})\}$. Let

$$\mathcal{N}(\omega - g^\star, \varpi - h^\star) = \frac{\delta^M}{\delta^m}\Bigg[\sum_{p=1}^m \sup_{-\sigma_2\leq\theta\leq 0}\Big(\omega_p(\theta) - g_p^\star\Big)^2 + \sum_{p=1}^m \sup_{-\sigma_1\leq\theta\leq 0}\Big(\varpi_p(\theta) - h_p^\star\Big)^2\Bigg],$$

then

$$\|v(t)\|^2 + \|w(t)\|^2 \leq \mathcal{N}(\omega - g^\star, \varpi - h^\star)\mathcal{E}_{\lambda,1}\Big(-\frac{\gamma^m}{\delta^M}(t-t_0)^\lambda\Big)\ \forall\ t \geq t_0,$$

where $\mathcal{N} \geq 0$ and $\mathcal{N} = 0$ satisfy only if $\omega(\theta) = g^\star$ for $-\sigma_2 \leq \theta \leq 0$ and $\varpi(\theta) = h^\star$ for $-\sigma_1 \leq \theta \leq 0$, respectively. Therefore, based on Definition 3.2, the existence of equilibrium point $(g^\star, h^\star)$ of FODGRNs (2.1) is globally Mittag-Leffler stable, and the proof of Theorem 3.5 is ended.

If $\sigma_1 = \sigma_2 = 0$, then system (2.1) becomes the following form:

$$\begin{cases} {}_0^C D_t^\lambda g_p(t) = -a_p g_p(t) + \sum_{q=1}^m b_{pq} f_q(h_q(t)) + F_p, \\ {}_0^C D_t^\lambda h_p(t) = -c_p h_p(t) + d_p g_p(t),\ p = 1, 2.., m. \end{cases} \tag{3.20}$$

Corollary 3.6 Under Assumption 1, the existence of equilibrium point $(g^\star, h^\star)$ of (3.20) is globally Mittag-Leffler stable if there exist two positive diagonal matrices $\mathcal{X} \in \mathbb{R}^{m\times m}$ and $\mathcal{Y} \in \mathbb{R}^{m\times m}$ such that

$$\Phi_M(\mathcal{X})\Bigg(-2\|A\| + \|B\|\|S\| + \frac{\Phi_M(\mathcal{Y})}{\Phi_M(\mathcal{X})}\|D\|\Bigg) < 0$$

$$\Phi_M(\mathcal{Y})\Bigg(-2\|C\| + \|D\| + \frac{\Phi_M(\mathcal{X})}{\Phi_M(\mathcal{Y})}\|B\|\|S\|\Bigg) < 0.$$

Remark 3.7 This study is a first attempt on the global Mittag-Leffler stability criterion FODGRNs. This analysis takes into account for feedback regulation time delay σ_1 and translation time delay σ_2. The main difficulties of this study is how to deal with time-delay terms. To overcome this difficulty, we adopt Razumikhin condition. In [25,28], many sufficient criteria guaranteeing the global Mittag-Leffler stability of FODGRNs are obtained in terms of algebraic inequalities. Compared with other researches by employing algebraic inequalities method to obtain the global Mittag-Leffler stability criteria in [25,28], our results, in terms of norm matrices, are very easy to verified with help of MATLAB toolbox in practice.

4. Synchronization criteria

FODGRNs (2.1) acts as the master system and the slave system is

$$\begin{cases} {}_0^C D_t^\lambda u_p(t) = -a_p u_p(t) + \sum_{q=1}^m b_{pq} f_q(z_q(t-\sigma_1)) + F_p + x_p(t), \\ {}_0^C D_t^\lambda z_p(t) = -c_p z_p(t) + d_p u_p(t-\sigma_2) + y_p(t), \end{cases} \tag{4.1}$$

where $p = 1, 2, .., m$, $u_p(t) \in \mathbb{R}^m$ and $v_p(t) \in \mathbb{R}^m$ indicate the concentrations of mRNA and protein of p-node at time t, respectively. $x_p(t)$ and $y_p(t)$ are suitable controller and all others are same as one of system (2.1). The initial values of FODGRNs (4.1) can be described as:

$$u_p(t) = \tilde{\omega}_p(t),\ z_p(t) = \tilde{\varpi}_p(t),\ t \in [-\tau = \max\{\sigma_1, \sigma_2\}, 0],$$

where $\tilde{\omega}_p(t),\ \tilde{\varpi}_p(t) \in \mathbb{C}([-\tau, 0], \mathbb{R}^m)$ and its norm is defined by

$$\|\tilde{\varpi}\| = \sum_{p=1}^m \sup_{-\sigma_1 \le \theta \le 0} \left\{ \left| \tilde{\varpi}_p(\theta) \right| \right\},\ \|\tilde{\omega}\| = \sum_{p=1}^m \sup_{-\sigma_2 \le \theta \le 0} \left\{ \left| \tilde{\omega}_p(\theta) \right| \right\}.$$

Two types control like linear feedback control and adaptive feedback control, respectively, are designed as follows:

$$\begin{cases} x_p(t) = -k_p(u_p(t) - g_p(t)) \\ y_p(t) = -l_p(z_p(t) - h_p(t)) \end{cases} \tag{4.2}$$

and

$$\begin{cases} x_p(t) = -\xi_p(t)(u_p(t) - g_p(t)) \\ y_p(t) = -\eta_p(t)(z_p(t) - h_p(t)) \end{cases} \tag{4.3}$$

with adaptive rule

$$\begin{cases} {}_0^C D_t^\lambda \xi_p(t) = \tilde{k}_p \left| u_p(t) - g_p(t) \right|^2 \\ {}_0^C D_t^\lambda \eta_p(t) = \tilde{l}_p \left| z_p(t) - h_p(t) \right|^2 \end{cases}$$

for $p = 1, 2, .., m$, $k_p > 0,\ l_p > 0, \tilde{k}_p > 0,\ \tilde{l}_p > 0$ are suitable constants, $\xi_p(t)$ and $\eta_p(t)$ are adaptive coupling weights.

Let $v_p(t) = u_p(t) - g_p(t)$ and $w_p(t) = z_p(t) - h_p(t)$, then based on (2.1) and (4.1), the synchronization error system is described by

$$\begin{cases} {}_0^C D_t^\lambda v_p(t) = -a_p v_p(t) + \sum_{q=1}^m b_{pq}\tilde{f}_q(w_q(t-\sigma_1)) + x_p(t), \\ {}_0^C D_t^\lambda w_p(t) = -c_p w_p(t) + d_p v_p(t-\sigma_2) + y_p(t), \end{cases} \tag{4.4}$$

where $p = 1, 2, .., m$ and $\tilde{f}_q(w_q(t-\sigma_1)) = f_q(z_q(t-\sigma_1)) - f_q(h_q(t-\sigma_1))$.

In the development of synchronization criteria, the following Definition's are significant.

Definition 4.1 Master-slave FODGRNs systems (2.1) and (4.1) are said to realize global Mittag-Leffler synchronization under linear feedback control (4.2), if there exists two positive constants $\zeta_1 > 0$ and $\zeta_2 > 0$ such that for any solution $(u(t) - g(t),\ z(t) - h(t))$ of FODGRNs systems (2.1) and (4.1) with initial conditions $(\tilde{\omega}(t) - \omega(t), \tilde{\varpi}(t) - \varpi(t))$ such that

$$\sum_{p=1}^m \Big(u_p(t) - g_p(t)\Big)^2 + \sum_{p=1}^m \Big(z_p(t) - h_p(t)\Big)^2$$
$$\leq \Big\{\mathcal{N}(\tilde{\omega}(t) - \omega(t), \tilde{\varpi}(t) - \varpi(t))\mathcal{E}_{\lambda,1}(\zeta_1(t-t_0)^\lambda)\Big\}^{\zeta_2}$$

for $t \geq t_0$, where t_0 is starting time, $\mathcal{N}(0,0) = 0$, $\mathcal{N}(\tilde{\omega} - \omega, \tilde{\varpi} - \varpi) \geq 0$, and $\mathcal{N}(\tilde{\omega} - \omega, \tilde{\varpi} - \varpi)$ refers to locally Lipschitz with respect to $\tilde{\varpi}, \varpi \in \mathbb{C}([-\sigma_1, 0], \mathbb{R}^m)$ and $\tilde{\omega}, \omega \in \mathbb{C}([-\sigma_2, 0], \mathbb{R}^m)$.

Definition 4.2 Master-slave FODGRNs systems (2.1) and (4.1) are said to realize global asymptotical synchronization under adaptive feedback control (4.3), if

$$\lim_{t\to+\infty} \sum_{p=1}^m \Big(u_p(t) - g_p(t)\Big)^2 + \lim_{t\to+\infty} \sum_{p=1}^m \Big(z_p(t) - h_p(t)\Big)^2 = 0.$$

Remark 4.3 Global synchronization in the sense of Mittag-Leffler leads to global asymptotic synchronization.

Theorem 4.4 Under Assumption 1, master-slave FODGRNs systems (2.1) and (4.1) realize globally Mittag-Leffler synchronization under linear feedback control (4.2) if there exist two positive diagonal matrices $\mathcal{X} \in \mathbb{R}^{m\times m}$ and $\mathcal{Y} \in \mathbb{R}^{m\times m}$ such that

$$\Phi_M(\mathcal{X})\bigg(-2\|A\| - 2\|K\| + \|B\|\|S\|\frac{\Phi_M(\mathcal{X}) + \Phi_m(\mathcal{Y})}{\Phi_m(\mathcal{Y})} + \frac{\Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})}\|D\|\bigg) < 0,$$
$$\Phi_M(\mathcal{Y})\bigg(-2\|C\| - 2\|L\| + \|D\|\frac{\Phi_m(\mathcal{X}) + \Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})} + \frac{\Phi_M(\mathcal{X})}{\Phi_m(\mathcal{Y})}\|B\|\|S\|\bigg) < 0,$$

where $S = diag\{\beta_1, ..., \beta_m\}$, $K = diag\{k_1, ..., k_m\}$ and $L = diag\{l_1, ..., l_m\}$.

Proof. The proof of Theorem 4.4 is similar to the proof Theorem 3.5. Hence the proof of the above Theorem 4.4 is skipped.

Corollary 4.5 Under Assumption 1, master-slave FODGRNs systems (2.1) and (4.1) with $\sigma_1 = \sigma_2 = 0$ realize globally Mittag-Leffler synchronization under linear feedback control (4.2) if there exist two positive diagonal matrices $\mathcal{X} \in \mathbb{R}^{m\times m}$ and $\mathcal{Y} \in \mathbb{R}^{m\times m}$ such that

$$\Phi_M(\mathcal{X})\Big(- 2||A|| - 2||K|| + ||B||||S|| + \frac{\Phi_M(\mathcal{Y})}{\Phi_M(\mathcal{X})}||D||\Big) < 0$$
$$\Phi_M(\mathcal{Y})\Big(- 2||C|| - 2||L|| + ||D|| + \frac{\Phi_M(\mathcal{X})}{\Phi_M(\mathcal{Y})}||B||||S||\Big) < 0.$$

Theorem 4.6 Under Assumption 1, master-slave FODGRNs systems (2.1) and (4.1) realize global asymptotic synchronization under adaptive feedback control (4.3) if there exist positive diagonal matrices $\mathcal{X} > 0,\ \mathcal{Y} > 0,\ \mathcal{R}_1 > 0$ and $\mathcal{R}_2 > 0$ such that the following LMIs are holds:

(i). There exist two constants $\zeta_1 > 0$ and $\zeta_2 > 0$ such that

$$\begin{bmatrix} -2\mathcal{X}A - 2\mathcal{X}\tilde{K} + \zeta_1\mathcal{X} & \mathcal{X}B \\ \maltese & -\mathcal{R}_1 \end{bmatrix} < 0,$$

$$\begin{bmatrix} -2\mathcal{Y}C - 2\mathcal{Y}\tilde{L} + \zeta_2\mathcal{Y} & \mathcal{Y} \\ \maltese & -\mathcal{R}_2 \end{bmatrix} < 0.$$

(ii). There exist two constants $\zeta_3 > 0$ and $\zeta_4 > 0$ such that

$$\mathcal{R}_1 S^2 - \zeta_3\mathcal{Y} < 0,\ \mathcal{R}_2 D^2 - \zeta_4\mathcal{X} < 0.$$

(iii). For some $\varsigma > 1$ and $\varrho > 1$ such that

$$\Omega > \varsigma\zeta_3 + \varrho\zeta_4,$$

where $\Omega = \min\{\zeta_1, \zeta_2\}$, $S = diag\{\beta_1, ..., \beta_m\}$, $\mathcal{X} = diag\{x_1, ..., x_m\}$, $\mathcal{Y} = diag\{y_1, ..., y_m\}$, $\tilde{K} = diag\{\tilde{k}_1^\star, ..., \tilde{k}_m^\star\}$ and $\tilde{L} = diag\{\tilde{l}_1^\star, ..., \tilde{l}_m^\star\}$.

Proof. Consider the following subsequent Lyapunov-Razumikhin functional:

$$\begin{aligned} H(v(t), w(t)) &= \underbrace{\big|v(t)\big|^T \mathcal{X}\big|v(t)\big| + \big|w(t)\big|^T \mathcal{Y}\big|w(t)\big|}_{H_1(v(t),w(t))} \\ &\quad \underbrace{+ \sum_{p=1}^{m} \frac{x_p}{\tilde{k}_p}\Big[\xi_p(t) - \tilde{k}_p^\star\Big]^2 + \sum_{p=1}^{m} \frac{y_p}{\tilde{l}_p}\Big[\eta_p(t) - \tilde{l}_p^\star\Big]^2}_{H_2(v(t),w(t))} \end{aligned} \tag{4.5}$$

where $\tilde{k}_p^\star$ and $\tilde{l}_p^\star$ holds the LMI of condition (i) in Theorem 4.6. Based on Lemma 2.3, Lemma 2.6, Lemma 2.9 and Assumption 1, we obtain

$${}_0^C D_t^\lambda H(v(t), w(t)) \leq 2\big|v(t)\big|^T \mathcal{X}\, {}_0^C D_t^\lambda \big|v(t)\big| + 2\big|w(t)\big|^T \mathcal{Y}\, {}_0^C D_t^\lambda \big|w(t)\big|$$

$$
\begin{aligned}
& +\sum_{p=1}^{m}\frac{2x_p}{\tilde{k}_p}\Big[\xi_p(t)-\tilde{k}_p^{\star}\Big]\,{}_0^CD_t^{\lambda}\xi_p(t)+\sum_{p=1}^{m}\frac{2y_p}{\tilde{l}_p}\Big[\eta_p(t)-\tilde{l}_p^{\star}\Big]\,{}_0^CD_t^{\lambda}\eta_p(t) \\
\leq\; & 2\sum_{p=1}^{m}x_p|v_p(t)|\,{}_0^CD_t^{\lambda}|v_p(t)|+2\sum_{p=1}^{m}y_p|w_p(t)|\,{}_0^CD_t^{\lambda}|w_p(t)| \\
& +\sum_{p=1}^{m}\frac{2x_p}{\tilde{k}_p}\Big[\xi_p(t)-\tilde{k}_p^{\star}\Big]\{\tilde{k}_p\big|v_p(t)\big|^2\}+\sum_{p=1}^{m}\frac{2y_p}{\tilde{l}_p}\Big[\eta_p(t)-\tilde{l}_p^{\star}\Big]\{\tilde{l}_p\big|w_p(t)\big|^2\} \\
\leq\; & 2\sum_{p=1}^{m}x_p|v_p(t)|sign\Big\{-a_pv_p(t)+\sum_{q=1}^{m}b_{pq}\tilde{f}_q(w_q(t-\sigma_1))-\xi_p(t)v_p(t)\Big\} \\
& +2\sum_{p=1}^{m}y_p|w_p(t)|sign\Big\{-c_pw_p(t)+d_pv_p(t-\sigma_2)-\eta_p(t)w_p(t)\Big\} \\
& +2\sum_{p=1}^{m}x_p\Big[\xi_p(t)-\tilde{k}_p^{\star}\Big]\big|v_p(t)\big|^2+2\sum_{p=1}^{m}y_p\Big[\eta_p(t)-\tilde{l}_p^{\star}\Big]\big|w_p(t)\big|^2 \\
\leq\; & -2\sum_{p=1}^{m}|v_p(t)|\Big(x_p(a_p+k_p^{\star})\Big)|v_p(t)|-2\sum_{p=1}^{m}|w_p(t)|\Big(y_p(c_p+l_p^{\star})\Big)|w_p(t)| \\
& +2\sum_{p=1}^{m}\sum_{q=1}^{m}|v_p(t)|x_p\big|b_{pq}\big|\beta_q\big|w_q(t-\sigma_1)\big|+2\sum_{p=1}^{m}|w_p(t)|y_pd_p\big|v_p(t-\sigma_2)\big| \\
=\; & -2\big|v(t)\big|^T\mathcal{X}(A+\tilde{K})\big|v(t)\big|-2\big|w(t)\big|^T\mathcal{Y}(C+\tilde{L})\big|w(t)\big| \\
& +2\big|v(t)\big|^T\mathcal{X}BS\big|w(t-\sigma_1)\big|+2\big|w(t)\big|^T\mathcal{Y}D\big|v(t-\sigma_2)\big| \\
\leq\; & -2\big|v(t)\big|^T\mathcal{X}(A+\tilde{K})\big|v(t)\big|-2\big|w(t)\big|^T\mathcal{Y}(C+\tilde{L})\big|w(t)\big| \\
& +\big|v(t)\big|^T\mathcal{X}B\mathcal{R}_1^{-1}B^T\mathcal{X}^T\big|v(t)\big|+\big|w(t-\sigma_1)\big|^T\mathcal{R}_1S^2\big|w(t-\sigma_1)\big| \\
& +\big|w(t)\big|^T\mathcal{Y}\mathcal{R}_2^{-1}\mathcal{Y}^T\big|w(t)\big|+\big|v(t-\sigma_2)\big|^T\mathcal{R}_2D^2\big|v(t-\sigma_2)\big| \\
=\; & \big|v(t)\big|^T\Big[-2\mathcal{X}A-2\mathcal{X}\tilde{K}+\mathcal{X}B\mathcal{R}_1^{-1}B^T\mathcal{X}^T+\zeta_1\mathcal{X}\Big]\big|v(t)\big| \\
& +\big|w(t)\big|^T\Big[-2\mathcal{Y}C-2\mathcal{Y}\tilde{L}+\mathcal{Y}\mathcal{R}_2^{-1}\mathcal{Y}^T+\zeta_2\mathcal{Y}\Big]\big|w(t)\big| \\
& -\zeta_1\big|v(t)\big|^T\mathcal{X}\big|v(t)\big|+\big|w(t-\sigma_1)\big|^T\Big[\mathcal{R}_1S^2-\zeta_3\mathcal{Y}\Big]\big|w(t-\sigma_1)\big| \\
& -\zeta_2\big|w(t)\big|^T\mathcal{Y}\big|w(t)\big|+\big|v(t-\sigma_2)\big|^T\Big[\mathcal{R}_2D^2-\zeta_4\mathcal{X}\Big]\big|v(t-\sigma_2)\big| \\
& +\zeta_3\big|w(t-\sigma_1)\big|^T\mathcal{Y}\big|w(t-\sigma_1)\big|+\zeta_4\big|v(t-\sigma_2)\big|^T\mathcal{X}\big|v(t-\sigma_2)\big| \qquad (4.6)
\end{aligned}
$$

Combined with condition (i) and (ii) from Theorem 4.6 of (4.6) and Razumikhin theorem for fractional systems, see Ref [4] that

$$
\begin{aligned}
{}_0^CD_t^{\lambda}H(v(t),w(t)) \;\leq\; & -\Omega H_1(v(t),w(t))+\zeta_3H_1(v(t-\sigma_1),w(t-\sigma_1)) \\
& +\zeta_4H_1(v(t-\sigma_2),w(t-\sigma_2)) \\
\leq\; & -[\Omega-\zeta\zeta_3-\rho\zeta_3]H_1(v(t),w(t))
\end{aligned}
$$

$$= \quad -\gamma\, H_1(v(t), w(t)), \tag{4.7}$$

where $\gamma = \Omega - \zeta\zeta_3 - \rho\zeta_3 > 0$. Taking integer order integration of (4.7) on both sides, we have

$$\begin{aligned}
-\gamma \int_{t_0}^{t} H_1(v(\theta), w(\theta))d\theta \;&\geq\; \int_{t_0}^{t} {}_{0}^{C}D_{\theta}^{\lambda}\, H(v(\ell), w(\ell))d\ell\; d\theta \\
&= \frac{1}{\Gamma(1-\lambda)} \int_{t_0}^{t}\int_{t_0}^{\theta} (\theta-\ell)^{-\lambda} H'(v(\ell), w(\ell))d\ell\; d\theta \\
&= \frac{1}{\Gamma(1-\lambda)} \int_{t_0}^{t}\int_{\ell}^{t} (\theta-\ell)^{-\lambda} H'(v(\ell), w(\ell))d\theta\; d\ell \\
&= \frac{1}{\Gamma(2-\lambda)} \int_{t_0}^{t} (t-\ell)^{1-\lambda} H'(v(\ell), w(\ell))d\ell \\
&= \frac{1}{\Gamma(1-\lambda)} \int_{t_0}^{t} (t-\ell)^{-\lambda} H(v(\ell), w(\ell))d\ell \\
&\quad -\frac{1}{\Gamma(2-\lambda)}(t-t_0)^{1-\lambda} H(v(t_0), w(t_0)) \\
&\geq -\frac{1}{\Gamma(2-\lambda)}(t-t_0)^{1-\lambda} H(v(t_0), w(t_0))
\end{aligned}$$

which leads to

$$\int_{t_0}^{t} H_1(v(\theta), w(\theta))d\theta \;\leq\; \frac{H(v(t_0), w(t_0))}{\gamma\Gamma(2-\lambda)}(t-t_0)^{1-\lambda}. \tag{4.8}$$

Noting that

$$\delta^m\Big(\|v(t)\|^2 + \|w(t)\|^2\Big) \leq H_1(v(t), w(t)) \leq \delta^M\Big(\|v(t)\|^2 + \|w(t)\|^2\Big), \tag{4.9}$$

where $\delta^m = \min\Big\{\Phi_m(\mathcal{X}),\ \Phi_m(\mathcal{Y})\Big\}$ and $\delta^M = \max\Big\{\Phi_M(\mathcal{X}),\ \Phi_M(\mathcal{Y})\Big\}$. From (4.8) and (4.9), we sustain

$$\int_{t_0}^{t} \Big(\|v(\theta)\|^2 + \|w(\theta)\|^2\Big)d\theta \;\leq\; \frac{H(v(t_0), w(t_0))}{\delta^m\gamma\Gamma(2-\lambda)}(t-t_0)^{1-\lambda}$$

hence

$$\lim_{t\to+\infty} \frac{\int_{t_0}^{t} \Big(\|v(\theta)\|^2 + \|w(\theta)\|^2\Big)d\theta}{(t-t_0)^{1-\lambda}} \;\leq\; \frac{H(v(t_0), w(t_0))}{\delta^m\gamma\Gamma(2-\lambda)}.$$

If $\lim_{t\to+\infty} \int_{t_0}^{t} \Big(\|v(\theta)\|^2 + \|w(\theta)\|^2\Big)d\theta < +\infty$, then

$$\lim_{t\to+\infty} \Big(\|v(t)\|^2 + \|w(t)\|^2\Big) = 0,$$

where Lemma 2.8 has been used.

If $\lim_{t\to+\infty} \int_{t_0}^{t} \Big(\|v(\theta)\|^2 + \|w(\theta)\|^2\Big)d\theta = +\infty$, then by using well known L'Hospital rule, we get

$$\lim_{t\to+\infty} (t-t_0)^{\lambda}\Big(\|v(t)\|^2 + \|w(t)\|^2\Big) \;\leq\; \frac{H(v(t_0), w(t_0))}{\delta^m\gamma\Gamma(2-\lambda)}. \tag{4.10}$$

Based on Lemma 2.4, we take the fractional integral of both sides of (4.7) from t_0 to t, one can get

$$H(v(t), w(t)) \leq H(v(t_0), w(t_0)) - \frac{\Omega}{\Gamma(\lambda)} \int_{t_0}^{t} (t-\theta)^{\lambda-1} H_1(v(\theta), w(\theta)) d\theta. \tag{4.11}$$

Combined with (4.5) and (4.11), we establish

$$\delta^m \Big(\|v(t)\|^2 + \|w(t)\|^2 \Big) \leq H_1(v(t), w(t)) \leq H(v(t), w(t)) \leq H(v(t_0), w(t_0)),$$

which proves that $\Big(\|v(t)\|^2 + \|w(t)\|^2 \Big)$ must be bounded, then it follows from (4.10), there exist a $\mathcal{T} > 0$ such that

$$\Big(\|v(t)\|^2 + \|w(t)\|^2 \Big) \leq \frac{H(v(t_0), w(t_0))}{\delta^m \gamma \Gamma(2-\lambda)(t-t_0)^{\lambda}}, \ \forall\, t \geq \mathcal{T}, \tag{4.12}$$

which means that

$$\lim_{t \to +\infty} \Big(\|v(t)\|^2 + \|w(t)\|^2 \Big) = 0.$$

Therefore, master-slave FODGRNs systems (2.1) and (4.1) can realize global asymptotic synchronization under adaptive feedback control (4.3) and the proof of Theorem 4.6 is ended.

Corollary 4.7 Under Assumption 1, master-slave FODGRNs systems (2.1) and (4.1) with $\sigma_1 = \sigma_2 = 0$ realize global asymptotic synchronization under adaptive feedback control (4.3) if there exist positive diagonal matrices $\mathcal{X} > 0$, $\mathcal{Y} > 0$, $\mathcal{R}_1 > 0$ and $\mathcal{R}_2 > 0$ such that the following LMIs are holds:

(i). There exist two constants $\zeta_1 > 0$ and $\zeta_2 > 0$ such that

$$\begin{bmatrix} -2\mathcal{X}A - 2\mathcal{X}\tilde{K} + \mathcal{R}_2 D^2 + \zeta_1 \mathcal{X} & \mathcal{X}B \\ \maltese & -\mathcal{R}_1 \end{bmatrix} < 0,$$

$$\begin{bmatrix} -2\mathcal{Y}C - 2\mathcal{Y}\tilde{L} + \mathcal{R}_1 S^2 + \zeta_2 \mathcal{Y} & \mathcal{Y} \\ \maltese & -\mathcal{R}_2 \end{bmatrix} < 0,$$

where $\mathcal{X}$, $\mathcal{Y}$, $\tilde{K}$ and $\tilde{L}$ are already defined in Theorem 4.6.

Remark 4.8 For time delayed GRNs, there are many findings regarding stability and synchronization criteria, see Ref [10, 20, 26, 30, 33]. Yet these results are discussed mainly in the case of integer-order. Consequently, the stability and synchronization of FODGRNs via 2-norm method properties are not studied by anyone. Therefore our research work, in the sense of innovation, is completely distinct from previous ones.

Remark 4.9 When $\lambda = 1$, the FODGRNs model (2.1) reduces into global exponential stability and synchronization for integer-order time-delayed gene regulatory networks.

Remark 4.10 In Theorem 3.5, Theorem 4.4, and Theorem 4.6, the sufficient condition ensuring global stability in sense of Mittag-Leffler, global synchronization in sense of Mittag-Leffler, and adaptive synchronization for Caputo sense FODGRNs are derived in form of LMIs. These results can be easily checked with the MATLAB LMI Control Toolbox.

5. Numerical examples

This section, providing three numerical simulations to verify the superiority and benefits of the presented main results.

Example 5.1 Consider a class of three dimensional FODGRNs:

$$\begin{cases} {}_0^C D_t^{\lambda} g_p(t) = -a_p g_p(t) + \sum_{q=1}^{3} b_{pq} f_q(h_q(t-\sigma_1)) + F_p, \\ {}_0^C D_t^{\lambda} h_p(t) = -c_p h_p(t) + d_p g_p(t-\sigma_2), \ p = 1,2,3, \end{cases} \tag{5.1}$$

where $\lambda = 0.98$, $g(t) = (g_1(t), g_2(t), g_3(t))^T$, $h(t) = (h_1(t), h_2(t), h_3(t))^T$, $a_1 = a_2 = a_3 = 6$, $c_1 = c_2 = c_3 = 5$, $d_1 = d_2 = d_3 = 1.5$, $\sigma_1 = \sigma_2 = 0.6$, $f_q(h_q(t)) = \frac{h_q^2(t)}{1+h_q^2(t)}$, $F_1 = F_2 = F_3 = 1.5$, and

$$B = (b_{pq})_{3\times 3} = \begin{bmatrix} 1.4 & -0.8 & 1.6 \\ -0.7 & 0.65 & 1.2 \\ 2 & -1.5 & -0.5 \end{bmatrix}.$$

From Assumption 1, we have $\beta_1 = \beta_2 = \beta_3 = 0.3$. Choose $\alpha_1 = \alpha_1 = \alpha_3 = 1$. According to the conditions of Theorem 3.3, it is easy to obtain $\xi_1 = 37.63$, $\xi_2 = 38.35$ and $\xi_3 = 37.69$. Then, let us take $\mathcal{X} = diag\{2,2,2\}$ and $\mathcal{Y} = diag\{1.5,1.5,1.5\}$. Based on the conditions of Theorem 3.5, we have

$$\Phi_M(\mathcal{X})\bigg(-2\|A\| + \|B\|\|S\|\frac{\Phi_M(\mathcal{X}) + \Phi_m(\mathcal{Y})}{\Phi_m(\mathcal{Y})} + \frac{\Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})}\|D\|\bigg) = -17.3844 < 0$$

$$\Phi_M(\mathcal{Y})\bigg(-2\|C\| + \|D\|\frac{\Phi_m(\mathcal{X}) + \Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})} + \frac{\Phi_M(\mathcal{X})}{\Phi_m(\mathcal{Y})}\|B\|\|S\|\bigg) = -8.0625 < 0.$$

Thus, all conditions of Theorem 3.3 and Theorem 3.5 holds. Therefore, the existence of the equilibrium point of FODGRNs (5.1) is globally Mittag-Leffler stable. Furthermore, the simulation results for FODGRNs (5.1) narrates in Figures 1 and 2 under that initial values $\omega(t) = (-0.65, 0.3, 1.3)^T$, $\varpi(t) = (0.25, 0.8, 1.5)^T$. The state trajectories for concerned FODGRNs (5.1) are displayed in Figure 1. Figure 2 indicates the state norm trajectories for the concerned system (5.1).

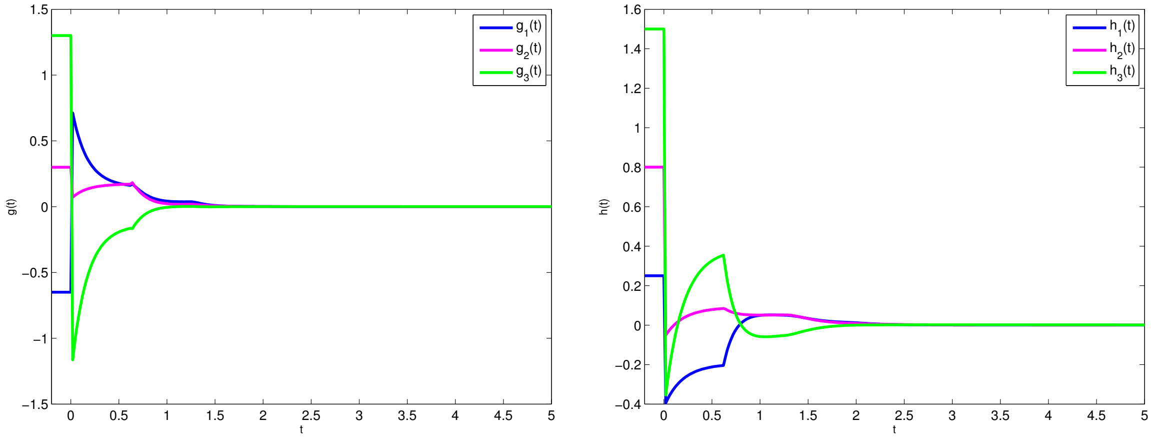

Figure 1. State trajectories $g(t)$ and $h(t)$ of FODGRNs (5.1) in Example 5.1.

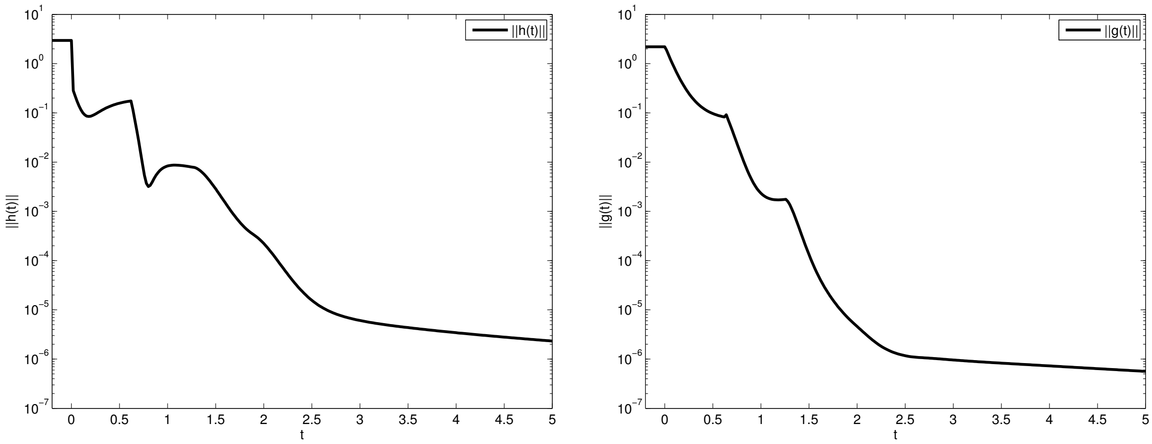

Figure 2. State norm trajectories $g(t)$ and $h(t)$ of FODGRNs (5.1) in Example 5.1.

where $\lambda = 0.99$, $g(t) = (g_1(t), g_2(t), g_3(t))^T$, $h(t) = (h_1(t), h_2(t), h_3(t))^T$, $a_1 = a_2 = a_3 = 1$, $c_1 = c_2 = c_3 = 2.5$, $d_1 = d_2 = d_3 = 3$, $\sigma_1 = \sigma_2 = 0.4$, $f_q(h_q(t)) = \frac{h_q^2(t)}{1+h_q^2(t)}$, $F_1 = F_2 = F_3 = 0$, and

$$B = (b_{pq})_{3\times 3}\begin{bmatrix} -2.5 & 3.2 & 1.5 \\ 1.3 & 1.6 & -2.5 \\ -2.4 & 1.6 & -1.8 \end{bmatrix}.$$

The three-dimensional slave system is given as:

$$\begin{cases} {}_0^C D_t^\lambda u_p(t) = -a_p u_p(t) + \sum_{q=1}^{3} b_{pq} f_q(z_q(t-\sigma_1)) + F_p + x_p(t), \\ {}_0^C D_t^\lambda z_p(t) = -c_p z_p(t) + d_p u_p(t-\sigma_2) + y_p(t), p = 1, 2, 3 \end{cases} \tag{5.2}$$

where $\lambda = 0.99$, $x_p(t)$ and $y_p(t)$ are linear feedback control, and others are same as FODGRNs (5.3). The initial conditions are chosen as: $\omega(t) = (1.7, -1.3, 2.5)^T$, $\varpi(t) = (-2, -1.4, -1.9)^T$, $\tilde{\omega}(t) = (-2, -1.6, -1.7)^T$ and $\tilde{\varpi}(t) = (1.4, 2.5, 2)^T$. When the controllers are not applied, the state trajectories of FODGRNs (5.3) and FODGRNs (5.2) are shown in Figure 3. In Figure 4 depicts the synchronization error trajectories of FODGRNs (5.3) and FODGRNs (5.2), without applying any control inputs. The chaotic behavior of FODGRNs (5.3) and FODGRNs (5.2), without using controller are shown in Figure 5.

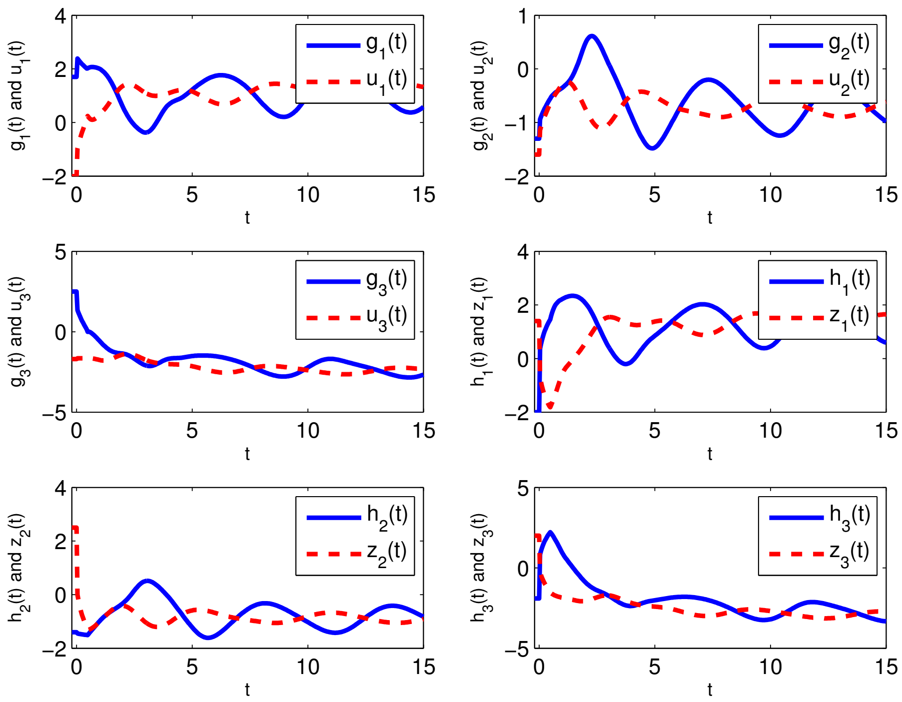

Figure 3. State trajectories of the FODGRNs (5.3) and (5.2) without control in Example 5.2.

Example 5.2 Consider a class of three-dimensional FODGRNs:

$$\begin{cases} {}_0^C D_t^\lambda g_p(t) = -a_p g_p(t) + \sum_{q=1}^{3} b_{pq} f_q(h_q(t-\sigma_1)) + F_p, \\ {}_0^C D_t^\lambda h_p(t) = -c_p h_p(t) + d_p g_p(t-\sigma_2), \;\; p = 1,2,3, \end{cases} \tag{5.3}$$

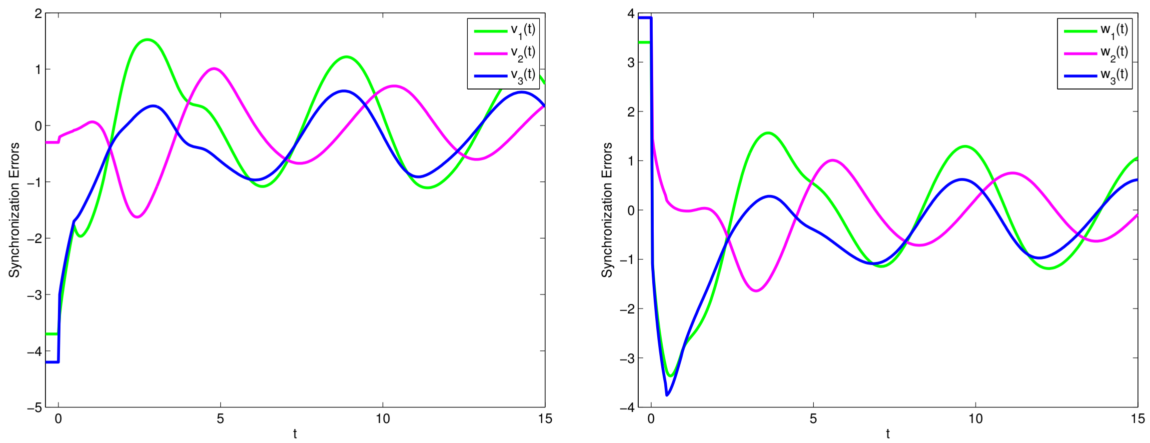

Figure 4. Time responses of the synchronization error without control inputs in Example 5.2.

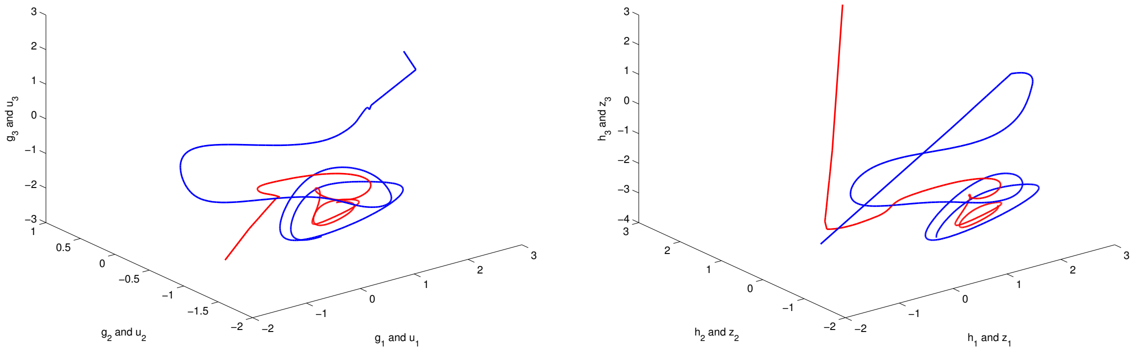

Figure 5. The phase trajectories of master system (5.3) and slave system (5.2) without control inputs in Example 5.2.

One simply to verify that Assumption 1 holds with $\beta_1 = \beta_2 = \beta_3 = 0.75$. By Theorem 4.4, we select $\mathcal{X} = diag\{1.75, 1.75, 1.75\}$ and $\mathcal{Y} = diag\{2, 2, 2\}$. If we choose the control gains of linear feedback controller (4.2) as: $k_1 = k_2 = k_3 = 14$ and $l_1 = l_2 = l_3 = 12$. Based on the above parameters, it is simply to get

$$\Phi_M(\mathcal{X})\Big(-2\|A\| - 2\|K\| + \|B\|\|S\|\frac{\Phi_M(\mathcal{X}) + \Phi_m(\mathcal{Y})}{\Phi_m(\mathcal{Y})} + \frac{\Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})}\|D\|\Big) = -34.37 < 0,$$

$$\Phi_M(\mathcal{Y})\Big(-2\|C\| - 2\|L\| + \|D\|\frac{\Phi_m(\mathcal{X}) + \Phi_M(\mathcal{Y})}{\Phi_m(\mathcal{X})} + \frac{\Phi_M(\mathcal{X})}{\Phi_m(\mathcal{Y})}\|B\|\|S\|\Big) = -40.75 < 0.$$

Hence, all the conditions of Theorem 4.4 hold. Therefore, the master system FODGRNs (5.3) synchronizes with the slave system FODGRNs (5.2) under linear feedback control.

When the linear feedback controllers are applied to the slave system, the state trajectories of FODGRNs (5.3) and FODGRNs (5.2) are exhibits in Figure 6. Figure 7 depicts the synchronization error trajectories of FODGRNs (5.3) and FODGRNs (5.2). The chaotic behavior of FODGRNs (5.3) and FODGRNs (5.2) with control inputs are shown in Figure 8. Synchronization error norm of FODGRNs (5.3) and FODGRNs (5.2) are displayed in Figure 9.

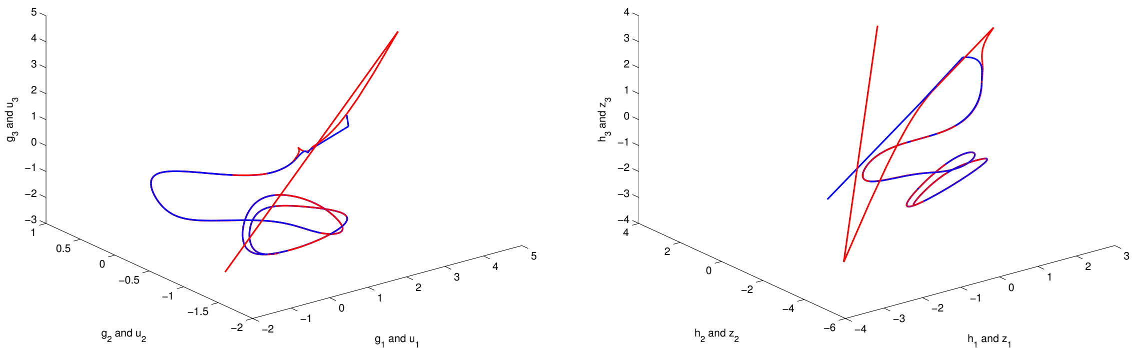

Figure 6. The phase trajectories of master system (5.3) and slave system (5.2) with control inputs in Example 5.2.

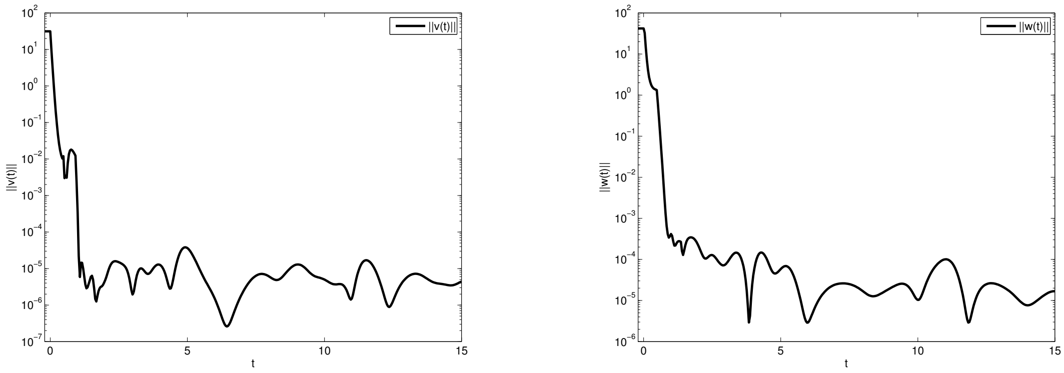

Figure 7. The synchronization error norms with control inputs in Example 5.2.

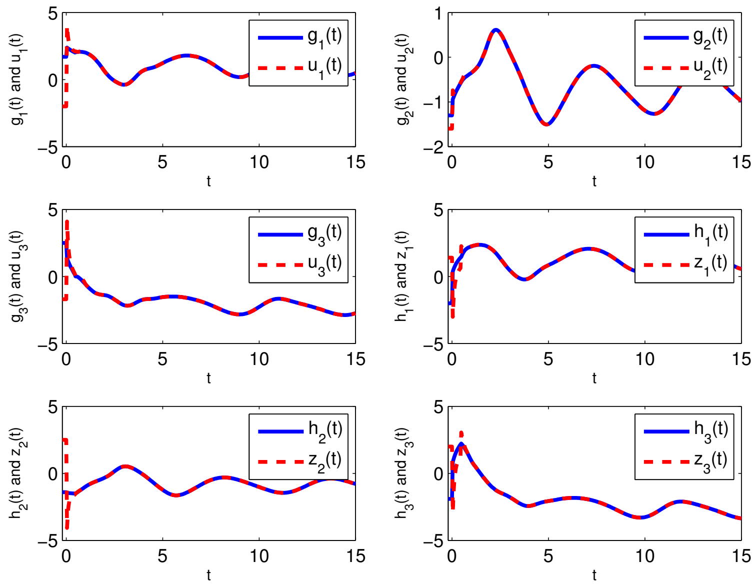

Figure 8. State trajectories of the FODGRNs (5.3) and (5.2) with control in Example 5.2.

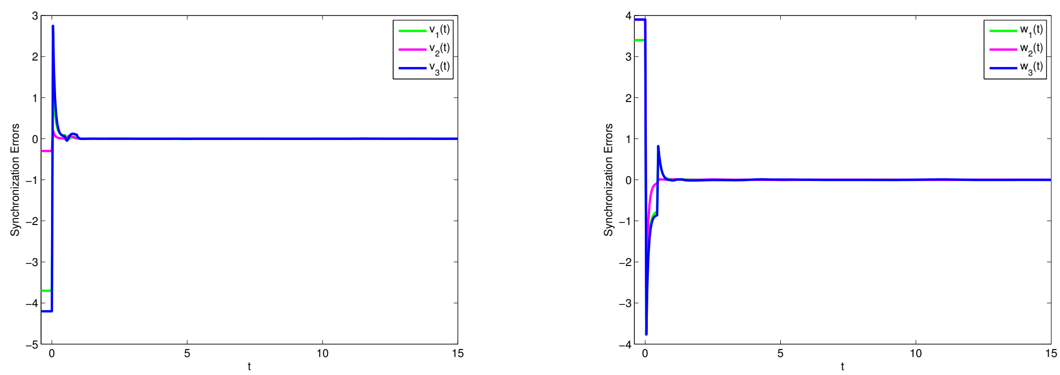

Figure 9. Time responses of the synchronization error with control inputs in Example 5.2.

Example 5.3 Consider a class of two dimensional FODGRNs:

$$\begin{cases} {}_0^C D_t^\lambda g_p(t) = -a_p g_p(t) + \sum_{q=1}^2 b_{pq} f_q(h_q(t-\sigma_1)) + F_p, \\ {}_0^C D_t^\lambda h_p(t) = -c_p h_p(t) + d_p g_p(t-\sigma_2), \ \ p = 1, 2, \end{cases} \tag{5.4}$$

where $\lambda = 0.98$, $g(t) = (g_1(t), g_2(t))^T$, $h(t) = (h_1(t), h_2(t))^T$, $a_1 = a_2 = 2$, $c_1 = c_2 = 2.5$, $d_1 = d_2 = 2.5$, $\sigma_1 = \sigma_2 = 0.9$, $f_q(h_q(t)) = \frac{h_q^2(t)}{1+h_q^2(t)}$, $F_1 = F_2 = 2.4$, and

$$B = (b_{pq})_{2\times 2} = \begin{bmatrix} -3.5 & 2.2 \\ -2.3 & -3.6 \end{bmatrix}.$$

The two-dimensional slave system is given as:

$$\begin{cases} {}_0^C D_t^\lambda u_p(t) = -a_p u_p(t) + \sum_{q=1}^2 b_{pq} f_q(z_q(t-\sigma_1)) + F_p + x_p(t), \\ {}_0^C D_t^\lambda z_p(t) = -c_p z_p(t) + d_p u_p(t-\sigma_2) + y_p(t), p = 1, 2, \end{cases} \tag{5.5}$$

where $\lambda = 0.98$, $x_p(t)$ and $y_p(t)$ are adaptive feedback control, the initial values are selected as: $\omega(t) = (1.8, -1.22)^T$, $\varpi(t) = (-1.14, 1.6)^T$, $\tilde{\omega}(t) = (-1.1, 1.6)^T$ and $\tilde{\varpi}(t) = (1.2, -2)^T$, and others are same as FODGRNs (5.4).

When the control inputs are not applied in FODGRNs (5.5), the state trajectories of (5.3) and (5.2) are shown in Figure 10. In Figure 11 depicts the time evolution of synchronization error, without using control inputs. The phase plot of FODGRNs (5.4) and FODGRNs (5.5), without using the controller is depicted in Figure 12. In Figures 10–12 demonstrate that FODGRNs (5.4) and FODGRNs (5.5) did not achieve synchronization if the control inputs are not applied.

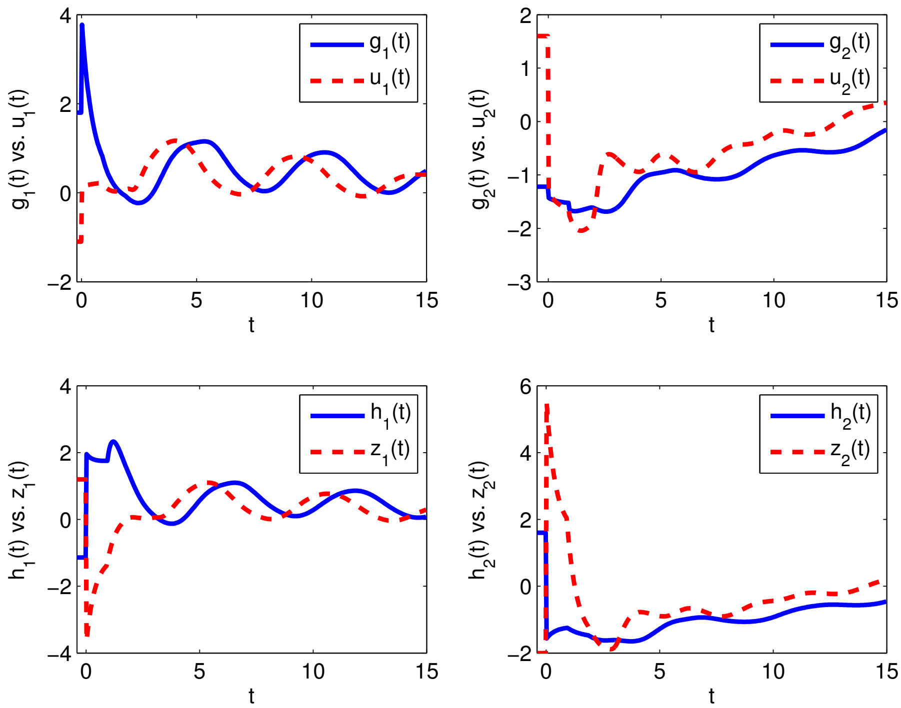

Figure 10. State trajectories of the FODGRNs (5.4) and (5.5) without control in Example 5.3.

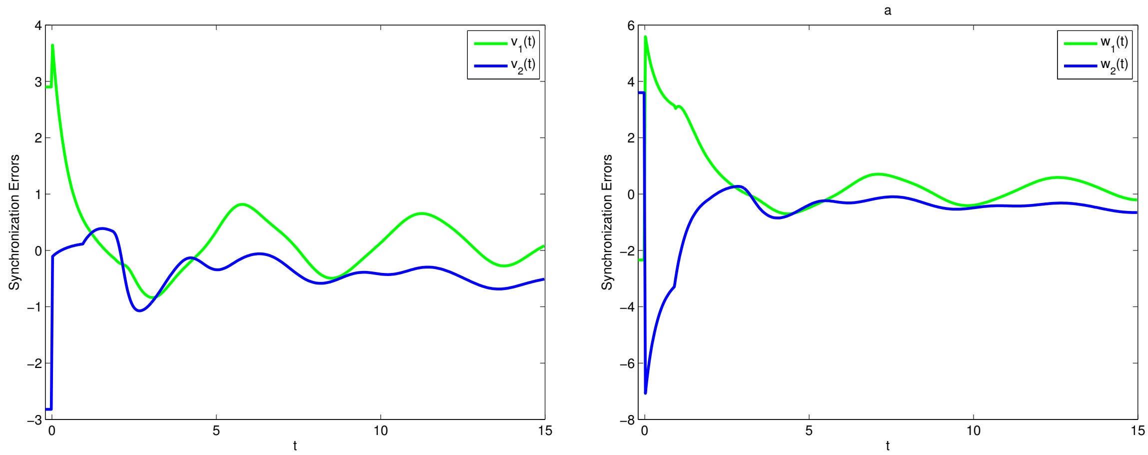

Figure 11. Time responses of the synchronization error without control inputs in Example 5.3.

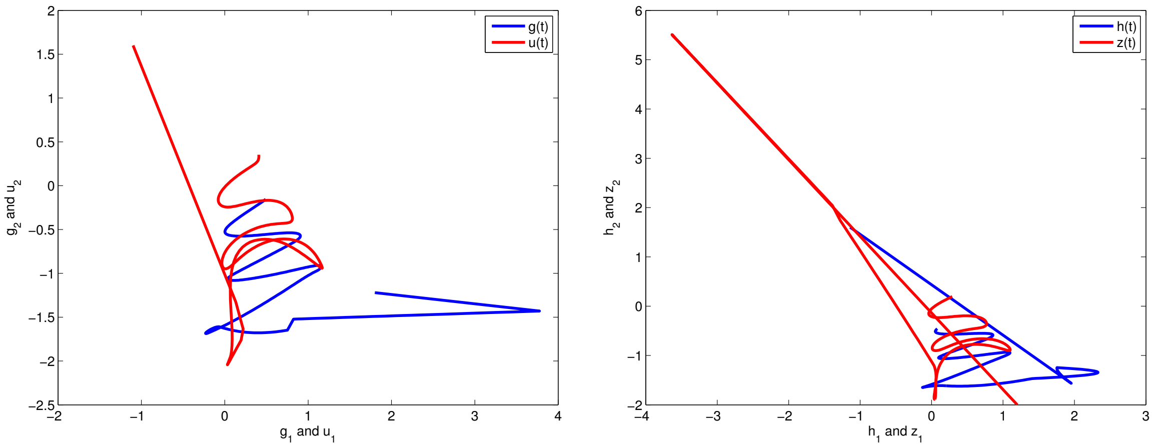

Figure 12. The change processes of $z_{l1}(t)$, $z_{l2}(t)$ and $\|m_l(t)\|_2$, $l = 1, 2..., 6$.

Through simple computation, we have $\beta_1 = \beta_2 = 0.1$. If the adaptive feedback controller $x_p(t)$ and $y_p(t)$ in (4.3) is designed as, select $\tilde{k}_1 = \tilde{k}_2 = 0.8$, $\tilde{k}_1 = \tilde{k}_2 = 0.5$, $\xi_1(0) = \xi_2(0) = 0.05$ and $\eta_1(0) = \eta_2(0) = 0.08$. Let $\zeta_1 = 3.5$, $\zeta_2 = 2.8$, $\zeta_3 = 0.5$, $\zeta_4 = 0.4$, $k_1^\star = k_2^\star = l_1^\star = l_2^\star = 0.9$. Then it is clear to find that the given LMIs (i) and (ii) of Theorem 4.6 is feasible, and these solutions are given as below:

$$\mathcal{X} = \begin{bmatrix} 0.769 & 0 \\ 0 & 0.796 \end{bmatrix}, \mathcal{Y} = \begin{bmatrix} 0.1875 & 0 \\ 0 & 0.1937 \end{bmatrix},$$
$$\mathcal{R}_1 = \begin{bmatrix} 4.5662 & 0 \\ 0 & 4.7104 \end{bmatrix}, \mathcal{R}_2 = \begin{bmatrix} 0.0232 & 0 \\ 0 & 0.0241 \end{bmatrix}.$$

Therefore, all conditions of Theorem 4.6 are holds, which indicates that master-slave FODGRNs systems (5.4) and (5.5) realize global asymptotic synchronization under adaptive feedback control (4.3). The time responses of states (5.3) and (5.2) with control inputs are displayed in Figure 13. The time evolution of synchronization errors with control inputs is illustrated in Figure14. The phase plot of FODGRNs (5.4) and FODGRNs (5.5) with control inputs are shown in Figure 15. Synchronization error norm of (5.3) and (5.2) with control inputs are presented in Figure 16. The adaptive control strengths are demonstrated in Figure 17, which it tends to some positive constants.

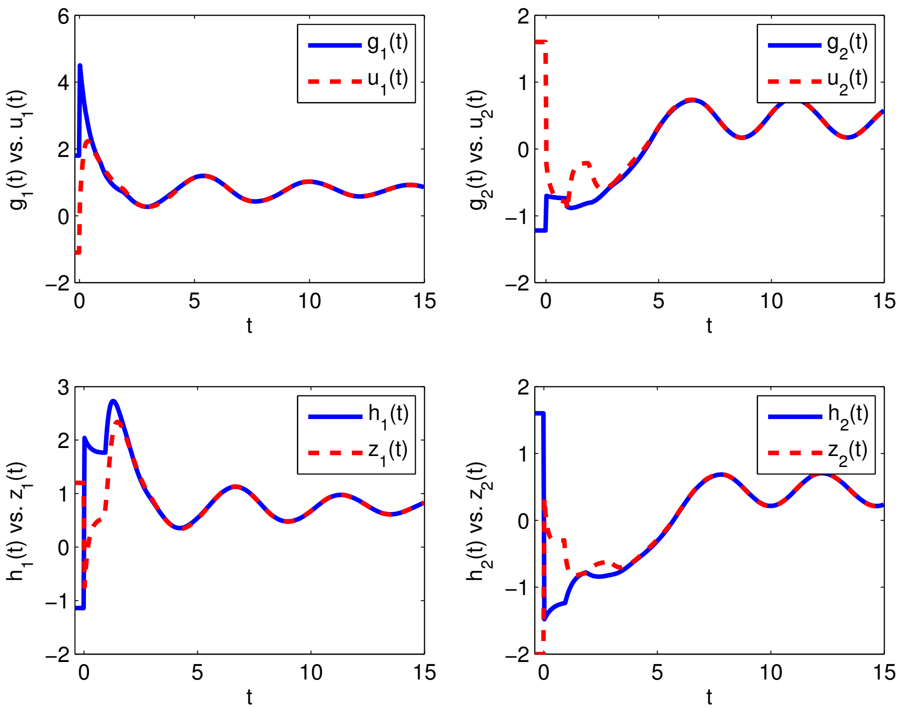

Figure 13. State trajectories of the FODGRNs (5.4) and (5.5) with control.

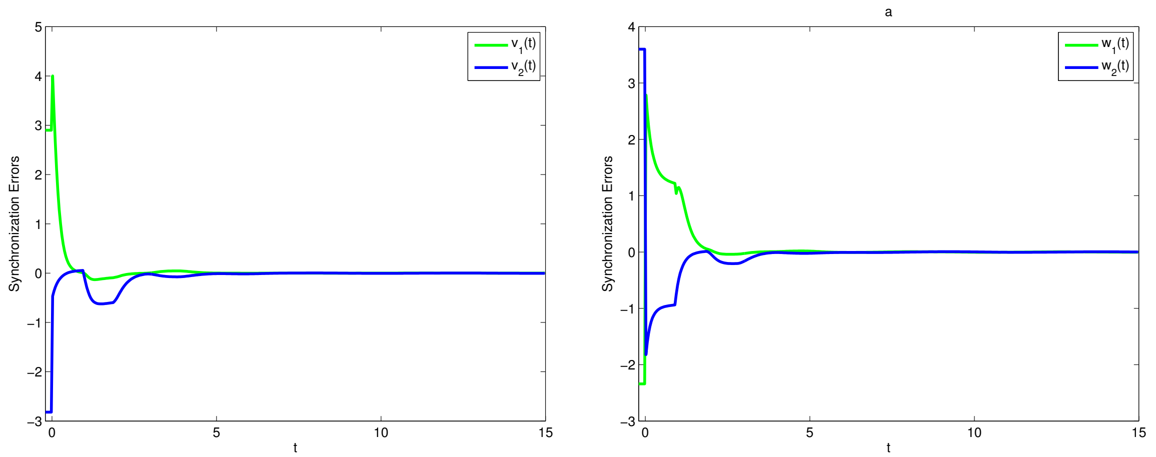

Figure 14. Time responses of the synchronization error with control inputs in Example 5.3.

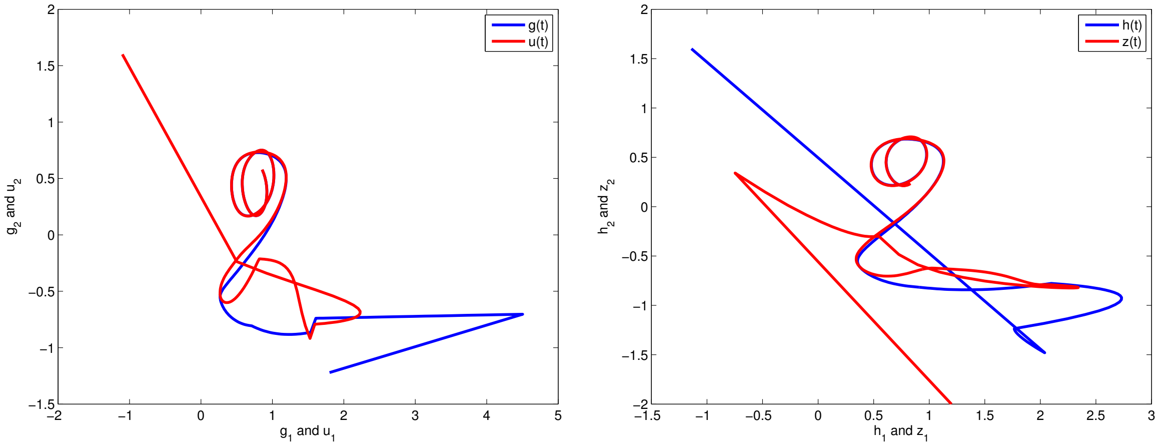

Figure 15. The change processes of $z_{l1}(t),\ z_{l2}(t)$ and $\|m_l(t)\|_2,\ l = 1, 2..., 6.$

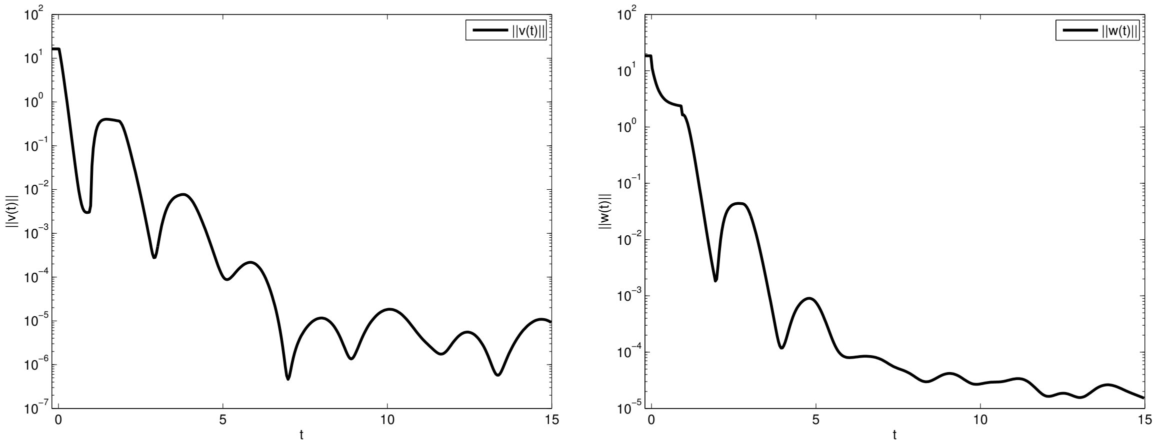

Figure 16. The synchronization error norms with control inputs.

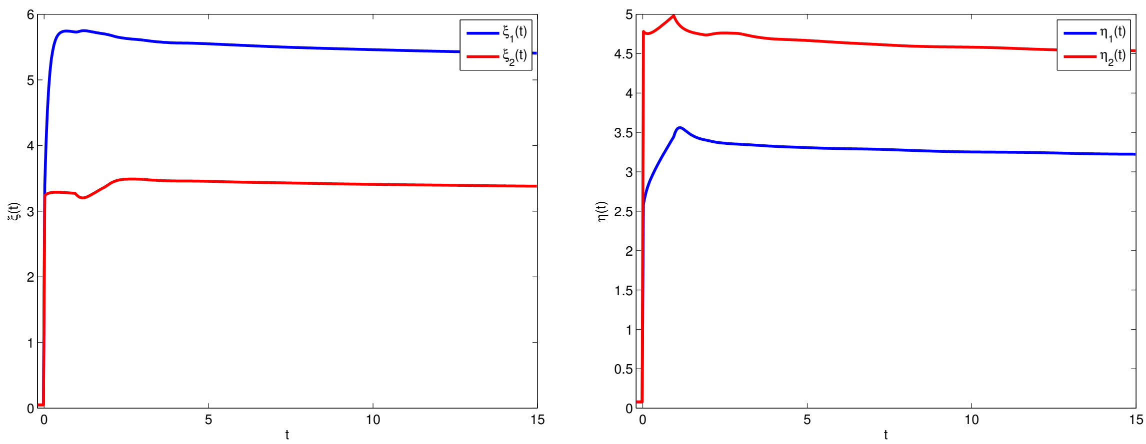

Figure 17. The change processes of $z_{l1}(t),\ z_{l2}(t)$ and $\|m_l(t)\|_2,\ l = 1, 2..., 6.$

Remark 5.4 As we can see that the linear feedback controller (4.2) is simpler than the adaptive feedback controller (4.3), but the control strengths of the adaptive feedback controller (4.3) is smaller than those of linear feedback controller (4.2). Adaptive synchronization is superior to the synchronization in general.

6. Conclusions

In this manuscript, the stability and synchronization for fractional-order gene regulatory networks wit time-delay effects has been investigated in brief. Under some inequality techniques, Razumikhin approach and fractional order Lyapunov method, the globally Mittag-Leffler stability of proposed FODGRNs is proved. Moreover, the suitable controllers were designed to ensure the several synchronization for addressing FODGRNs in terms of LMIs. Further, three numerical simulations are provided. Our future research work will be generalized to state estimator design for fuzzy non-integer order gene regulatory networks with time delays and impulsive effects.

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Conflict of interest

The authors declare no conflict of interest.

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