OPEN ACCESS
Fractals, Vol. 28, No. 8 (2020) 2040029 (18 pages)

DOI: 10.1142/S0218348X20400290

APPROXIMATE ENDPOINT SOLUTIONS FOR A CLASS OF FRACTIONAL *q*-DIFFERENTIAL INCLUSIONS BY COMPUTATIONAL RESULTS

S. M. AYDOGAN,* J. F. GÓMEZ AGUILAR,†,††,§§ D. BALEANU,‡,¶,**,‡‡,§§
SH. REZAPOUR§,¶ and M. E. SAMEI‖

**Department of Mathematics*
Istanbul Technical University, Istanbul, Turkey

†CONACyT-Tecnológico Nacional de México/CENIDET
Interior Internado Palmira S/N, Col. Palmira
C.P. 62490, Cuernavaca, Morelos, México

‡Department of Mathematics, Cankaya University
Ogretmenler Cad. 14 06530, Balgat, Ankara, Turkey

§Department of Mathematics
Azarbaijan Shahid Madani University, Tabriz, Iran

¶Department of Medical Research
China Medical University Hospital
China Medical University, Taichung, Taiwan

‖Department of Mathematics
Faculty of Basic Science, Bu-Ali Sina University
Hamedan 65178, Iran

***Institute of Space Sciences, Magurele, Bucharest, Romania*

††jose.ga@cenidet.tecnm.mx
‡‡aumitru.baleanu@gmail.com

Received December 19, 2019
Accepted February 16, 2020
Published June 13, 2020

§§Corresponding authors.

Abstract

By using the notion of endpoints for set-valued functions and some classical fixed point techniques, we investigate the existence of solutions for two fractional q-differential inclusions under some integral boundary value conditions. By providing an example, we illustrate our main result about endpoint. Also, we give some related algorithms and numerical results.

Keywords: Approximate Endpoint; Boundary Value Condition; Caputo q-Derivation; Fractional q-Differential Inclusion.

1. INTRODUCTION

In 1910, Jackson introduced the subject of q-difference equations.[1] Later, many researchers studied q-difference equations.[2–13] On the other hand, many modern works recently appeared on integro-differential equations by using different views and fractional derivatives which young researchers could use the works (see, for example Refs. 14–27).

In 2012, Ahmad *et al.* studied the existence and uniqueness of solutions for the fractional q-difference equations ${}^cD_q^\alpha u(t) = T(t,u(t))$ with boundary conditions $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$ and $\alpha_2 u(1) - \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, where $\alpha \in (1,2]$, $\alpha_i, \beta_i, \gamma_i,\ \eta_i$ are real numbers, for $i = 1,2$ and $T \in C(J \times \mathbb{R}, \mathbb{R})$.[6] In 2013, Zhao *et al.* reviewed the q-integral problem $(D_q^\alpha u)(t) + f(t,u(t)) = 0$ with boundary conditions $u(1) = \mu I_q^\beta u(\eta)$ and $u(0) = 0$ for almost all $t \in (0,1)$, where $q \in (0,1)$, $\alpha \in (1,2]$, $\beta \in (0,2]$, $\eta \in (0,1)$, μ is positive real number, D_q^α is the q-derivative of Riemann–Liouville and real-values continuous map u defined on $I \times [0,\infty)$.[13] In 2014, Ahmad *et al.* investigated the problem ${}^cD_q^\beta({}^cD_q^\gamma + \lambda)u(t) = pf(t,u(t)) + kI_q^\xi g(t,u(t))$ with boundary conditions $\alpha_1 u(0) - \beta_1 (t^{(1-\gamma)} D_q u(0))\big|_{t=0} = \sigma_1 u(\eta_1)$ and $\alpha_2 u(1) + \beta_2 D_q u(1) = \sigma_2 u(\eta_2)$, where $t, q \in [0,1]$, ${}^cD_q^\beta$ is the fractional Caputo q-derivative, $0 < \beta, \gamma \le 1$, $I_q^\xi(.)$ denotes the Riemann–Liouville integral with $\xi \in (0,1)$, f and g are given continuous functions, λ and p, k are real constants, $\alpha_i, \beta_i, \sigma_i \in \mathbb{R}$ and $\eta_i \in (0,1)$ for $i = 1,2$.[5] In 2019, Samei *et al.* reviewed the existence of solutions for some multi-term q-integro-differential equations with nonseparated and initial boundary conditions.[11]

Now, by using main idea of the papers, we investigate the fractional q-differential inclusion

$$ {}^cD_q^\alpha u(t) \in T(t, u(t), u'(t), u''(t)), \tag{1.1} $$

with integral boundary conditions

$$ \begin{aligned} &u(0) + u(p) + u(1) \\ &\quad = \int_0^1 f_0(s,u(s))ds, \\ &{}^cD_q^\beta u(0) + {}^cD_q^\beta u(p) + {}^cD_q^\beta u(1) \\ &\quad = \int_0^1 f_1(s,u(s))ds, \\ &{}^cD_q^\gamma u(0) + {}^cD_q^\gamma u(p) + {}^cD_q^\gamma u(1) \\ &\quad = \int_0^1 f_2(s,u(s))ds, \end{aligned} \tag{1.2} $$

where $t \in J = [0,1]$, $\alpha \in (2,3]$, $0 < q, p, \beta < 1$, $\gamma \in (1,2)$, the maps $f_1, f_2, f_3 : J \times \mathbb{R} \to \mathbb{R}$ are continuous, $T\colon J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ is a multifunction and ${}^cD_q^\beta$ is the fractional Caputo q-derivation. Denote the set of all compact subsets of $\mathbb{R}$ by $P_{cp}(\mathbb{R})$. Also, we study the existence of solutions for the fractional q-differential inclusion problem

$$ \begin{aligned} &{}^cD_q^\alpha u(t) \\ &\quad \in T(t, u(t), {}^cD_q^{\gamma_1} u(t), \ldots, {}^cD_q^{\gamma_n} u(t)), \end{aligned} \tag{1.3} $$

with boundary conditions

$$ \begin{aligned} u'(0) + a_1 u'(1) &= \sum_{i=1}^n {}^cD_q^{\gamma_i} u(p), \\ u(0) + a_2 u(1) &= \sum_{i=1}^n I_q^{\gamma_i} u(p), \end{aligned} \tag{1.4} $$

where $t \in J$, $\alpha \in (1,2]$, $0 < q, p, \gamma_i < 1$, $\alpha - \gamma_i \in [1,\infty)$ for all $1 \le i \le n$, $a_1 > \sum_{i=1}^n \frac{p^{1-\gamma_i}}{\Gamma_q(2-\gamma_i)}$, $a_2 > \sum_{i=1}^n \frac{p^{\gamma_i+1}}{\Gamma_q(\gamma_i+2)}$ and $T : J \times \mathbb{R}^{n+1} \to P(\mathbb{R})$ is a multifunction.

Let $q \in (0,1)$ and $a \in \mathbb{R}$. Define $[a]_q = \frac{1-q^a}{1-q}$.[1] The power function $(x-y)_q^n$ with $n \in \mathbb{N}_0$ is defined by $(x-y)_q^{(n)} = \prod_{k=0}^{n-1}(x - yq^k)$ for $n \ge 1$ and

$(x-y)_q^{(0)} = 1$, where x and y are real numbers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$.[2] Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have $(x-y)_q^{(\alpha)} = x^\alpha \prod_{k=0}^{\infty}(x - yq^k)/(x - yq^{\alpha+k})$. If $y = 0$, then it is clear that $x^{(\alpha)} = x^\alpha$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(z) = (1-q)^{(z-1)}/(1-q)^{z-1}$, where $z \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}$.[1] Note that, $\Gamma_q(z+1) = [z]_q\Gamma_q(z)$. The value of q-Gamma function, $\Gamma_q(z)$, for input values q and z is derived by counting the number of sentences n in summation by simplifying analysis (check Tables 1–3). For this design, we prepare a pseudo-code description of the technique for estimating q-Gamma function of order n which show in Algorithm 2. The q-derivative of function f, is defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ which is shown in Algorithm 3.[2] Also, the higher order q-derivative of a function f is defined by $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \geq 1$, where $(D_q^0 f)(x) = f(x)$.[2,3] The q-integral of a function f defined on $[0,b]$ is defined by

$$I_q f(x) = \int_0^x f(s)d_q s = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k),$$

Algorithm 1. The proposed method for calculated $(a-b)_q^{(\alpha)}$

Input: a, b, α, n, q
$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * (a - b * a^k)/(a - b * q^{\alpha+k})$
  end for
  $p \leftarrow a^\alpha * s$
end if
Output: $(a-b)^{(\alpha)}$

Algorithm 2. The proposed method for calculated $\Gamma_q(x)$

Input: n, $q \in (0,1)$, $x \in \mathbb{R}\backslash\{0,-1,2,\ldots,\}$
$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1-q^{k+1})(1-q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$
Output: $\Gamma_q(x)$

for $0 \leq x \leq b$, provided the series absolutely converges.[2,3] The q-derivative of function f is

Algorithm 3. The proposed method for calculated $(D_q f)(x)$

Input: $q \in (0,1)$, $f(x)$, x
syms z
if $x = 0$ **then**
  $g \leftarrow \lim((f(z) - f(q*z))/((1-q)z), z, 0)$
else
  $g \leftarrow (f(x) - f(q*x))/((1-q)x)$
end if
Output: $(D_q f)(x)$

defined by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ which is shown in Algorithm 3.[2,3] If a in $[0,b]$, then $\int_a^b f(u)d_q u = I_q f(b) - I_q f(a) = (1-q)\sum_{k=0}^{\infty} q^k[bf(bq^k) - af(aq^k)]$ whenever the series exists. The operator I_q^n is given by $(I_q^0 h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q(I_q^{n-1}h))(x)$ for $n \geq 1$ and $g \in C([0,b])$.[2,3] It has been proved that $(D_q(I_q f))(x) = f(x)$ and $(I_q(D_q f))(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$.[2,3] The fractional Riemann–Liouville-type q-integral of the function f on J for $\alpha \geq 0$ is defined by $(I_q^0 f)(t) = f(t)$ and $(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} f(s)d_q s$ for $t \in J$ and $\alpha > 0$.[8] Also, the Caputo fractional q-derivative of a function f is defined by

$$\begin{aligned}({}^c D_q^\alpha f)(t) &= (I_q^{[\alpha]-\alpha}(D_q^{[\alpha]} f))(t) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \\ &\quad \times \int_0^t (t-qs)^{([\alpha]-\alpha-1)} \\ &\quad \times (D_q^{[\alpha]} f)(s)d_q s, \qquad (1.5)\end{aligned}$$

where $t \in J$ and $\alpha > 0$.[8] It has been proved that $(I_q^\beta(I_q^\alpha f))(x) = (I_q^{\alpha+\beta} f)(x)$ and $\left(D_q^\alpha(I_q^\alpha f)\right)(x) = f(x)$, where $\alpha, \beta \geq 0$.[8] By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 4.

Algorithm 4. The proposed method for calculated $(I_q^\alpha f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x
$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1-q^{i+1})^{\alpha-1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow (x^\alpha * (1-q) * s)/(\Gamma_q(x))$
Output: $(I_q^\alpha f)(x)$

Algorithm 5. The proposed method for calculated $\int_a^b f(r)d_qr$

Input: $q \in (0,1)$, α, n, $f(x)$, a, b

```
1: s ← 0
2: for i = 0 : n do
3:     s ← s + q^i * (b * f(b * q^i) − a * f(a * q^i))
4: end for
5: g ← (1 − q) * s
```

Output: $\int_a^b f(r)d_qr$

We say that a multifunction $G: J \to P_{cl}(\mathbb{R})$ is measurable if the map $t \mapsto d(y, G(t))$ is measurable for each real number y.[14] The Pompeiu–Hausdorff metric $H_d\colon 2^X \times 2^X \to [0,\infty)$ on a metric space (X,ρ) is defined by $H_\rho(A,B) = \max\{\sup_{a\in A} \rho(a,B), \sup_{b\in B} \rho(A,b)\}$, where $\rho(A,b) = \inf_{a\in A} \rho(a,b)$.[14,28] Denote the set of closed and bounded and the set of closed subsets of X by $CB(X)$ and $C(X)$, respectively. Then, $(CB(X), H_\rho)$ is a metric space and $(C(X), H_\rho)$ is a generalized metric space.[28] An element $z \in X$ is called an endpoint of multifunction $T: X \to 2^X$ whenever $Tz = \{z\}$.[29] Also, multifunction T has approximate endpoint property whenever $\inf_{x\in X} \sup_{y\in Tx} \rho(x,y) = 0$.[29] A function $\theta: \mathbb{R} \to \mathbb{R}$ is called upper semi-continuous whenever $\limsup_{n\to\infty} \theta(\lambda_n) \le \theta(\lambda)$ for all sequence $\{\lambda_n\}_{n\ge1}$ with $\lambda_n \to \lambda$.[29] In 2010, Amini–Harandi proved the next result.

Lemma 1.1 (Ref. 29). *Consider an upper semi-continuous function $\theta: [0,\infty) \to [0,\infty)$ such that $\theta(t) < t$ and $\liminf_{t\to\infty}(t - \theta(t)) > 0$, for all $t > 0$. Also, Assume that (X,ρ) is a complete metric space and $T\colon X \to CB(X)$ a multifunction such that $H_d(Tx,Ty) \le \theta(\rho(x,y))$, for all $x,y \in X$. Then T has a unique endpoint if and only if T has approximate endpoint property.*

2. MAIN RESULTS

Now, we are ready to state and prove our main results. First, we provide our key result.

Lemma 2.1. *Let $v \in C(J,\mathbb{R})$, $\alpha \in (2,3]$, $0 < \beta, q, p < 1$, $\gamma \in (1,2)$ and $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ be continuous functions. The unique solution of the fractional q-differential problem*

$$ {}^cD_q^\alpha u(t) = v(t), \tag{2.1}$$

with boundary conditions (1.2) *is given by*

$$\begin{aligned} u(t) &= I_q^\alpha v(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))ds \\ &\quad - \frac{1}{3}[I_q^\alpha v(1) + I_q^\alpha v(p)] \\ &\quad + a_1(t)\int_0^1 f_1(s,u(s))ds \\ &\quad + a_2(t)[I_q^{\alpha-\beta} v(1) + I_q^{\alpha-\beta} v(p)] \\ &\quad + (b_1 + a_3(t))\int_0^1 g_2(s,u(s))ds \\ &\quad + (b_2 + a_4(t))[I_q^{\alpha-\gamma} v(1) + I_q^{\alpha-\gamma} v(p)], \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} a_1(t) &= \frac{3t\Gamma_q(2-\beta) - (p+1)\Gamma_q(2-\beta)}{3(p^{1-\beta}+1)}, \\ a_2(t) &= \frac{(p+1)\Gamma_q(2-\beta) - 3\Gamma_q(2-\beta)t}{3(p^{1-\beta}+1)}, \\ a_3(t) &= \frac{-6(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)t}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\ &\quad + \frac{3(p^{1-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(3-\beta)t^2}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\ a_4(t) &= \frac{6(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)t}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\ &\quad - \frac{3\Gamma_q(3-\gamma)\Gamma_q(3-\beta)(p^{1-\beta}+1)t^2}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\ b_1 &= \frac{2(p+1)(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\ &\quad - \frac{(p^2+1)\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}, \\ b_2 &= \frac{(p^2+1)\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} \\ &\quad - \frac{2(p+1)(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{6(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}. \end{aligned} \tag{2.3}$$

Proof. It is known that the general solution of Eq. (2.1) is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} v(s)d_qs \\ &\quad + c_0 + c_1 t + c_2 t^2, \end{aligned} \tag{2.4}$$

where c_0, c_1 and c_2 are real constants.[14,28] Thus, ${}^cD_q^{\beta}u(t) = I_q^{\alpha-\beta}v(t) + \frac{c_1t^{1-\beta}}{\Gamma_q(2-\beta)} + \frac{2c_2t^{2-\beta}}{\Gamma_q(3-\beta)}$ and ${}^cD_q^{\gamma}u(t) = I_q^{\alpha-\gamma}v(t) + \frac{2c_2t^{2-\gamma}}{\Gamma_q(3-\gamma)}$. Thus, we get

$$\begin{aligned}u(0)+u(p)+u(1) &= 3c_0 + (1+p)c_1 + (1+p^2)c_2\\ &\quad + I_q^{\alpha}v(1) + I_q^{\alpha}v(p),\end{aligned}$$

and

$$\begin{aligned}&{}^cD_q^{\beta}u(0) + {}^cD_q^{\beta}u(p) + {}^cD_q^{\beta}u(1)\\ &\quad = c_1\frac{p^{1-\beta}+1}{\Gamma_q(2-\beta)} + c_2\frac{2(p^{2-\beta}+1)}{\Gamma_q(3-\beta)}\\ &\qquad + I_q^{\alpha-\beta}v(1) + I_q^{\alpha-\beta}v(p),\\ &{}^cD_q^{\gamma}u(0) + {}^cD_q^{\gamma}u(p) + {}^cD_q^{\gamma}x(1)\\ &\quad = c_2\frac{2(p^{2-\gamma}+1)}{\Gamma_q(3-\gamma)} + I_q^{\alpha-\gamma}v(1) + I_q^{\alpha-\gamma}v(p).\end{aligned}$$

By using the boundary conditions, we obtain

$$\begin{aligned}&3c_0 + (1+p)c_1 + (1+p^2)c_2\\ &\quad = \int_0^1 f_0(s,u(s))ds - I_q^{\alpha}v(1) - I_q^{\alpha}v(p),\\ &c_1\frac{p^{1-\beta}+1}{\Gamma_q(2-\beta)} + c_2\frac{2(p^{2-\beta}+1)}{\Gamma_q(3-\beta)}\\ &\quad = \int_0^1 f_1(s,x(s))ds - I_q^{\alpha-\beta}v(1)\\ &\qquad - I_q^{\alpha-\beta}v(p),\\ &c_2\frac{2(p^{2-\gamma}+1)}{\Gamma_q(3-\gamma)}\\ &\quad = \int_0^1 f_2(s,x(s))ds - I_q^{\alpha-\gamma}v(1)\\ &\qquad - I_q^{\alpha-\gamma}v(p),\end{aligned}$$

and so

$$\begin{aligned}c_0 &= \frac{1}{3}\int_0^1 f_0(s,u(s))ds - \frac{1}{3}[I_q^{\alpha}v(1) + I_q^{\alpha}v(p)]\\ &\quad - \frac{\Gamma_q(2-\beta)(p+1)}{3(p^{1-\beta}+1)}\int_0^1 f_1(s,u(s))ds\\ &\quad + \frac{(p+1)\Gamma_q(2-\beta)}{3(p^{1-\beta}+1)}[I_q^{\alpha-\beta}v(1) + I_q^{\alpha-\beta}v(p)]\\ &\quad + b_1\int_0^1 f_2(s,u(s))ds\\ &\quad + b_2[I_q^{\alpha-\gamma}v(1) + I_q^{\alpha-\gamma}v(p)],\\ c_1 &= \frac{\Gamma_q(2-\beta)}{(p^{1-\beta}+1)}\int_0^1 f_1(s,u(s))ds - \frac{\Gamma_q(2-\beta)}{(p^{1-\beta}+1)}\\ &\quad \times [I_q^{\alpha-\beta}v(1) + I_q^{\alpha-\beta}v(p)]\\ &\quad - \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}\\ &\quad \times \int_0^1 f_2(s,u(s))ds\\ &\quad + \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)}\\ &\quad \times [I_q^{\alpha-\gamma}v(1) + I_q^{\alpha-\gamma}v(p)],\\ c_2 &= \frac{\Gamma_q(3-\gamma)}{2(p^{2-\gamma}+1)}\int_0^1 f_2(s,u(s))ds\\ &\quad - \frac{\Gamma_q(3-\gamma)}{2(p^{2-\gamma}+1)}[I_q^{\alpha-\gamma}v(1) + I_q^{\alpha-\gamma}v(p)].\end{aligned}$$

On the other hand, by some calculations, one could get the given map u is a solution for the problem. This completes the proof. □

Assume that $\mathcal{X} = C^2(J)$ is endowed with the norm $\|u\| = \sup_{t\in J}|u(t)| + \sup_{t\in J}|u'(t)| + \sup_{t\in J}|u''(t)|$. Then $(\mathcal{X}, \|.\|)$ is a Banach space. For $u \in \mathcal{X}$, we define the selection set $S_{T,u}$ by the set of all $v \in L^1(J)$ somehow that $v(t) \in T(t,u(t),u'(t),u''(t))$ for all $t \in\in J$ (see Ref. 14). To study the problem (1.1) and (1.2), we consider the next conditions.

(C1) The multifunction $T : J\times\mathbb{R}^3 \to P_{cp}(\mathbb{R})$ is integrable bounded and $T(.,x_1,x_2,x_3) : J \to P_{cp}(\mathbb{R})$ is measurable for all $x_i \in \mathbb{R}$.

(C2) The functions f_1, f_2 and $f_3 : J\times\mathbb{R}\to\mathbb{R}$ are continuous and the map $\theta : [0,\infty) \to [0,\infty)$ is nondecreasing and upper semi-continuous with $\liminf_{t\to\infty}(t-\theta(t)) > 0$ and $\theta(t) < t$ for all $t > 0$.

(C3) There exist $m, m_0, m_1, m_2 \in C(J,[0,\infty))$ such that

$$\begin{aligned}&H_d(T(t,x_1,x_2,x_3),\\ &\quad T(t,x_1',x_2',x_3')) \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}m(t)\theta\\ &\qquad \times\left(\sum_{k=1}^{3}|x_i - x_i'|\right),\end{aligned}$$

and $|f_j(t,x) - f_j(t,x')| \leq \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m_j(t)\,\psi(|x-x'|)$, for all $t \in J$, $x, x', x_i, x_i' \in \mathbb{R}$, where

$$\Lambda_1 = \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha+1)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma_q(\alpha+1)} + \frac{5\Gamma_q(2-\beta)\|m_1\|_\infty}{3} + \frac{10\Gamma_q(2-\beta)\|m\|_\infty}{3\Gamma_q(\alpha-\beta+1)} + 10(2\Gamma_q(2-\beta)+\Gamma_q(3-\beta)) \times \left(\frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty \times \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{3\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)} \right) \right],$$

$$\Lambda_2 = \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha)} + \frac{2\Gamma_q(2-\beta)\|m\|_\infty}{\Gamma_q(\alpha-\beta+1)} + (2\Gamma_q(2-\beta)+\Gamma_q(3-\beta)) \times \left(\frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty \times \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{\Gamma(3-\beta)\Gamma_q(\alpha-\gamma+1)} \right) \right],$$

$$\Lambda_3 = \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty\Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{\Gamma_q(\alpha-\gamma+1)} \right].$$

Define the multifunction $N : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$N(u) = \{h \in \mathcal{X} \mid \exists\, v \in S_{T,u} \text{ such that } h(t) = w(t) \text{ for all } t \in J\},$$

where

$$\begin{aligned} w(t) = {} & I_q^\alpha v(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))ds \\ & - \frac{1}{3}[I_q^\alpha v(1) + I_q^\alpha v(p)] \\ & + a_1(t)\int_0^1 f_1(s,u(s))ds \\ & + a_2(t)[I_q^{\alpha-\beta} v(1) + I_q^{\alpha-\beta} v(p)] \\ & + (b_1 + a_3(t))\int_0^1 f_2(s,u(s))ds \\ & + (b_2 + a_4(t))[I_q^{\alpha-\gamma} v(1) + I_q^{\alpha-\gamma} v(p)]. \end{aligned}$$

Theorem 2.2. *The boundary value q-differential inclusion problems* (1.1) *and* (1.2) *has a solution whenever the multifunction $N : \mathcal{X} \to P(\mathcal{X})$ has the approximate endpoint property and the conditions* (C1)–(C3) *hold.*

Proof. We show that the multifunction N has an endpoint which is a solution for the problems (1.1) and (1.2). Since the multivalued map $t \vdash T(t, u(t), u'(t), u''(t))$ is measurable, N has closed values and so has measurable selection. This implies that $S_{T,u}$ is nonempty for all $u \in \mathcal{X}$. Now, we show that $N(u) \subset \mathcal{X}$ is closed for all $u \in \mathcal{X}$. Let $u \in \mathcal{X}$ and $\{x_n\}_{n\geq 1}$ be a sequence in $N(u)$ with $u_n \to x$. For each $n \in \mathbb{N}$, choose $v_n \in S_{T,u}$ such that

$$\begin{aligned} x_n(t) = {} & I_q^\alpha v_n(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))ds \\ & - \frac{1}{3}[I_q^\alpha v_n(1) + I_q^\alpha v_n(p)] \\ & + a_1(t)\int_0^1 f_1(s,u(s))ds \\ & + a_2(t)[I_q^{\alpha-\beta} v_n(1) + I_q^{\alpha-\beta} v_n(p)] \\ & + (b_1 + a_3(t))\int_0^1 f_2(s,u(s))ds \\ & + (b_2 + a_4(t))[I_q^{\alpha-\gamma} v_n(1) + I_q^{\alpha-\gamma} v_n(p)]. \end{aligned}$$

Since T has compact values, $\{v_n\}_{n\geq 1}$ has a subsequence which converges to some $v \in L^1(J)$. Denote this subsequence by $\{v_n\}_{n\geq 1}$ again. It is easy to check that $v \in S_{T,u}$ and $x_n \to x$, where

$$\begin{aligned} x(t) = {} & I_q^\alpha v(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))ds \\ & - \frac{1}{3}[I_q^\alpha v(1) + I_q^\alpha v(p)] \end{aligned}$$

$$+ a_1(t)\int_0^1 f_1(s,u(s))ds + a_2(t) \times [I_q^{\alpha-\beta}v(1) + I_q^{\alpha-\beta}v(p)] + (b_1+a_3(t))\int_0^1 f_2(s,u(s))ds + (b_2+a_4(t))[I_q^{\alpha-\gamma}v(1) I_q^{\alpha-\gamma}v(p)],$$

for all $t \in J$. This implies that $x \in N(u)$ and so N has closed values. Since T has compact values, it is easy to see that $N(u)$ is a bounded set for all $u \in \mathcal{X}$. Now, we show that $H_d(N(u), N(v)) \le \theta(\|u-v\|)$. Let $u, v \in \mathcal{X}$ and $h_1 \in N(v)$. Choose $w_1 \in S_{T,v}$ such that

$$h_1(t) = I_q^\alpha w_1(t) + \frac{1}{3}\int_0^1 f_0(s,v(s))ds - \frac{1}{3}[I_q^\alpha w_1(1) + I_q^\alpha w_1(p)] + a_1(t)\int_0^1 f_1(s,v(s))ds + a_2(t)[I_q^{\alpha-\beta}w_1(1) + I_q^{\alpha-\beta}w_1(p)] + (b_1+a_3(t))\int_0^1 f_2(s,v(s))ds + (b_2+a_4(t))[I_q^{\alpha-\gamma}w_1(1) + I_q^{\alpha-\gamma}w_1(p)],$$

for almost all $t \in J$. Put $\tilde{T}_{u(t)} = T(t,u(t),u'(t), u''(t))$ and $\tilde{T}_{v(t)} = T(t,v(t),v'(t),v''(t))$. Since

$$H_d(\tilde{T}_{u(t)}, \tilde{T}_{v(t)}) \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m(t)\theta(|u(t)-n(t)| + |u'(t)-v'(t)| + |u''(t)-v''(t)|),$$

for all $t \in J$, there exists $w \in \tilde{T}_{u(t)}$ such that

$$|w_1(t) - w| \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m(t) \times \theta(|u(t)-v(t)| + |u'(t)-v'(t)| + |u''(t)-v''(t)|), \tag{2.5}$$

for all $t \in J$. Consider the multivalued map $G : J \to P(\mathbb{R})$ defined the set of all $w \in \mathbb{R}$ such that w satisfies in (2.5). Since w_1 and $\varphi = m\theta(|u-v|+|u'-v'|+|u''-v''|)\left(\frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\right)$ are measurable, the multifunction $G(.) \cap T(., u(.), u'(.), u''(.))$, is measurable. Now, we can choose $w_2(t) \in T(t,u(t),u'(t),u''(t))$ such that

$$w_1(t) - w_2(t)| \le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m(t) \times \psi(|u(t)-v(t)| + |u'(t)-v'(t)| + |u''(t)-v''(t)|),$$

for all $t \in J$. Consider the element $h_2 \in N(u)$ defined by

$$h_2(t) = I_q^\alpha w_2(t) + \frac{1}{3}\int_0^1 f_0(s,u(s))ds - \frac{1}{3}[I_q^\alpha w_2(1) + I_q^\alpha w_2(p)] + a_1(t)\int_0^1 f_1(s,u(s))ds + a_2(t)[I_q^{\alpha-\beta}w_2(1) + I_q^{\alpha-\beta}w_2(p)] + (b_1+a_3(t))\int_0^1 f_2(s,u(s))ds + (b_2+a_4(t))[I_q^{\alpha-\gamma}w_2(1) + I_q^{\alpha-\gamma}w_2(p)],$$

for all $t \in J$. Then, we have

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &\le I_q^\alpha|w_1(t)-w_2(t)| + \frac{1}{3}\int_0^1 |f_0(s,v(s)) - f_0(s,u(s))|ds \\ &\quad + \frac{1}{3}[I_q^\alpha|w_1(1)-w_2(1)| I_q^{\alpha-1}|w_1(p) - w_2(p)|ds] + |a_1(t)| \times \int_0^1 |f_1(s,v(s)) - f_1(s,u(s))|ds \\ &\quad + |a_2(t)|[I_q^{\alpha-\beta}|w_1(1)-w_2(1)| + I_q^{\alpha-\beta}|w_1(p)-w_2(p)|] + |b_1+a_3(t)| \times \int_0^1 |f_2(s,v(s)) - f_2(s,u(s))|ds \\ &\quad + |b_2+a_4(t)|[I_q^{\alpha-\gamma}|w_1(1)-w_2(1)|ds + I_q^{\alpha-\gamma}|w_1(p)-w_2(p)|ds] \\ &\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\theta(\|u-v\|) \end{aligned}$$

$$\times \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha+1)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma_q(\alpha+1)} + \frac{5\Gamma_q(2-\beta)\|m_1\|_\infty}{3} + \frac{10\Gamma_q(2-\beta)\|m\|_\infty}{3\Gamma_q(\alpha-\beta+1)} + 10(2\Gamma_q(2-\beta)+\Gamma_q(3-\beta)) \times \left(\frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty \times \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{3\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)} \right) \right]$$

$$= \frac{\Lambda_1}{\Lambda_1+\Lambda_2+\Lambda_3}\psi(\|u-v\|),$$

$$\begin{aligned}
|h_1'(t) - h_2'(t)| &\le I_q^{\alpha-1}|w_1(t) - w_2(t)| + \frac{\Gamma(2-\beta)}{(p^{1-\beta}+1)}[I_q^{\alpha-\beta}|w_1(1) - w_2(1)| + I_q^{\alpha-\beta}|w_1(p) - w_2(p)|] + |a_5(t)| \\
&\quad \times \int_0^1 |f_2(s, v(s)) - f_2(s, u(s))| ds + |a_6(t)|[I_q^{\alpha-\gamma}|w_1(1) - w_2(1)| + I_q^{\alpha-\gamma-1}|w_1(p) - w_2(p)|] \\
&\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\theta(\|u-v\|) \times \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha)} + \frac{2\Gamma_q(2-\beta)\|m\|_\infty}{\Gamma_q(\alpha-\beta+1)} + (2\Gamma_q(2-\beta)+\Gamma_q(3-\beta)) \times \left(\frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty \times \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma+1)} \right) \right] \\
&= \frac{\Lambda_2}{\Lambda_1+\Lambda_2+\Lambda_3}\theta(\|u-v\|),
\end{aligned}$$

and

$$\begin{aligned}
|h_1''(t) - h_2''(t)| &\le I_q^{\alpha-2}|w_1(t) - w_2(t)| + \frac{\Gamma_q(3-\gamma)}{(p^{2-\gamma}+1)} \int_0^1 |f_2(s, v(s)) - f_2(s, u(s))| ds \\
&\quad + \frac{\Gamma_q(3-\gamma)}{(p^{2-\gamma}+1)}[I_q^{\alpha-\gamma}|w_1(1) - w_2(1)| + I_q^{\alpha-\gamma}|w_1(p) - w_2(p)|] \\
&\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\psi(\|u-v\|) \times \left[\frac{\|m\|_\infty}{\Gamma_q(\alpha-1)} + \frac{\Gamma_q(3-\gamma)(\|m_2\|_\infty \times \Gamma_q(\alpha-\gamma+1) + 2\|m\|_\infty)}{\Gamma_q(\alpha-\gamma+1)} \right] \\
&= \frac{\Lambda_3}{\Lambda_1+\Lambda_2+\Lambda_3}\psi(\|u-v\|),
\end{aligned}$$

where

$$a_5(t) = \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)} + \frac{\Gamma_q(3-\gamma)(p^{1-\beta}+1)\Gamma_q(3-\beta)t}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)},$$

$$a_6(t) = \frac{(p^{2-\beta}+1)\Gamma_q(3-\gamma)\Gamma_q(2-\beta)}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma)} - \frac{\Gamma_q(3-\gamma)\Gamma_q(3-\beta)(p^{1-\beta}+1)t}{(p^{1-\beta}+1)(p^{2-\gamma}+1)\Gamma_q(3-\beta)\Gamma_q(\alpha-\gamma)}.$$

Hence,

$$\begin{aligned}
\|h_1 - h_2\| &= \sup_{t\in J}|h_1(t) - h_2(t)| + \sup_{t\in J}|h_1'(t) - h_2'(t)| + \sup_{t\in J}|h_1''(t) - h_2''(t)| \\
&\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}\theta \times (\|x-y\|)(\Lambda_1+\Lambda_2+\Lambda_3) \\
&= \theta(\|x-y\|).
\end{aligned}$$

Thus, $H_d(N(u), N(v)) \le \theta(\|u-v\|)$ for all $u, v \in \mathcal{X}$. On the other hand, the multifunction N has approximate endpoint property. By using Lemma 1.1, there exists $u^* \in \mathcal{X}$ such that $N(u^*) = \{u^*\}$. By using Lemma 2.1, u^* is a solution for the problems (1.1) and (1.2). □

Now, we investigate the existence of solution for the fractional q-differential inclusion problem with integral boundary value conditions

$$\begin{aligned} {}^cD_q^{\alpha}u(t) &\in T(t, u(t), {}^cD_q^{\gamma_1}u(t), \\ &\quad \ldots, {}^cD_q^{\gamma_n}u(t)), \\ u'(0) + a_1 u'(1) &= \sum_{i=1}^{n} {}^cD_q^{\gamma_i}u(p), \\ u(0) + a_2 u(1) &= \sum_{i=1}^{n} I_q^{\gamma_i}u(p), \end{aligned} \tag{2.6}$$

where $T : J \times \mathbb{R}^{n+1} \to P(\mathbb{R})$ is a multifunction, $t \in J$, $\alpha \in (1, 2]$, $n \geq 2$, $0 < q, p, \gamma_i < 1$, $\alpha - \gamma_i \geq 1$ for all $1 \leq i \leq n$, $a_1 > \sum_{i=1}^{n} \frac{p^{1-\gamma_i}}{\Gamma_q(2-\gamma_i)}$ and $a_2 > \sum_{i=1}^{n} \frac{p^{\gamma_i+1}}{\Gamma_q(\gamma_i+2)}$.

Lemma 2.3. *Let $v \in C(J, \mathbb{R})$, $1 < \alpha \geq 2$, $0 < q, p < 1$, $n \geq 2$ and $0 < \beta_i < 1$ for $i = 1, \ldots, n$. The unique solution of the fractional q-differential problem with the boundary value conditions*

$$\begin{cases} {}^cD_q^{\alpha}u(t) = v(t), \\ x'(0) + a_1 u'(1) = \sum_{i=1}^{n} {}^cD_q^{\beta_i}u(p), \\ x(0) + a_2 u(1) = \sum_{i=1}^{n} I_q^{\beta_i}u(p), \end{cases} \tag{2.7}$$

is given by $u(t) = \int_0^1 G(t,s)v(s)ds$, where $a_1 > \sum_{i=1}^{n} \frac{p^{1-\beta_i}}{\Gamma_q(2-\beta_i)}$, $a_2 > \sum_{i=1}^{n} \frac{p^{\beta_i+1}}{\Gamma_q(\beta_i+2)}$ and $G(t,s)$ is the Green function given by

$$\begin{aligned} G(t,s) &= \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\ &\quad + \frac{1}{A}\sum_{i=1}^{n} \frac{(p-qs)^{(\alpha+\beta_i-1)}}{\Gamma_q(\alpha+\beta_i)} \\ &\quad - \frac{a_2}{A\Gamma_q(\alpha)}(1-qs)^{(\alpha-1)} - \frac{1}{AB} \\ &\quad \times \left(a - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right) \\ &\quad \times \sum_{i=1}^{n} \frac{(p-qs)^{(\alpha-\beta_i-1)}}{\Gamma_q(\alpha-\beta_i)} \\ &\quad - \frac{a_1}{AB}\left(a - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right)\frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \\ &\quad + \frac{t}{B}\sum_{i=1}^{n} \frac{(p-qs)^{(\alpha-\beta_i-1)}}{\Gamma_q(\alpha-\beta_i)} \\ &\quad - \frac{a_1 t}{B\Gamma_q(\alpha-1)}(1-qs)^{(\alpha-2)}, \end{aligned}$$

whenever $0 \leq s \leq p \leq t \leq 1$,

$$\begin{aligned} G(t,s) &= \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\ &\quad - \frac{a_2}{A\Gamma_q(\alpha)}(1-qs)^{(\alpha-1)} \\ &\quad - \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right) \\ &\quad \times \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \\ &\quad - \frac{a_1 t}{B\Gamma_q(\alpha-1)}(1-qs)^{(\alpha-2)}, \end{aligned}$$

whenever $0 \leq p \leq s \leq t \leq 1$,

$$\begin{aligned} G(t,s) &= -\frac{a_2}{A\Gamma_q(\alpha)}(1-qs)^{(\alpha-1)} \\ &\quad - \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right) \\ &\quad \times \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \\ &\quad - \frac{a_1 t}{B\Gamma_q(\alpha-1)}(1-qs)^{(\alpha-2)}, \end{aligned}$$

whenever $0 \leq p \leq s \leq t \leq 1$,

$$\begin{aligned} G(t,s) &= \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \\ &\quad + \frac{1}{A}\sum_{i=1}^{n} \frac{(p-qs)^{(\alpha+\beta_i-1)}}{\Gamma_q(\alpha+\beta_i)} \\ &\quad - \frac{a_2}{A\Gamma_q(\alpha)}(1-qs)^{(\alpha-1)} \\ &\quad - \frac{1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right) \end{aligned}$$

$$\times \sum_{i=1}^{n} \frac{(p-qs)^{(\alpha-\beta_i-1)}}{\Gamma_q(\alpha-\beta_i)}$$
$$- \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma(\beta_i+3)}\right)$$
$$\times \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}$$
$$+ \frac{t}{B}\sum_{i=1}^{n} \frac{(p-qs)^{(\alpha-\beta_i-1)}}{\Gamma_q(\alpha-\beta_i)}$$
$$- \frac{a_1 t}{B\Gamma_q(\alpha-1)}(1-qs)^{(\alpha-2)},$$

whenever $0 \le s \le t \le p \le 1$,

$$G(t,s) = \frac{1}{A}\sum_{i=1}^{n} \frac{(p-qs)^{(\alpha+\beta_i-1)}}{\Gamma_q(\alpha+\beta_i)}$$
$$- \frac{a_2}{A\Gamma_q(\alpha)}(1-qs)^{(\alpha-1)}$$
$$- \frac{1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right)$$
$$\times \sum_{i=1}^{n} \frac{(p-qs)^{(\alpha-\beta_i-1)}}{\Gamma_q(\alpha-\beta_i)}$$
$$- \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right)$$
$$\times \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} + \frac{t}{B}\sum_{i=1}^{n}$$
$$\times \frac{(p-qs)^{(\alpha-\beta_i-1)}}{\Gamma_q(\alpha-\beta_i)}$$
$$- \frac{a_1 t}{B\Gamma_q(\alpha-1)}(1-qs)^{(\alpha-2)},$$

whenever $0 \le s \le t \le p \le 1$ *and*

$$G(t,s) = -\frac{a_2}{A\Gamma_q(\alpha)}(1-qs)^{(\alpha-1)}$$
$$- \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right)$$
$$\times \frac{(1-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}$$
$$- \frac{a_1 t}{B\Gamma_q(\alpha-1)}(1-qs)^{(\alpha-2)},$$

whenever $0 \le t \le p \le s \le 1$. *Here,* $A = 1 + a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+1}}{\Gamma_q(\beta_i+2)}$ *and* $B = 1 + a_1 - \sum_{i=1}^{n} \frac{p^{1-\beta_i}}{\Gamma_q(2-\beta_i)}$.

Proof. It is known that the solution of ${}^cD_q^{\alpha} u(t) = v(t)$ is given by

$$u(t) = I_q^{\alpha} v(t) + c_0 + c_1 t$$
$$= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} v(s) ds + c_0 + c_1 t,$$

where $c_0, c_1, \in \mathbb{R}$ are real constants.[14,28] Thus, we have ${}^cD_q^{\beta_i} u(t) = I_q^{\alpha-\beta_i} v(t) + \frac{c_1 t^{1-\beta_i}}{\Gamma_q(2-\beta_i)}$, $I_q^{\beta_i} u(t) = I_q^{\alpha+\beta_i} v(t) + \frac{c_0 t^{1+\beta_i}}{\Gamma_q(2+\beta_i)} + \frac{c_1 t^{2+\beta_i}}{\Gamma_q(3+\beta_i)}$ and $u'(t) = I_q^{\alpha-1} v(t) + c_1$. Hence, $u(0) + a_2 u(1) = (a_2+1)c_0 + a_2 c_1 + a_2 I_q^{\alpha} v(1)$ and $u'(0) + a_1 u'(1) = (1 + a_1)c_1 + a_1 I_q^{\alpha-1} v(1)$. By using the boundary conditions, we get $c_0\big(1 + a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+1}}{\Gamma_q(\beta_i+2)}\big) + c_1\big(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\big) = \sum_{i=1}^{n} I_q^{\alpha+\beta_i} v(p) - a_2 I_q^{\alpha} v(1)$ and $c_1\big(1 + a_1 - \sum_{i=1}^{n} \frac{p^{1-\beta_i}}{\Gamma_q(2-\beta_i)}\big) = \sum_{i=1}^{n} I_q^{\alpha-\beta_i} v(p) - a_1 I_q^{\alpha-1} v(1)$. Thus,

$$c_0 = \frac{1}{A}\sum_{i=1}^{n} I_q^{\alpha+\beta_i} v(p) - \frac{a_2}{A} I_q^{\alpha} v(1)$$
$$- \frac{1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right) \sum_{i=1}^{n} I_q^{\alpha-\beta_i} v(p)$$
$$- \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right) I_q^{\alpha-1} v(1),$$

and $c_1 = \frac{1}{B}\sum_{i=1}^{n} I_q^{\alpha-\beta_i} v(p) - \frac{a_1}{B} I_q^{\alpha-1} v(1)$. Hence,

$$u(t) = I_q^{\alpha} v(t) + \frac{1}{A}\sum_{i=1}^{n} I_q^{\alpha+\beta_i} v(p) - \frac{a_2}{A} I_q^{\alpha} v(1)$$
$$- \frac{1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right)$$
$$\times \sum_{i=1}^{n} I_q^{\alpha-\beta_i} v(p)$$
$$- \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\beta_i+2}}{\Gamma_q(\beta_i+3)}\right)$$

$$\times I_q^{\alpha-1}v(1) + \frac{t}{B}\sum_{i=1}^{n} I_q^{\alpha-\beta_i}v(p) - \frac{ta_1}{B}I_q^{\alpha-1}v(1) = \int_0^1 G(t,s)v(s)ds.$$

The converse part concludes by some calculation. This completes the proof. □

Consider the Banach space $\mathcal{A} = \{u \,|\, u, {}^cD_q^{\gamma_i}u \in C(J,R) : \forall\, i = 1,2,\ldots,n\}$ endowed with the norm

$$\|u\| = \sup_{t\in J}|u(t)| + \sum_{i=1}^{n}\sup_{t\in J}|{}^cD_q^{\gamma_i}u(t)|.$$

Define $S_{T,u} = \{v \in L^1(J) \,|\, v(t) \in T(t,u(t), {}^cD_q^{\gamma_1}u(t),\ldots,{}^cD_q^{\gamma_n}u(t))\}$ for all $t\in J$ and $u\in\mathcal{A}$. For $1\le j\le n$, put

$$L_1 = \frac{1}{\Gamma_q(\alpha+1)} + \frac{1}{A}\sum_{i=1}^{n}\frac{p^{\alpha+\gamma_i}}{\Gamma_q(\alpha+\gamma_i+1)} + \frac{a_2}{A\Gamma_q(\alpha+1)} + \frac{1}{AB}\left|a - \sum_{i=1}^{n}\frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right| \times \sum_{i=1}^{n}\frac{p^{\alpha-\gamma_i}}{\Gamma_q(\alpha-\gamma_i+1)} + \frac{a_1}{AB\Gamma(\alpha)} \times \left|a_2 - \sum_{i=1}^{n}\frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right| + \frac{1}{B}\sum_{i=1}^{n}\frac{p^{\alpha-\gamma_i}}{\Gamma_q(\alpha-\gamma_i+1)} + \frac{a_1}{B\Gamma_q(\alpha)},$$

and $L_2^j = \frac{1}{\Gamma_q(\alpha-\gamma_j+1)} + \frac{1}{B\Gamma_q(2-\gamma_j)}\sum_{i=1}^{n}\frac{p^{\alpha-\gamma_i}}{\Gamma_q(\alpha-\gamma_i+1)} + \frac{a_1}{B\Gamma_q(2-\gamma_j)\Gamma_q(\alpha)}$.

Theorem 2.4. *Let the nondecreasing map* $\theta : [0,\infty)\to[0,\infty)$ *be an upper semi-continuous such that* $\theta(t) < t$ *and* $\liminf_{t\to\infty}(t-\theta(t)) > 0$ *for all* $t > 0$ *and* $T : J\times\mathbb{R}^{n+1}\to P_{cp}(\mathbb{R})$ *a multifunction such that* $T(.,u_1,u_2,\ldots,u_{n+1}) : J \to P_{cp}(\mathbb{R})$ *is measurable and bounded integrable for all* $u_1,u_2,\ldots,u_{n+1}\in\mathbb{R}$. *Put* $\tilde{T}_{t,u_i} = T(t,u_1,u_2,\ldots,u_{n+1})$, $\tilde{T}_{t,v_i} = T(t,v_1,v_2,\ldots,v_{n+1})$ *and assume that there exists* $m\in C(J,[0,\infty))$ *such that*

$$H_d(\tilde{T}_{t,u_i} - \tilde{T}_{t,v_i}) \le m(t)\psi\left(\sum_{i=1}^{n+1}|u_i - v_i|\right) \times \left(\frac{1}{\|m\|_\infty(L_1 + \sum_{j=1}^{n}L_2^j)}\right).$$

Then the inclusion problem (1.3), (1.4) *has a solution whenever multifunction* $\Omega : \mathcal{A}\to 2^{\mathcal{A}}$ *define by* $\Omega(u) = \{h\in\mathcal{A}\,|\,\exists\, v\in S_{T,u} : h(t) = \int_0^1 G(t,s)v(s)ds, \forall\, t\in J\}$ *has the approximate endpoint property.*

Proof. We show that the multifunction $\Omega : \mathcal{A}\to P(\mathcal{A})$ has an endpoint which is a solution for the problem (1.3) and (1.4). First, we show that $\Omega(u)$ is closed for all $u\in\mathcal{A}$. Let $u\in\mathcal{A}$ and $\{x_n\}_{n\ge1}$ be a sequence in $\Omega(u)$ with $x_n\to x$. For each n, choose $w_n\in S_{T,u}$ such that

$$x_n(t) = \int_0^1 G(t,s)w_n(s)ds,$$

for all $t\in J$. Since F has compact values, $\{w_n\}_{n\ge1}$ has a subsequence which converges to some $w\in L^1(J)$. We denote this subsequence again by $\{w_n\}_{n\ge1}$. It is easy to check that $w\in S_{T,u}$ and $x_n(t)\to x(t) = \int_0^1 G(t,s)w(s)ds$ for all $t\in J$. This implies that $x\in\Omega(u)$ and so Ω has closed values. Since T is a compact multivalued map, it is easy to check that $\Omega(u)$ is a bounded set for all $u\in\mathcal{A}$. Now, we show that $H_d(\Omega(u),\Omega(v)) \le \theta(\|u-v\|)$ for all $u,v\in\mathcal{A}$. Suppose that $u,v\in X$ and $h_1\in\Omega(v)$. Choose $w_1\in S_{T,v}$ such that $h_1(t) = \int_0^1 G(t,s)w_1(s)ds$ for almost all $t\in J$. Put $\tilde{T}_{t,{}^cD_q^{\gamma}u} = T(t,u(t),{}^cD_q^{\gamma_1}u(t),\ldots,{}^cD_q^{\gamma_n}u(t))$ and

$$\tilde{T}_{t,{}^cD_q^{\gamma}v} = F(t,v(t),{}^cD_q^{\gamma_1}v(t),\ldots,{}^cD_q^{\gamma_n}v(t)).$$

Let $t\in J$. Since

$$H_d(\tilde{T}_{t,{}^cD_q^{\gamma}u},\tilde{T}_{t,{}^cD_q^{\gamma}v}) \le m(t)\psi\left(|u(t)-v(t)| + \sum_{i=1}^{n}|{}^cD_q^{\gamma_i}u(t) - {}^cD_q^{\gamma_i}v(t)|\right) \times \left(\frac{1}{\|m\|_\infty\left(L_1 + \sum_{j=1}^{n}L_2^j\right)}\right),$$

there exists $w \in T(t, u(t), {}^cD_q^{\gamma_1}u(t), \ldots, {}^cD_q^{\gamma_n}u(t))$ such that

$$|w_1(t) - w| \le m(t)\psi\left(|u(t) - v(t)| + \sum_{i=1}^{n} |{}^cD_q^{\gamma_i}u(t) - {}^cD_q^{\gamma_1}v(t)|\right) \times \left(\frac{1}{\|m\|_\infty \left(L_1 + \sum_{j=1}^{n} L_2^j\right)}\right). \tag{2.8}$$

Consider the multivalued map $U : J \to P(\mathbb{R})$, where $U(t)$ is the set of all $w \in \mathbb{R}$ such that w satisfies in (2.8). Since w_1 and

$$\varphi = m\psi\left(|u - v| + \sum_{i=1}^{n} |{}^cD_q^{\gamma_i}u - {}^cD_q^{\gamma_1}v|\right) \times \left(\frac{1}{\|m\|_\infty \left(L_1 + \sum_{j=1}^{n} L_2^j\right)}\right)$$

are measurable, the multifunction $U(.) \cap T(t, u(.), {}^cD_q^{\gamma_1}u(.), \ldots, {}^cD_q^{\gamma_n}u(.))$ is measurable. Choose $w_2(t) \in T(t, u(t), {}^cD_q^{\gamma_1}u(t), \ldots, {}^cD_q^{\gamma_n}u(t))$ such that

$$|w_1(t) - w_2(t)| \le m(t)\theta\left(|u(t) - v(t)| + \sum_{i=1}^{n} |{}^cD_q^{\gamma_1}u(t) - {}^cD_q^{\gamma_1}v(t)|\right) \times \left(\frac{1}{\|m\|_\infty \left(L_1 + \sum_{j=1}^{n} L_2^j\right)}\right).$$

Now, consider the element $h_2 \in \Omega(u)$ which is defined by $h_2(t) = \int_0^1 G(t,s)w_2(s)ds$. Thus,

$$\begin{aligned}
&|h_1(t) - h_2(t)| \\
&= \left|\int_0^1 G(t,s)w_1(s)ds - \int_0^1 G(t,s)w_2(s)ds\right| \\
&= \Bigg| I_q^\alpha w_1(t) + \frac{1}{A}\sum_{i=1}^{n} I_q^{\alpha+\gamma_i} w_1(p) - \frac{a_2}{A} I_q^\alpha w_1(1) \\
&\quad - \frac{1}{AB}\left(a_1 - \sum_{i=1}^{n} \frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right) \sum_{i=1}^{n} I_q^{\alpha-\gamma_i} w_1(p) \\
&\quad - \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right) I_q^{\alpha-1} w_1(1) + \frac{t}{B} \\
&\quad \times \sum_{i=1}^{n} I_q^{\alpha-\gamma_i} w_1(p) - \frac{ta_2}{B} I_q^{\alpha-1} w_1(1) \\
&\quad - I_q^\alpha w_2(1) - \frac{1}{A}\sum_{i=1}^{n} I_q^{\alpha+\gamma_i} w_2(p) + \frac{a_1}{A} I_q^\alpha w_2(1) \\
&\quad + \frac{1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right) \sum_{i=1}^{n} I_q^{\alpha-\gamma_i} w_2(p) \\
&\quad + \frac{a_1}{AB}\left(a_2 - \sum_{i=1}^{n} \frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right) \\
&\quad \times I_q^{\alpha-1} w_2(1) - \frac{t}{B}\sum_{i=1}^{n} I_q^{\alpha-\gamma_i} w_2(p) \\
&\quad + \frac{ta_1}{B} I_q^{\alpha-1} w_2(1) \Bigg| \\
&\le I_q^\alpha |w_1(t) - w_2(t)| \\
&\quad + \frac{1}{A}\sum_{i=1}^{n} I_q^{\alpha+\gamma_i} |w_1(p) - w_2(p)| \\
&\quad + \frac{a}{A} I_q^\alpha |w_1(1) - w_2(1)| \\
&\quad + \frac{1}{AB}\left|a_2 - \sum_{i=1}^{n} \frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right| \\
&\quad \times \sum_{i=1}^{n} I_q^{\alpha-\gamma_i} |w_1(p) - w_2(p)| \\
&\quad + \frac{a_1}{AB}\left|a_2 - \sum_{i=1}^{n} \frac{p^{\gamma_i+2}}{\Gamma_q(\gamma_i+3)}\right| \\
&\quad \times I_q^{\alpha-1} |w_1(1) - w_2(1)| \\
&\quad + \frac{t}{B}\sum_{i=1}^{n} I_q^{\alpha-\gamma_i} |w_1(p) - w_2(p)| \\
&\quad + \frac{ta_1}{B} I_q^{\alpha-1} |w_1(1) - w_2(1)| \\
&\le \left(\frac{L_1}{L_1 + \sum_{j=1}^{n} L_2^j}\right)\psi(\|u - v\|),
\end{aligned}$$

and

$$\begin{aligned}
&|{}^cD_q^{\gamma_j}h_1(t) - {}^cD_q^{\gamma_j}h_2(t)| \\
&\quad \le I_q^{\alpha-\gamma_j}|w_1(t) - w_2(t)| + \frac{t^{1-\gamma_j}}{B\Gamma(2-\gamma_j)}
\end{aligned}$$

$$\times \sum_{i=1}^{n} I_q^{\alpha-\gamma_i}|w_1(p) - w_2(p)| + \frac{a_1 t^{1-\gamma_j}}{B\Gamma(2-\gamma_j)} I_q^{\alpha-1}|w_1(1) - w_2(1)| \leq \left(\frac{L_2^j}{L_1 + \sum_{j=1}^n \Lambda_2^j}\right)\psi(\|u-v\|),$$

for all $1 \leq j \leq n$. Hence,

$$\|h_1 - h_2\| = \sup_{t\in J}|h_1(t) - h_2(t)| + \sup_{t\in J}\sum_{i=1}^{n}|{}^cD_q^{\gamma_i}h_1(t) - {}^cD_q^{\gamma_i}h_2(t)| \leq \theta(\|u-v\|) \times \left(\frac{L_1}{L_1 + \sum_{j=1}^n L_2^j} + \sum_{i=1}^{n}\frac{L_2^i}{L_1 + \sum_{j=1}^n L_2^j}\right) = \theta(\|u-v\|).$$

By interchanging the roles of u, v, we conclude $H_d(\Omega(u), \Omega(v)) \leq \theta(\|u-v\|)$. Since the multifunction Ω has approximate endpoint property, by using Lemma 2.3 there exists $u^* \in \mathcal{A}$ such that $\Omega(u^*) = \{u^*\}$. This completes the proof. □

Here, we provide an example to illustrate our first main result. In this way, we give a computational technique for checking the problems (1.1) and (1.2). We need to present a simplified analysis could be executed values of the q-Gamma function. To this aim, we consider a pseudo-code description of the method for calculation of the q-Gamma function of order n in Algorithm 2.

Example 2.5. Consider the fractional q-differential inclusion problem

$$ {}^cD_q^{\frac{9}{4}}u(t) \in \left[0, \frac{t^2}{100}\sin u(t) + \frac{1}{100}\cos u'(t) + \frac{1}{100} \times \left(\frac{|u''(t)|}{1+|u''(t)|}\right)\right], \tag{2.9}$$

with the integral boundary conditions

$$\begin{cases} u(0) + u(\frac{3}{4}) + u(1) = \int_0^1 \frac{s^2}{20}\cos u(s)ds, \\ {}^cD_q^{\frac{2}{3}}u(0) + {}^cD_q^{\frac{2}{3}}u\left(\frac{3}{4}\right) + {}^cD_q^{\frac{2}{3}}u(1) = \int_0^1 \frac{e^{s^2-1}}{20}\cos u(s)ds, \\ {}^cD_q^{\frac{5}{3}}u(0) + {}^cD_q^{\frac{5}{3}}u\left(\frac{3}{4}\right) + {}^cD_q^{\frac{5}{3}}u(1) = \int_0^1 \frac{2s^3+1}{20\pi}\cos u(s)ds, \end{cases} \tag{2.10}$$

where $t \in J = [0,1]$, $\alpha = \frac{9}{4}$, $\beta = \frac{2}{3}$, $\gamma = \frac{5}{3}$ and $p = \frac{3}{4}$. Consider the map $T : J \times \mathbb{R}^3 \to P(\mathbb{R})$ define by $T(t, x_1, x_2, x_3) = \left[0, \frac{t^2}{100}\sin x_1 + \frac{1}{100}\cos x_2 + \frac{1}{100}\left(\frac{|x_3|}{1+|x_3|}\right)\right]$. Also, define the maps $f_0, f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$ define by $f_0(t,x) = \frac{t^2}{20}\cos x$, $f_1(t,x) = \frac{e^{t^2-1}}{20}\cos x$ and $f_2(t,x) = \frac{2t^3+1}{300\pi}\cos x$. Consider $N : C^2(J) \to 2^{C^2(J)}$ defined by

$$N(u) = \{h \in C^2(J) \mid \exists\, v \in S_{T,u} : h(t) = w(t) \text{ for all } t \in J\},$$

where

$$w(t) = I_q^{\frac{9}{4}}v(t) + \frac{1}{3}\int_0^1 \frac{s^2}{20}\cos u(s)ds - \frac{1}{3}\left[I_q^{\frac{9}{4}}v(1) + I_q^{\frac{9}{4}}v\left(\frac{3}{4}\right)\right] + a_1(t)\int_0^1 \frac{e^{s^2-1}}{20}\cos u(s)ds + a_2(t)\left[I_q^{\frac{19}{12}}v(1) + I_q^{\frac{19}{12}}v\left(\frac{3}{4}\right)\right] + (b_1 + a_3(t))\int_0^1 \frac{2s^3+1}{300\pi}\cos u(s)ds + (b_2 + a_4(t))\left[I_q^{\frac{7}{12}}v(1) + I_q^{\frac{7}{12}}v\left(\frac{3}{4}\right)\right].$$

Put

$$a_1(t) = \frac{3\Gamma_q\left(\frac{4}{3}\right)t - \frac{7}{4}\Gamma_q\left(\frac{4}{3}\right)}{3\left(\left(\frac{3}{4}\right)^{\frac{1}{3}} + 1\right)},$$

$$a_2(t) = \frac{\frac{7}{4}\Gamma_q\left(\frac{4}{3}\right) - 3\Gamma_q\left(\frac{4}{3}\right)t}{3\left(\left(\frac{3}{4}\right)^{\frac{1}{3}} + 1\right)},$$

$$a_3(t) = \frac{\begin{array}{c}-6\left(\left(\frac{3}{4}\right)^{\frac{4}{3}}+1\right)\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{4}{3}\right)t \\ +3\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{7}{3}\right)t^2\end{array}}{6\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\Gamma_q\left(\frac{7}{3}\right)},$$

$$a_4(t) = \frac{\begin{array}{c}6\left(\left(\frac{3}{4}\right)^{\frac{4}{3}}+1\right)\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{4}{3}\right)t \\ -3\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{7}{3}\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)t^2\end{array}}{6\left(\left(\frac{1}{3}\right)^{\frac{1}{3}}+1\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\Gamma_q\left(\frac{7}{3}\right)},$$

$$b_1 = \frac{\begin{array}{c}\frac{7}{2}\left(\left(\frac{3}{4}\right)^{\frac{4}{3}}+1\right)\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{4}{3}\right) \\ -\left(\left(\frac{3}{4}\right)^{2}+1\right)\Gamma_q\left(\frac{4}{3}\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\Gamma_q\left(\frac{7}{3}\right)\end{array}}{6\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\Gamma_q\left(\frac{7}{3}\right)},$$

$$b_2 = \frac{\begin{array}{c}\left(\left(\frac{3}{4}\right)^{2}+1\right)\Gamma_q\left(\frac{4}{3}\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right) \\ \times\Gamma_q\left(\frac{7}{3}\right)-\frac{8}{3}\left(\left(\frac{3}{4}\right)^{\frac{4}{3}}+1\right)\Gamma_q\left(\frac{4}{3}\right)\Gamma_q\left(\frac{4}{3}\right)\end{array}}{6\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\left(\left(\frac{3}{4}\right)^{\frac{1}{3}}+1\right)\Gamma_q\left(\frac{7}{3}\right)}, \tag{2.11}$$

$m(t) = \frac{3t}{20}$, $m_0(t) = \frac{t^2}{20}$, $m_1(t) = \frac{e^{t^2-1}}{20}$, $m_2(t) = \frac{2t^3+1}{300\pi}$ and $\psi(t) = \frac{t}{5}$. Then, we have

$$\Lambda_1 = \left[\frac{\|m\|_\infty}{\Gamma_q\left(\frac{13}{4}\right)} + \frac{\|m_0\|_\infty}{3} + \frac{2\|m\|_\infty}{3\Gamma_q\left(\frac{13}{4}\right)} + \frac{5\Gamma_q\left(\frac{4}{3}\right)\|m_1\|_\infty}{3} + \frac{10\Gamma_q\left(\frac{4}{3}\right)\|m\|_\infty}{3\Gamma_q\left(\frac{31}{12}\right)} + \frac{\begin{array}{c}10\left(2\Gamma_q\left(\frac{4}{3}\right)+\Gamma_q\left(\frac{7}{3}\right)\right) \\ \times\Gamma_q\left(\frac{4}{3}\right)\left(\|m_2\|_\infty\Gamma_q\left(\frac{19}{12}\right)+2\|m\|_\infty\right)\end{array}}{3\Gamma_q\left(\frac{7}{3}\right)\Gamma_q\left(\frac{19}{12}\right)}\right],$$

$$\Lambda_2 = \left[\frac{\|m\|_\infty}{\Gamma_q\left(\frac{9}{4}\right)} + \frac{2\Gamma_q\left(\frac{4}{3}\right)\|m\|_\infty}{\Gamma_q\left(\frac{31}{12}\right)} + \frac{\begin{array}{c}\left(2\Gamma_q\left(\frac{4}{3}\right)+\Gamma_q\left(\frac{7}{3}\right)\right)\Gamma_q\left(\frac{4}{3}\right) \\ \times\left(\|m_2\|_\infty\Gamma_q\left(\frac{19}{12}\right)+2\|m\|_\infty\right)\end{array}}{\Gamma\left(\frac{7}{3}\right)\Gamma_q\left(\frac{19}{12}\right)}\right],$$

$$\Lambda_3 = \left[\frac{\|m\|_\infty}{\Gamma_q\left(\frac{5}{4}\right)} + \frac{\Gamma_q\left(\frac{4}{3}\right)\left(\|m_2\|_\infty\Gamma_q\left(\frac{19}{12}\right)+2\|m\|_\infty\right)}{\Gamma_q\left(\frac{19}{12}\right)}\right].$$

Table 1 Some Numerical Results for Calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}$ Which is Constant, for $x = 9.5, 65, 110, 780$ in Algorithm 2.

n	$x = 9.5$	$x = 65$	$x = 110$	$x = 780$
1	2.679786	4432.545834	1804225.634753	$1.29090809480473E+45$
2	2.674552	4423.888518	1800701.756560	$1.28838678993206E+45$
3	2.673899	4422.808467	1800262.132108	$1.28807224237593E+45$
4	2.673818	4422.673494	1800207.192468	$1.28803293353064E+45$
5	2.673808	4422.656623	1800200.325222	$1.28802802007493E+45$
6	2.673806	4422.654514	1800199.466820	$1.28802740589531E+45$
7	2.673806	4422.654250	1800199.359519	$1.28802732912289E+45$
8	2.673806	4422.654217	1800199.346107	$1.28802731952634E+45$
9	2.673806	4422.654213	1800199.344430	$1.28802731832677E+45$
10	2.673806	4422.654213	1800199.344221	$1.28802731817683E+45$
11	2.673806	4422.654212	1800199.344195	$1.28802731815808E+45$
12	2.673806	4422.654212	1800199.344191	$1.28802731815574E+45$
13	2.673806	4422.654212	1800199.344191	$1.28802731815545E+45$
14	2.673806	4422.654212	1800199.344191	$1.28802731815541E+45$
15	2.673806	4422.654212	1800199.344191	$1.28802731815541E+45$
16	2.673806	4422.654212	1800199.344191	$1.28802731815541E+45$
17	2.673806	4422.654212	1800199.344191	$1.28802731815541E+45$
18	2.673806	4422.654212	1800199.344191	$1.28802731815541E+45$
19	2.673806	4422.654212	1800199.344191	$1.28802731815541E+45$

Table 2 Some Numerical Results for Calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}$ for $x = 9.5$ of Algorithm 2.

n	$q = \frac{1}{8}$	$q = \frac{1}{2}$	$q = \frac{4}{5}$	$q = \frac{8}{9}$
1	2.679786	136.046206	79062.138227	6301918.338883
2	2.674552	119.081545	41793.335091	2528395.395827
3	2.673899	111.658224	26290.733638	1232715.590371
4	2.673818	108.178242	18589.881264	689176.848061
5	2.673808	106.492553	14278.326587	426538.394173
6	<u>2.673806</u>	105.662861	11650.586796	285518.687713
7	2.673806	105.251251	9946.3508930	203363.796571
⋮	⋮	⋮	⋮	⋮
26	2.673806	104.841780	5522.283831	25842.863721
27	2.673806	104.841780	5513.202433	25230.371788
28	2.673806	<u>104.841779</u>	5505.949683	24699.649904
29	2.673806	104.841779	5500.155385	24238.446645
⋮	⋮	⋮	⋮	⋮
106	2.673806	104.841779	5477.048235	20879.606269
107	2.673806	104.841779	<u>5477.048234</u>	20879.566792
108	2.673806	104.841779	5477.048234	20879.531702
⋮	⋮	⋮	⋮	⋮
118	2.673806	104.841779	5477.048234	20879.337427
119	2.673806	104.841779	5477.048234	20879.327822
120	2.673806	104.841779	5477.048234	<u>20879.319284</u>

Table 3 Some Numerical Results for Calculation of $\Gamma_q(x)$ with $q = \frac{1}{8}, \frac{1}{2}, \frac{4}{5}, \frac{8}{9}$ for $x = 110$ of Algorithm 2.

n	$q = \frac{1}{8}$	$q = \frac{1}{2}$	$q = \frac{4}{5}$	$q = \frac{8}{9}$
1	1804225.634753	$2.43388915243820E+32$	$1.10933564801075E+75$	$2.3996994906237E+102$
2	1800701.756560	$2.12965300838343E+32$	$5.41355796236824E+74$	$7.1431517307455E+101$
3	1800262.132108	$1.99654969535946E+32$	$3.19616462101800E+74$	$2.6837217226512E+101$
4	1800207.192468	$1.93415751737948E+32$	$2.14884539802207E+74$	$1.1944485864825E+101$
5	1800200.325222	$1.90393630617042E+32$	$1.58553847001434E+74$	$6.0526350536381E+100$
6	1800199.466820	$1.88906180377847E+32$	$1.25302695267477E+74$	$3.3987862057282E+100$
7	1800199.359519	$1.88168265610746E+32$	$1.04280391429109E+74$	$2.0741306563269E+100$
8	1800199.346107	$1.87800749466975E+32$	$9.02841142168746E+73$	$1.3555712905453E+100$
9	1800199.344430	$1.87617350297573E+32$	$8.05899312693661E+73$	$9.38129101307050E+99$
10	1800199.344221	$1.87525740263248E+32$	$7.36673088857628E+73$	$6.81335603265770E+99$
11	1800199.344195	$1.87479957611817E+32$	$6.86049299667128E+73$	$5.15556440821410E+99$
12	<u>1800199.344191</u>	$1.87457071874804E+32$	$6.48333340557523E+73$	$4.04051908444650E+99$
⋮	⋮	⋮	⋮	⋮
48	1800199.344191	<u>$1.87434189862553E+32$</u>	$5.18960499065178E+73$	$6.66324790738213E+98$
⋮	⋮	⋮	⋮	⋮
90	1800199.344191	$1.87434189862553E+32$	<u>$5.18923469131315E+73$</u>	$6.50025876524830E+98$
91	1800199.344191	$1.87434189862553E+32$	$5.18923468501255E+73$	$6.50013085733126E+98$
92	1800199.344191	$1.87434189862553E+32$	$5.18923467997207E+73$	$6.50001716364224E+98$
93	1800199.344191	$1.87434189862553E+32$	$5.18923467593968E+73$	$6.49991610435300E+98$
⋮	⋮	⋮	⋮	⋮
118	1800199.344191	$1.87434189862553E+32$	$5.18923465987107E+73$	$6.49915022957670E+98$
119	1800199.344191	$1.87434189862553E+32$	$5.18923465985889E+73$	$6.49914550293450E+98$
120	1800199.344191	$1.87434189862553E+32$	$5.18923465984914E+73$	<u>$6.49914130147782E+98$</u>

Table 4 **Some Numerical Results of $\Gamma_q(\alpha)$ in Example 2.5 with Different Values of q by Algorithm 2.**

n	$\alpha = \frac{3}{4}$	$\alpha = \frac{5}{4}$	$\alpha = \frac{4}{3}$	$\alpha = \frac{19}{12}$	$\alpha = \frac{9}{4}$	$\alpha = \frac{7}{3}$	$\alpha = \frac{31}{12}$	$\alpha = \frac{13}{4}$
				$q = \frac{1}{3}$				
1	1.1174	0.9592	0.9559	0.9673	1.1055	1.1315	1.2199	1.5327
2	1.1311	0.9505	0.9448	0.9500	1.0747	1.0990	1.1824	1.4805
3	1.1356	0.9476	0.9411	0.9444	1.0647	1.0886	1.1703	1.4638
4	1.1371	0.9467	0.9400	0.9425	1.0615	1.0851	1.1664	1.4583
5	1.1376	0.9464	0.9396	0.9419	1.0604	1.0840	1.1651	1.4564
6	1.1377	0.9463	0.9394	0.9417	1.0600	1.0836	1.1646	1.4558
7	1.1378	0.9462	0.9394	0.9417	1.0599	1.0835	1.1645	1.4556
8	1.1378	0.9462	0.9394	0.9416	1.0599	1.0834	1.1644	1.4556
9	1.1378	0.9462	0.9394	0.9416	1.0598	1.0834	1.1644	1.4555
10	1.1378	0.9462	0.9394	0.9416	1.0598	1.0834	1.1644	1.4555
				$q = \frac{1}{2}$				
1	1.1069	0.9743	0.9772	1.0122	1.2620	1.3087	1.4715	2.1039
2	1.1377	0.9526	0.9493	0.9662	1.1655	1.2049	1.3437	1.8906
3	1.1522	0.9426	0.9364	0.9453	1.1221	1.1583	1.2865	1.7960
4	1.1593	0.9378	0.9302	0.9353	1.1015	1.1362	1.2594	1.7514
5	1.1628	0.9355	0.9272	0.9303	1.0915	1.1254	1.2463	1.7297
6	1.1645	0.9343	0.9257	0.9279	1.0865	1.1201	1.2398	1.7190
7	1.1654	0.9337	0.9249	0.9267	1.0841	1.1175	1.2365	1.7137
8	1.1658	0.9334	0.9245	0.9261	1.0828	1.1161	1.2349	1.7111
9	1.1660	0.9333	0.9244	0.9258	1.0822	1.1155	1.2341	1.7098
10	1.1662	0.9332	0.9243	0.9257	1.0819	1.1152	1.2337	1.7091
				$q = \frac{4}{5}$				
1	0.9665	1.1206	1.1787	1.4118	2.6441	2.8906	3.8168	8.5184
2	1.0284	1.0602	1.0963	1.2516	2.1063	2.2761	2.9085	6.0237
3	1.0710	1.0218	1.0443	1.1539	1.8020	1.9312	2.4107	4.7288
4	1.1018	0.9954	1.0091	1.0891	1.6109	1.7160	2.1053	3.9658
5	1.1248	0.9766	0.9840	1.0438	1.4826	1.5723	1.9040	3.4780
6	1.1421	0.9628	0.9657	1.0111	1.3927	1.4718	1.7646	3.1483
7	1.1555	0.9523	0.9519	0.9868	1.3275	1.3992	1.6647	2.9162
8	1.1659	0.9444	0.9414	0.9685	1.2793	1.3455	1.5913	2.7481
9	1.1740	0.9383	0.9334	0.9545	1.2429	1.3051	1.5363	2.6235
10	1.1803	0.9335	0.9271	0.9436	1.2151	1.2743	1.4945	2.5297

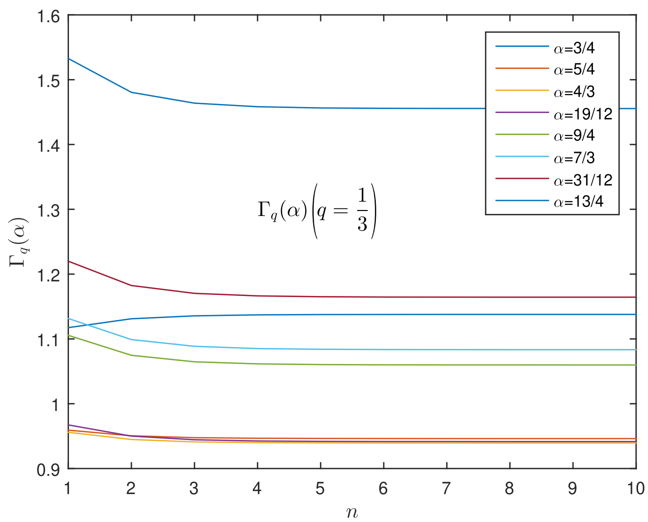

Fig. 1 Numerical results of $\Gamma_q(\alpha)$, where $q = \frac{1}{3}$ in Table 4.

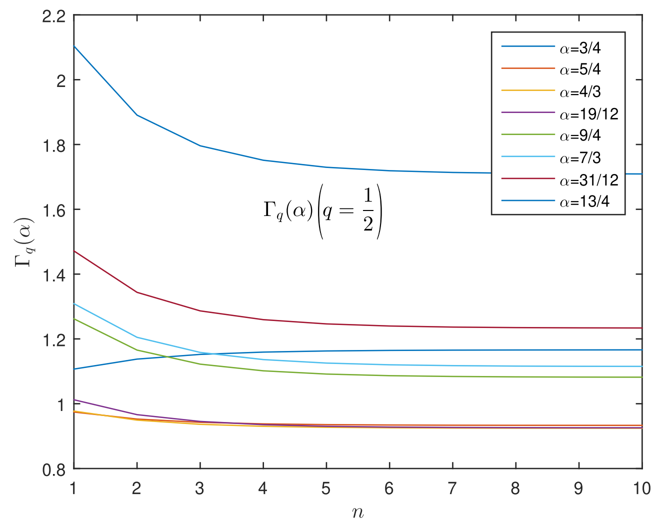

Fig. 2 Numerical results of $\Gamma_q(\alpha)$, where $q = \frac{1}{2}$ in Table 4.

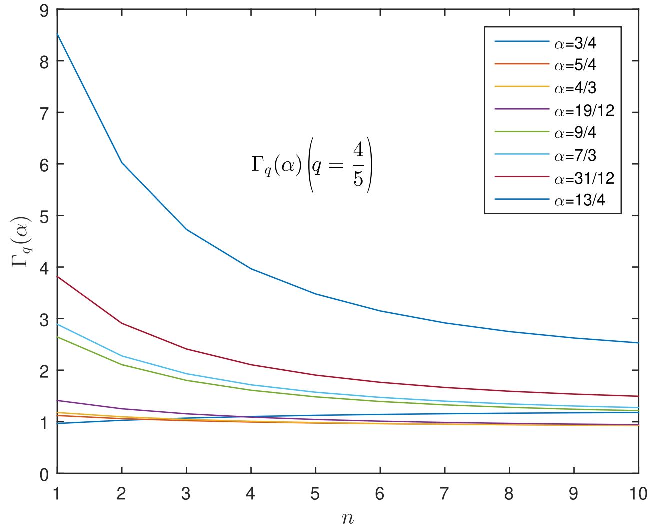

Fig. 3 Numerical results of $\Gamma_q(\alpha)$, where $q = \frac{4}{5}$ in Table 4.

Table 5 Some Numerical Results of Λ_1, Λ_2 and Λ_3 in Example 2.5 with Different Values of q.

n	Λ_1	Λ_2	Λ_3	$\frac{1}{\Lambda_1+\Lambda_2+\Lambda_3}$
		$q = \frac{1}{3}$		
1	3.3364	1.1763	0.4559	0.2013
2	3.3955	1.1988	0.4592	0.1979
3	3.4147	1.2061	0.4602	0.1968
4	3.4219	1.2088	0.4606	0.1964
5	3.4240	1.2096	0.4608	0.1963
6	3.4244	1.2098	0.4608	0.1963
7	3.4246	1.2099	0.4608	0.1963
8	3.4251	1.2101	0.4608	0.1962
9	3.4251	1.2101	0.4608	0.1962
10	3.4251	1.2101	0.4608	0.1962
		$q = \frac{1}{2}$		
1	2.9820	1.0480	0.4467	0.2234
2	3.1379	1.1076	0.4552	0.2127
3	3.2160	1.1375	0.4593	0.2078
4	3.2553	1.1525	0.4613	0.2054
5	3.2754	1.1601	0.4623	0.2042
6	3.2852	1.1639	0.4628	0.2036
7	3.2898	1.1656	0.4630	0.2033
8	3.2923	1.1666	0.4631	0.2032
9	3.2939	1.1671	0.4632	0.2031
10	3.2943	1.1673	0.4632	0.2031
		$q = \frac{4}{5}$		
1	1.8371	0.6109	0.3881	0.3526
2	2.0805	0.7071	0.4077	0.3130
3	2.2800	0.7853	0.4216	0.2868
4	2.4430	0.8488	0.4319	0.2686
5	2.5751	0.9001	0.4395	0.2554
6	2.6823	0.9415	0.4454	0.2457
7	2.7686	0.9748	0.4499	0.2385
8	2.8380	1.0016	0.4534	0.2329
9	2.8943	1.0232	0.4562	0.2286
10	2.9392	1.0404	0.4584	0.2253

By checking the data in Tables 4 and 5, it is easy to check that

$$\begin{aligned} &H_d(T(t,u_1,u_2,u_3),F(t,v_1,v_2,v_3)) \\ &\quad\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m(t)\theta \\ &\qquad\times\left(\sum_{k=1}^{3}|u_k-v_k|\right), \end{aligned}$$

and

$$\begin{aligned} &|f_j(t,u)-f_j(t,v)| \\ &\quad\le \frac{1}{\Lambda_1+\Lambda_2+\Lambda_3} m_j(t)\psi(|u-v|), \end{aligned}$$

for $t \in J$ and $j = 0,1,2$. Since $\sup_{u\in N(0)}\|u\| = 0$, we have $\inf_{u\in C^2(J)}(\sup_{v\in N(u)}\|u-v\|) = 0$. Thus, N has the approximate endpoint property (please check Figs. 1–3). Now, by using Theorem 2.2, the fractional q-differential inclusion problem (2.9), (2.10) has at least one solution.

ACKNOWLEDGMENTS

José Francisco Gómez Aguilar acknowledges the support provided by CONACyT: Cátedras CONACyT para jóvenes investigadores 2014 and SNI-CONACyT.

REFERENCES

1. F. H. Jackson, q-difference equations, *Am. J. Math.* **32** (1910) 305–314.
2. C. R. Adams, The general theory of a class of linear partial q-difference equations, *Trans. Amer. Math. Soc.* **26**(2) (1924) 283–312.
3. C. R. Adams, Note on the integro-q-difference equations, *Trans. Amer. Math. Soc.* **31**(4) (1929) 861–867.
4. B. Ahmad, S. Etemad, M. Ettefagh and Sh. Rezapour, On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions, *Bull. Math. Soc. Sci. Math. Roumanie* **59**(2)(107) (2016) 119–134.
5. B. Ahmad, J. J. Nieto, A. Alsaedi and H. Al-Hutami, Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions, *J. Franklin Inst.* **351** (2014) 2890–2909.
6. B. Ahmad, S. K. Ntouyas and I. K. Purnaras, Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations, *Adv. Diff. Equ.* **2012** (2012) 140.

7. N. Balkani, Sh. Rezapour and R. H. Haghi, Approximate solutions for a fractional q-integro-difference equation, *J. Math. Extension* **13**(3) (2019) 109–119.
8. R. A. C. Ferreira, Nontrivials solutions for fractional q-difference boundary value problems, *Elect. J. Qual. Theory Diff. Equ.* **70** (2010) 1–101.
9. V. Kalvandi and M. E. Samei, New stability results for a sum-type fractional q-integro-differential equation, *J. Adv. Math. Stud.* **12**(2) (2019) 201–209.
10. S. K. Ntouyas and M. E. Samei, Existence and uniqueness of solutions for multi-term fractional q-integro-differential equations via quantum calculus, *Adv. Diff. Equ.* **2019** (2019) 475, doi:10.1186/s13662-019-2414-8.
11. M. E. Samei, Gh. Khalilzadeh Ranjbar and V. Hedayati, Existence of solutions for equations and inclusions of multi-term fractional q-integro-differential with non-separated and initial boundary conditions, *J. Ineq. Appl.* **2019** (2019) 273.
12. M. E. Samei and G. Khalilzadeh Ranjbar, Some theorems of existence of solutions for fractional hybrid q-difference inclusion, *J. Adv. Math. Stud.* **12**(1) (2019) 63–76.
13. Y. Zhao, H. Chen and Q. Zhang, Existence results for fractional q-difference equations with nonlocal q-integral boundary conditions, *Adv. Diff. Equ.* **2013** (2013) 48.
14. R. P. Agarwal, D. Baleanu, V. Hedayati and Sh. Rezapour, Two fractional derivative inclusion problems via integral boundary condition, *Appl. Math. Comput.* **257** (2015) 205–212.
15. N. Aghazadeh, E. Ravash and Sh. Rezapour, Existence results and numerical solutions for a multi-term fractional integro-differential equation, *Kragujevac J. Math.* **43**(3) (2019) 413–426.
16. E. Akbari Kojabad and Sh. Rezapour, Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials, *Adv. Diff. Equ.* **2017** (2017) 351.
17. D. Baleanu, V. Hedayati, Sh. Rezapour and M. M. Al-Qurashi, On two fractional differential inclusions, *Springerplus* **5**(1) (2016) 882.
18. D. Baleanu, Sh. Rezapour and Z. Saberpour, On fractional integro-differential inclusions via the extended fractional Caputo–Fabrizio derivation, *Bound. Value Probl.* **2019** (2019) 79.
19. M. De La Sena, V. Hedayati, Y. Gholizade Atani and Sh. Rezapour, The existence and numerical solution for a k-dimensional system of multi-term fractional integro-differential equations, *Nonlinear Anal. Model. Control* **22**(2) (2017) 188–209.
20. V. Hedayati and M. E. Samei, Positive solutions of fractional differential equation with two pieces in chain interval and simultaneous dirichlet boundary conditions, *Bound. Val. Probl.* **2019** (2019) 141, doi:10.1186/s13661-019-1251-8.
21. A. Mohammadi, N. Aghazadeh and Sh. Rezapour, Haar wavelet collocation method for solving singular and nonlinear fractional time-dependent Emden–Fowler equations with initial and boundary conditions, *Math. Sci.* **13** (2019) 255–265.
22. Sh. Rezapour and V. Hedayati, On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex-compact valued multifunctions, *Kragujevac J. Math.* **41**(1) (2017) 143–158.
23. M. E. Samei, V. Hedayati and Sh. Rezapour, Existence results for a fractional hybrid differential inclusion with Caputo-Hadamard type fractional derivative, *Adv. Differ. Equ.* **2019** (2019) 163.
24. M. Shabibi, M. Postolache, Sh. Rezapour and S. M. Vaezpour, Investigation of a multi-singular pointwise defined fractional integro-differential equation, *J. Math. Anal.* **7**(5) (2016) 61–77.
25. M. Shabibi, Sh. Rezapour and S. M. Vaezpour, A singular fractional integro-differential equation, *Sci. Bull. Univ. Politeh. Buch. A* **79**(1) (2017) 109–118.
26. S. Liang and M. E. Samei, New approach to solutions of a class of singular fractional q-differential problem via quantum calculus, *Adv. Diff. Equ.* **2019** (2019) 163, doi:10.1186/s13662-019-2489-2.
27. M. E. Samei, Existence of solutions for a system of singular sum fractional q-differential equations via quantum calculus, *Adv. Diff. Equ.* **2019** (2019) 163, doi:10.1186/s13662-019-2480-y.
28. M. Kisielewicz, *Differential Inclusions and Optimal Control* (Kluwer, Dordrecht, 1991).
29. A. Amini-Harandi, Endpoints of set-valued contractions in metric spaces, *Nonlinear Anal.* **72** (2010) 132–134.
30. M. E. Samei, V. Hedayati and G. K. Ranjbar, The existence of solution for k-dimensional system of Langevin Hadamard-type fractional differential inclusions with $2k$ different fractional orders, *Mediterranean J. Math.* **17** (2020) 37, doi:10.1007/s00009-019-1471-2.