Research Article

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Approximate analytical fractional view of convection–diffusion equations

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Abstract: In this article, a modified variational iteration method along with Laplace transformation is used for obtaining the solution of fractional-order nonlinear convection–diffusion equations (CDEs). The proposed technique is applied for the first time to solve fractional-order nonlinear CDEs and attain a series-form solution with the quick rate of convergence. Tabular and graphical representations are presented to confirm the reliability of the suggested technique. The solutions are calculated for fractional as well as for integer orders of the problems. The solution graphs of the solutions at various fractional derivatives are plotted. The accuracy is measured in terms of absolute error. The higher degree of accuracy is observed from the table and figures. It is further investigated that fractional solutions have the convergence behavior toward the solution at integer order. The applicability of the present technique is verified by illustrative examples. The simple and effective procedure of the current technique supports its implementation to solve other nonlinear fractional problems in different areas of applied science.

Keywords: variational iteration method, homotopy perturbation method, convection–diffusion equations, Laplace transform method, Mittag–Leffler function

1 Introduction

Fractional calculus (FC) is the branch of mathematics which can be used to analyze various problems in science and engineering more accurately as compared to ordinary calculus. In the last few decades, significant interest has been shown by the researchers to FC in different areas, such as edge detection, electromagnetic, engineering, viscoelasticity, electrochemistry, cosmology, turbulence, diffusion, signal processing material science, physics and acoustics. Many other problems in applied sciences are modeled by fractional-order partial differential equations (PDEs) [1–3]. Various dynamical systems in physics and engineering are also modeled by using fractional-order differential equations. A number of researchers have contributed a lot to provide an outstanding history of fractional-order derivative and integration operators such as Caputo [4], Yin et al. [5], Rashid et al., Arife et al. [6] and Oldham and Spanier [7].

Over the last decade, the study of nonlinear PDEs modeling different physical processes has become a significant tool. Nonlinear processes are of fundamental interest in the diverse fields of science and engineering. Most of the nonlinear phenomena are the best representations of our real-world problems. Fractional PDEs are important mathematical models which can model many complicated phenomena more accurately in various areas of sciences such as diffusion equations [8], heat equations, wave equations [9], telegraph equations [10,11], local fractional dissipative and damped wave equations [12], time-fractional Zakharov–Kuznetsov

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equation [13], nonlinear Schrodinger equation [14], homogeneous Smoluchowski's coagulation equation [15], third-order dispersive fractional-order PDEs [16], Kortewege–De Vries equations [17], local fractional transport and Fokker Planck equations [18,19], nonlinear predator–prey biological population dynamical system [20], fractional wave equation and dynamical model [21,22], fractional-order Helmholtz equations [23] and Navier–Stokes equation [24].

In this article, convection–diffusion equations (CDEs) of fractional-order are solved by the homotopy perturbation method (HPM) and variational iteration technique along with Laplace transform (VHPTM).

$$\frac{\partial^{\beta}\upsilon}{\partial t^{\beta}} = \frac{\partial^2\upsilon}{\partial x^2} - \bar{c}\frac{\partial\upsilon}{\partial x} + \phi(\upsilon) + \bar{g}(x,t),$$
$$0 < x \leq 1, \quad 0 < \beta \leq 1, \quad t > 0,$$

initial condition is

$$\upsilon(x,0) = f(x),$$

where $\phi(\upsilon)$ is a sensible nonlinear operator, that is selected as an energy capacity, $\bar{c}$ is a constant parameter and β representing the time fractional-order derivative.

The CDE is a mixture of the equations of diffusion and convection (advection) and explains physical phenomena in which particles, electricity or other physical quantities are transmitted within a physical structure through two procedures: convection and diffusion. The CDEs are commonly used as mathematical models for computational simulations in engineering and science, for example, in models of oil reservoirs, mass and energy transport and worldwide climate manufacturing, where the originally discontinuous model is reproduced by diffusion and convection, the latter at $\bar{c}$ velocity. Depending on the situation that the same equation can be named the CDE or drift-diffusion equation and fractional diffusion equations and anomalous diffusion [25,26].

Fractional-order CDEs (FCDEs) are the extended form of ordinary CDEs. FCDEs can express physical problems more accurately as compared to ordinary CDEs. In this regard, the numerical and analytical solutions for FCDEs are the focus point for the researchers, and therefore different techniques have been established such as adomian decomposition method [27], Sumudu transform method and homotopy analysis transform method were used by Singh et al. [28]; HPM was applied by Yildrim and Momani [29]; variational iteration technique was used by Merdan [30]; and Irandoust-pakchin et al. successfully implemented the flatlet oblique multiwavelet and found a mathematical approach for the class of FCDEs [31].

The VHPTM is a mixture of three techniques, namely, HPM, variational iteration technique and Laplace transform (LT). VHPTM [34–39] is a hybrid technique and carry the beneficial features of both HPM and varational iteration method (VIM) and is very consistent with various physical problems. The proposed technique provides the closed and series-form solution having easily computable and convergent terms [40].

2 Basic concepts

2.1 Definition

LT of $g(t)$, $t > 0$ is denoted as [42]

$$Q(s) = \mathcal{L}[g(t)] = \int_0^{\infty} e^{-st} g(t)\,dt.$$

2.2 Theorem

LT in the forms of convolution [42]

$$\mathcal{L}[g_1 \times g_2] = \mathcal{L}[g_1(t)] \times \mathcal{L}[g_2(t)],$$

where $g_1 \times g_2$ defines the convolution between g_1 and g_2,

$$(g_1 \times g_2)t = \int_0^{\tau} g_1(\tau) g_2(t-\tau)\,dt.$$

LT of the fractional derivative

$$\mathcal{L}(D_t^{\beta} g(t)) = s^{\beta} Q(s) - \sum_{k=0}^{n-1} s^{\beta-1-k} g^{(k)}(0),$$
$$m - 1 < \beta < m,$$

where $Q(s)$ is the LT of $g(t)$.

2.3 Definition

The Riemann–Liouville definition of fractional integral is [34]

$$I_x^{\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-s)^{\beta-1} g(s)\,ds,$$

where

$$\Gamma(\beta) = \int_0^{\infty} e^{-x} x^{\beta-1}\,dx, \quad \beta \in \mathbb{C}.$$

2.4 Definition

The Caputo definition of fractional derivative of order β is given as follows:

$$D^{\beta}g(t) = \frac{\partial^{\beta}g(t)}{\partial t^{\beta}} = \begin{cases} I^{m-\beta}\left[\frac{\partial^{\beta}g(t)}{\partial t^{\beta}}\right], & \text{if } m-1<\beta<m, m\in\mathbb{N}, \\ \frac{\partial^{\beta}g(t)}{\partial t^{\beta}}, \end{cases}$$

with the following properties

$$I^{\beta}I^{a}g(x) = I^{\beta+a}g(x), \quad a,\beta \geq 0.$$

$$I^{\beta}x^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(y+\lambda+1)}x^{\beta+\lambda}, \quad \beta>0, \lambda>-1, x>0.$$

$$I^{\beta}D^{\beta}g(x) = g(x) - \sum_{k=0}^{m-1}g^{(k)}(0^{+})\frac{x^{k}}{k!},$$

for $x>0, m-1<\beta\leq m.$

3 General implementation of VHPTM

To illustrate the basic principle of VHPTM [34,35], we consider the following equation:

$$D_{t}^{\beta}\upsilon(x,t) + \mathcal{R}\upsilon(x,t) + \mathcal{N}\upsilon(x,t) = g(x,t), \tag{1}$$

with the initial solution

$$\upsilon(x,0) = g(x),$$

where the linear and nonlinear terms are represented by $\mathcal{R}$ and $\mathcal{N}$ and inhomogeneous term is $g(x,t)$.

Applying LT to equation (1), we get

$$s^{\beta}\pounds\{\upsilon(x,t)\} - \sum_{k=0}^{m-1}s^{\beta-1-k}\frac{\partial^{k}\upsilon(x,t)}{\partial^{k}t}\bigg|_{t=0} = -\pounds\{\mathcal{R}\upsilon(x,t) + \mathcal{N}\upsilon(x,t) - f(x,t)\}.$$

Using the variation iteration method

$$\pounds_{t}\{\upsilon_{j+1}(x,t)\} = \pounds\{\upsilon_{j}(x,t)\} + \lambda(s)[s^{\beta}\pounds\{\upsilon_{t}(x,t)\} - \sum_{k=0}^{m-1}s^{\beta-1-k}\frac{\partial^{k}\upsilon(x,t)}{\partial^{k}t}\bigg|_{t=0} + \pounds\{\mathcal{R}\upsilon(x,t) + \mathcal{N}\upsilon(x,t) - f(x,t)\}], \tag{2}$$

where $\lambda(s) = \frac{-1}{s^{\beta}}$ is the Lagrange multiplier [35].

Applying inverse LT to equation (2)

$$\upsilon_{j+1}(x,t) = \upsilon_{j}(x,t) - \pounds^{-1}\left[\frac{1}{s^{\beta}}\left\{s^{\beta}\pounds\{\upsilon_{t}(x,t)\} - \sum_{k=0}^{m-1}s^{\beta-1-k}\frac{\partial^{k}\upsilon(x,t)}{\partial^{k}t}\bigg|_{t=0} + \pounds\{\mathcal{R}\upsilon(x,t) + \mathcal{N}\upsilon(x,t) - f(x,t)\}\right\}\right]. \tag{3}$$

The basic HPM approximation is

$$\upsilon(x,t) = \sum_{j=0}^{\infty}p^{j}\upsilon_{j}(x,t) = \upsilon_{0} + p\upsilon_{1} + p^{2}\upsilon_{2} + p^{3}\upsilon_{3} + \cdots, \tag{4}$$

and the nonlinear functional can be written as

$$\mathcal{N}\upsilon(x,t) = \sum_{j=0}^{\infty}p^{j}\mathcal{H}(\upsilon). \tag{5}$$

$\mathcal{H}_{j}$ is He's polynomial,

$$\mathcal{H}_{j}(\upsilon_{0} + \upsilon_{1} + \cdots + u_{j}) = \frac{1}{j!}\frac{\partial^{j}}{\partial p^{j}}\left[\mathcal{N}\left(\sum_{i=0}^{\infty}p^{i}\upsilon_{i}\right)\right]. \tag{6}$$

VHPTM solution of equation (3) along with He's polynomial is

$$\sum_{j=0}^{\infty}p^{j}\upsilon_{j+1}(x,t) = \sum_{j=0}^{\infty}p^{j}\upsilon_{j}(x,t) + \pounds^{-1}\left[\frac{1}{s^{\beta}}\left\{s^{\beta}\pounds\left\{\sum_{j=0}^{\infty}p^{j}\frac{\partial\upsilon_{j}}{\partial t}(x,s) + \sum_{j=0}^{\infty}p^{j}\mathcal{R}\upsilon_{j}(x,t) + \sum_{j=0}^{\infty}p^{j}\mathcal{H}_{j}(\upsilon) - f(x,t)\right\}\right\}\right]. \tag{7}$$

The coefficient resulting from powers of p.

$$\upsilon_{0}(x,t) = g(x),$$

$$\upsilon_{1}(x,t) = \upsilon_{0}(x,t) + \pounds^{-1}\left[\frac{1}{s^{\beta}}\left\{s^{\beta}\pounds\left\{\frac{\partial\upsilon_{0}}{\partial t}(x,s) + \mathcal{R}\upsilon_{0}(x,t) + \mathcal{H}_{0}(\upsilon) - f(x,t)\right\}\right\}\right]. \tag{8}$$

Equation (8) represents the generalized scheme for VHPTM to solve fractional PDEs.

4 Numerical examples

4.1 Example 1

The nonlinear homogeneous CDE of fractional order is

$$\frac{\partial^{\beta}\upsilon}{\partial t^{\beta}} = \frac{\partial^{2}\upsilon}{\partial x^{2}} - \frac{\partial\upsilon}{\partial x} + \upsilon\frac{\partial^{2}\upsilon}{\partial x^{2}} - \upsilon^{2} + \upsilon, \quad 0<x\leq 1, \quad 0<\beta\leq 1, \quad t>0, \tag{9}$$

with boundary conditions

$$\upsilon(0,t)=e^{t},\quad \upsilon(1,t)=e^{t+1}, \tag{10}$$

and initial condition

$$\upsilon(x,0)=e^{x}. \tag{11}$$

For the following fractional PDEs, the functional correction is given by

$$\begin{aligned}\upsilon_{j+1}(x,t)=\upsilon_j(x,t)\quad&+£^{-1}\bigg[\lambda(s)£\bigg\{s^{\beta}\frac{\partial\upsilon}{\partial t}(x,t)\\&-\frac{\partial^2\upsilon_j}{\partial x^2}(x,t)+\frac{\partial\upsilon_j}{\partial x}(x,t)\\&-\upsilon_j(x,t)\frac{\partial^2\upsilon_j}{\partial x^2}(x,t)+\upsilon_j^2(x,t)-\upsilon_j(x,t)\bigg\}\bigg],\end{aligned} \tag{12}$$

where $\lambda(s)$ is the Lagrange multiplier

$$\lambda(s)=\frac{-1}{s^{\beta}}.$$

Using He's polynomial, equation (12) can be written as:

$$\begin{aligned}&p^0\upsilon_1(x,t)+p^1\upsilon_2(x,t)+p^2\upsilon_3(x,t)+\cdots\\&=\sum_{j=0}^{\infty}p^j\upsilon_j(x,t)-£^{-1}\bigg[\frac{1}{s^{\beta}}£\bigg\{s^{\beta}\bigg(p^0\frac{\partial\upsilon_0}{\partial t}+p\frac{\partial\upsilon_1}{\partial t}\\&+p^2\frac{\partial\upsilon_1}{\partial t}\bigg)-\bigg(\frac{p^0\partial^2\upsilon_0}{\partial x^2}+p\frac{\partial^2\upsilon_1}{\partial x^2}+p^2\frac{\partial^2\upsilon_2}{\partial x^2}+\cdots\bigg)\\&+\bigg(p^0\frac{\partial\upsilon_0}{\partial x}+p\frac{\partial\upsilon_1}{\partial x}+p^2\frac{\partial\upsilon_2}{\partial x}+\cdots\bigg)-\bigg\{p^0\upsilon_0\frac{\partial^2\upsilon_0}{\partial x^2}\\&+p\bigg(\upsilon_0\frac{\partial^2\upsilon_1}{\partial x^2}+\upsilon_1\frac{\partial^2\upsilon_0}{\partial x^2}\bigg)+p^2\bigg(\upsilon_2\frac{\partial^2\upsilon_0}{\partial x^2}+\upsilon_1\frac{\partial^2\upsilon_1}{\partial x^2}\bigg)\\&+\upsilon_0\frac{\partial^2\upsilon_2}{\partial x^2}+\cdots\bigg\}+(p^0\upsilon_0^2+p(2\upsilon_0\upsilon_1)+p^2(2\upsilon_0\upsilon_2\\&+\upsilon_1^2)+\cdots)-(p^0\upsilon_0+p\upsilon_1+p^2\upsilon_2+\cdots)\bigg\}\bigg].\end{aligned} \tag{13}$$

Comparing the coefficients of the same power of p, we get

$$\upsilon_0(x,t)=e^{x}$$

$$\begin{aligned}p^0\upsilon_1(x,t)&=p^0\upsilon_0(x,t)-£^{-1}\bigg[\frac{1}{s^{\beta}}£\bigg\{p^0s^{\beta}\frac{\partial\upsilon_0}{\partial t}\\&-p^0\frac{\partial^2\upsilon_0}{\partial x^2}+p^0\frac{\partial\upsilon_0}{\partial x}-p^0\upsilon_0\frac{\partial^2\upsilon_0}{\partial x^2}+p^0\upsilon_0^2-p^0\upsilon_0\bigg)\bigg\}\bigg],\end{aligned}$$

$$\upsilon_1(x,t)=e^{x}+e^{x}\frac{t^{\beta}}{\Gamma(\beta+1)},$$

$$\begin{aligned}p^1\upsilon_2(x,t)&=p^1\upsilon_1(x,t)-£^{-1}\bigg[\frac{1}{s^{\beta}}£\bigg\{p^1s^{\beta}\frac{\partial\upsilon_1}{\partial t}-p^1\frac{\partial^2\upsilon_1}{\partial x^2}\\&\times p^1\frac{\partial\upsilon_1}{\partial x}-p^1\bigg(\upsilon_0\frac{\partial^2\upsilon_1}{\partial x^2}+\upsilon_1\frac{\partial^2\upsilon_0}{\partial x^2}+p^12\upsilon_0\upsilon_1-p^1\upsilon_1\bigg)\bigg\}\bigg],\end{aligned}$$

$$\upsilon_2(x,t)=e^{x}+e^{x}\frac{t^{\beta}}{\Gamma(\beta+1)}+e^{x}\frac{t^{2\beta}}{\Gamma(2\beta+1)},$$

$$\begin{aligned}p^2\upsilon_3(x,t)&=p^2\upsilon_2(x,t)-£^{-1}\bigg[\frac{1}{s^{\beta}}£\bigg\{p^2s^{\beta}\frac{\partial\upsilon_2}{\partial t}\\&-p^2\frac{\partial^2\upsilon_2}{\partial x^2}(x,t)+p^2\frac{\partial\upsilon_2}{\partial x}(x,t)-p^2\bigg(\upsilon_2\frac{\partial^2\upsilon_0}{\partial x^2}\\&+\upsilon_1\frac{\partial^2\upsilon_1}{\partial x^2}+\upsilon_0\frac{\partial^2\upsilon_2}{\partial x^2}\bigg)+p^2(2\upsilon_0(x,t)\upsilon_1(x,t)\\&+\upsilon_1^2(x,t))-p^2\upsilon_2(x,t)\bigg\}\bigg],\end{aligned}$$

$$\begin{aligned}\upsilon_3(x,t)&=e^{x}+e^{x}\frac{t^{\beta}}{\Gamma(\beta+1)}+e^{x}\frac{t^{2\beta}}{\Gamma(2\beta+1)}\\&+e^{x}\frac{t^{3\beta}}{\Gamma(3\beta+1)},\end{aligned}$$

$$\vdots$$

$$p^j:\upsilon_j(x,t)=e^{x}\frac{t^{j\beta}}{\Gamma(j\beta+1)}.$$

The VHPTM solution of Example 1 is

$$\begin{aligned}\upsilon(x,t)\;&=e^{x}\bigg[1+\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\frac{t^{3\beta}}{\Gamma(3\beta+1)}\\&+\cdots+\frac{t^{j}\beta}{\Gamma(j\beta+1)}\bigg].\end{aligned} \tag{14}$$

The series obtained in equation (14) at $\beta=1$ is as follows:

$$\upsilon(x,t)=e^{x}\bigg[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\bigg]. \tag{15}$$

The actual solution is

$$\upsilon(x,t)=e^{x+t}. \tag{16}$$

4.2 Example 2

The nonhomogeneous nonlinear fractional CDE is

$$\begin{aligned}\frac{\partial^{\beta}\upsilon}{\partial t^{\beta}}&=\frac{\partial^2\upsilon}{\partial x^2}-\frac{\partial\upsilon}{\partial x}+\frac{\partial}{\partial t}\bigg(\upsilon\frac{\partial^2\upsilon}{\partial x^2}\bigg)-2x,\\&0<x\le 1,\quad 0<\beta\le 1,\quad t>0,\end{aligned} \tag{17}$$

with boundary conditions

$$\upsilon(0,t)=2t,\quad \upsilon(1,t)=1+2t, \tag{18}$$

and initial condition

$$\upsilon(x,0)=x^2. \tag{19}$$

For the following fractional PDEs, the functional correction is given by

$$\upsilon_{j+1}(x,t) = \upsilon_j(x,t) + £^{-1}\left[\lambda(s)£\left\{\frac{\partial^\beta \upsilon}{\partial t^\beta}(x,t) - \frac{\partial^2 \upsilon_j}{\partial x^2}(x,t) + \frac{\partial \upsilon_j}{\partial x}(x,t) - \frac{\partial}{\partial t}\left(\upsilon_j(x,t)\frac{\partial^2 \upsilon_j}{\partial x^2}(x,t)\right) + 2x\right\}\right]. \tag{20}$$

The Lagrange multiplier is

$$\lambda(s) = \frac{-1}{s^\beta}.$$

Using He's polynomial, equation (20) can be written as

$$\begin{aligned}
&p^0\upsilon_1 + p^1\upsilon_2 + p^2\upsilon_3 + \cdots \\
&= \sum_{j=0}^{\infty} p^j \upsilon_j(x,t) \\
&- £^{-1}\left[\frac{1}{s^\beta}£\left\{s^\beta\left(p^0\frac{\partial \upsilon_0}{\partial t} + p\frac{\partial \upsilon_1}{\partial t} + p^2\frac{\partial \upsilon_2}{\partial t}\right)\right.\right. \\
&- \left(p^0\frac{\partial^2 \upsilon_0}{\partial x^2} + p\frac{\partial^2 \upsilon_1}{\partial x^2} + p^2\frac{\partial^2 \upsilon_2}{\partial x^2} + \cdots\right) \\
&+ \left(p^0\frac{\partial \upsilon_0}{\partial x} + p\frac{\partial \upsilon_1}{\partial x} + p^2\frac{\partial \upsilon_2}{\partial x} + \cdots\right) \\
&- \frac{\partial}{\partial t}\left\{p^0\upsilon_0\frac{\partial^2 \upsilon_0}{\partial x^2} + p\left(\upsilon_0\frac{\partial^2 \upsilon_1}{\partial x^2} + \upsilon_1\frac{\partial^2 \upsilon_0}{\partial x^2}\right)\right. \\
&\left.\left.\left. + p^2\left(\upsilon_2\frac{\partial^2 \upsilon_0}{\partial x^2} + \upsilon_1\frac{\partial^2 \upsilon_1}{\partial x^2} + \upsilon_0\frac{\partial^2 \upsilon_2}{\partial x^2}\right) + \cdots\right\} + 2x\right\}\right].
\end{aligned} \tag{21}$$

Comparing the coefficients of the same power of p, we get

$$\upsilon_0(x,t) = x^2,$$

$$p^0\upsilon_1(x,t) = p^0\upsilon_0(x,t) - £^{-1}\left[p^0\frac{1}{s^\beta}£\left\{s^\beta\frac{\partial \upsilon_0}{\partial t} - \frac{\partial^2 \upsilon_0}{\partial x^2} + \frac{\partial \upsilon_0}{\partial x} - \frac{\partial}{\partial t}\left\{\upsilon_0\frac{\partial^2 \upsilon_0}{\partial x^2}\right\} + 2x\right\}\right],$$

$$\upsilon_1(x,t) = x^2 + (2-4x)\frac{t^\beta}{\Gamma(\beta+1)},$$

$$p^1\upsilon_2(x,t) = p^1\upsilon_1(x,t) - £^{-1}\left[p^1\frac{1}{s^\beta}£\left\{s^\beta\frac{\partial \upsilon_1}{\partial t} - \frac{\partial^2 \upsilon_1}{\partial x^2} + \frac{\partial \upsilon_1}{\partial x} - \frac{\partial}{\partial t}\left\{\upsilon_0\frac{\partial^2 \upsilon_1}{\partial x^2} + \upsilon_1\frac{\partial^2 \upsilon_0}{\partial x^2}\right\} + 2x\right\}\right],$$

$$\upsilon_2(x,t) = x^2 + (2-4x)\frac{t^\beta}{\Gamma(\beta+1)} + 4\frac{t^{2\beta}}{\Gamma(2\beta+1)} - 4x(3x-1)\frac{t^{2\beta-1}}{\Gamma(2\beta)}$$

$$p^2\upsilon_3(x,t) = p^2\upsilon_1(x,t) - £^{-1}\left[p^2\frac{1}{s^\beta}£\left\{s^\beta\frac{\partial \upsilon_2}{\partial t} - \frac{\partial^2 \upsilon_2}{\partial x^2} + \frac{\partial \upsilon_2}{\partial x} - \frac{\partial}{\partial t}\left\{\upsilon_2\frac{\partial^2 \upsilon_0}{\partial x^2} + \upsilon_1\frac{\partial^2 \upsilon_1}{\partial x^2} + \upsilon_0\frac{\partial^2 \upsilon_2}{\partial x^2}\right\} + 2x\right\}\right],$$

$$\begin{aligned}
\upsilon_3(x,t) &= x^2 + (2-4x)\frac{t^\beta}{\Gamma(\beta+1)} + 4\frac{t^{2\beta}}{\Gamma(2\beta+1)} \\
&- 4x(3x-1)\frac{t^{2\beta-1}}{\Gamma(2\beta)} - 24\frac{t^{3\beta-1}}{\Gamma(3\beta)} \\
&-4(6x+1)\frac{t^{3\beta-1}}{\Gamma(3\beta)} - 4x^2(6x-1)\frac{t^{3\beta-2}}{\Gamma(3\beta-1)} \\
&-8(1-2x)\frac{\Gamma(2\beta+1)t^{3\beta-1}}{\Gamma(3\beta)(\Gamma(\beta+1))^2} + 8x\frac{t^{3\beta-1}}{\Gamma(3\beta)} \\
&+ 8x^2(1-3x)\frac{t^{3\beta-2}}{\Gamma(3\beta-1)},
\end{aligned}$$

$$\vdots$$

Therefore, obtained analytical result in the following form:

$$\begin{aligned}
\upsilon(x,t) &= x^2 + (2-4x)\frac{t^\beta}{\Gamma(\beta+1)} + 4\frac{t^{2\beta}}{\Gamma(2\beta+1)} \\
&- 4x(3x-1)\frac{t^{2\beta-1}}{\Gamma(2\beta)} - 24\frac{t^{3\beta-1}}{\Gamma(3\beta)} \\
&- 4(6x+1)\frac{t^{3\beta-1}}{\Gamma(3\beta)} - 4x^2(6x-1)\frac{t^{3\beta-2}}{\Gamma(3\beta-1)} \\
&- 8(1-2x)\frac{\Gamma(2\beta+1)t^{3\beta-1}}{\Gamma(3\beta)(\Gamma(\beta+1))^2} + 8x\frac{t^{3\beta-1}}{\Gamma(3\beta)} \\
&+ 8x^2(1-3x)\frac{t^{3\beta-2}}{\Gamma(3\beta-1)} + \cdots.
\end{aligned} \tag{22}$$

The exact solution of $\beta = 1$ is

$$\upsilon(x,t) = x^2 + 2t. \tag{23}$$

5 Discussion on graphs and tables

In this section, the graphical representation and analysis are discussed to highlight the novelty of the present research work. In this connection, Figure 1 shows the solution graphs of actual and VHPTM solutions at $\beta = 1$. Figure 1 reveals that the graphs of both solutions are very close and confirms the higher efficiency of the suggested technique. In Figure 2, the solutions at different values of β are calculated and a very committed relation can be seen among the solutions of example 1. The error

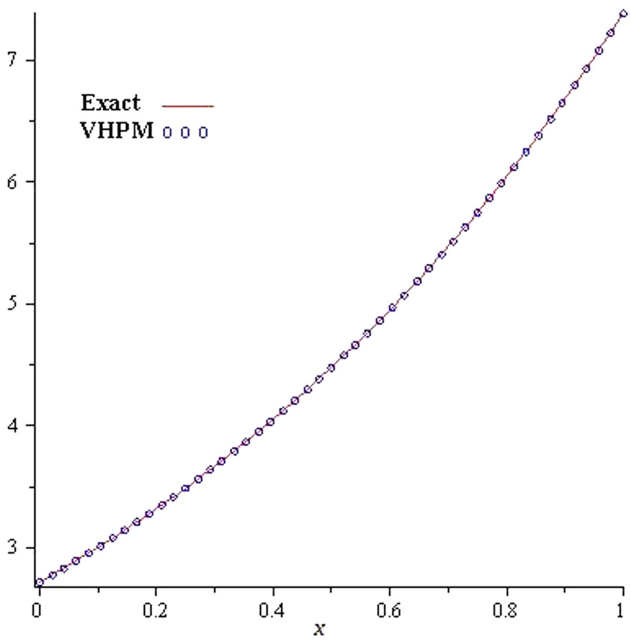

Figure 1: Exact and VHPTM solution plot of example 1 at $\beta = 1$.

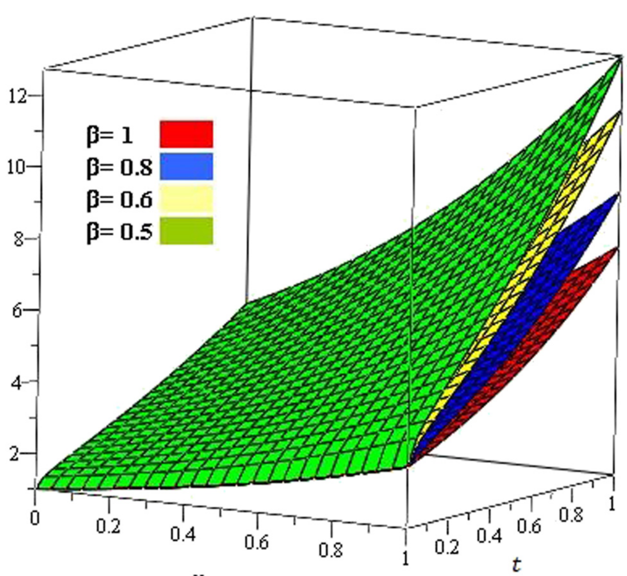

Figure 2: 3-D plot of VHPTM solution of example 1 at different fractional orders $\beta = 0.5, 0.6, 0.8, 1$.

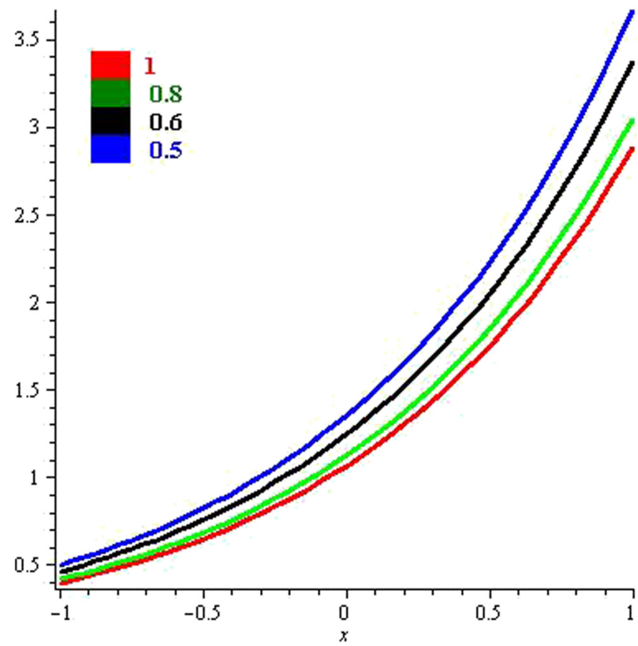

Figure 3: VHPTM solutions of example 1 at different fractional orders $\beta = 0.5, 0.6, 0.8, 1$.

graph is presented in Figure 3, which shows that the accuracy of the suggested method is sufficient. The results of example 2 are presented by graphs in Figures 4 and 5. The sub-plots in Figure 4 express the actual and VHPTM results and are shown to be very close in relation. Various fractional behaviors of the model given in example 2 are displayed in Figure 4. The values of beta = 0.6 and 0.8 are used for graphical representation of the derived results. Besides the graphs, Table 1 is used to compare the results of VHPTM and VIM. The overall, graphical and tabular analyses have justified the accurate and effective implementation of the present technique (Figures 6–8).

Table 1: VHPTM and HPM [28] solutions of example 1

	VHPTM			HPM	Exact	Error
x	$\beta = 0.50$	$\beta = 0.75$	$\beta = 1$	$\beta = 1$	$\beta = 1$	$\beta = 1$
0.0	4.934171	3.484061	2.718253	2.718155	2.718281	2.78×10^{-5}
0.2	6.026610	4.255441	3.320082	3.320840	3.320116	3.40×10^{-5}
0.4	7.360918	5.197608	4.055158	4.055862	4.055199	4.15×10^{-5}
0.6	8.990646	6.348373	4.952981	4.952820	4.953032	5.07×10^{-5}
0.8	10.98120	7.753920	6.049585	6.049543	6.049647	6.20×10^{-5}
1	13.41246	9.470660	7.388980	7.388441	7.389056	7.57×10^{-5}

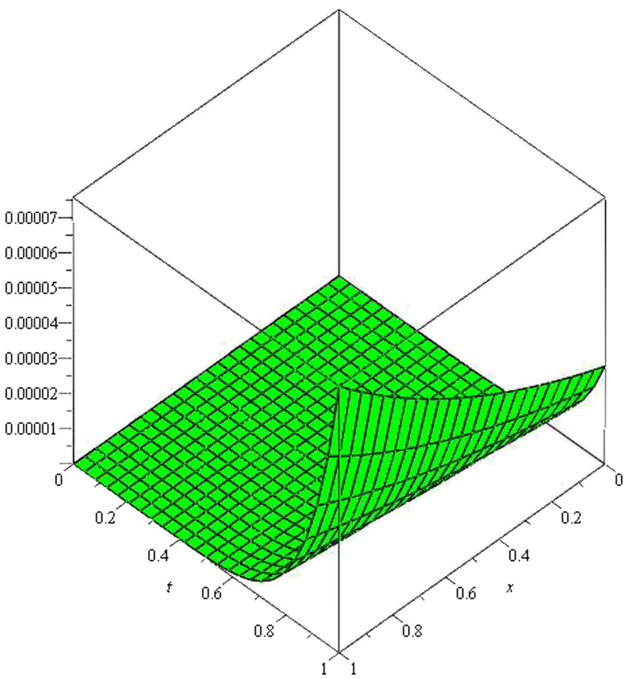

Figure 4: VHPTM-error plot of example 1 at $\beta = 1$.

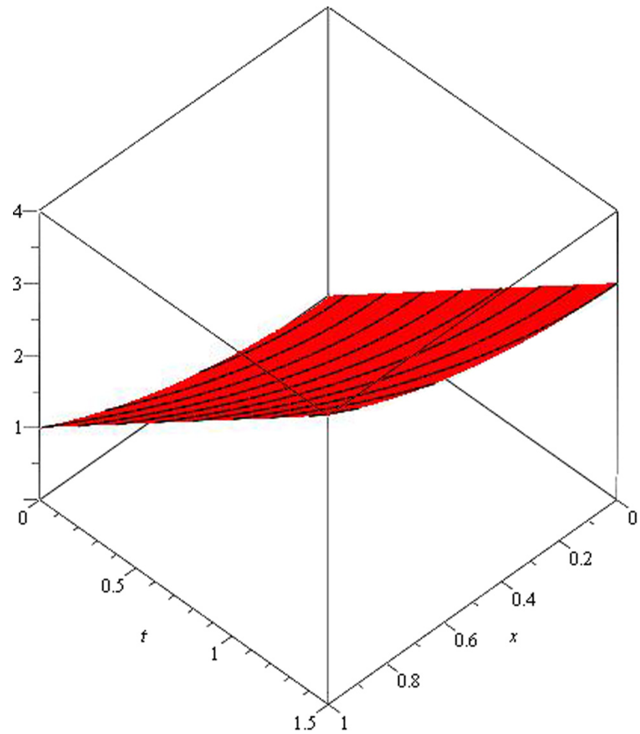

Figure 6: VHPTM solution plot of example 2 at $\beta = 1$.

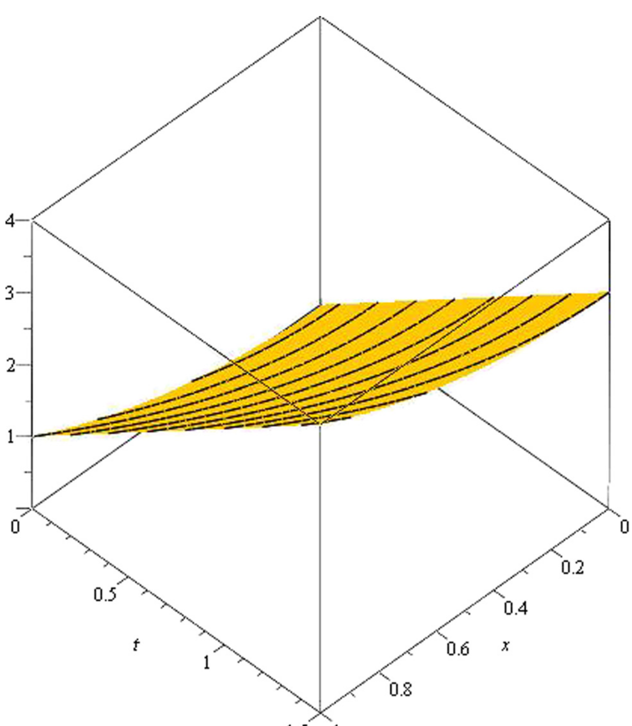

Figure 5: Exact solution plot of example 2.

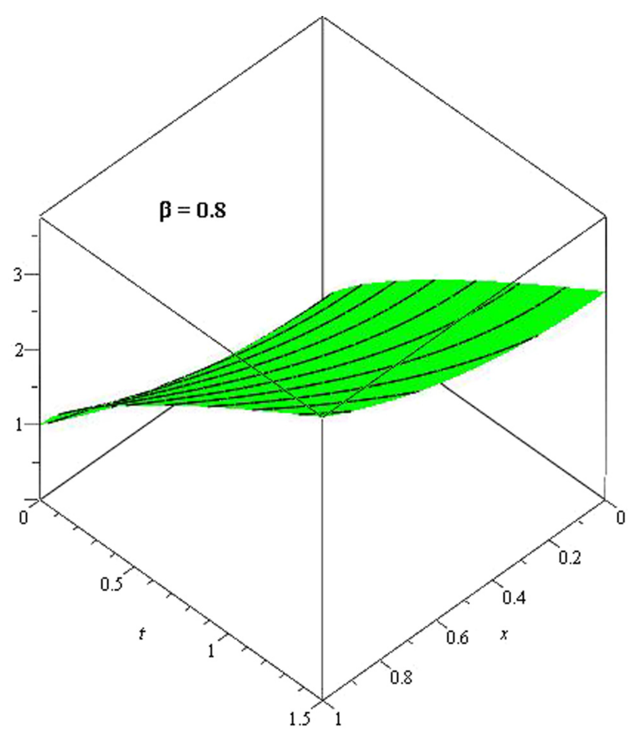

Figure 7: VHPTM solution plot of example 2 at $\beta = 0.8$.

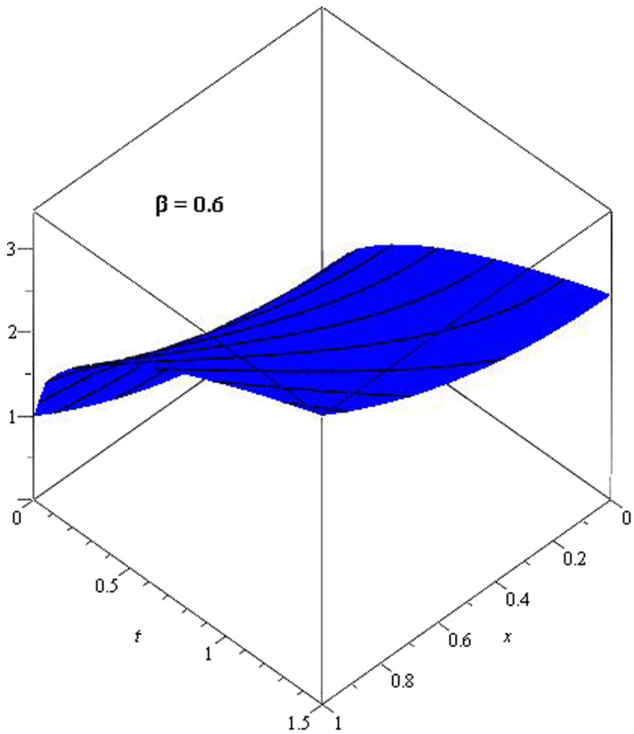

Figure 8: VHPTM solution plot of example 2 at $\beta = 0.6$.

6 Conclusions

In this article, an efficient technique is used to solve FCDEs. The proposed technique is the mixture of the variational iteration method, HPM and LT method. The nonlinear terms in the targeted problems are expressed in terms of He's polynomials. The suggested hybrid method has an easier and straightforward procedure to obtain the solution of fractional problems. For understanding, some numerical examples are solved to determine the reliability and applicability of VHPTM. The obtained results are plotted by using its graphical representation. Through graphs, a very strong relation is shown between the actual and VHPTM solutions. The fractional solutions are plotted to show the behavior of various dynamics of the given physical phenomena. A sufficient rate of convergence of the fractional solutions toward integer order solution is achieved. The higher rate of convergence is achieved by using Laplace Homotopy Perturbation Transform Method (LHPTM). In conclusion, the current method has simple and straightforward implementation to attain the actual solution, and therefore VHPTM is preferred to solve other nonlinear fractional problems in various areas of applied science.

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