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Research article

Comparison principles of fractional differential equations with non-local derivative and their applications

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Abstract: In this paper, we derive and prove a maximum principle for a linear fractional differential equation with non-local fractional derivative. The proof is based on an estimate of the non-local derivative of a function at its extreme points. A priori norm estimate and a uniqueness result are obtained for a linear fractional boundary value problem, as well as a uniqueness result for a nonlinear fractional boundary value problem. Several comparison principles are also obtained for linear and nonlinear equations.

Keywords: fractional differential equations; maximum principle; fractional derivatives
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1. Introduction

In this paper, we consider the linear and nonlinear fractional boundary value problems

$$P_\alpha(u) = ({}^{RC}D^\alpha_{a,b})u(t) + r(t)u(t) = g(t),\ t \in (a,b),\ 0 < \alpha < 1, \tag{1.1}$$

$$N_\alpha(u) = ({}^{RC}D^\alpha_{a,b})u(t) = h(t,u),\ t \in (a,b), \tag{1.2}$$

$$u(a) = u_a,\ \ u(b) = u_b, \tag{1.3}$$

where $r, g \in C[a,b]$, $h(t,u)$ is a smooth function, and $({}^{RC}D^\alpha_{a,b}u)(t)$ is the non-local fractional derivative. The non-local fractional derivative of order $0 < \alpha < 1$, is defined by

$$({}^{RC}D^\alpha_{a,b}f)(t) = \frac{1}{2\Gamma(1-\alpha)}\Big(\int_a^t (t-s)^{-\alpha} f'(s)ds - \int_t^b (s-t)^{-\alpha} f'(s)ds\Big). \tag{1.4}$$

For the definition of higher order non-local derivatives and their properties we refer the reader to [1].

Remark 1.1. *The non-local fractional derivatives defined in Eq (1.4) is known in the literature by the Riesz-Caputo derivative. As the definition has no direct connection to the Riesz derivative, we call it the non-local derivative.*

Fractional calculus is an emerging field in mathematics and it has many important applications in several fields of science and engineering [5]. Fractional differential equations with different types of fractional derivatives have been studied extensively. The existence of solutions of the problem (1.2), (1.3) was established in [6]. In [7] a new maximum principle was derived and implemented to study a multi-term time-space non-local fractional differential equation over an open bounded domain. Fractional variation principles were derived using several types of non-local fractional operators, and their applications were illustrated, see [2–4]. All the above mentioned results make non-local fractional operator an interesting operator to be further investigated. Maximum principles are commonly used to study the qualitative behavior of various types of functional equations. In recent years there many studies devoted to extend the idea of maximum principles to fractional differential equations. Several comparison principles were derived and used to analyze the solutions of fractional equations with different types of fractional derivative, see [7–11]. In this paper, we extend the idea of maximum principle to analyze the solutions of the linear and non-linear boundary value problems (1.1)–(1.3). In Section 2, we derive a new estimate of the non-local derivative of order $0 < \alpha < 1$, of a function at its extreme points. We then, use the result to formulate and prove a maximum principle for a linear fractional equation in Section 3. In Section 4, we analyze the solutions of the associated linear and nonlinear fractional boundary value problems. Finally, we close up with some conclusions in Section 5.

2. Extremum principles for the non-local derivative

In the following we use the space $CW^1([a,b]) = C[a,b] \cap W^1(a,b)$, where $W^1(a,b)$ is the space of functions $f \in C^1(a,b)$ such that $f' \in L^1(a,b)$. The space $CW^1([a,b])$ is less restrictive than $C^1[a,b]$. For instance, $f(t) = \sqrt{t} \in CW^1([a,b])$ but not in $C^1[0,1]$. We have the following extremum principles for the non-local derivative. Analogous results are obtained for several types of fractional derivatives, and we refer the reader to [12, 13], among the first papers discussing this issue.

Lemma 2.1. *Let a function $f \in CW^1([a,b])$ attain its maximum at a point $t_0 \in (a,b)$ and $0 < \alpha < 1$. Then the inequality*

$$({}^{RC}D^{\alpha}_{a,b}f)(t_0) \geq \frac{1}{2\Gamma(1-\alpha)}\left(\frac{f(t_0)-f(a)}{(t_0-a)^{\alpha}} + \frac{f(t_0)-f(b)}{(b-t_0)^{\alpha}}\right) \geq 0, \tag{2.1}$$

holds true.

Proof. We define the auxiliary function $g(t) = f(t_0) - f(t),\ t \in [a,b]$. Then it follows that $g(t) \geq 0$, on $[a,b]$, $g(t_0) = g'(t_0) = 0$ and $({}^{RC}D^{\alpha}_{a,b}g)(t) = -({}^{RC}D^{\alpha}_{a,b}f)(t)$. We have

$$2\Gamma(1-\alpha)({}^{RC}D^{\alpha}_{a,b}g)(t_0) = \int_a^{t_0}(t_0-s)^{-\alpha}g'(s)ds - \int_{t_0}^{b}(s-t_0)^{-\alpha}g'(s)ds.$$

Integrating by parts yields

$$
\begin{aligned}
2\Gamma(1-\alpha)({}^{RC}D^{\alpha}_{a,b}g)(t_0) &= (t_0-s)^{-\alpha}g(s)|_a^{t_0} - \alpha\int_a^{t_0}(t_0-s)^{-\alpha-1}g(s)ds \\
&\quad -(s-t_0)^{-\alpha}g(s)|_{t_0}^{b} - \alpha\int_{t_0}^{b}(s-t_0)^{-\alpha-1}g(s)ds.
\end{aligned}
$$

Since $g(t_0) = 0$, we have

$$
\lim_{s\to t_0}\frac{g(s)}{(t_0-s)^{\alpha}} = \lim_{s\to t_0}\frac{g'(t)}{-\alpha(t_0-s)^{\alpha-1}} = -\lim_{s\to t_0}\frac{1}{\alpha}(t_0-s)^{1-\alpha}g'(t) = 0,
$$

and

$$
\lim_{s\to t_0}\frac{g(s)}{(s-t_0)^{\alpha}} = 0.
$$

Thus,

$$
\begin{aligned}
2\Gamma(1-\alpha)({}^{RC}D^{\alpha}_{a,b}g)(t_0) &= -(t_0-a)^{-\alpha}g(a) - \alpha\int_a^{t_0}(t_0-s)^{-\alpha-1}g(s)ds \\
&\quad -(b-t_0)^{-\alpha}g(b) - \alpha\int_{t_0}^{b}(s-t_0)^{-\alpha-1}g(s)ds \\
&\le -(t_0-a)^{-\alpha}g(a) - (b-t_0)^{-\alpha}g(b). \qquad (2.2)
\end{aligned}
$$

The last equation yields

$$
2\Gamma(1-\alpha)({}^{RC}D^{\alpha}_{a,b}(-g))(t_0) \ge (t_0-a)^{-\alpha}g(a) + (b-t_0)^{-\alpha}g(b),
$$

or

$$
2\Gamma(1-\alpha)({}^{RC}D^{\alpha}_{a,b}f)(t_0) \ge (t_0-a)^{-\alpha}\big(f(t_0)-f(a)\big) + (b-t_0)^{-\alpha}\big(f(t_0)-f(b)\big) \ge 0,
$$

which proves the result. □

Remark 2.1. *Since $g \in CW^1([a,b])$ and $g(t_0) = 0$, then $g(t) = (t_0-t)h(t)$, for some $h \in C^1(a,b)$, thus the integral*

$$
\int_a^{t_0}(t_0-s)^{-\alpha-1}g(s)ds = \int_a^{t_0}(t_0-s)^{-\alpha}h(s)ds,
$$

exists and is nonnegative.

Remark 2.2. *The following extremum principle was obtained in [7]. Let a function $f \in C^1[a,b]$ attain its maximum at a point $t_0 \in (a,b)$ and $0 < \alpha < 1$, then it holds that*

$$
({}^{RC}D^{\alpha}_{a,b}f)(t_0) \ge 0. \qquad (2.3)
$$

However, the extremum principle in (2.1) is more general and defined in a wider space $CW^1([a,b])$.

By applying analogous steps to $-f$ we have

Lemma 2.2. *Let a function $f \in CW^1([a,b])$ attain its minimum at a point $t_0 \in (a,b)$ and $0 < \alpha < 1$. Then the inequality*

$$
({}^{RC}D^{\alpha}_{a,b}f)(t_0) \le \frac{1}{2\Gamma(1-\alpha)}\Big(\frac{f(t_0)-f(a)}{(t_0-a)^{\alpha}} + \frac{f(t_0)-f(b)}{(b-t_0)^{\alpha}}\Big), \qquad (2.4)
$$

holds true.

3. A maximum principle

We implement the results in Section 2 to obtain a maximum principle for a linear fractional equation. We have

Lemma 3.1. *(Maximum Principle) Let a function $u \in CW^1([a,b])$ satisfy the fractional inequality*

$$P_\alpha(u) = ({}^{RC}D^\alpha_{a,b})u(t) + r(t)u(t) \le 0,\ t \in (a,b),\ 0 < \alpha < 1, \tag{3.1}$$

where $r(t) \ge 0$ is continuous on $[a,b]$. Then

$$u(t) \le \max\{u(a), u(b), 0\}. \tag{3.2}$$

Proof. Since $u(t)$ is continuous on $[a,b]$, then u attains a maximum at $t_0 \in [a,b]$. Assume by contradiction that the result in Eq (3.2) is not true, then it holds that

$$t_0 \in (a,b),\ u(t_0) > 0,\ u(t_0) > u(a) \text{ and } u(t_0) > u(b).$$

Applying the result of Lemma 2.1 we have

$$({}^{RC}D^\alpha_{a,b}u)(t_0) \ge \frac{1}{2\Gamma(1-\alpha)}\Big(\frac{u(t_0)-u(a)}{(t_0-a)^\alpha} + \frac{u(t_0)-u(b)}{(b-t_0)^\alpha}\Big) > 0. \tag{3.3}$$

Thus,

$$\begin{aligned} P_\alpha(u)(t_0) &= ({}^{RC}D^\alpha_{a,b})u(t_0) + r(t_0)u(t_0), \\ &\ge ({}^{RC}D^\alpha_{a,b})u(t_0) > 0, \end{aligned}$$

which contradicts the fractional inequality (3.1), and completes the proof.

□

4. Applications to linear and nonlinear equations

We apply the maxim principle to derive several comparison principles for linear and nonlinear fractional equations. We also obtain uniqueness result to the fractional boundary value problems (1.1)–(1.3) and a norm estimate of solutions to the linear fractional boundary value problem (1.1) and (1.3). We have

4.1. The linear case

Lemma 4.1. *Let $u_1, u_2 \in CW^1([a,b])$ be two possible solutions to*

$$\begin{aligned} P_\alpha(u_1) = ({}^{RC}D^\alpha_{a,b})u_1(t) + r(t)u_1(t) &= g_1(t),\ t \in (a,b), \\ P_\alpha(u_2) = ({}^{RC}D^\alpha_{a,b})u_2(t) + r(t)u_2(t) &= g_2(t),\ t \in (a,b), \end{aligned}$$

where $r(t) \ge 0, g_1(t), g_2(t)$ are continuous on $[a,b]$, and $u_1(a) = u_2(a), u_1(b) = u_2(b)$. If $g_1(t) \le g_2(t)$, then it holds that

$$u_1(t) \le u_2(t), \quad t \in [a,b].$$

Proof. Let $z = u_1 - u_2$, then $z \in CW^1([a, b])$, and it holds that

$$\begin{aligned} P_\alpha(z) &= ({}^{RC}D^\alpha_{a,b})z(t) + r(t)z(t), \\ &= ({}^{RC}D^\alpha_{a,b})(u_1 - u_2) + r(t)(u_1 - u_2), \\ &= g_1(t) - g_2(t) \leq 0,\ t \in (a, b). \end{aligned} \tag{4.1}$$

By virtue of Lemma 3.1 we have $z(t) \leq \max\{z(a), z(b), 0\}$. Since $z(a) = z(b) = 0$, then it holds that $z(t) \leq 0$, on $[a, b]$ which proves the result. □

Lemma 4.2. *Let $u \in CW^1([a, b])$ be a possible solution to*

$$P_\alpha(u) = ({}^{RC}D^\alpha_{a,b})u(t) + r(t)u(t) \quad = \quad g(t),\ t \in (a, b),$$

where $r(t) \geq \delta_0$ for some $\delta_0 > 0$, is continuous on $[a, b]$. Then it holds that

$$||u||_{[a,b]} = \max_{t\in[a,b]} |u(t)| \leq M = \max_{t\in[a,b]}\{|\frac{g(t)}{r(t)}|, u(a), u(b)\}.$$

Proof. We have $M \geq |\frac{g(t)}{r(t)}|$, or $Mr(t) \geq |g(t)|,\ t \in [a, b]$. Let $v_1 = u - M$, then $v_1 \in CW^1([a, b])$ and it holds that

$$\begin{aligned} P_\alpha(v_1) &= ({}^{RC}D^\alpha_{a,b})v_1(t) + r(t)v_1(t) \\ &= ({}^{RC}D^\alpha_{a,b})u(t) + r(t)u(t) - r(t)M \\ &= g(t) - r(t)M \leq |g(t)| - r(t)M \leq 0. \end{aligned}$$

Thus by virtue of Lemma 3.1 we have $v_1 = u - M \leq \max\{v_1(a), v_1(b), 0\}$. Since $v_1(a) = u(a) - M \leq 0$, and $v_1(b) = u(b) - M \leq 0$, we have

$$v_1 \leq 0, \quad \text{or} \quad u \leq M. \tag{4.2}$$

Analogously, let $v_2 = -M - u$, then $v_2 \in CW^1([a, b]),\ v_2(a) \leq 0, v_2(b) \leq 0$, and

$$\begin{aligned} P_\alpha(v_2) &= ({}^{RC}D^\alpha_{a,b})v_2(t) + r(t)v_2(t) \\ &= -({}^{RC}D^\alpha_{a,b})u(t) - r(t)u(t) - r(t)M \\ &= -g(t) - r(t)M \leq |g(t)| - r(t)M \leq 0. \end{aligned}$$

Thus by the result of Lemma 3.1 we have $v_2 = -u - M \leq 0$, or

$$u \geq -M. \tag{4.3}$$

By combining the results of Eqs (4.2) and (4.3) we have $|u(t)| \leq M,\ t \in [a, b]$ and hence the result. □

Lemma 4.3. *The linear fractional boundary value problem (1.1) and (1.3) admits at most one solution $u \in CW^1([a, b])$.*

Proof. Let $u_1, u_2 \in CW^1([a, b])$ be two possible solutions, and define $v = u_1 - u_2,\ t \in [a, b]$. We have $v \in CW^1([a, b])$ and it holds that

$$P_\alpha(v) = ({}^{RC}D^\alpha_{a,b})(u_1 - u_2) + r(t)(u_1 - u_2) = 0,$$
$$v(a) = v(b) = 0.$$

Thus by virtue of Lemma 4.2 we have

$$\|v\|_{[a,b]} \leq 0, \text{ or } \|v\|_{[a,b]} = 0, \text{ or } v = 0.$$

Thus $u_1 = u_2$ which completes the proof.

□

4.2. The nonlinear case

Lemma 4.4. *If $h(t, u)$ is non-increasing with respect to u, then the nonlinear fractional boundary value problem (1.2) and (1.3) admits at most one solution $u \in CW^1([a, b])$.*

Proof. Assume that $u_1, u_2 \in CW^1([a, b])$ be two solutions of (1.2) and (1.3), and let $z = u_1 - u_2$. Then $z \in CW^1([a, b])$, $z(a) = z(b) = 0$, and

$$N_\alpha(u_1) - N_\alpha(u_2) = ({}^{RC}D^\alpha_{a,b})(u_1 - u_2) - [h(t, u_1) - h(t, u_2)] = 0.$$

Since $h(t, u)$ is a smooth function, applying the mean value theorem we have

$$h(t, u_1) - h(t, u_2) = \frac{\partial h}{\partial u}(u^*)(u_1 - u_2),$$

for some u^* between u_1 and u_2. Thus,

$$\begin{aligned} N_\alpha(u_1) - N_\alpha(u_2) &= ({}^{RC}D^\alpha_{a,b})(u_1 - u_2) - \frac{\partial h}{\partial u}(u^*)(u_1 - u_2) = 0, \\ &= ({}^{RC}D^\alpha_{a,b})z - \frac{\partial h}{\partial u}(u^*)z = 0. \end{aligned} \tag{4.4}$$

Since $-\frac{\partial h}{\partial u}(u^*) > 0$, and $z(a) = z(b) = 0$, we have $z(t) \leq 0$, by virtue of Lemma 3.1. Also, Eq (4.4) holds true for $-z$ and thus $-z \leq 0$, by virtue of Lemma 3.1. Thus, $z = 0$ which proves that $u_1 = u_2$. □

Lemma 4.5. *Let $u, v \in CW^1([a, b])$ be possible solutions to the nonlinear fractional inequalities*

$$({}^{RC}D^\alpha_{a,b})u \leq h(t, u), \ t \in (a, b) \tag{4.5}$$
$$({}^{RC}D^\alpha_{a,b})v \geq h(t, v), \ t \in (a, b), \tag{4.6}$$

with $u(a) \leq v(a)$ and $u(b) \leq v(b)$. If $h(t, u)$ is a smooth function and is non-increasing with respect to u, then

$$u(t) \leq v(t), \ t \in [a, b].$$

Proof. Let $z = u - v$, then $z(a), z(b) \leq 0$ and it holds that

$$({}^{RC}D^\alpha_{a,b})z \leq h(t, u) - h(t, v) = \frac{\partial h}{\partial u}(u^*)(u - v) = \frac{\partial h}{\partial u}(u^*)z,$$

for some u^* between u and v. The last equation yields

$$({}^{RC}D^{\alpha}_{a,b})z - \frac{\partial h}{\partial u}(u^*)z \leq 0,$$

which together with $z(a), z(b) \leq 0$, proves that $z \leq 0, t \in [a, b]$, and completes the proof.

□

We illustrate the results with the following examples. Consider the linear boundary value problem

$$\left({}^{RC}D^{\alpha}_{0,1}u\right)(t) \quad + \quad u(t) = g(t),\ t \in (0,1), \tag{4.7}$$

$$u(0) = 0, \qquad u(1) = 1. \tag{4.8}$$

For $u(t) = t$, direct calculations lead to

$$\begin{aligned}({}^{RC}D^{\alpha}_{0,1}t)(t) &= \frac{1}{2\Gamma(1-\alpha)}\Big(\int_0^t (t-s)^{-\alpha}ds - \int_t^1 (s-t)^{-\alpha}ds\Big) \\ &= \frac{1}{2\Gamma(2-\alpha)}(t^{1-\alpha} - (1-t)^{1-\alpha}).\end{aligned} \tag{4.9}$$

Thus, $u(t) = t$ is a solution to the fractional boundary value problem (4.7), (4.8), provided that $g(t) = \frac{1}{2\Gamma(2-\alpha)}(t^{1-\alpha} - (1-t)^{1-\alpha}) + t$, for arbitrary $0 < \alpha < 1$. By Lemma 4.3 this is the unique solution $u \in CW^1([0,1])$ to (4.7), (4.8). Analogously, $u(t) = t$ is a solution to the nonlinear fractional boundary value problem

$$\left({}^{RC}D^{\alpha}_{0,1}u\right)(t) \quad = \quad h(t,u),\ t \in (0,1), \tag{4.10}$$

$$u(0) = 0, \qquad u(1) = 1, \tag{4.11}$$

where $h(t,u) = \frac{1}{2\Gamma(2-\alpha)}(t^{1-\alpha} - (1-t)^{1-\alpha}) - u^2 + t^2$, for arbitrary $0 < \alpha < 1$. Since $\frac{\partial h}{\partial u} = -2u = -2t \leq 0$, $t \in (0,1)$, then by virtue of the result in Lemma 4.4 we have, $u(t) = t$, is the unique solution to (4.10), (4.11) in the space $CW^1([0,1])$.

5. Concluding remarks

In this paper, we have proved two extremum principles for the non-local fractional derivative of order $0 < \alpha < 1$. Based on these extremum principles, a maximum principle is derived for a linear fractional equation. We have formulated and proved several comparison principles for the linear and nonlinear fractional equations, and obtained uniqueness results for the associated fractional boundary value problems. A norm estimate of solutions of the linear boundary value problem is also derived. The obtained results are extendable to the linear and nonlinear multi-term fractional boundary value problems

$$
\begin{aligned}
P_\alpha(u) &= \left({}^{RC}D_{a,b}^{\alpha_m} + \sum_{i=1}^{m-1} c_i \, {}^{RC}D_{a,b}^{\alpha_i}\right)u(t) + r(t)u(t) = g(t), \ t \in (a,b), \\
N_\alpha(u) &= \left({}^{RC}D_{a,b}^{\alpha_m} + \sum_{i=1}^{m-1} c_i \, {}^{RC}D_{a,b}^{\alpha_i}\right)u(t) = h(t,u), \ t \in (a,b), \\
u(a) &= u_a, \ \ u(b) = u_b,
\end{aligned}
$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$, $c_i \geq 0, i = 1, \cdots, m-1$, $r, g \in C[a,b]$, $h(t,u)$ is a smooth function. However, the existence results of the above systems are the main challenging, and we leave these problems for a future study.

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Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. A. M. A. El-Sayed, M. Gaber, On the finite Caputo and finite Riesz derivatives, *Electron. J. Theor. Phys.,* **3** (2006), 81–95.
2. A. B. Malinowska, D. F. Torres, Fractional calculus of variations for a combined Caputo derivative, *Fract. Calc. Appl. Anal.,* **14** (2011), 523–537.
3. T. Odzijewics, A. B. Malinowska, D. F. Torres, Fractional variable calculus with classical and combined Caputo derivatives, *Nonlinear Anal.: Theory, Method. Appl.,* **75** (2012), 1507–1515.
4. T. Odzijewics, A. B. Malinowska, D. F. Torres, Fractional calculus of variations in terms of a generalized fractional integral with applications to physics, *Abstr. Appl. Anal.,* **2012** (2012), 1–24.
5. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications,* Gordon and Breach, Yverdon, 1993.
6. F. Chen, D. Baleanu, G. C. Wu, Existence results of fractional differential equations with Riesz-Caputo derivative, *Eur. Phys. J. Spec. Top.,* **226** (2017), 3411–3425.
7. H. Ye, F. Liu, V. Anh, I. Turner, Maximum principle and numerical method for the multi-term time space Riesz-Caputo fractional differential equations, *Appl. Math. Comput.,* **227** (2014), 531–540.
8. M. Al-Refai, Basic results on nonlinear eigenvalue problems of fractional order, *Electron. J. Differ. Equ.,* **2012** (2012), 1–12.

9. M. Al-Refai, Y. Luchko, Maximum principles for the fractional diffusion equations with the Riemann-Liouville fractional derivative and their applications, *Fract. Calc. Appl. Anal.,* **17** (2014), 483–498.
10. M. Al-Refai, Y. Luchko, Analysis of fractional diffusion equations of distributed order: maximum principles and its applications, *Analysis,* **36** (2016), 123–133.
11. M. Al-Refai, K. Pal, A maximum principle for a fractional boundary value problem with convection term and applications, *Math. Model. Anal.,* **24** (2019), 62–71.
12. M. Al-Refai, On the fractional derivative at extreme points, *Elect. J. Qual. Theory Differ. Equ.,* **2012** (2012), 1–5.
13. Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.,* **351** (2009), 218–223.

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