*mathematics*

Article

# A Result on a Pata-Ćirić Type Contraction at a Point

**Erdal Karapınar [1,2,*,†], Andreea Fulga [3,†], Vladimir Rakočević [4,†]**

1 Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
2 Department of Mathematics, Çankaya University, 06790 Etimesgut, Ankara, Turkey
3 Department of Mathematics and Computer Science, Transilvania University of Brasov, 500036 Brasov, Romania; afulga@unitbv.ro
4 Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia; vrakoc@sbb.rs
* Correspondence: karapinar@mail.cmuh.org.tw or erdalkarapinar@yahoo.com
† These authors contributed equally to this work.

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**Abstract:** In this manuscript, we define a new contraction mapping, Pata-Ćirić type contraction at a point, that merges distinct contractions defined by Seghal, Pata and Ćirić. We proved that in a complete space, each Pata-Ćirić type contraction at a point possesses a fixed point. We express an example to illustrate the observed result.

**Keywords:** pata type contraction; iterate at a point; fixed point; metric space

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## 1. Introduction and Preliminaries

In recent decades, the fixed point theory has not only been one of the most interesting research topics in nonlinear functional analysis, but also one of the most dynamic and productive research areas. Although the first published work on this subject was conducted a century ago, the fixed point theory continues to be the center of attention. The reason why the fixed point theory is still the attraction center can be explained as follows: different problems in various disciplines can be turned into fixed point problems. Mostly, solving the fixed problem is easier than the original representation. Note that in the metric fixed point theory, not only the existence (mostly, also uniqueness) of the fixed point (which corresponds to the solution) is guaranteed, but also, it is shown "how to reach" the mentioned fixed point and hence the solution.

Before giving the main results of the manuscript, we shall introduce and fix the notations that we deal with it. Throughout this article, we use the letter $\mathcal{S}$ to denote a non-empty set. We presume that $d$ forms a metric over $\mathcal{S}$. Thereafter, the pair $(\mathcal{S}, d)$ denotes metric space. Unless otherwise stated in the particular conditions, the pair $(\mathcal{S}^*, d)$ expresses that the corresponding metric space is complete. The expressions $\mathbb{R}, \mathbb{R}^+, \mathbb{R}_0^+$ stand for the set of all "real numbers", "positive real numbers", and "non-negative real numbers". In addition, the letters $\mathbb{N}$ and $\mathbb{N}_0$ are restricted to indicate the set of all positive integers and all non-negative integers, respectively.

In the following theorem, Bryant [1] proved the analog of Banach's fixed point theorem [2] for not the mapping itself but for its iterated form of the given mapping.

**Theorem 1** ([1]). *A self-mapping $\mathcal{T}$ on $(\mathcal{S}^*, d)$ admits a unique fixed point $u \in \mathcal{S}$, if there exist $\kappa \in [0, 1)$ and $n \in \mathbb{N}$ so that*

$$d(\mathcal{T}^n z, \mathcal{T}^n w) \leq \kappa d(z, w), \tag{1}$$

*for all $w \in \mathcal{S}$.*

Inspired from the idea of Bryant [1], we give the following example to illustrate the necessity of the result of Bryant [1]

**Example 1.** *Let $\mathcal{T} : [0,1] \to [0,1]$ be defined by*

$$\mathcal{T}(u) = \begin{cases} 0 & \text{if } u \in [0, \frac{1}{2}], \\ \frac{1}{2} & \text{if } u \in (\frac{1}{2}, 1]. \end{cases}$$

*The inequality (1) holds for $n = 2$. Despite T not being continuous, $T0 = 0$.*

**Theorem 2** ([3]). *A continuous self-mapping $\mathcal{T}$ on $(\mathcal{S}^*, d)$ admits a unique fixed point $u \in \mathcal{S}$, if it satisfies the condition: there exists a constant $k > 1$ such that, for each $z \in \mathcal{S}$, there is a positive integer $n(z)$ such that*

$$d(\mathcal{T}^{n(z)} z, \mathcal{T}^{n(z)} w) \leq k d(z, w), \tag{2}$$

*for all $w \in \mathcal{S}$.*

Immediately after the publication of Seghal's article, Guseman [4] showed that the theorem of continuity was unnecessary, see also [5,6].

Throughout this work, we note as $\Upsilon$ the set of all auxiliary functions $\psi : [0,1] \to [0,+\infty)$ which are increasing, continuous at zero, and $\psi(0) = 0$. Furthermore, for an arbitrary point $z_0$ in a $(\mathcal{S}, d)$, we consider the function

$$\|z\| = d(z, z_0), \text{ for all } z \in \mathcal{S},$$

that will be called "the zero of $\mathcal{S}$".

**Theorem 3** ([7]). *A self-mapping $\mathcal{T}$, defined on $(\mathcal{S}, d)$, admits a fixed point $u \in \mathcal{S}$, if for every $\varepsilon \in [0,1]$ and all $z, w \in \mathcal{S}$ the inequality below holds:*

$$d(\mathcal{T}z, \mathcal{T}w) \quad \leq (1-\varepsilon) d(z, w) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) \left[1 + \|z\| + \|w\|\right]^{\beta}, \tag{3}$$

*where $\psi \in \Upsilon$ and $\Lambda \geq 0$, $\lambda \geq 1$ and $\beta \in [0, \lambda]$ are fixed constants.*

**Definition 1** ([8]). *A self-mapping $\mathcal{T}$, defined on $(\mathcal{S}, d)$, is called Pata type contraction at a point if for every $\varepsilon \in [0,1]$ and for any $z \in \mathcal{S}$, there exists a positive integer $n(z)$ such that*

$$d(\mathcal{T}^{n(z)} z, \mathcal{T}^{n(z)} w) \leq (1-\varepsilon) d(z, w) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) \left[1 + \|z\| + \|w\|\right]^{\beta}, \tag{4}$$

*for all $w \in \mathcal{S}$, where $\psi \in \Upsilon$ and $\Lambda \geq 0$, $\lambda \geq 1$, $\beta \in [0, \lambda]$ are fixed constants.*

**Theorem 4** ([8]). *Suppose that a self-mapping $\mathcal{T}$ on $(\mathcal{S}, d)$ is a Pata type contraction at a point. Then, $\mathcal{T}$ admits a unique fixed point.*

## 2. Main Results

**Definition 2.** *A self-mapping $\mathcal{T}$, defined on $(\mathcal{S}, d)$, is called Pata-Ćirić type contraction at a point if for every $\varepsilon \in [0,1]$ and for each $z \in \mathcal{S}$ there exists a positive integer $n(z)$ such that the following inequality is fulfilled:*

$$d(\mathcal{T}^{n(z)} z, \mathcal{T}^{n(z)} w) \leq \mathcal{P}(z, w) \tag{5}$$

*for all $w \in \mathcal{S}$, where $\psi \in \Upsilon$ and $\Lambda \geq 0$, $\lambda \geq 1$ and $\beta \in [0, \lambda]$ are fixed constants, with*

$$\begin{aligned} \mathcal{P}(z,w) &= (1-\varepsilon)\max\left\{d(z,w), d(z,\mathcal{T}^{n(z)}z), d(z,\mathcal{T}^{n(z)}w), \tfrac{d(w,\mathcal{T}^{n(z)}z)}{2}, \tfrac{d(w,\mathcal{T}^{n(z)}w)}{2}\right\} \\ &\quad +\Lambda\varepsilon^{\lambda}\psi(\varepsilon)\left[1+\|z\|+\|w\|+\left\|\mathcal{T}^{n(z)}z\right\|+\left\|\mathcal{T}^{n(z)}w\right\|\right]^{\beta}. \end{aligned}$$

**Theorem 5.** *Every Pata-Ćirić type contraction at a point over $(\mathcal{S}^*, d)$ admits a unique fixed point $u \in \mathcal{S}$. Moreover, $\mathcal{T}^m z \to u$ as $m \to +\infty$ for each $z \in \mathcal{S}$. In addition, there is $n(u) \in \mathbb{N}$ so that $\mathcal{T}^{n(u)}$ is continuous at $u$.*

**Proof.** Take an arbitrary point $z_0 \in \mathcal{S}$. We presume that $\mathcal{T}z_0 \neq z_0$, because, on the contrary, $z_0$ forms a fixed point for $\mathcal{T}$ and it terminates the proof.

Starting with such a point $z_0 \in \mathcal{S}$, we construct a sequence $\{z_k\}$ like this:

$$z_{k+1} = \mathcal{T}^{n_k} z_k \text{ for all } k \in \mathbb{N}, \tag{6}$$

where, $n_k = n(z_k)$. Iteratively, we find

$$z_1 = \mathcal{T}^{n_0} z_0, z_2 = \mathcal{T}^{n_1} z_1 = \mathcal{T}^{n_1+n_0} z_0,$$

and thereby we have

$$\begin{aligned} z_k &= \mathcal{T}^{n_{k-1}+\ldots+n_1+n_0} z_0, \\ z_{k+l} &= \mathcal{T}^{n_k+n_{k-1}+\ldots+n_{n+k-1}} z_k, \end{aligned} \tag{7}$$

for any $l \in \mathbb{N}$. By (5), for $\varepsilon = 0$ we have

$$\begin{aligned} d(z_k, z_{k+1}) &= d(z_k, \mathcal{T}^{n_k} z_k) = d(\mathcal{T}^{n_{k-1}} z_{k-1}, \mathcal{T}^{n_{k-1}}(\mathcal{T}^{n_k} z_{k-1})) \leq \mathcal{P}(z_{k-1}, \mathcal{T}^{n_k} z_{k-1}) \\ &\leq \max\left\{\begin{array}{c} d(z_{k-1}, \mathcal{T}^{n_k} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}}(\mathcal{T}^{n_k} z_{k-1})), \\ \frac{d(\mathcal{T}^{n_k} z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}}{2}, \frac{d(\mathcal{T}^{n_k} z_{k-1}, \mathcal{T}^{n_{k-1}}(\mathcal{T}^{n_k} z_{k-1}))}{2} \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d(z_{k-1}, \mathcal{T}^{n_k} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}+n_k} z_{k-1}), \\ \frac{d(\mathcal{T}^{n_k} z_{k-1}, z_{k-1}) + d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1})}{2}, \\ \frac{d(\mathcal{T}^{n_k} z_{k-1}, z_{k-1}) + d(z_{k-1}, \mathcal{T}^{n_{k-1}+n_k} z_{k-1})}{2} \end{array}\right\} \\ &\leq \max\left\{d(z_{k-1}, \mathcal{T}^{n_k} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}+n_k} z_{k-1})\right\}. \end{aligned} \tag{8}$$

Let $q_1 \in \{n_{k-1}, n_k, n_{k-1}+n_k\}$ such that

$$d(z_{k-1}, \mathcal{T}^{q_1} z_{k-1}) = \max\left\{d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_k} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}+n_k} z_{k-1})\right\}.$$

Then

$$d(z_k, z_{k+1}) = d(z_k, \mathcal{T}^{n_k} z_k) \leq d(z_{k-1}, \mathcal{T}^{q_1} z_{k-1})$$

and further

$$\begin{aligned}
d(z_{k-1}, \mathcal{T}^{q_1} z_{k-1}) &= d(\mathcal{T}^{n_{k-2}} z_{k-2}, \mathcal{T}^{q_1}(\mathcal{T}^{n_{k-2}} z_{k-2}) = d(\mathcal{T}^{n_{k-2}} z_{k-2}, \mathcal{T}^{n_{k-2}}(\mathcal{T}^{q_1} z_{k-2})) \\
&\leq \max \left\{ \begin{array}{c} d(z_{k-2}, \mathcal{T}^{q_1} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}}(\mathcal{T}^{q_1} z_{k-2})) \\ \frac{d(\mathcal{T}^{q_1} z_{k-2}, \mathcal{T}^{n_{k-2}} z_{k-2})}{2}, \frac{d(\mathcal{T}^{q_1} z_{k-2}, \mathcal{T}^{n_{k-2}}(\mathcal{T}^{q_1} z_{k-2}))}{2} \end{array} \right\} \\
&\leq \max \left\{ \begin{array}{c} d(z_{k-2}, \mathcal{T}^{q_1} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}+q_1} z_{k-2}) \\ \frac{d(\mathcal{T}^{q_1} z_{k-2}, z_{k-2}) + d(z_{k-2}, \mathcal{T}^{n_{k-2}} z_{k-2})}{2}, \\ \frac{d(\mathcal{T}^{q_1} z_{k-2}, z_{k-2}) + d(z_{k-2}, \mathcal{T}^{n_{k-2}+q_1} z_{k-2})}{2} \end{array} \right\} \\
&\leq \max \{ z_{k-2}, \mathcal{T}^{q_1} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}+q_1} z_{k-2}) \} \\
&= d(z_{k-2}, \mathcal{T}^{q_2} z_{k-2}),
\end{aligned}$$

where $q_2 \in \{q_1, n_{k-2}, q_1 + n_{k-2}\}$ is chosen so that

$$d(z_{k-2}, \mathcal{T}^{q_2} z_{k-2}) = \max \left\{ d(z_{k-2}, \mathcal{T}^{q_1} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}} z_{k-2}), d(z_{k-2}, \mathcal{T}^{n_{k-2}+q_1} z_{k-2}) \right\}.$$

Continuing in this way, we deduce that

$$d(z_k, z_{k+1}) = d(z_k, \mathcal{T}^{n_k} z_k)) \leq d(z_{k-1}, \mathcal{T}^{q_1} z_{k-1}) \leq ... \leq d(z_0, \mathcal{T}^{q_k} z_0) \tag{9}$$

for $q_1, q_2, ..., q_k \in \mathbb{N}$.

Let $q$ be an arbitrary but fixed positive integer and $k$ a positive integer, depending on $z_0$ and $q$. (We can suppose that $q > n_0$ and also $k > n_0$). We denote this as

$$\rho_0 = d(z_0, \mathcal{T}^k z_0) = \max \{ d(z_0, \mathcal{T}^s z_0) : 0 < s \leq q \}$$

Let $m \in \mathbb{N}$ such that $m = in_0 + j$ with $i \geq 1$, $0 \leq j \leq n_0 - 1$ and $\Lambda \geq 0$, $\lambda \geq 1$, $\beta \in [0, \lambda]$ be fixed constants. Using the triangle inequality, by (5) we have

$$\begin{aligned}
d(z_0, \mathcal{T}^m z_0) &\leq d(z_0, \mathcal{T}^{n_0} z_0) + d(\mathcal{T}^{n_0} z_0, \mathcal{T}^{m+n_0} z_0) + d(\mathcal{T}^{m+n_0} z_0, \mathcal{T}^m z_0) \\
&\leq d(z_0, \mathcal{T}^{n_0} z_0) + d(\mathcal{T}^{n_0} z_0, \mathcal{T}^{n_0}(\mathcal{T}^m z_0)) + d(\mathcal{T}^m(\mathcal{T}^{n_0} z_0), \mathcal{T}^m z_0) \\
&\leq d(z_0, \mathcal{T}^{n_0} z_0) + \mathcal{P}(z_0, \mathcal{T}^m z_0) + d(z_0, \mathcal{T}^k z_0) \\
&\leq 2\rho_0 + \mathcal{P}(z_0, \mathcal{T}^m z_0),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\mathcal{P}(z_0, \mathcal{T}^m z_0) &= \mathcal{P}(z_0, \mathcal{T}^{in_0+j} z_0) \\
&= (1-\varepsilon)\max\left\{ \begin{array}{c} d(z_0, \mathcal{T}^{in_0+j} z_0), d(z_0, \mathcal{T}^{n_0} z_0), d(z_0, \mathcal{T}^{n_0}(\mathcal{T}^{in_0+j} z_0), \\ \frac{d(\mathcal{T}^{in_0+j} z_0, \mathcal{T}^{n_0} z_0)}{2}, \frac{d(\mathcal{T}^{in_0+j} z_0, \mathcal{T}^{n_0}(\mathcal{T}^{in_0+j} z_0))}{2} \end{array} \right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|z_0\| + \left\|\mathcal{T}^{in_0+j} z_0\right\| + \|\mathcal{T}^{n_0} z_0\| + \left\|\mathcal{T}^{(i+1)n_0+j} z_0\right\|]^{\beta} \\
&= (1-\varepsilon)\max\left\{ \begin{array}{c} d(z_0, \mathcal{T}^{in_0+j} z_0), d(z_0, \mathcal{T}^{n_0} z_0), \\ d(z_0, \mathcal{T}^{in_0+j} z_0) + d(\mathcal{T}^{in_0+j} z_0, \mathcal{T}^{n_0}(\mathcal{T}^{in_0+j} z_0), \\ \frac{d(\mathcal{T}^{in_0+j} z_0, z_0) + d(z_0, \mathcal{T}^{n_0} z_0)}{2}, \\ \frac{d(\mathcal{T}^{in_0+j} z_0, \mathcal{T}^{(i+1)n_0+j} z_0)}{2} \end{array} \right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \left\|\mathcal{T}^{in_0+j} z_0\right\| + \|\mathcal{T}^{n_0} z_0\| + \left\|\mathcal{T}^{(i+1)n_0+j} z_0\right\|]^{\beta} \\
&\leq (1-\varepsilon)\max\left\{ \begin{array}{c} d(z_0, \mathcal{T}^{in_0+j} z_0), d(z_0, \mathcal{T}^{n_0} z_0), \\ d(z_0, \mathcal{T}^{in_0+j} z_0) + d(\mathcal{T}^{in_0+j} z_0, \mathcal{T}^{(i+1)n_0+j} z_0), \\ \frac{d(\mathcal{T}^{in_0+j} z_0, z_0) + d(z_0, \mathcal{T}^{n_0} z_0)}{2}, \frac{d(z_0, \mathcal{T}^{k} z_0))}{2} \end{array} \right\} .+ \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \left\|\mathcal{T}^{in_0+j} z_0\right\| + \|\mathcal{T}^{n_0} z_0\| + \left\|\mathcal{T}^{(i+1)n_0+j} z_0\right\|]^{\beta}
\end{aligned}$$

Let $\varrho_i = d(z_0, \mathcal{T}^{in_0+j} z_0)$. Thus, coming back in (10) and taking into account (9) we have

$$\begin{aligned}
\varrho_i &\leq 2\rho_0 + (1-\varepsilon)\max\left\{\varrho_i, \rho_0, \varrho_i + \rho_0, \tfrac{\rho_0+\varrho_i}{2}, \tfrac{\rho_0}{2}\right\} + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \varrho_i + \rho_0 + \varrho_{i+1}]^{\beta} \\
&\leq 2\rho_0 + (1-\varepsilon)\max\{\varrho_i, \rho_0, \varrho_i + \rho_0\} + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\varrho_i + 2\rho_0]^{\beta} \qquad (11) \\
&= 2\rho_0 + (1-\varepsilon)(\varrho_i + \rho_0) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\varrho_i + 2\rho_0]^{\beta}.
\end{aligned}$$

Our next purpose is to prove that the sequence $\{\varrho_i\}$ is bounded. Taking into account that $\beta \in [0, \lambda])$, the above inequality becomes

$$\begin{aligned}
\varrho_i &\leq 2\rho_0 + (1-\varepsilon)(\varrho_i + \rho_0) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\varrho_i + 2\rho_0]^{\beta} \\
&\leq 2\rho_0 + (1-\varepsilon)(\varrho_i + \rho_0) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\varrho_i]^{\lambda}[1 + 2\rho_0]^{\lambda} \\
&\leq 2\rho_0 + (1-\varepsilon)(\varrho_i + \rho_0) + 2^{\lambda}\varrho_i^{\lambda}\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \tfrac{1}{2\varrho_i}]^{\lambda}[1 + 2\rho_0]^{\lambda}
\end{aligned}$$

and therefore,

$$\varepsilon\varrho_i \leq (3-\varepsilon)\rho_0 + 2^{\lambda}\varrho_i^{\lambda}\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \frac{1}{2\varrho_i}]^{\lambda}[1 + 2\rho_0]^{\lambda}.$$

We assert that the sequence $\{\varrho_i\}$ is bounded. To use the method of *Reductio ad Absurdum*, we suppose that the sequence $\{\varrho_i\}$ is unbounded. Attendantly, we can find a sub-sequence $\{\varrho_{i_n}\}$ such that $\lim_{n\to\infty} \varrho_{i_n} = \infty$. For this reason, there is number $N_1 \in \mathbb{N}$ such that $\varrho_{i_n} > 1 + 3\rho_0$ for each $n \geq N_1$. Thus,

$$\varepsilon\varrho_{i_n} \leq 3\rho_0 + 2^{2\lambda}\varrho_{i_n}^{\lambda}\Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\rho_0]^{\lambda}$$

and letting $\varepsilon = \varepsilon_n = \frac{1+3\rho_0}{\varrho_{i_n}} < 1$ we have

$$\begin{aligned} 1 &\leq 2^{2\lambda} \varrho_{i_n}^{\lambda} \Lambda \left(\frac{1+3\rho_0}{\varrho_{i_n}}\right)^{\lambda} \psi(\varepsilon_n)[1+2\rho_0]^{\lambda} \\ &= 2^{2\lambda} \Lambda [1+2\rho_0]^{\lambda} [1+3\rho_0]^{\lambda} \psi(\varepsilon_n). \end{aligned}$$

Since $\varepsilon_n \to 0$ as $n \to 0$, taking into account the properties of the function $\psi$, we obtain

$$1 \leq 2^{2\lambda} \Lambda [1+3\rho_0]^{2\lambda} \psi(\varepsilon_n) \to 0,$$

which is a contradiction. Therefore, our assumption is false, so that the set $\{d(z_0, \mathcal{T}^{in_0+j} z_0) : i \in \mathbb{N}\}$ is bounded and varying $j \in \{0, 1, 2, ...n_0 - 1\}$ we get that the set $\{d(z_0, \mathcal{T}^m z_0) : m \in \{0, 1, 2, ...\}\}$ is bounded. Thus, taking into account (7), there exists a positive number $\mathcal{K}$ such that

$$c_m = d(z_0, z_m) = \|z_m\| \leq \mathcal{K},$$

for $m \geq m_0$. In order to prove that the sequence $\{z_n\}$ is Cauchy, let $k, l \in \mathbb{N}$. Denoting by $p_0 = n_{k+l-1} + n_{k+l-2} + ... + n_k$, we have

$$\begin{aligned} d(z_k, z_{k+l}) &= d(z_k, \mathcal{T}^{p_0} z_k) = d\left(\mathcal{T}^{n_{k-1}} z_{k-1}, \mathcal{T}^{n_{k-1}}(\mathcal{T}^{p_0} z_{k-1})\right) \leq \mathcal{P}(z_{k-1}, \mathcal{T}^{p_0} z_{k-1}) \\ &\leq (1-\varepsilon) \max \left\{ \begin{array}{c} d(z_{k-1}, \mathcal{T}^{p_0} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{p_0+n_{k-1}} z_{k-1}), \\ \frac{d(\mathcal{T}^{p_0} z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1})}{2}, \frac{d(\mathcal{T}^{p_0} z_{k-1}, \mathcal{T}^{p_0+n_{k-1}} z_{k-1})}{2} \end{array} \right\} + \\ &\quad + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \|z_{k-1}\| + \|\mathcal{T}^{p_0} z_{k-1}\| + \|\mathcal{T}^{n_{k-1}} z_{k-1}\| + \|\mathcal{T}^{p_0+n_{k-1}} z_{k-1}\|]^{\beta} \\ &\leq (1-\varepsilon) \max \left\{ \begin{array}{c} d(z_{k-1}, \mathcal{T}^{p_0} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{p_0+n_{k-1}} z_{k-1}), \\ \frac{d(\mathcal{T}^{p_0} z_{k-1}, z_{k-1}) + d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1})}{2}, \\ \frac{d(\mathcal{T}^{p_0} z_{k-1}, \mathcal{T}^{p_0+n_{k-1}} z_{k-1}) + d(\mathcal{T}^{p_0} z_{k-1}, \mathcal{T}^{p_0+n_{k-1}} z_{k-1})}{2} \end{array} \right\} + \\ &\quad + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + \|z_{k-1}\| + \|\mathcal{T}^{p_0} z_{k-1}\| + \|\mathcal{T}^{n_{k-1}} z_{k-1}\| + \|\mathcal{T}^{p_0+n_{k-1}} z_{k-1}\|]^{\beta} \\ &\leq (1-\varepsilon) d(z_{k-1}, \mathcal{T}^{p_1} z_{k-1}) + \Lambda \varepsilon^{\lambda} \psi(\varepsilon) [1 + 4\mathcal{K}]^{\beta}, \end{aligned}$$

where $p_1$ in $\{p_0, n_{k-1}, p_0 + n_{k-1}\}$ is chosen such that

$$d(z_{k-1}, \mathcal{T}^{p_1} z_{k-1}) = \max \left\{ d(z_{k-1}, \mathcal{T}^{p_0} z_{k-1}), d(z_{k-1}, \mathcal{T}^{n_{k-1}} z_{k-1}), d(z_{k-1}, \mathcal{T}^{p_0+n_{k-1}} z_{k-1}) \right\}.$$

Let $\varepsilon = 1 - \left(\frac{k-1}{k}\right)^{\lambda} < \frac{\lambda}{k}$. We have

$$\begin{aligned} k^{\lambda} d(z_k, z_{k+l}) &\leq k^{\lambda} \left(\frac{k-1}{k}\right)^{\lambda} d(z_{k-1}, \mathcal{T}^{p_1} z_{k-1}) + k^{\lambda} \Lambda \left(\frac{\lambda}{k}\right)^{\lambda} \psi(\tfrac{\lambda}{k}) [1 + 4\mathcal{K}]^{\beta} \\ &= (k-1)^{\lambda} d(z_{k-1}, \mathcal{T}^{p_1} z_{k-1}) + \mathcal{C} \lambda^{\lambda} \psi(\tfrac{\lambda}{k}), \end{aligned}$$

where $\mathcal{C} = \Lambda [1 + 4\mathcal{K}]^{\beta}$. Continuing in this way, we found that

$$\begin{aligned} k^{\lambda} d(z_k, z_{k+l}) &\leq (k-1)^{\lambda} d(z_{k-2}, \mathcal{T}^{p_2} z_{k-2}) + \mathcal{C} \lambda^{\lambda} \psi(\tfrac{\lambda}{k-1}) + \mathcal{C} \lambda^{\lambda} \psi(\tfrac{\lambda}{k}) \\ &\quad \ldots \\ &\leq \mathcal{C} \lambda^{\lambda} \sum_{j=1}^{k} \psi\left(\frac{\lambda}{j}\right) \end{aligned}$$

and then

$$k^{\lambda} d(z_k, z_{k+l}) \leq C\lambda^{\lambda} \sum_{j=1}^{k} \psi\left(\frac{\lambda}{j}\right).$$

Dividing by $k^{\lambda}$, the previous inequality gives

$$d(z_k, z_{k+l}) \leq C\left(\frac{\lambda}{k}\right)^{\lambda} \sum_{j=1}^{k} \psi\left(\frac{\lambda}{j}\right) \to 0.$$

Thereof, the sequence $\{z_n\}$ is Cauchy on $(\mathcal{S}^*, d)$. On account of completeness, there is $u \in \mathcal{S}$ so that $\lim_{n\to\infty} z_n = u$.

The first goal is to indicate that $u = \mathcal{T}^{n(u)}u$. Otherwise, it turns to $\mathcal{T}^{n(u)}u \neq u$. Accordingly, we have

$$\begin{aligned}
0 &< d(\mathcal{T}^{n(u)}u, u) \leq d(\mathcal{T}^{n(u)}u, \mathcal{T}^{n(u)}(\mathcal{T}^{q}z_0)) + d(\mathcal{T}^{n(u)}(\mathcal{T}^{q}z_0), u) \\
&\leq \mathcal{P}(u, \mathcal{T}^{q}z_0) + d(\mathcal{T}^{n(u)+q}z_0), u) \\
&\leq (1-\varepsilon)\max\left\{\begin{array}{c} d(u, \mathcal{T}^{q}z_0), d(u, \mathcal{T}^{n(u)}u), d(u, \mathcal{T}^{n(u)+q}z_0), \\ \frac{d(\mathcal{T}^{q}z_0, \mathcal{T}^{n(u)}u)}{2}, \frac{d(\mathcal{T}^{q}z_0, \mathcal{T}^{n(u)+q}z_0)}{2} \end{array}\right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + \|\mathcal{T}^{q}z_0\| + \left\|\mathcal{T}^{n(u)}u\right\| + \left\|\mathcal{T}^{n(u)+q}z_0\right\|]^{\beta} + d(\mathcal{T}^{n(u)+q}z_0, u) \\
&\leq (1-\varepsilon)\max\left\{\begin{array}{c} d(u, \mathcal{T}^{q}z_0), d(u, \mathcal{T}^{n(u)}u), d(u, \mathcal{T}^{n(u)+q}z_0), \\ \frac{d(\mathcal{T}^{q}z_0, u) + ad(u, \mathcal{T}^{n(u)}u)}{2}, \frac{d(\mathcal{T}^{q}z_0, u) + d(u, \mathcal{T}^{n(u)+q}z_0)}{2} \end{array}\right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + \|\mathcal{T}^{q}z_0\| + \left\|\mathcal{T}^{n(u)}u\right\| + \left\|\mathcal{T}^{n(u)+q}z_0\right\|]^{\beta} + d(\mathcal{T}^{n(u)+q}z_0, u)
\end{aligned}$$

By the triangle inequality, we have $\left\|\mathcal{T}^{n(u)}u\right\| \leq d(\mathcal{T}^{n(u)}u, u) + d(u, z_0) = d(\mathcal{T}^{n(u)}u, u) + \|u\|$, and letting $q \to +\infty$ in the above inequality we have

$$d(\mathcal{T}^{n(u)}u, u) \leq (1-\varepsilon)d(u, \mathcal{T}^{n(u)}u) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\|u\| + 3\mathcal{K}]^{\beta}, \tag{12}$$

but this implies that

$$d(\mathcal{T}^{n(u)}u, u) \leq \Lambda\varepsilon^{\lambda-1}\psi(\varepsilon)[1 + 2\|u\| + 3\mathcal{K}]^{\beta}.$$

Taking $\varepsilon \to 0$, since $\lambda \geq 1$ and using the property of function $\psi$ we have that

$$0 < d(\mathcal{T}^{n(u)}u, u) \leq 0,$$

which is a contradiction. Therefore, $d(u, \mathcal{T}^{n(u)}u) = 0$, that is $u$ is a fixed point of $\mathcal{T}^{n(u)}$.

Suppose that there exists another point $v \in \mathcal{S}$ such that $\mathcal{T}^{n(u)}v = v$ and $v \neq u$, by (5) we have

$$\begin{aligned}
0 &< d(u,v) = d(\mathcal{T}^{n(u)}u, \mathcal{T}^{n(u)}v) \leq \mathcal{P}(u,v) \\
&\leq (1-\varepsilon)\max\left\{ d(u,v), d(u,\mathcal{T}^{n(u)}u), d(u,\mathcal{T}^{n(u)}v), \tfrac{d(v,\mathcal{T}^{n(u)}u)}{2}, \tfrac{d(v,\mathcal{T}^{n(u)}v)}{2} \right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + \|v\| + \left\|\mathcal{T}^{n(u)}u\right\| + \left\|\mathcal{T}^{n(u)}v\right\|]^{\beta} \\
&= (1-\varepsilon)\max\left\{ d(u,v), d(u,u), d(u,v), \tfrac{d(v,u)}{2}, \tfrac{d(v,v)}{2} \right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\|u\| + 2\|v\|]^{\beta} \\
&= (1-\varepsilon)d(u,v) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 2\|u\| + 2\|v\|]^{\beta},
\end{aligned}$$

or,

$$d(u,v) \leq \Lambda\varepsilon^{\lambda-1}\psi(\varepsilon)[1 + 2\|u\| + 2\|v\|]^{\beta}.$$

Letting $\varepsilon \to 0$ and keeping in mind the property of $\psi$, we have $d(u,v) = 0$, that is, the fixed point of $\mathcal{T}^{n(u)}$ is unique.

Finally, because

$$\mathcal{T}^{n(u)}(\mathcal{T}u) = \mathcal{T}(\mathcal{T}^{n(u)}u) = \mathcal{T}u$$

taking into account this uniqueness, we have $\mathcal{T}u = u$.

We are now concerned with the second part of the theorem. Let $u$ be a fix point of $\mathcal{T}$ in $\mathcal{S}$. For $j \in \{0, 1, ..., n(u) - 1\}$ we define

$$a_i = d(u, \mathcal{T}^{in(u)+j}z)$$

for $i \in \mathbb{N}_0$ and reasoning by contradiction, we will proof that $a_i \leq a_{i-1}$ for $i \geq 1$. Presuming that there is $i_0 \in \mathbb{N}$ such that $a_{i_0} > a_{i_0-1} \geq 1$, since $\mathcal{T}^{n(u)}u = u$, we have,

$$\begin{aligned}
a_{i_0} &= d(u, \mathcal{T}^{i_0 n(u)+j}z) = d(\mathcal{T}^{n(u)}u, \mathcal{T}^{n(u)}(\mathcal{T}^{(i_0-1)n(u)+j}z)) \leq \mathcal{P}(u, \mathcal{T}^{(i_0-1)n(u)+j}z) \\
&\leq (1-\varepsilon)\max\left\{ \begin{array}{c} d(u,\mathcal{T}^{(i_0-1)n(u)+j}z), d(u,\mathcal{T}^{n(u)}), d(u,\mathcal{T}_z^{i_0 n(u)+j}), \\ \frac{d(\mathcal{T}^{(i_0-1)n(u)+j}z, \mathcal{T}^{n(u)}u)}{2}, \frac{d(\mathcal{T}^{(i_0-1)n(u)+j}z, \mathcal{T}^{i_0 n(u)+j}z)}{2} \end{array} \right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + \left\|\mathcal{T}^{(i_0-1)n(u)+j}z\right\| + \left\|\mathcal{T}^{nu)}u\right\| + \left\|\mathcal{T}_z^{i_0 n(u)+j}\right\|]^{\beta}.
\end{aligned}$$

Since, by the triangle inequality,

$$\begin{aligned}
d(\mathcal{T}^{(i_0-1)n(u)+j}z, \mathcal{T}_z^{i_0 n(u)+j}) &\leq d(\mathcal{T}^{(i_0-1)n(u)+j}z, u) + d(u, \mathcal{T}_z^{i_0 n(u)+j}) \\
&= a_{i_0-1} + a_{i_0}
\end{aligned}$$

and

$$\begin{aligned}
\left\|\mathcal{T}^{(i_0-1)n(u)+j}z\right\| &= d(\mathcal{T}^{(i_0-1)n(u)+j}z, z_0) \leq d(\mathcal{T}^{(i_0-1)n(u)+j}z, u) + d(u, z_0) \\
&= a_{i_0-1} + \|u\|,
\end{aligned}$$

we get

$$\begin{aligned}
a_{i_0} &\leq (1-\varepsilon)\max\left\{ a_{i_0-1}, 0, a_{i_0}, \tfrac{a_{i_0-1}}{2}, \tfrac{a_{i_0-1}+a_{i_0}}{2} \right\} + \\
&\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + a_{i_0-1} + \|u\| + \|u\| + a_{i_0} + \|u\|]^{\beta} \\
&\leq (1-\varepsilon)a_{i_0} + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\|u\| + a_{i_0-1} + a_{i_0}]^{\beta}
\end{aligned}$$

and then

$$a_{i_0} \leq \Lambda \varepsilon^{\lambda-1} \psi(\varepsilon)[1 + 4\|u\| + a_{i_0-1} + a_{i_0}]^{\beta}.$$

Keeping, the continuity of the function $\psi$ in mind, together with the property $\psi(0) = 0$, we find that $a_{i_0} = 0$, letting $\varepsilon \to 0$ in the above inequality. It contradicts our assumption $a_{i_0} > a_{i_0-1} \geq 0$. We have proved that the sequence $\{a_i\}$ is non-increasing, so that there exists $a \geq 0$ such that $\lim_{i\to\infty} a_i = a$. For all $i \in \mathbb{N}$ we have

$$\begin{aligned} a_i &= d(u, \mathcal{T}^{in(u)+j}z) = d(\mathcal{T}^{n(u)}u, \mathcal{T}^{n(u)}(\mathcal{T}^{(i-1)n(u)+j}z)) \leq \mathcal{P}(u, \mathcal{T}^{(i-1)n(u)+j}z) \\ &\leq (1-\varepsilon)\max\left\{ \begin{array}{c} d(u, \mathcal{T}^{(i-1)n(u)+j}z), d(u, \mathcal{T}^{n(u)}), d(u, \mathcal{T}_z^{in(u)+j}), \\ \frac{d(\mathcal{T}^{(i-1)n(u)+j}z, \mathcal{T}^{n(u)}u)}{2}, \frac{d(\mathcal{T}^{(i-1)n(u)+j}z, \mathcal{T}^{in(u)+j}z)}{2} \end{array} \right\} + \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + \left\|\mathcal{T}^{(i-1)n(u)+j}z\right\| + \left\|\mathcal{T}^{n(a(u))}u\right\| + \left\|\mathcal{T}^{in(u)+j}z\right\|]^{\beta} \\ &\leq (1-\varepsilon)\max\left\{a_{i-1}, 0, a_i, \tfrac{a_{i-1}}{2}, \tfrac{a_{i-1}+a_i}{2}\right\} + \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + a_{i_0-1} + \|u\| + \|u\| + a_{i_0} + \|u\|]^{\beta}. \end{aligned}$$

Thus,

$$a_i \leq (1-\varepsilon)a_{i-1} + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\|u\| + a_{i-1} + a_i]^{\beta}$$

or, letting $i \to +\infty$

$$a \leq (1-\varepsilon)a + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\|u\| + 2a]^{\beta}.$$

By repeating the above arguments, when $\varepsilon \to 0$ we get $a = 0$. Since $z, i, j$ were fixed, we find that $\lim_{m\to\infty} d(\mathcal{T}^m z, u) = 0$.

For the last part, let $\{w_m\} \subseteq \mathcal{S}$ be an arbitrary sequence, with $\lim_{m\to\infty} d(w_m, u) = 0$. We shall indicate that $\lim_{m\to\infty} d(\mathcal{T}^{n(u)}w_m, u) = 0$ by the method of *Reductio ad Absurdum*. Suppose, on the contrary, that there is $\delta > 0$ so that $\lim_{m\to\infty} d(u, \mathcal{T}^{n(u)}w_m) = \delta$. By (5), we have

$$\begin{aligned} d(u, \mathcal{T}^{n(u)}w_m) &= d(\mathcal{T}^{n(u)}u, \mathcal{T}^{n(u)}w_m) \leq \mathcal{P}(u, w_n) \\ &\leq (1-\varepsilon)\max\left\{ \begin{array}{c} d(u, w_m), d(u, \mathcal{T}^{n(u)}u), d(u, \mathcal{T}^{n(u)}w_m), \\ \frac{d(w_m, \mathcal{T}^{n(u)}u)}{2}, \frac{d(w_m, \mathcal{T}^{n(u)}w_m)}{2} \end{array} \right\} + \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|u\| + \|w_n\| + \left\|\mathcal{T}^{n(u)}u\right\| + \left\|\mathcal{T}^{n(u)}w_m\right\|]^{\beta} \\ &\leq (1-\varepsilon)\max\left\{ \begin{array}{c} d(u, w_m), 0, d(u, \mathcal{T}^{n(u)}w_m), \\ \frac{d(w_m, u)}{2}, \frac{d(w_m, u) + d(u, \mathcal{T}^{n(u)}w_m)}{2} \end{array} \right\} + \\ &\quad + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\|u\| + d(w_m, u) + d(u, \mathcal{T}^{n(u)}w_m)]^{\beta}. \end{aligned}$$

Therefore, as $m \to +\infty$ we have

$$\delta \leq (1-\varepsilon)\delta + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + 4\|u\| + \delta]^{\beta}.$$

Furthermore,

$$\delta \leq \Lambda\varepsilon^{\lambda-1}\psi(\varepsilon)[1 + 4\|u\| + \delta]^{\beta}$$

and letting $\varepsilon \to 0$, we get $\delta \leq 0$, which is a contradiction. This means that $\mathcal{T}^{n(u)}w_n \to u$ and since $\mathcal{T}^{n(u)}u = u$, we conclude that $\mathcal{T}^{n(u)}$ is continuous at $u$. □

**Example 2.** *On the set $\mathcal{S} = \{a, b, c, e, g\}$ we define the distance $d$ as follows:*

| $d(z,w)$ | $a$ | $b$ | $c$ | $e$ | $g$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1 | 2 | 3 | 1 |
| $b$ | 1 | 0 | 1 | 2 | 2 |
| $c$ | 2 | 1 | 0 | 1 | 3 |
| $e$ | 3 | 2 | 1 | 0 | 4 |
| $g$ | 1 | 2 | 3 | 4 | 0 |

*Let the mapping $\mathcal{T} : \mathcal{S} \to \mathcal{S}$, where*

| $z$ | $a$ | $b$ | $c$ | $e$ | $g$ |
|---|---|---|---|---|---|
| $\mathcal{T}z$ | $c$ | $e$ | $b$ | $a$ | $g$ |

*It easy to see that the Theorem 3 it cannot be applied, because for $z = b$, $w = c$ and $\varepsilon = 0$ we have*

$$d(\mathcal{T}b, \mathcal{T}c) = d(e, b) = 2 > 1 = d(b, c).$$

*On the other hand, since*

| $z$ | $a$ | $b$ | $c$ | $e$ | $g$ |
|---|---|---|---|---|---|
| $\mathcal{T}^3 z$ | $e$ | $c$ | $a$ | $b$ | $g$ |

*choosing $\lambda = \beta = 1$, $\psi(\varepsilon) = \varepsilon^2$ and $\Lambda = 4 + \frac{1}{1+\varepsilon^3}$ we have:*

$$d(z, w) \leq \mathcal{P}(z, w),$$

*for every $z, w \in \mathcal{S}$, so that the mapping $\mathcal{T}$ has a unique fixed point.*

## 3. Application

We will consider in this section the problem of the existence and uniqueness of solutions for a second-order nonlinear integrodifferential equation, as an application of our main results.

Let $\mathcal{B}$ be a Banach space endowed with the norm $\|\cdot\|$. We denote by $E$ the Banach space of all continuous functions from $[0,1]$ in $\mathcal{B}$, with the norm

$$\|z\|_E = \sup\{\|z(t)\| : t \in [0,1]\}.$$

and by $\mathcal{S} = C([0,1], B_\rho)$ the complete metric space with

$$d(z, w) = \|z - w\|_{\mathcal{S}} = \sup_{t\in[0,1]} \{\|z(t) - w(t)\| : z, w \in \mathcal{S}\}.$$

(Here, $B_\rho = \{x : \|x\| \leq \rho\}$.)

Moreover, we recall here that the set $\{\mathcal{C}(t) : t \in [0,1]\}$ of bounded linear operators in $\mathcal{B}$ is said to be strongly continuous cosine family if and only if:

($c_1$) $\mathcal{C}(0) = I$;
($c_2$) $\mathcal{C}(t)z$ is strongly continuous in $t \in \mathbb{R}$, for any fixed $z \in \mathcal{B}$;
($c_3$) $\mathcal{C}(s+t) + \mathcal{C}(s-t) = 2\mathcal{C}(s)\mathcal{C}(t)$, for $s, t \in \mathbb{R}$.

Let the $\left\{\tilde{S}(t) : t \in \mathbb{R}\right\}$ be the associated sine family of that cosine family, defined as follows

$$\tilde{S}(t)z = \int_0^t \mathcal{C}(s)z ds, \quad z \in \mathcal{B}, \quad t \in \mathbb{R}$$

and two positive constants $\Theta_1 \geq 1, \Theta_2$ such that

$$\|\mathcal{C}(t)\| \leq \Theta_1, \qquad \|\tilde{S}(t)\| \leq \Theta_2,$$

for all $t \in [0, 1]$. Let $f : [0, 1] \times \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, $\mathcal{k} : [0, 1] \times [0, 1] \times \mathcal{B} \to \mathcal{B}$ be continuous function and the nonlinear integro-differential equation

$$z''(t) = Az(t) + f(t, z(t), \int_0^t \mathcal{k}(t, \tau, z(\tau)d\tau, \text{ with } t \in [0, 1], \tag{13}$$

with the initial value

$$z(t_0) = z_0, \qquad z'(t_0) = z_1, \tag{14}$$

where $z_0, z_1 \in \mathcal{B}$ are given. Here $A : \mathcal{B} \to \mathcal{B}$, defined by

$$Az = \frac{d^2}{dt^2}\mathcal{C}(t)z|_{t=0}, \text{ for } z \in \mathcal{B} \text{ such that } \mathcal{C}(\cdot)z \in C^2(\mathbb{R}, \mathcal{B})$$

is the infinitesimal generator of a cosine family $\{\mathcal{C}(t) : t \in \mathbb{R}\}$.

The function $z \in \mathcal{S}$, defined for $t \in [0, 1]$ by

$$z(t) \quad = \mathcal{C}(t)z_0 + \tilde{S}(t)z_1 + \textstyle\int_0^t \tilde{S}(t-s)f(s, z(s), \int_0^s \mathcal{k}(s, \theta, z(\theta))d\theta)ds, \tag{15}$$

is a mild solution of the Equation (13) with the initial values (14).

**Theorem 6.** *The Equation (13) admits an unique mild solution $z \in \mathcal{S}$ on $[0, 1]$ provided that the following hypothesis is satisfied:*

- *there exist a continuous function $\phi : [0, 1] \to \mathbb{R}$ such that $|\phi(t)| < t$ for every $t \in [0, 1]$ and a nonnegative constant $\Upsilon$with $6/7 < \Theta_2\Upsilon \leq 6/5$, such that*

  $$\|f(t, z_1, w_1) - f(t, z_2, w_2)\| \leq \Upsilon[\|z_1 - z_2\| + \|w_1 - w_2\|]$$

  *and*

  $$\|\mathcal{k}(t, s, z_1) - \mathcal{k}(t, s, z_2)\| \leq |\phi(t)| \, \|z_1 - z_2\| \, ,$$

  *for all $t, s \in [0, 1]$ and $z, w \in \mathcal{S}$.*

**Proof.** Considering the mapping $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ be defined by

$$\mathcal{T}z(t) = \mathcal{C}(t)z_0 + \tilde{S}(t)z_1 + \int_0^t \tilde{S}(t-s)f(s, z(s), \int_0^s \mathcal{k}(s, \theta, z(\theta))d\theta)ds,$$

the Equation (13) becomes $\mathcal{T}z = z$, that is, the fixed point of $\mathcal{T}$ is the mild solution for (13)–(14). In this case, we have

$$\begin{aligned}
\|(\mathcal{T}z)t - (\mathcal{T}w)t\| &\leq \int_0^t \|\tilde{S}(t-s)\| \, \|f(s, z(s), \int_0^s k(s,\theta,z(\theta))d\theta) - \\
&\quad -f(s, w(s), \int_0^s k(s,\theta,w(\theta))d\theta) \, \| ds \\
&\leq \Theta_2 \int_0^t \Upsilon \left[ \|z-w\| + \int_0^s |\phi(\theta)| \, \|z-w\| \, d\theta \right] ds \\
&\leq \Theta_2 \int_0^t \Upsilon \left[ \sup_{t\in[0,1]} \|z(t)-w(t)\| + \int_0^s \theta \sup_{t\in[0,1]} \|z(t)-w(t)\| \, d\theta \right] ds,
\end{aligned}$$

so that,

$$\begin{aligned}
d(\mathcal{T}z, \mathcal{T}w) &\leq \Theta_2 \Upsilon d(z,w) \int_0^t [1 + \int_0^s \theta d\theta] ds \\
&\leq \Theta_2 \Upsilon d(z,w) [t + \tfrac{t^3}{6}].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left\|(\mathcal{T}^2 z)t - (\mathcal{T}^2 w)t\right\| &= \|\mathcal{T}(\mathcal{T}z)t - \mathcal{T}(\mathcal{T}w)t\| \leq \int_0^t \left\|\tilde{S}(t-s)\right\| \left\|f(s, (\mathcal{T}z)s, \int_0^s k(s,\theta,(\mathcal{T}z)\theta)d\theta) - \right. \\
&\quad -f(s, (\mathcal{T}w)s, \int_0^s k(s,\theta,(\mathcal{T}w)\theta)d\theta) \, \| ds \\
&\leq \Theta_2 \int_0^t \Upsilon \left[ \|\mathcal{T}z - \mathcal{T}w\| + \int_0^s |\phi(\theta)| \, \|\mathcal{T}z - \mathcal{T}w\| \, d\theta \right] ds
\end{aligned}$$

or, equivalent

$$\begin{aligned}
d(\mathcal{T}^2 z, \mathcal{T}^2 w) &\leq \Theta_2 \int_0^t \Upsilon \left[ d(\mathcal{T}z, \mathcal{T}w) + d(\mathcal{T}z, \mathcal{T}w) \int_0^s |\phi(\theta)| \, d\theta \right] ds \\
&\leq (\Theta_2 \Upsilon)^2 d(z,w) \int_0^t \left[ s + \tfrac{s^3}{6} + \int_0^s \theta(\theta + \tfrac{\theta^3}{6}) d\theta \right] ds \\
&\leq (\Theta_2 \Upsilon)^2 d(z,w) [\tfrac{t^2}{2} + \tfrac{t^4}{24} + \tfrac{t^4}{12} + \tfrac{t^6}{180}] \\
&\leq \tfrac{227}{360} (\Theta_2 \Upsilon)^2 d(z,w).
\end{aligned}$$

Taking into account the conditions imposed in the hypothesis of the theorem regarding the constant $\Upsilon$, we can find $\varepsilon \in [0,1]$ such that $\frac{227}{360}(\Theta_2\Upsilon)^2 = \varepsilon^2$. We have, then,

$$\begin{aligned}
d(\mathcal{T}^2 z, \mathcal{T}^2 w) &\leq \varepsilon^2 d(z,w) \leq \varepsilon^2 d(z,w) + \varepsilon d(z,w) \\
&= \varepsilon d(z,w) - \varepsilon^2 d(z,w) + 2\varepsilon^2 d(z,w) \\
&\leq (1-\varepsilon)\varepsilon d(z,w) + 2\varepsilon^2 (d(z,z_0) + d(z_0,w)) \\
&\leq (1-\varepsilon) d(z,w) + 2\varepsilon^2 [1 + \|z\| + \|w\|].
\end{aligned}$$

Therefore, considering $\psi(\varepsilon) = \varepsilon$, $\Lambda = 2$, $\lambda = \beta = 1$, we can easily see that for $z$ arbitrary in $\mathcal{E}$, there exists $n(z) = 2$ such that

$$d(\mathcal{T}^2 z, \mathcal{T}^2 w) \leq \mathcal{P}(z,w), \text{ for every } w \in \mathcal{E},$$

that is, the mapping $\mathcal{T}$ is a Pata-Ćirić type contraction at a point. Therefore, since all the assumption of Theorem 5 are fulfilled, the problem (13)–(14) admits an unique solution. (Moreover, we mention that in the case $n = 1$, the presumptions of the Theorem 5 are not satisfied.) □

## 4. Conclusions

In this paper, we combine and extend Pata type contractions and Ćirić type contraction at a point. With this work, we underline the contribution of V. Pata [7] at the fixed point theory by defining an auxiliary distance function $\|z\| = d(z, x)$ where $x$ is an arbitrary but fixed point. Indeed, in all metric fixed point theory proofs, we choose an arbitrary point and built a constructive (iterative) sequence based on the given mapping that is called Picard sequence:
For a self-mapping $f$ on a metric space $X$ and arbitrary point "$x$" (renamed as "$x_0$"). Then, $x_1 = Tx_0$,

$$x_n = fx_{n-1} \text{ for all positive integers.}$$

Note that in the original proof of Banach (and many others in the consecutive papers on the metric fixed point theory) for any point "$x$", this sequence converges to the fixed point of $T$. Here, V.Pata, suggest such auxiliary distance function by initiating from an arbitrary point "$x$", construct a sequence to refine Banach's fixed point theorem.

In this note, we employ the approach of Pata in a more general case to generalize and unify several existing results in the literature. Thus, several consequences of our results can be observed by using the examples that have been introduced in [8–12].

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## References

1. Bryant, V.W. A remark on a fixed point theorem for iterated mappings. *Am. Math. Mon.* **1968**, *75*, 399–400. [CrossRef]
2. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
3. Sehgal, V.M. A fixed point theorem for mappings with a contractive iterate. *Proc. Am. Math. Soc.* **1969**, *23*, 631–634. [CrossRef]
4. Guseman, L.F., Jr. Fixed point theorems for mappings with a contractive iterate at a point. *Proc. Am. Math. Soc.* **1970**, *26*, 615–618. [CrossRef]
5. Ćirić, L. *Some Recent Results in Metrical Fixed Point Theory*; University of Belgrade: Beograd, Serbia, 2003.
6. Ćirić, L. Generalized contractions andfixed-point theorems. *Publ. Inst. Math.* **1971**, *12*, 19–26.
7. Pata, V. A fixed point theorem in metric spaces. *J. Fixed Point Theory Appl.* **2011**, *10*, 299–305. [CrossRef]
8. Karapınar, E.; Fulga, A.; Rakočević, V. A discussion on a Pata type contraction via iterate at a point. *Filomat*, in press.
9. Alqahtani, B.; Fulga, A.; Karapınar, E. Fixed Point Results on Δ-Symmetric Quasi-Metric Space via Simulation Function with an Application to Ulam Stability. *Mathematics* **2018**, *6*, 208. [CrossRef]
10. Alqahtani, B.; Fulga, A.; Karapinar, E. A fixed point result with a contractive iterate at a point. *Mathematics* **2019**, *7*, 606. [CrossRef]
11. Alqahtani, B.; Fulga, A.; Karapinar, E.; Kumari, P.S. Sehgal Type Contractions on Dislocated Spaces. *Mathematics* **2019**, *7*, 153. [CrossRef]
12. Karapinar, E.; Fulga, A.; Alghamdi, M. A Common Fixed Point Theorem For Iterative Contraction of Sehgal Type. *Symmetry* **2019**, *11*, 470. [CrossRef]