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A study of symmetric contractions with an application to generalized fractional differential equations

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Abstract

This article proposes four distinct kinds of symmetric contraction in the framework of complete F-metric spaces. We examine the condition to guarantee the existence and uniqueness of a fixed point for these contractions. As an application, we look for the solutions to fractional boundary value problems involving a generalized fractional derivative known as the fractional derivative with respect to another function.

Keywords: Boundary value problem; Generalized fractional derivative; Fixed point; Measure of noncompactness

1 Introduction

Among the most interesting and published research topics in the last few decades, we can count the metric fixed-point theory, see, e.g., [1–8], and fractional differential/integral equations, see, e.g., [9–25]. In [20], the authors considered certain fractional and ordinary differential equations and provided solutions by using the metric fixed-point theory techniques. In this paper, we follows the same direction as in [20] and propose solutions for certain fractional differential equations which are based on the new fixed-point theory approaches.

One of the interesting approaches was introduced in [26], where the authors initiated the idea of interpolative-type contractions and established brand new fixed-point results; see also [27–32]. It is worth mentioning that the abstract space structure is as important as the conditions of the contraction in the fixed-point theory. One of the recent exciting results in this direction was introduced by Jleli and Samet [8], who gave a new extension of a metric notion called $\mathcal{F}$-metric space (abbreviated as $\mathcal{F}$-MS).

Let $\mathcal{F}$ be a set of functions $f:(0,+\infty)\to(-\infty,+\infty)$ such that

$(\mathcal{F}_1)$ f is nondecreasing, that is, for all $0<c<d$, we have $f(c)\le f(d)$;

$(\mathcal{F}_2)$ for each sequence $\{d_n\}\subset(0,+\infty)$, we have

$$\lim_{n\to+\infty} d_n = 0 \quad \text{if and only if} \quad \lim_{n\to+\infty} f(d_n) = -\infty.$$

Definition 1.1 ([8]) Let $A \neq \emptyset$ with $D : A \times A \to [0, +\infty)$ be a given mapping. Suppose there exists $(f, \mu) \in \mathcal{F} \times [0, +\infty)$ such that

(D_1) $(w, \upsilon) \in A \times A$, $D(w, \upsilon) = 0 \iff w = \upsilon$;

(D_2) $D(w, \upsilon) = D(\upsilon, w)$ for all $(w, \upsilon) \in A \times A$;

(D_3) For every $(w, \upsilon) \in A \times A$, for each natural number $N \geq 2$, and for every $(u_i)_{i=1}^{N} \subset A$ with $(u_1, u_N) = (w, \upsilon)$, we have that

$$D(w, \upsilon) > 0 \quad \text{implies} \quad f\big(D(w, \upsilon)\big) \leq f\left(\sum_{i=1}^{N-1} d(u_i, u_{i+1})\right) + \mu.$$

Then, D is said to be an $\mathcal{F}$-metric on A.

Here, the pair (A, D) is called an $\mathcal{F}$-MS.

A sequence $\{w_n\}$ in (A, D) is $\mathcal{F}$-Cauchy if $\lim_{n,m \to \infty} D(w_n, w_m) = 0$. Furthermore, (A, D) is $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence is $\mathcal{F}$-convergent in A.

The following example was stated in [8].

Example 1.2 The set of natural numbers $\mathbb{N} = X$ is an $\mathcal{F}$-MS if we define D by

$$D(w, \upsilon) = \begin{cases} (w - \upsilon)^2, & \text{if } (w, \upsilon) \in [0, 3] \times [0, 3], \\ |w - \upsilon|, & \text{if } (w, \upsilon) \notin [0, 3] \times [0, 3], \end{cases}$$

for all $(w, \upsilon) \in A \times A$, $f(t) = \ln(t)$ and $\mu = \ln(3)$. Notice that D is not a metric but (X, D) is an $\mathcal{F}$-MS.

Jleli and Samet [8] proposed a simple fixed-point theorem as follows.

Theorem 1.3 ([8]) *Let (A, D) be an $\mathcal{F}$-MS. Let $g : A \to A$ be a self mapping. Suppose the following conditions are met*:

(i) *(A, D) is $\mathcal{F}$-complete*;

(ii) *there exists a constant $k \in (0, 1)$ such that*

$$D\big(g(w), g(\upsilon)\big) \leq kD(w, \upsilon), \quad (w, \upsilon) \in A \times A.$$

Then, g attains a unique fixed point $w^* \in A$.

In 2012, Samet et al. introduced a class of α-admissible mappings as follows:

Definition 1.4 ([33]) Let $T : A \to A$ and $\alpha : A \times A \to [0, +\infty)$. Then T is said to be α-admissible if $w, \upsilon \in A$, $\alpha(w, \upsilon) \geq 1$ implies that $\alpha(Tw, T\upsilon) \geq 1$.

Next, Salimi et al. [34] modified the concept of α-admissible mappings as follows:

Definition 1.5 ([34]) Let $T : A \to A$ and $\alpha, \eta : A \times A \to [0, +\infty)$ two functions. Then T is called an α-admissible mapping with respect to η if $w, \upsilon \in A$, $\alpha(w, \upsilon) \geq \eta(w, \upsilon)$ implies that $\alpha(Tw, T\upsilon) \geq \eta(Tw, T\upsilon)$.

Definition 1.6 ([35]) Consider a metric space (A,d), a mapping $T:A\to A$, and let $\alpha,\eta:A\times A\to[0,\infty)$ be two functions. Then the mapping T is called an α–η-continuous mapping in (A,d) whenever given $w\in A$ and a sequence $\{w_n\}$ with

$$w_n\to w \quad \text{at } \infty, \qquad \alpha(w_n,w_{n+1})\geq\eta(w_n,w_{n+1}), \quad \forall n\in\mathbb{N} \quad \text{implies} \quad Tw_n\to Tw.$$

For more details, see, e.g., [36, 37].

A mapping $T:A\to A$ is called orbitally continuous at $\nu\in A$ if $\lim_{n\to\infty}T^n w=\nu$ implies that $\lim_{n\to\infty}T^n w=T\nu$. A mapping T is orbitally continuous on A if T is orbitally continuous $\forall\nu\in A$.

2 Main results

In this part, we firstly present a new symmetric fractional α–η-contraction of type I.

Definition 2.1 Let $T:A\to A$ be a mapping on an $\mathcal{F}$-metric space (A,D) and consider two functions $\alpha,\eta:A\times A\to[0,+\infty)$. We say that T is a symmetric fractional α–η-contraction of type I along with constants $\lambda\in[0,1)$ and $\beta,\hat{w},\gamma\in(0,1)$ if, whenever $\alpha(w,\upsilon)\geq\eta(w,\upsilon)$, we have

$$D(Tw,T\upsilon)\leq\lambda\bigl(\check{S}_1(w,\upsilon)\bigr), \tag{2.1}$$

where

$$\begin{aligned}\check{S}_1(w,\upsilon)={}&D(w,\upsilon)\cdot\bigl[D(w,Tw)\bigr]^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\cdot\bigl[D(\upsilon,T\upsilon)\bigr]^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\\&\cdot\bigl[D(w,Tw)+D(\upsilon,T\upsilon)\bigr]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\\&\cdot\bigl[D(w,T\upsilon)+D(\upsilon,Tw)\bigr]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\end{aligned}$$

for all $w,\upsilon\in A\setminus\mathrm{Fix}(T)$.

Example 2.2 Let $A=\{0,1,2,3\}$ be endowed with an $\mathcal{F}$-metric D defined by

$$D(w,\upsilon)=\begin{cases}(w-\upsilon)^2, & \text{if } (w,\upsilon)\in A\times A,\\ |w-\upsilon|, & \text{if } (w,\upsilon)\notin A\times A.\end{cases}$$

Consider $f(t)=\ln(t)$ and $\mu=\ln(3)$. Define $T:A\to A$ by

$$T0=0,\qquad T1=1,\qquad T2=T3=0,$$

and $\alpha,\eta:A\times A\to[0,+\infty)$ by

$$\alpha(w,\upsilon)=\begin{cases}1, & \text{if } w,\upsilon\in A,\\ 0, & \text{otherwise,}\end{cases}\qquad \eta(w,\upsilon)=\begin{cases}\frac{1}{2}, & \text{if } w,\upsilon\in A,\\ 0, & \text{otherwise,}\end{cases}$$

if $w, \upsilon \in A$. Clearly, $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$ and

$$\begin{aligned}
D(T2,T3) &= 0 \\
&\leq \lambda\Big[D(2,3)\cdot D(2,T2)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\cdot D(3,T3)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big(D(2,T2)+D(3,T3)\big)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\big(D(2,T3)+D(3,T2)\big)^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda\Big[1\cdot D(2,0)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} D(3,0)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\big(D(2,0)+D(3,0)\big)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big(D(2,0)+D(3,0)\big)^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda\Big[(4)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\cdot(9)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\cdot(4+9)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\cdot(4+9)^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&\leq \lambda\Big[(4)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\cdot(9)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}\cdot(4\cdot 9)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\cdot(4\cdot 9)^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda\big[(4)\cdot(9)\big]^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}+\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}+\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} = \lambda.
\end{aligned}$$

Clearly, (2.1) holds for all $w, \upsilon \in A \setminus \mathrm{Fix}(T)$, if one takes any values of the constants $\lambda \in [0,1)$, $\beta, \hat{w}, \gamma \in (0,1)$. Note that T has two fixed points, which are 0 and 1.

Now, we state brand new fixed-point theorems for symmetric fractional α–η-contraction of type I in an $\mathcal{F}$-complete $\mathcal{F}$-MS setting.

Theorem 2.3 *Let* (A,D) *be a complete* $\mathcal{F}$*-metric space and* T *be a symmetric fractional* α–η*-contraction of type I satisfying the following*:

(i) T *is a* α*-admissible mapping with respect to* η;

(ii) *there exists a* $w_0 \in A$ *such that* $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$;

(iii) T *is* α–η*-continuous.*

Then, T *possesses a fixed point at* A.

Proof Let w_0 be in A such that $\alpha(w_0,Tw_0) \geq \eta(w_0,Tw_0)$. For $w_0 \in A$, we build a sequence $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1 = Tw_0$, $w_2 = Tw_1 = T^2 w_0$. Proceeding like this, we obtain $w_{n+1} = Tw_n = T^{n+1}w_0$, for every $n \in \mathbb{N}$. Now, since the mapping T is α-admissible with respect to η, we have $\alpha(w_0,w_1) = \alpha(w_0,Tw_0) \geq \eta(w_0,Tw_0) = \eta(w_0,w_1)$. Carrying on this way, we get

$$\alpha(w_{n-1},w_n) \geq \eta(w_{n-1},w_n) = \eta(w_{n-1},Tw_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

If $w_{n+1} = w_n$ for some $n \in \mathbb{N}$ then $w_n = w^*$ is a fixed point of T. So, we assume that $w_n \neq w_{n+1}$, accompanied by

$$D(Tw_{n-1},Tw_n) = D(w_n,Tw_n) > 0, \quad \text{for all } n \in \mathbb{N}.$$

As T is a symmetric fractional α–η-contraction of type I, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
D(w_n,w_{n+1}) &= D(Tw_{n-1},Tw_n) \\
&\leq \lambda\Big[D(w_{n-1},w_n)\cdot D(w_{n-1},Tw_{n-1})^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot D(w_n, Tw_n)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \left[D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n)\right]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\cdot \left[D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1})\right]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
= {}& \lambda\Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\cdot \left[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\right]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\cdot \left[D(w_{n-1}, w_{n+1}) + D(w_n, w_n)\right]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
\leq {}& \lambda\Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\cdot \left[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\right]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\cdot \left[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\right]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
= {}& \lambda\Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\cdot \left[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\right]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
\leq {}& \lambda\Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\cdot \left[D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})\right]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
= {}& \lambda D(w_{n-1}, w_n)^{1+\frac{1}{(\beta-\hat{w})(\beta-\gamma)} + \frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \\
&\cdot D(w_n, w_{n+1})^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)} + \frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \\
= {}& \lambda D(w_{n-1}, w_n).
\end{aligned}
$$

We deduce that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n) \tag{2.3}$$

and $\{D(w_{n-1}, w_n)\}$ is a nonincreasing sequence with nonnegative terms. Thus, there is a nonnegative constant ϱ such that $\lim_{n\to\infty} D(w_{n-1}, w_n) = \varrho$. Note that $\varrho \geq 0$. From (2.3), we have

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n) \leq \lambda^n D(w_0, w_1),$$

which provides that

$$\sum_{i=n}^{m-1} D(w_i, w_{i+1}) \leq \frac{\lambda^n}{1-\lambda} D(w_0, w_1), \quad m > n.$$

In the limit we reach

$$\lim_{n\to+\infty} \frac{\lambda^n}{1-\lambda} D(w_0, w_1) = 0,$$

that is, there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1-\lambda} D(w_0, w_1) < \delta, \quad n \geq N.$$

Let $\epsilon > 0$ be fixed, $(f, \mu) \in \mathcal{F} \times [0, \infty)$, and let (D_3) be satisfied. By $(\mathcal{F}_2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \quad \text{implies} \quad f(t) < f(\epsilon) - \mu. \tag{2.4}$$

Hence by (2.4) and $(\mathcal{F}_1)$, we get

$$f\left(\sum_{i=n}^{m-1} D(w_i, w_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} D(w_0, w_1)\right) < f(\epsilon) - \mu, \tag{2.5}$$

where $m, n \in \mathbb{N}$ with $m > n \geq N$ and $D(w_n, w_m) > 0$. Therefore, by using (D_3) and (2.5), we have

$$f\bigl(D(w_m, w_n)\bigr) \leq f\left(\sum_{i=n}^{m-1} \bigl(D(w_i, w_{i+1})\bigr)\right) + \mu < f(\epsilon),$$

which by $(\mathcal{F}_1)$ implies

$$D(w_m, w_n) < \epsilon, \quad \text{for } m > n \geq N.$$

Consequently, $\{w_n\}$ is an $\mathcal{F}$-Cauchy sequence. Since (A, D) is an $\mathcal{F}$-complete metric space, there exists an $w^* \in A$ such that w_n is $\mathcal{F}$-convergent to w^*, i.e.,

$$\lim_{n\to\infty} D\bigl(w_n, w^*\bigr) = 0. \tag{2.6}$$

Now T is α–η-continuous, $\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n)$, for each $n \in \mathbb{N}$, and so $w_{n+1} = Tw_n \to Tw^*$ as $n \to \infty$, in other words, $w^* = Tw^*$. Now we are going to prove that w^* is a fixed point of T. We argue by contradiction by supposing that $D(Tw^*, w^*) > 0$. By (D_3), we have

$$f\bigl(D\bigl(Tw^*, w^*\bigr)\bigr) \leq f\bigl(D\bigl(Tw^*, Tw_n\bigr) + D\bigl(Tw_n, w^*\bigr)\bigr) + \mu, \quad n \in \mathbb{N}.$$

Using $(\mathcal{F}_1)$ and the contractivity condition gives

$$f\bigl(D\bigl(Tw^*, w^*\bigr)\bigr) \leq f\left(\lambda \begin{pmatrix} D\bigl(w^*, w_n\bigr) \cdot D\bigl(Tw^*, w^*\bigr)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D\bigl(w_n, Tw_n\bigr)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ \cdot \bigl[D\bigl(Tw^*, w^*\bigr) + D\bigl(w_n, Tw_n\bigr)\bigr]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \cdot \bigl[D\bigl(Tw_n, w^*\bigr) + D\bigl(w_n, Tw^*\bigr)\bigr]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} + D\bigl(w_{n+1}, w^*\bigr) \end{pmatrix}\right) + \mu,$$

for all $n \in \mathbb{N}$. On the other hand, by using $(\mathcal{F}_2)$ and (2.6), we get

$$\lim_{n\to\infty} f\bigl(\lambda D\bigl(w^*, w_n\bigr) + D\bigl(w_{n+1}, w^*\bigr)\bigr) + \mu = -\infty,$$

which gives a contradiction. Therefore, $D(Tw^*, w^*) = 0$, and hence w^* possesses a fixed point of T. □

Theorem 2.4 *Let (A,D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be symmetric fractional α–η-contraction of type I fulfilling the following conditions*:

(i) *T is a α-admissible mapping with respect to η;*

(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$;*

(iii) *there is a sequence $\{w_n\}$ in A such that $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$ with $w_n \to w^*$ as $n \to \infty$ and that $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$.*

Then, T possesses a fixed point in A.

Proof Along the lines of the proof of Theorem 2.3, we acquire $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ each $n \in \mathbb{N}$. Using (D_3), we have

$$f\big(D\big(Tw^*, w^*\big)\big) \leq f\big(D\big(Tw^*, Tw_n\big) + D\big(w_n, w^*\big)\big) + \mu.$$

From (2.1) and $(\mathcal{F}_1)$, we have

$$\begin{aligned} f\big(D\big(Tw^*, w^*\big)\big) &\leq f\big(\big(D\big(Tw^*, Tw_n\big)\big) + D\big(Tw_n, w^*\big)\big) + \mu \\ &\leq f\left(\lambda\left(\begin{array}{l} D\big(w^*, w_n\big) \cdot D\big(Tw^*, w^*\big)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, Tw_n)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ \quad \cdot \big[D\big(Tw^*, w^*\big) + D(w_n, Tw_n)\big]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \quad \cdot \big[D\big(Tw_n, w^*\big) + D\big(w_n, Tw^*\big)\big]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} + D\big(w_{n+1}, w^*\big) \end{array}\right)\right) \\ &\quad + \mu. \end{aligned}$$

Employing (2.6) and

$$\lim_{n\to\infty} D\big(w_n, w^*\big) = 0, \quad \text{together with} \quad \lim_{n\to\infty} D\big(w_{n+1}, w^*\big) = 0,$$

we obtain

$$f\big(D\big(w^*, Tw^*\big)\big) \leq f\big(D\big(w^*, Tw^*\big)\big) + \mu.$$

Making use of $(\mathcal{F}_2)$, we find that

$$\lim_{n\to\infty} f\big(D\big(w^*, Tw^*\big)\big) + \mu = -\infty,$$

which is a contradiction. Therefore $D(w^*, Tw^*) = 0$. In other words, w^* is a fixed point of T. □

Example 2.5 Consider $A = \mathbb{R} \supset \mathbb{N}$ with an $\mathcal{F}$-metric $D : A \times A \to [0, \infty)$ defined by

$$D(w, \upsilon) = \begin{cases} (w-\upsilon)^2, & \text{if } (w, \upsilon) \in \mathbb{N} \times \mathbb{N}, \\ |w-\upsilon|, & \text{if } (w, \upsilon) \notin \mathbb{N} \times \mathbb{N}, \end{cases}$$

accompanied by $f(t) = \ln(t)$ and $\mu = \ln(100)$. Define $T : A \to A$ by

$$Tw = \begin{cases} 1 - \frac{w}{2}, & \text{if } w \in \mathbb{N}, \\ 0, & \text{if } w \notin \mathbb{N}, \end{cases}$$

and $\alpha, \eta : A \times A \to [0, +\infty)$ by

$$\alpha(w,\upsilon) = \begin{cases} 2, & \text{if } w,\upsilon \in [0,\infty), \\ 0, & \text{otherwise}, \end{cases} \qquad \eta(w,\upsilon) = \begin{cases} 1, & \text{if } w,\upsilon \in [0,\infty), \\ 0, & \text{otherwise}. \end{cases}$$

Case I. If $w = \upsilon$ then, clearly, $D(w,\upsilon) = 0$. Hence all conditions of Theorem 2.3 are satisfied.

Case II. If w, υ are in $\mathbb{N}$, but $Tw \notin \mathbb{N}$, $T\upsilon \notin \mathbb{N}$, then

$$D(Tw, T\upsilon) = D\left(1 - \frac{w}{2}, 1 - \frac{\upsilon}{2}\right) = \left[\frac{1}{2}|w - \upsilon|\right].$$

Clearly, T is an α-admissible mapping with respect to η, whenever $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$, so that

$$D(Tw, T\upsilon) = \frac{1}{2}|w - \upsilon| \leq \lambda \begin{bmatrix} (\upsilon - w)^2 \cdot \left|\frac{3}{2}w - 1\right|^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \left|\frac{3}{2}\upsilon - 1\right|^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ \cdot \left(\left|\frac{3}{2}w - 1\right| + \left|\frac{3}{2}\upsilon - 1\right|\right)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \cdot \left(\left|w - \frac{1}{2}\upsilon\right| + \left|\upsilon - \frac{1}{2}w\right|\right)^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \end{bmatrix},$$

by taking constants $\lambda \in [0,1)$, and $\beta, \hat{w}, \gamma \in (0,1)$, for all $w, \upsilon \in \mathbb{N} \setminus \operatorname{Fix}(T)$.

Case III. When both w, υ are not in $\mathbb{N}$, we obtain

$$D(Tw, T\upsilon) = 0,$$

and then, clearly, T is α-admissible mapping with respect to η, whenever $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$, so that

$$D(Tw, T\upsilon) = 0 \leq \lambda\left[|w - \upsilon| \cdot |w|^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot |\upsilon|^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \left(|w| + |\upsilon|\right)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\right],$$

where $\lambda \in [0,1)$, and $\beta, \hat{w}, \gamma \in (0,1)$, for all $w, \upsilon \in \mathbb{N} \setminus \operatorname{Fix}(T)$.

Case IV. If one of w, v is in $\mathbb{N}$ and the other is not in $\mathbb{N}$, we obtain

$$D(Tw, T\upsilon) = D\left(1 - \frac{w}{2}, 0\right) = \left|1 - \frac{w}{2}\right|.$$

Clearly, T is α-admissible mapping with respect to η, whenever $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$, so that

$$D(Tw, T\upsilon) = \left|1 - \frac{w}{2}\right| \leq \lambda \begin{bmatrix} |w - \upsilon| \cdot \left|\frac{3}{2}w - 1\right|^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot |\upsilon|^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ \cdot \left(\left|\frac{3}{2}w - 1\right| + |\upsilon|\right)^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \cdot \left(|w| + \left|\upsilon + \frac{1}{2}w - 1\right|\right)^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}} \end{bmatrix}.$$

Therefore, all conditions of Theorem 2.3 are satisfied. Hence T is a symmetric fractional $\alpha{-}\eta$-contraction of type I.

Definition 2.6 Let (A,D) be an $\mathcal{F}$-metric space (A,D) and $\alpha,\eta : A \times A \to [0,+\infty)$ two functions. Then $\mathcal{F}$ is said to be $\alpha{-}\eta$-complete on A if and only if every $\mathcal{F}$-Cauchy sequence $\{w_n\}$, satisfying

$$\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1}) \quad \text{for each } n \in \mathbb{N},$$

$\mathcal{F}$-converges in A.

Remark 2.7 Theorems 2.3 and 2.4 also hold for an $\alpha{-}\eta$-complete $\mathcal{F}$-metric space instead of $\mathcal{F}$-complete $\mathcal{F}$-metric space (for details, see [7]).

3 Symmetric fractional $\alpha{-}\eta$-contraction of type II

In this section, a symmetric fractional $\alpha{-}\eta$-contraction of type II is introduced in the setting of an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Using this notion, we shall provide a fixed point theorem.

Definition 3.1 Consider a self-mapping $T : A \to A$ on an $\mathcal{F}$-metric space (A,D) and let two functions $\alpha,\eta : A \times A \to [0,+\infty)$ be given. We say that T is a symmetric fractional $\alpha{-}\eta$-contraction of type II provided there are constants $\lambda \in [0,1)$ and $\beta,\hat{w},\gamma \in (0,1)$ such that, whenever $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$, we have

$$D(Tw, T\upsilon) \leq \lambda\bigl(\check{S}_2(w,\upsilon)\bigr), \tag{3.1}$$

where

$$\check{S}_2(w,\upsilon) = \left\{ \begin{array}{l} D(w,\upsilon) \cdot \bigl[D(w,Tw)\bigr]^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \bigl[D(\upsilon,T\upsilon)\bigr]^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\ \quad \cdot \bigl[D(w,Tw) + D(\upsilon,T\upsilon)\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \bigl[D(w,T\upsilon) + D(\upsilon,Tw)\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \end{array} \right\},$$

for all $w,\upsilon \in A \setminus \operatorname{Fix}(T)$.

Now we show and demonstrate our next theorem.

Theorem 3.2 *Let (A,D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and T be a symmetric fractional $\alpha{-}\eta$-contraction of type II fulfilling the following conditions*:

(i) *T is a α-admissible mapping with respect to η*;
(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$*;
(iii) *T is $\alpha{-}\eta$-continuous.*

Then, T possesses a fixed point in A.

Proof Let w_0 in A be sucht that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$. For this $w_0 \in A$, we build a sequence $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1 = Tw_0$, $w_2 = Tw_1 = T^2 w_0$. Proceeding this way, $w_{n+1} = Tw_n = T^{n+1} w_0$, for all $n \in \mathbb{N}$. Since the mapping T is α-admissible with respect to η, $\alpha(w_0, w_1) = \alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0) = \eta(w_0, w_1)$. Carrying on in this way, we obtain

$$\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n) = \eta(w_{n-1}, Tw_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

If $w_{n+1} = w_n$ for some $n \in \mathbb{N}$ then $w_n = w^*$ is a fixed point of T. So, we assume that $w_n \neq w_{n+1}$ and that

$$D(Tw_{n-1}, Tw_n) = D(w_n, Tw_n) > 0, \quad \text{for every } n \in \mathbb{N}.$$

As T is a symmetric fractional α–η-contraction of type II, for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
D(w_n, w_{n+1}) &= D(Tw_{n-1}, Tw_n) \\
&\leq \lambda \Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, Tw_{n-1})^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot D(w_n, Tw_n)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \big[D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n) \big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1}) \big]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
&= \lambda \Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1}) \big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_{n+1}) + D(w_n, w_n) \big]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
&\leq \lambda \Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1}) \big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1}) \big]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
&= \lambda \Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1}) \big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
&\leq \lambda \Big[D(w_{n-1}, w_n) \cdot D(w_{n-1}, w_n)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1}) \big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \Big] \\
&= \lambda [D(w_{n-1}, w_n)^{1+\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)} + \frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \\
&\quad \cdot D(w_n, w_{n+1})^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)} + \frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \\
&= \lambda D(w_{n-1}, w_n)
\end{aligned}$$

and deduce

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n). \tag{3.3}$$

We conclude that $\{D(w_{n-1}, w_n)\}$ is a nonincreasing sequence with nonnegative terms. As a result, there is a nonnegative constant ρ such that $\lim_{n\to\infty} D(w_{n-1}, w_n) = \rho$. We shall show that $\rho > 0$. Indeed, from (3.3), we derive that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n) \leq \lambda^n D(w_0, w_1). \tag{3.4}$$

The rest of the proof follows along the same lines as the proof of Theorem 2.3. □

Theorem 3.3 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A,D) and let T be a symmetric fractional α–η-contraction of type II satisfying the following conditions*:

(i) *T is an α-admissible mapping with respect to η;*

(ii) *there exists an $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$;*

(iii) *there is a sequence $\{w_n\}$ in A such that $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$, $w_n \to w^*$ as $n \to \infty$, and $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$.*

Then, T possesses a fixed point in A.

Proof We follow the lines of the proof of Theorem 2.4. Since, by (iii), $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for every $n \in \mathbb{N}$, using (D_3), we get

$$f\bigl(D\bigl(Tw^*, w^*\bigr)\bigr) \leq f\bigl(D\bigl(Tw^*, Tw_n\bigr) + D\bigl(w_n, w^*\bigr)\bigr) + \mu.$$

From (3.1) and $(\mathcal{F}_1)$, we have

$$\begin{aligned} f\bigl(D\bigl(Tw^*, w^*\bigr)\bigr) &\leq f\bigl(\bigl(D\bigl(Tw^*, Tw_n\bigr)\bigr) + D\bigl(Tw_n, w^*\bigr)\bigr) + \mu \\ &\leq f\left(\lambda\left(\begin{array}{l} D\bigl(w^*, w_n\bigr) \cdot D\bigl(Tw^*, w^*\bigr)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, Tw_n)^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\ \quad \cdot \bigl[D\bigl(Tw^*, w^*\bigr) + D(w_n, Tw_n)\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \quad \cdot \bigl[D\bigl(Tw_n, w^*\bigr) + D\bigl(w_n, Tw^*\bigr)\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} + D\bigl(w_{n+1}, w^*\bigr) \end{array}\right)\right) \\ &\quad + \mu. \end{aligned}$$

Making use of (2.6) and that

$$\lim_{n\to\infty} D\bigl(w_n, w^*\bigr) = 0 \quad \text{together} \quad \lim_{n\to\infty} D\bigl(w_{n+1}, w^*\bigr) = 0,$$

we obtain

$$f\bigl(D\bigl(w^*, Tw^*\bigr)\bigr) \leq f\bigl(D\bigl(w^*, Tw^*\bigr)\bigr) + \mu.$$

Using $(\mathcal{F}_2)$, we have

$$\lim_{n\to\infty} f\bigl(D\bigl(w^*, Tw^*\bigr)\bigr) + \mu = -\infty,$$

which is a logical inconsistency. Hence $D(w^*, Tw^*) = 0$, that is, w^* is a fixed point of T. □

4 Symmetric fractional α–η-contraction of type III

In this section, a symmetric fractional α–η-contraction of type III is considered in the setting of an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Before stating a fixed-point theorem for such maps, we define a symmetric fractional α–η-contraction of type III as follows:

Definition 4.1 Let (A,D) be an $\mathcal{F}$-metric space with a self-mapping $T: A \to A$ and two functions $\alpha, \eta: A \times A \to [0, +\infty)$. We say that T is a symmetric fractional α–η-contraction of type III if there are constants $\lambda \in [0,1)$ and $\beta, \hat{w}, \gamma \in (0,1)$ such that, whenever $\alpha(w, \upsilon) \geq \eta(w, \upsilon)$, we have

$$D(Tw, T\upsilon) \leq \lambda\bigl(\check{S}_3(w, \upsilon)\bigr), \tag{4.1}$$

where

$$\check{S}_3(w,\upsilon)=\lambda\max\left\{\begin{array}{c} D(w,\upsilon),\left[D(w,Tw)\right]^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}}\cdot\left[D(\upsilon,T\upsilon)\right]^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}}\\ \cdot\left[D(w,Tw)+D(\upsilon,T\upsilon)\right]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\\ \cdot\left[D(w,T\upsilon)+D(\upsilon,Tw)\right]^{\frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\end{array}\right\},$$

for all $w,\upsilon\in A\setminus\operatorname{Fix}(T)$.

Now we state and prove our next theorem.

Theorem 4.2 *Let (A,D) be an $\mathcal{F}$-complete $\mathcal{F}$-metric space and consider a symmetric fractional α–η-contraction T of type III that satisfies the following conditions*:

(i) *T is a α-admissible mapping with respect to η*;

(ii) *there exist a $w_0\in A$ such that $\alpha(w_0,Tw_0)\geq\eta(w_0,Tw_0)$*;

(iii) *T is α–η-continuous.*

Then, T possesses a fixed point in A.

Proof Let w_0 in A be such that $\alpha(w_0,Tw_0)\geq\eta(w_0,Tw_0)$. Using this $w_0\in A$, we define $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1=Tw_0$, $w_2=Tw_1=T^2w_0$. Continuing this way, we get $w_{n+1}=Tw_n=T^{n+1}w_0$, for every $n\in\mathbb{N}$. Since the mapping T is α-admissible with respect to η, $\alpha(w_0,w_1)=\alpha(w_0,Tw_0)\geq\eta(w_0,Tw_0)=\eta(w_0,w_1)$. Carrying on in this way, we find

$$\alpha(w_{n-1},w_n)\geq\eta(w_{n-1},w_n)=\eta(w_{n-1},Tw_{n-1}),\quad \text{for all } n\in\mathbb{N}. \tag{4.2}$$

If $w_{n+1}=w_n$ for some $n\in\mathbb{N}$ then $w_n=w^*$ is a fixed point of T. So, we assume that $w_n\neq w_{n+1}$ and that

$$D(Tw_{n-1},Tw_n)=D(w_n,Tw_n)>0,\quad \text{for each } n\in\mathbb{N}.$$

As T is a symmetric fractional α–η-contraction of type III, for any $n\in\mathbb{N}$, we get

$$\begin{aligned}
D(w_n,w_{n+1})&=D(Tw_{n-1},Tw_n)\\
&\leq\lambda\max\Big[D(w_{n-1},w_n),D(w_{n-1},Tw_{n-1})^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}}\\
&\quad\cdot D(w_n,Tw_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}}\\
&\quad\cdot\left[D(w_{n-1},Tw_{n-1})+D(w_n,Tw_n)\right]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\\
&\quad\cdot\left[D(w_{n-1},Tw_n)+D(w_n,Tw_{n-1})\right]^{\frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\Big]\\
&=\lambda\max\Big[D(w_{n-1},w_n),D(w_{n-1},w_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}}\cdot D(w_n,w_{n+1})^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}}\\
&\quad\cdot\left[D(w_{n-1},w_n)+D(w_n,w_{n+1})\right]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)}}\\
&\quad\cdot\left[D(w_{n-1},w_{n+1})+D(w_n,w_n)\right]^{\frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \lambda \max\Big[D(w_{n-1},w_n), D(w_{n-1},w_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n,w_{n+1})^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1},w_n) + D(w_n,w_{n+1})\big]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1},w_n) + D(w_n,w_{n+1})\big]^{\frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda \max\Big[D(w_{n-1},w_n), D(w_{n-1},w_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n,w_{n+1})^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1},w_n) + D(w_n,w_{n+1})\big]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&\le \lambda \max\Big[D(w_{n-1},w_n), D(w_{n-1},w_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n,w_{n+1})^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1},w_n) \cdot D(w_n,w_{n+1})\big]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda \max\Big[D(w_{n-1},w_n), D(w_{n-1},w_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)} + \frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}} \\
&\quad \cdot D(w_n,w_{n+1})^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)} + \frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda \max\big\{D(w_{n-1},w_n), D(w_n,w_{n+1})\big\}.
\end{aligned}
$$

If $\max\{D(w_n,w_{n+1}), D(w_{n-1},w_n)\} = D(w_n,w_{n+1})$ then

$$D(w_n,w_{n+1}) \le \lambda D(w_n,w_{n+1}),$$

which is a contradiction. Thus we deduce that

$$D(w_n,w_{n+1}) \le \lambda D(w_{n-1},w_n), \tag{4.3}$$

and conclude that $\{D(w_{n-1},w_n)\}$ is a nonincreasing sequence with nonnegative terms. So there is a nonnegative constant ρ such that $\lim_{n\to\infty} D(w_{n-1},w_n) = \rho$. We shall establish that $\rho > 0$. Indeed, from (4.3), we derive that

$$D(w_n,w_{n+1}) \le \lambda D(w_{n-1},w_n) \le \lambda^n D(w_0,w_1). \tag{4.4}$$

The rest of the argument follows the lines of the proof of Theorem 2.3. □

Theorem 4.3 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A,D) and let T be a symmetric fractional α–η-contraction of type III satisfying the following conditions*:

(i) *T is an α-admissible mapping with respect to η;*
(ii) *there exists an $w_0 \in A$ such that $\alpha(w_0,Tw_0) \ge \eta(w_0,Tw_0)$;*
(iii) *there is a sequence $\{w_n\}$ in A such that $\alpha(w_n,w_{n+1}) \ge \eta(w_n,w_{n+1})$, $w_n \to w^*$ as $n \to \infty$, and that $\alpha(w_n,w^*) \ge \eta(w_n,w^*)$ holds for each $n \in \mathbb{N}$.*

Then T possesses a fixed point in A.

Proof Much as in the proof of Theorem 2.4, considering (iii), we have $\alpha(w_n,w^*) \ge \eta(w_n,w^*)$ for all $n \in \mathbb{N}$. By (D_3), we obtain

$$f\big(D\big(Tw^*,w^*\big)\big) \le f\big(D\big(Tw^*,Tw_n\big) + D\big(w_n,w^*\big)\big) + \mu.$$

Using (3.1) along with $(\mathcal{F}_1)$, we have

$$f\big(D\big(Tw^*,w^*\big)\big) \le f\big(\big(D\big(Tw^*,Tw_n\big)\big) + D\big(Tw_n,w^*\big)\big) + \mu$$
$$\le f\left(\lambda\left(\begin{array}{l} D\big(w^*,w_n\big), D\big(Tw^*,w^*\big)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n,Tw_n)^{\frac{\beta^2}{(\beta-\hat{w})(\beta-\gamma)}} \\ \cdot \big[D\big(Tw^*,w^*\big) + D(w_n,Tw_n)\big]^{\frac{\hat{w}^2}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \cdot \big[D\big(Tw_n,w^*\big) + D\big(w_n,Tw^*\big)\big]^{\frac{\gamma^2}{(\gamma-\beta)(\gamma-\hat{w})}} + D\big(w_{n+1},w^*\big) \end{array}\right)\right)$$
$$+ \mu.$$

Using (2.6) implies

$$\lim_{n\to\infty} D\big(w_n,w^*\big) = 0 \quad \text{as long as} \quad \lim_{n\to\infty} D\big(w_{n+1},w^*\big) = 0,$$

so we obtain

$$f\big(D\big(w^*,Tw^*\big)\big) \le f\big(D\big(w^*,Tw^*\big)\big) + \mu.$$

Utilizing $(\mathcal{F}_2)$, we have

$$\lim_{n\to\infty} f\big(D\big(w^*,Tw^*\big)\big) + \mu = -\infty,$$

which is a logical inconsistency. Hence $D(w^*,Tw^*) = 0$, that is, w^* a fixed point of T. □

5 Symmetric fractional α–η-contraction of type IV

In this part, we propose a new notion of symmetric fractional α–η-contraction of type IV in the framework of an $\mathcal{F}$-complete $\mathcal{F}$-metric space.

Definition 5.1 Consider an $\mathcal{F}$-metric space (A,D) with a self-mapping $T : A \to A$ and two functions $\alpha,\eta : A\times A \to [0,+\infty)$. We name T a symmetric fractional α–η-contraction of type IV if there are constants $\lambda \in [0,1)$ and $\beta,\hat{w},\gamma \in (0,1)$ with $\beta + \hat{w} + \gamma < 1$ such that, whenever $\alpha(w,\upsilon) \ge \eta(w,\upsilon)$, we have

$$D(Tw,T\upsilon) \le \lambda\big(\check{S}_4(w,\upsilon)\big), \tag{5.1}$$

where

$$\check{S}_4(w,\upsilon) = \lambda\left\{\begin{array}{l} D(w,\upsilon)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w,Tw)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \big[D(w,Tw) + D(\upsilon,T\upsilon)\big]^{\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \cdot \big[D(w,T\upsilon) + D(\upsilon,Tw)\big]^{\frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}} \end{array}\right\},$$

for all $w,\upsilon \in A\setminus \operatorname{Fix}(T)$.

Now we state and prove our next theorem.

Theorem 5.2 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A,D) along with a symmetric fractional α–η-contraction T of type IV that satisfies the following conditions*:

(i) *T is a α-admissible mapping with respect to η;*
(ii) *there exist a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$;*
(iii) *T is α–η-continuous.*

Then T possesses a fixed point in A.

Proof Let w_0 in A be such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$. For this $w_0 \in A$, we build a sequence $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1 = Tw_0$, $w_2 = Tw_1 = T^2 w_0$. Proceeding this way, $w_{n+1} = Tw_n = T^{n+1}w_0$, for every $n \in \mathbb{N}$. Since the mapping T is α-admissible with respect to η, $\alpha(w_0, w_1) = \alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0) = \eta(w_0, w_1)$. Carrying on in this way, we get

$$\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n) = \eta(w_{n-1}, Tw_{n-1}), \quad \text{for each } n \in \mathbb{N}. \tag{5.2}$$

If $w_{n+1} = w_n$ for some $n \in \mathbb{N}$ then $w_n = w^*$ is a fixed point of T. So, we assume that $w_n \neq w_{n+1}$ and

$$D(Tw_{n-1}, Tw_n) = D(w_n, Tw_n) > 0, \quad \text{for all } n \in \mathbb{N}.$$

As T is a symmetric fractional α–η-contraction of type IV, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
D(w_n, w_{n+1}) &= D(Tw_{n-1}, Tw_n) \\
&\leq \lambda\Big[D(w_{n-1}, w_n)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_{n-1}, Tw_{n-1})^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n)\big]^{\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1})\big]^{\frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda\Big[D(w_{n-1}, w_n)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\big]^{\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_{n+1}) + D(w_n, w_n)\big]^{\frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&\leq \lambda\Big[D(w_{n-1}, w_n)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\big]^{\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\big]^{\frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&= \lambda\Big[D(w_{n-1}, w_n)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \big[D(w_{n-1}, w_n) + D(w_n, w_{n+1})\big]^{\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}}\Big] \\
&\leq \lambda\left[\begin{array}{l} D(w_{n-1}, w_n)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(w_n, w_{n+1})^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}} \\ \quad \cdot \big[D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})\big]^{\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}} \end{array}\right] \\
&= \lambda\Big[D(w_{n-1}, w_n)^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)} + \frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)} + \frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}}
\end{aligned}$$

$$\left.\cdot D(w_n, w_{n+1})^{\frac{\beta^3}{(\beta-\hat{w})(\beta-\gamma)}+\frac{\hat{w}^3}{(\hat{w}-\beta)(\hat{w}-\gamma)}+\frac{\gamma^3}{(\gamma-\beta)(\gamma-\hat{w})}}\right]$$
$$= \lambda\big\{D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})\big\}^{\beta+\hat{w}+\gamma}$$
$$\leq \lambda \max\big\{D(w_{n-1}, w_n), D(w_n, w_{n+1})\big\}.$$

If $\max\{D(w_n, w_{n+1}), D(w_{n-1}, w_n)\} = D(w_n, w_{n+1})$ then

$$D(w_n, w_{n+1}) \leq \lambda D(w_n, w_{n+1}),$$

which is a contradiction. So we deduce that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n). \tag{5.3}$$

The rest of the argument follows the proof of Theorem 2.3. □

Theorem 5.3 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A, D) and let T be a symmetric fractional α–η-contraction of type IV fulfilling the following conditions*:

(i) *T is a α-admissible mapping with respect to η*;

(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$*;

(iii) *there is a sequence $\{w_n\}$ in A satisfying $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$, such that $w_n \to w^*$ as $n \to \infty$, and $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$.*

Then T possesses a fixed point in A.

Taking $\eta(w, \upsilon) = 1$ in Theorems 2.3, 2.4, 3.2, and 3.3, we obtain the following corollaries.

Corollary 5.4 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A, D) and let T be a symmetric fractional α–η-contraction of type I fulfilling the following conditions*:

(i) *T is a α-admissible mapping*;

(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq 1$*;

(iii) *T is α–η-continuous.*

Then T has a fixed point in A.

Corollary 5.5 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A, D) and let T be a symmetric fractional α–η-contraction of type I fulfilling the following conditions*:

(i) *T is a α-admissible mapping*;

(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq 1$*;

(iii) *there is a sequence $\{w_n\}$ in A satisfying $\alpha(w_n, w_{n+1}) \geq 1$, such that $w_n \to w^*$ as $n \to \infty$, and $\alpha(w_n, w^*) \geq 1$ holds for each $n \in \mathbb{N}$.*

Then T possesses a fixed point in A.

Corollary 5.6 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A, D) and let T be a symmetric fractional α–η-contraction of type II satisfying the following conditions*:

(i) *T is a α-admissible mapping*;

(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq 1$*;

(iii) *T is α–η-continuous.*

Then T has a fixed point in A.

Corollary 5.7 *Consider an $\mathcal{F}$-complete $\mathcal{F}$-metric space (A,D) and let T be a symmetric fractional α–η-contraction of type II satisfying the following conditions*:

(i) *T is a α-admissible mapping*;

(ii) *there exists a $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq 1$*;

(iii) *there is a sequence $\{w_n\}$ in A satisfying $\alpha(w_n, w_{n+1}) \geq 1$ such that $w_n \to w^*$ as $n \to \infty$, and $\alpha(w_n, w^*) \geq 1$ holds for each $n \in \mathbb{N}$.*

Then T possesses a fixed point in A.

In a similar fashion, we can deduce analogues of Corollaries 5.4, 5.5, 5.6, and 5.7 for a symmetric fractional α–η-contraction of type III and IV, respectively.

6 Consequences

As a consequence of our results, we derive some results for Suzuki-type contractions, orbitally T-complete and orbitally continuous mappings in $\mathcal{F}$-metric spaces.

Theorem 6.1 *Consider an $\mathcal{F}$-metric space (A,D) and let T be a continuous self-mapping on A. Assume that there are $r \in [0,1)$ and $\beta, \hat{w}, \gamma \in (0,1)$ such that*

$$D(w,Tw) \leq D(w,\upsilon)) \quad \textit{implies} \quad D(Tw,T\upsilon) \leq r\big(\check{S}_1(w,\upsilon)\big), \tag{6.1}$$

where

$$\check{S}_1(w,\upsilon) = D(w,\upsilon) \cdot D(w,Tw)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(\upsilon,T\upsilon)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ \cdot \big[D(w,Tw) + D(\upsilon,T\upsilon)\big]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w,T\upsilon) + D(\upsilon,Tw)\big]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}$$

for all $w, \upsilon \in A \setminus \operatorname{Fix}(T)$.

Then T possesses a fixed point in A.

Proof Describe $\alpha, \eta : A \times A \to [0,+\infty)$ by

$$\alpha(w,\upsilon) = D(w,\upsilon) \quad \text{and} \quad \eta(w,\upsilon) = D(w,Tw), \quad \text{for all } w,\upsilon \in A,$$

and $\beta, \hat{w}, \gamma \in (0,1)$, as well as $r \in [0,1)$. It is clear that

$$\eta(w,\upsilon) \leq \alpha(w,\upsilon), \quad \text{for all } w,\upsilon \in A,$$

that is, conditions (i)–(iii) of our Theorem 2.3 hold true. If

$$\eta(w,Tw) \leq \alpha(w,\upsilon) \quad \text{then } D(w,Tw) \leq D(w,\upsilon),$$

which implies the contractive condition

$$D(Tw,T\upsilon) \leq r\big(\check{S}_1(w,\upsilon)\big).$$

Finally, every assumption of Theorem 2.3 holds true. Hence T possesses a fixed point in A. □

Theorem 6.2 *Consider an $\mathcal{F}$-metric space (A, D) and let T be a self-mapping of A. Suppose the following assumptions hold*:

(i) *(A, D) is an orbitally T-complete $\mathcal{F}$-metric space*;

(ii) *there exist $r \in [0, 1)$ and $\beta, \hat{w}, \gamma \in (0, 1)$ such that*

$$D(Tw, T\upsilon) \leq r\big(\check{S}_1(w, \upsilon)\big),$$

where

$$\begin{aligned}\check{S}_1(w, \upsilon) = {} & D(w, \upsilon) \cdot D(w, Tw)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(\upsilon, T\upsilon)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ & \cdot \big[D(w, Tw) + D(\upsilon, T\upsilon)\big]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w, T\upsilon) + D(\upsilon, Tw)\big]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\end{aligned}$$

for all $w, \upsilon \in O(\omega)$ for some $\omega \in A$, where $O(\omega)$ is an orbit of ω;

(iii) *if $\{\upsilon_n\}$ is a sequence such that $\{\upsilon_n\} \subseteq O(\omega)$ with $\upsilon_n \to \upsilon^*$ as $n \to \infty$ then $\upsilon^* \in O(\omega)$.*

Then T possesses a fixed point.

Proof Describe $\alpha, \eta : A \times A \to [0, +\infty)$ by setting $\alpha(w, \upsilon) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(w, \upsilon) = 0$ otherwise, and $\eta(w, \upsilon) = 1$ for all $w, \upsilon \in A$ (see [7, Remark 6]). Then (A, D) is an α–η-complete $\mathcal{F}$-metric space and T is an α-admissible mapping with respect to η. If $\alpha(w, \upsilon) \geq \eta(w, \upsilon)$ then $w, \upsilon \in O(\omega)$, and so from (ii) we have

$$D(Tw, T\upsilon) \leq r\big(\check{S}_1(w, \upsilon)\big),$$

where

$$\begin{aligned}\check{S}_1(w, \upsilon) = {} & D(w, \upsilon) \cdot D(w, Tw)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(\upsilon, T\upsilon)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ & \cdot \big[D(w, Tw) + D(\upsilon, T\upsilon)\big]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w, T\upsilon) + D(\upsilon, Tw)\big]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}},\end{aligned}$$

that is, T is a symmetric fractional α–η-contraction of type I. Let $\{\upsilon_n\}$ be a sequence such that $\alpha(\upsilon_n, \upsilon_{n+1}) \geq \eta(\upsilon_n, \upsilon_{n+1})$ and $\upsilon_n \to \upsilon^*$ for $n \to \infty$. So, $\{\upsilon_n\} \subseteq O(\omega)$. From (iii), $\upsilon^* \in O(\omega)$, that is, $\alpha(\upsilon_n, \upsilon^*) \geq \eta(\upsilon_n, \upsilon^*)$. Hence every assumption of Theorem 2.4 holds true. Thus T possesses a fixed point. □

Theorem 6.3 *Consider an $\mathcal{F}$-metric space (A, D) and let T be a self-mapping of A. Suppose the following conditions hold*:

(i) *for all $w, \upsilon \in O(\omega)$, there are $r \in [0, 1)$ and $\beta, \hat{w}, \gamma \in (0, 1)$ such that*

$$D(Tw, T\upsilon) \leq r\big(\check{S}_1(w, \upsilon)\big),$$

where

$$\begin{aligned}\check{S}_1(w, \upsilon) = {} & D(w, \upsilon) \cdot \big[D(w, Tw)\big]^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \big[D(\upsilon, T\upsilon)\big]^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ & \cdot \big[D(w, Tw) + D(\upsilon, T\upsilon)\big]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w, T\upsilon) + D(\upsilon, Tw)\big]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}}\end{aligned}$$

for some $\omega \in A$;

(ii) *the operator T is orbitally continuous.*
Then T possesses a fixed point.

Proof Describe $\alpha, \eta : A \times A \to [0, +\infty)$ by setting $\alpha(w, \upsilon) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(w, \upsilon) = 0$ otherwise, with $\eta(w, \upsilon) = 1$ (see [38, Remark 1.1]). We know that T is an α–η-continuous mapping. If $\alpha(w, \upsilon) \geq \eta(w, \upsilon)$ then $w, \upsilon \in O(\omega)$. So $Tw, T\upsilon \in O(\omega)$, that is, $\alpha(Tw, T\upsilon) \geq \eta(Tw, T\upsilon)$. Therefore T is an α-admissible mapping with respect to η. From (i) we have

$$D(Tw, T\upsilon) \leq r\big(\check{S}_1(w, \upsilon)\big),$$

where

$$\begin{aligned}\check{S}_1(w, \upsilon) = {} & D(w, \upsilon) \cdot D(w, Tw)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \cdot D(\upsilon, T\upsilon)^{\frac{1}{(\beta-\hat{w})(\beta-\gamma)}} \\ & \cdot \big[D(w, Tw) + D(\upsilon, T\upsilon)\big]^{\frac{1}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w, T\upsilon) + D(\upsilon, Tw)\big]^{\frac{1}{(\gamma-\beta)(\gamma-\hat{w})}},\end{aligned}$$

implying that T is a symmetric fractional α–η-contraction of type I. Hence each assumption of Theorem 2.3 holds true. Thus T has a fixed point. □

Theorem 6.4 *Consider an $\mathcal{F}$-metric space (A, D) and let T be a self-mapping of A. Suppose the following conditions hold*:

(i) *(A, D) is an orbitally T-complete $\mathcal{F}$-metric space*;

(ii) *there exist $r \in [0, 1)$ and $\beta, \hat{w}, \gamma \in (0, 1)$ such that*

$$D(Tw, T\upsilon) \leq r\big(\check{S}_2(w, \upsilon)\big),$$

where

$$\begin{aligned}\check{S}_2(w, \upsilon) = {} & D(w, \upsilon) \cdot \big[D(w, Tw)\big]^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \big[D(\upsilon, T\upsilon)\big]^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\ & \cdot \big[D(w, Tw) + D(\upsilon, T\upsilon)\big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w, T\upsilon) + D(\upsilon, Tw)\big]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}\end{aligned}$$

for all $w, \upsilon \in O(\omega)$, for some $\omega \in A$, where $O(\omega)$ is an orbit of ω;

(iii) *if $\{\upsilon_n\}$ is a sequence such that $\{\upsilon_n\} \subseteq O(\omega)$ with $\upsilon_n \to \upsilon^*$ as $n \to \infty$ then $\upsilon^* \in O(\omega)$.*

Then, T possesses a fixed point.

Theorem 6.5 *Consider an $\mathcal{F}$-metric space (A, D) and let T be a self-mapping of A. Suppose the following conditions hold*:

(i) *for all $w, \upsilon \in O(\omega)$, there exist $r \in [0, 1)$ and $\beta, \hat{w}, \gamma \in (0, 1)$ such that*

$$D(Tw, T\upsilon) \leq r\big(\check{S}_2(w, \upsilon)\big),$$

where

$$\begin{aligned}\check{S}_2(w, \upsilon) = {} & D(w, \upsilon) \cdot \big[D(w, Tw)\big]^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \big[D(\upsilon, T\upsilon)\big]^{\frac{\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\ & \cdot \big[D(w, Tw) + D(\upsilon, T\upsilon)\big]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \big[D(w, T\upsilon) + D(\upsilon, Tw)\big]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}\end{aligned}$$

for some $\omega \in A$;

(ii) *the operator T is an orbitally continuous.*

Then T possesses a fixed point.

Theorems 6.1–6.3 can be derived easily for symmetric fractional contractions of type III and IV, respectively.

7 Application to fractional differential equations

The local and nonlocal fractional differential equations have recently proved to be significant tools in the modeling of many phenomena in numerous fields of science and engineering. The fractional-order differential equations have numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetics, etc. For more details, see [9–25]. Our aim is to show the existence and uniqueness of a bounded solution to a boundary value problem involving a generalized fractional derivative in the Riemann–Liouville sense.

Actually, the left Riemann–Liouville fractional integral of a Lebesgue-integrable function f with respect to an increasing function g is given as follows [10]:

$$ {}_aI_g^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \bigl(g(t)-g(s)\bigr)^{\alpha-1} f(s)g'(s)\,ds, \quad \text{where } \alpha > 0. \tag{7.1} $$

The associated left Riemann–Liouville fractional derivative of f with respect to the same increasing function g is given by [10]

$$ \begin{aligned} {}_aD_g^{\alpha} f(t) &= \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n I^{(n-\alpha)} f(t) \\ &= \left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n \frac{1}{\Gamma(\alpha)} \int_a^t \bigl(g(t)-g(s)\bigr)^{n-\alpha-1} f(s)g'(s)\,ds, \end{aligned} \tag{7.2} $$

where $\alpha \geq 0$, $n = [\alpha]+1$, and $[\alpha]$ is the integer part of α. The following theorem combines the fractional integral and derivative.

Theorem 7.1 ([11]) *Let $\alpha > 0$, $n = -[-\alpha]$, $f \in L[a,b]$, and ${}_aI_g^{\alpha} f \in AC_g^n[a,b]$. Then*

$$ {}_aI_{ga}^{\alpha}D_g^{\alpha} f(t) = f(t) - \sum_{k=1}^{n} c_k \bigl(g(t)-g(a)\bigr)^{\alpha-k}. $$

We are considering the following boundary value problem:

$$ {}_aD_g^{\alpha} y(t) + f\bigl(t, y(t)\bigr) = 0, \qquad y(a) = y(b) = 0, \quad 1 < \alpha \leq 2. \tag{7.3} $$

Lemma 7.2 *Let $\alpha > 0$, $n = -[-\alpha]$, $f \in L[a,b]$, and ${}_aI_g^{\alpha} f \in AC_g^n[a,b]$. Then, y is a solution of the boundary value problem (7.3) if and only if*

$$ y(t) = \int_a^b G(s,t) f\bigl(s, y(s)\bigr) g'(s)\,ds, $$

where the Green's function

$$G(s,t)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\frac{(g(b)-g(s))(g(t)-g(a))}{(g(b)-g(a))})^{\alpha-1}-(g(t)-g(s))^{\alpha-1}, & a<s\leq t,\\ (\frac{(g(b)-g(s))(g(t)-g(a))}{(g(b)-g(a))})^{\alpha-1}, & t\leq s<b,\end{cases}$$

satisfies the following:

- $G(s,t)\geq 0$;
- $\max_{a\leq s,t\leq b}G(s,t)=\frac{1}{\Gamma(\alpha)}(\frac{g(b)-g(a)}{4})^{\alpha-1}$.

Proof Applying the integral in (7.1) to (7.3), we get

$${}_aI^{\alpha}_{g\,a}D^{\alpha}_{g}y(t)=-{}_aI^{\alpha}_{g}f\bigl(t,y(t)\bigr)=-\frac{1}{\Gamma(\alpha)}\int_a^t\bigl(g(t)-g(s)\bigr)^{\alpha-1}f(s)g'(s)\,ds.$$

Now using Theorem 7.1, we obtain

$$y(t)=c_1\bigl(g(t)-g(a)\bigr)^{\alpha-1}+c_2\bigl(g(t)-g(a)\bigr)^{\alpha-2}-\frac{1}{\Gamma(\alpha)}\int_a^t\bigl(g(t)-g(s)\bigr)^{\alpha-1}f(s)g'(s)\,ds.$$

Then $y(a)=0$ gives $c_2=0$, while $y(b)=0$ gives

$$c_1=\frac{(g(b)-g(a))^{1-\alpha}}{\Gamma(\alpha)}\int_a^b\bigl(g(b)-g(s)\bigr)^{\alpha-1}f\bigl(s,y(s)\bigr)g'(s)\,ds.$$

Therefore

$$\begin{aligned}y(t)&=\frac{1}{\Gamma(\alpha)}\int_a^b\frac{(g(b)-g(s))(g(t)-g(a))^{\alpha-1}}{(g(b)-g(a))}f\bigl(s,y(s)\bigr)g'(s)\,ds\\&\quad-\frac{1}{\Gamma(\alpha)}\int_a^t\bigl(g(t)-g(s)\bigr)^{\alpha-1}f\bigl(s,y(s)\bigr)g'(s)\,ds.\end{aligned}$$

Hence

$$y(t)=\int_a^b G(s,t)f\bigl(s,y(s)\bigr)g'(s)\,ds,$$

$$\text{where } G(s,t)=\frac{1}{\Gamma(\alpha)}\begin{cases}(\frac{(g(b)-g(s))(g(t)-g(a))}{(g(b)-g(a))})^{\alpha-1}-(g(t)-g(s))^{\alpha-1}, & a<s\leq t,\\ (\frac{(g(b)-g(s))(g(t)-g(a))}{(g(b)-g(a))})^{\alpha-1}, & t\leq s<b.\end{cases}$$

It is clear that $G(s,t)\geq 0$, when $s\geq t$.

For $a\leq s<t$, one can prove that

$$\begin{aligned}G(s,t)&=\left(\frac{(g(t)-g(a))}{(g(b)-g(a))}\right)^{\alpha-1}\\&\quad\cdot\left[\bigl(g(b)-g(a)\bigr)^{\alpha-1}-\left(g(b)-\left(g(a)+\frac{(g(s)-g(a))(g(b)-g(a))}{(g(t)-g(a))}\right)^{\alpha-1}\right)\right].\end{aligned}$$

Since $g(a)+\frac{(g(s)-g(a))(g(b)-g(a))}{(g(t)-g(a))}\geq g(s)$, one can deduce that $G(s,t)\geq 0$, for $s\leq t$.

For $t \leq s$, we have

$$\frac{\partial G}{\partial t} = \frac{1}{\Gamma(\alpha)}\left(\frac{g(b)-g(s)}{(g(b)-g(a))}\right)^{\alpha-1} \cdot (\alpha-1)\big(g(t)-g(a)\big)^{\alpha-2} g'(t) \geq 0,$$

thus $G(s,t)$ is increasing as a function of t.

For $s \leq t$,

$$\begin{aligned}
\frac{\partial G}{\partial t} &= \frac{g'(t)(\alpha-1)}{\Gamma(\alpha)} \cdot \left[-\big(g(t)-g(s)\big)^{\alpha-2} + \left(\frac{g(b)-g(s)}{g(b)-g(a)}\right)^{\alpha-1}\big(g(t)-g(a)\big)^{\alpha-2}\right] \\
&= \frac{g'(t)}{\Gamma(\alpha-1)} \cdot \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\alpha-2} \\
&\quad \cdot \left[\left(\frac{g(b)-g(s)}{g(b)-g(a)}\right)^{\alpha-1} - \left(\frac{(g(b)-g(a))(g(t)-g(s))}{g(b)-g(a)}\right)^{\alpha-2}\right] \\
&\leq \frac{g'(t)}{\Gamma(\alpha-1)} \cdot \left(\frac{g(t)-g(a)}{g(b)-g(a)}\right)^{\alpha-2} \\
&\quad \cdot \left[\big(g(b)-g(a)\big)^{\alpha-2} - \left(g(b) - \left(g(a) + \frac{g(b)-g(a)}{g(t)-g(a)}\big(g(s)-g(a)\big)\right)^{\alpha-2}\right)\right] \\
&< 0.
\end{aligned}$$

Thus $G(s,t)$ is decreasing when $s \leq t$. Hence $G(s,t)$ attains its maximum when $s = t$, and

$$\begin{aligned}
G(s,s) &= \frac{1}{\Gamma(\alpha)} \frac{(g(b)-g(s))^{\alpha-1}(g(s)-g(a))^{\alpha-1}}{(g(b)-g(a))^{\alpha-1}} = \hat{G}(s), \\
\hat{G}'(s) &= -\frac{1}{\Gamma(\alpha)}(\alpha-1)\frac{(g(b)-g(s))^{\alpha-2}}{(g(b)-g(a))^{\alpha-1}} g'(s) \cdot \big(g(s)-g(a)\big)^{\alpha-1} \\
&\quad + \frac{1}{\Gamma(\alpha)} \frac{(g(b)-g(s))^{\alpha-1}(\alpha-1)(g(s)-g(a))^{\alpha-2} g'(s)}{(g(b)-g(a))^{\alpha-1}} \\
&= 0
\end{aligned}$$

yield $g(s) = \frac{g(a)+g(b)}{2}$, or that the critical point is

$$s^* = g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

Thus, the maximum of $G(s,t)$ is

$$\begin{aligned}
\hat{G}(s^*) &= \frac{1}{\Gamma(\alpha)}\left(\frac{g(b)-g(a)}{4}\right)^{\alpha-1}, \\
|G(s,t)| &\leq \frac{1}{\Gamma(\alpha)}\left(\frac{g(b)-g(a)}{4}\right)^{\alpha-1}.
\end{aligned}$$

□

Here we denote the Riemann–Stieltjes integrable function w with respect to s and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $C(I)$ be the linear space of all continuous functions defined on $I = [0,1]$, and let $D(w,\upsilon) = \|w-\upsilon\|_\infty^2 = \max_{t\in I} |w(t)-\upsilon(t)|^2$ for all $w,\upsilon \in C(I)$. Then $(C(I),D)$ is an $\mathcal{F}$-complete metric space.

We consider the following conditions:

(a) there exist $r \in [0,1)$ and $\zeta\colon \mathbb{R}^2 \to \mathbb{R}$, a function such that for each $a,b \in \mathbb{R}$ with $\zeta(a,b) \geq \xi(a,b)$ satisfies

$$\begin{aligned} &\bigl|f\bigl(s,w(s)\bigr) - f\bigl(s,\upsilon(s)\bigr)\bigr| \\ &\quad \leq \bigl|w(s)-\upsilon(s)\bigr|^2 \cdot \bigl|w(s)-Tw(s)\bigr|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \bigl|\upsilon(s)-T\upsilon(s)\bigr|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\ &\qquad \cdot \bigl[\bigl|w(s)-Tw(s)\bigr| + \bigl|\upsilon(s)-T\upsilon(s)\bigr|\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ &\qquad \cdot \bigl[\bigl|w(s)-T\upsilon(s)\bigr| + \bigl|\upsilon(s)-Tw(s)\bigr|\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}, \end{aligned}$$

where $\beta, \hat{w}, \gamma \in (0,1)$;

(b) there exists $w_1 \in C(I)$ such that

$$\zeta\left(w_1(t), \int_a^b G(t,s)f\bigl(s,w_1(s)\bigr)g'(s)\,ds\right) \geq \xi\left(w_1(t), \int_a^b G(t,s)f\bigl(s,w_1(s)\bigr)g'(s)\,ds\right),$$

for all $t \in I$;

(c) for each $w, \upsilon \in C(I)$, there exist $w_1, v_1 \in C(I)$ such that $\zeta(w(t),\upsilon(t)) \geq \xi(w(t),\upsilon(t))$ implies

$$\begin{aligned} &\zeta\left(\int_a^b G(t,s)f\bigl(s,w_1(s)\bigr)g'(s)\,ds, \int_a^b G(t,s)f\bigl(s,\upsilon_1(s)\bigr)g'(s)\,ds\right) \\ &\quad \geq \xi\left(\int_a^b G(t,s)f\bigl(s,w_1(s)\bigr)g'(s)\,ds, \int_a^b G(t,s)f\bigl(s,\upsilon_1(s)\bigr)g'(s)\,ds\right), \end{aligned}$$

for all $t \in I$;

(d) for any cluster point w of a sequence $\{w_n\}$ of points in $C(I)$ with

$$\zeta(w_n, w_{n+1}) \geq \xi(w_n, w_{n+1}), \qquad \lim_{n\to\infty} \inf \zeta(w_n, w) \geq \lim_{n\to\infty} \inf \xi(w_n, w).$$

Theorem 7.3 *Suppose that conditions* (a)–(d) *are satisfied. Then* (7.3) *has at least one solution* $w^* \in C(I)$.

Proof We know that $w \in C(I)$ is a solution of (7.3) if and only if $w \in C(I)$ is a solution of the fractional-order integral equation

$$w(t) = \lambda \int_a^b G(t,s)f\bigl(s,w(s)\bigr)g'(s)\,ds \quad \text{for all } t \in I,$$

where $\lambda, \in [0,1)$. We define a map $T : C(I) \to C(I)$ by

$$Tw(t) = \lambda \int_a^b G(t,s)f\bigl(s,w(s)\bigr)g'(s)\,ds \quad \text{for all } t \in I.$$

Then problem (7.3) is equivalent to finding $w^* \in C(I)$ that is a fixed point of T. Let $w, \upsilon \in C(I)$ be such that $\zeta(w(t), \upsilon(t)) \geq 0$ for all $t \in I$. For using condition (a), we get

$$\begin{aligned}
\left|Tw(t) - T\upsilon(t)\right| &= \left|\lambda \int_a^b G(t,s)\bigl[f\bigl(s, w(s)\bigr) - f\bigl(s, \upsilon(s)\bigr)\bigr] g'(s)\, ds\right| \\
&\leq |\lambda| \int_a^b \left|G(t,s)\right| \left|f\bigl(s, w(s)\bigr) - f\bigl(s, \upsilon(s)\bigr) g'(s)\, ds\right| \\
&\leq |\lambda| \int_a^b \left|G(t,s)\right| g'(s) r\, ds \left|w(s) - \upsilon(s)\right|^2 \\
&\quad \cdot \left|w(s) - Tw(s)\right|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \left|\upsilon(s) - T\upsilon(s)\right|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \bigl[\left|w(s) - Tw(s)\right| + \left|\upsilon(s) - T\upsilon(s)\right|\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \bigl[\left|w(s) - T\upsilon(s)\right| + \left|\upsilon(s) - Tw(s)\right|\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}] \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\frac{g(b) - g(a)}{4}\right)^{\alpha-1} \bigl(g(b) - g(a)\bigr) \left\|w(s) - \upsilon(s)\right\|_\infty^2 \\
&\quad \cdot \left\|w(s) - Tw(s)\right\|_\infty^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \left\|\upsilon(s) - T\upsilon(s)\right\|_\infty^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \bigl[\left\|w(s) - Tw(s)\right\|_\infty^2 + \left\|\upsilon(s) - T\upsilon(s)\right\|_\infty^2\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \bigl[\left\|w(s) - T\upsilon(s)\right\|_\infty^2 + \left\|\upsilon(s) - Tw(s)\right\|_\infty^2\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}} \\
&\leq r \left\|w(s) - \upsilon(s)\right\|_\infty^2 \cdot \left\|w(s) - Tw(s)\right\|_\infty^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
&\quad \cdot \left\|\upsilon(s) - T\upsilon(s)\right\|_\infty^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \left\|w(s) - T\upsilon(s)\right\|_\infty^{2(p\gamma - q\gamma)} \\
&\quad \cdot \bigl[\left\|w(s) - Tw(s)\right\|_\infty^2 + \left\|\upsilon(s) - T\upsilon(s)\right\|_\infty^2\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\
&\quad \cdot \bigl[\left\|w(s) - T\upsilon(s)\right\|_\infty^2 + \left\|\upsilon(s) - Tw(s)\right\|_\infty^2\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}.
\end{aligned}$$

Thus

$$\begin{aligned}
D(Tw, T\upsilon) < {} & \left|w(s) - \upsilon(s)\right|^2 \cdot \left|w(s) - Tw(s)\right|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\
& \cdot \left|\upsilon(s) - T\upsilon(s)\right|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot \bigl[\left|w(s) - Tw(s)\right| \\
& + \left|\upsilon(s) - T\upsilon(s)\right|\bigr]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \cdot \bigl[\left|w(s) - T\upsilon(s)\right| + \left|\upsilon(s) - Tw(s)\right|\bigr]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}
\end{aligned}$$

for all $w, \upsilon \in C(I)$ such that $\zeta(w(t), \upsilon(t)) \geq \xi(w(t), \upsilon(t))$ for all $t \in I$. We define $\alpha : C(I) \times C(I) \to [0, \infty)$ by

$$\alpha(w, \upsilon) = \begin{cases} 1 & \text{if } \zeta(w(t), \upsilon(t)) \geq 0, t \in I, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and}$$

$$\eta(w, \upsilon) = \begin{cases} \frac{1}{2} & \text{if } \xi(w(t), \upsilon(t)) \geq 0, t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $w, \upsilon \in C(I)$, $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$, we have

$$D(Tw, T\upsilon) \leq r \begin{cases} |w(s)-\upsilon(s)|^2 \cdot |w(s)-Tw(s)|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \cdot |\upsilon(s)-T\upsilon(s)|^{\frac{2\beta}{(\beta-\hat{w})(\beta-\gamma)}} \\ \quad \cdot [|w(s)-Tw(s)| + |\upsilon(s)-T\upsilon(s)|]^{\frac{\hat{w}}{(\hat{w}-\beta)(\hat{w}-\gamma)}} \\ \quad \cdot [|w(s)-T\upsilon(s)| + |\upsilon(s)-Tw(s)|]^{\frac{\gamma}{(\gamma-\beta)(\gamma-\hat{w})}}. \end{cases}$$

Obviously, $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$ for all $w,\upsilon \in C(I)$. If $\alpha(w,\upsilon) \geq \eta(w,\upsilon)$ for each $w,\upsilon \in C(I)$ then $\zeta(w(t),\upsilon(t)) \geq \xi(w(t),\upsilon(t))$. From condition (c), we have $\zeta(Tw(t),T\upsilon(t)) \geq \xi(Tw(t),T\upsilon(t))$, and so $\alpha(Tw,T\upsilon) \geq \eta(Tw,T\upsilon)$. Thus, T is an α-admissible map with respect to η. From condition (b), there exists $w_1 \in C(I)$ such that $\alpha(w_1,Tw_1) = \eta(w_1,Tw_1)$. By condition (d), we have that for any cluster point w of a sequence $\{w_n\}$ of points in $C(I)$ with $\alpha(w_n,w_{n+1}) = \eta(w_n,w_{n+1})$, $\lim_{n\to\infty} \inf \alpha(w_n,w) = \lim_{n\to\infty} \inf \eta(w_n,w)$. By applying Theorem 2.3, T has a fixed point in $C(I)$, i.e., there exists $w^* \in C(I)$ such that $Tw^* = w^*$, and w^* is a solution of (7.3). □

Applications The fractional-order differential equations emerge in various areas of engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, control theory, biology, economics, blood flow phenomena, signal and image processing, biophysics, aerodynamics, fitting of experimental data.

8 Conclusions

The aim of this paper was to introduce four classes of symmetric fractional contraction. This research focuses on a new idea of symmetric fractional $\alpha-\eta$-contraction of type I, II, III, and IV in the setting of an $\mathcal{F}$-metric space, which is different and more general than an ordinary metric space. This paper will open a new domain of fixed-point theory. We develop here Suzuki-type fixed point results in orbitally complete $\mathcal{F}$-metric spaces. These new investigations and applications will enhance the impact of the new setup.

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