Mathematics

Research article

Delay-coupled fractional order complex Cohen-Grossberg neural networks under parameter uncertainty: Synchronization stability criteria

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Abstract: This paper inspects the issues of synchronization stability and robust synchronization stability for fractional order coupled complex interconnected Cohen-Grossberg neural networks under linear coupling delays. For investigation of synchronization stability results, the comparison theorem for multiple delayed fractional order linear system is derived at first. Then, by means of given fractional comparison principle, some inequality methods, Kronecker product technique and classical Lyapunov-functional, several asymptotical synchronization stability criteria are addressed in the voice of linear matrix inequality (LMI) for the proposed model. Moreover, when parameter uncertainty exists, we also the investigate on the robust synchronization stability criteria for complex structure on linear coupling delayed Cohen-Grossberg type neural networks. At last, the validity of the proposed analytical results are performed by two computer simulations.

1. Introduction

In 1695, the foundation of non-integer order calculus become first of all discussed through Guillaume de Leibnitz and Gottfried Wilhelm Leibnitz, and its development were very gradual for long period. In practice, many real-world objects need to be described by fractional order models due to the fact dynamics of fractional order models are more accurate than integer order models [14–18]. Currently, fractional order calculus has been very promising areas of research and thus successfully applied in numerous fields such as circuit systems [9], market dynamics [21], biological models [26], dielectric polarization [28], and so on. Recently, many researchers have investigated asymptotic behavior of fractional-order dynamical systems and some interesting and important results were accounted. In [12], the authors discussed the asymptotic stability of fractional order time delayed systems based on the fractional order Halanay inequality. In [33], the author studied the global asymptotic stability of Caputo-Liouville generalized fractional order electrical RLC circuit model based on the Lyapunov stability theory.

Moreover, fractional order calculus has been integrated into artificial neural networks, and fractional order neural network (FNNs) are a kind of potentially applicable networks. In past few years, there has been an increasing interest in the investigation of dynamical behaviours of neural networks, and some important scientific results were obtained [4–7, 37, 39]. In [1], authors discussed the stability and synchronization of memristive FNNs with multiple delays, comparison principle, set valued maps and the Lyapunov function method were utilized to assure the stability and synchronization of memristive FNNs. In [7], synchronization with discontinuous activation have been discussed. Differential inclusion theory, Lyapunov stability theory and some novel sufficient criteria were applied to ensure the finite-time stabilization for the addressed model. In [42], authors studied the stability of fractional-order multiple time varying delayed competitive type neural networks. By means of Lyapunov method and graph theory techniques, some novel conditions were derived to achieve global asymptotic stability for the addressed model. In [43], authors studied the synchronization of FNNs in complex field. Based on the fractional comparison theorem, some novel conditions were obtained to achieve asymptotical synchronization for the addressed model.

Over the past few years, a lot of consideration has been paid to complex systems in light of their potential applications to different fields such as metabolic systems, communication networks, global economic markets, and so forth, and lots of significant results about the synchronization analysis of complex networks with fractional order derivative have been extensively investigated, see instance [22–24, 31, 40]. As is known to all, coupled neural networks (CNNs) is a generalization of complex networks. Due to widespread applications in many research fields like classification [2], harmonic oscillation generation [11], pattern recognition [36], image encryption [47] and secure communication [48], a complex structure on CNNs has been broadly investigated by many researchers, see example [41, 44, 47]. Yanli et al. [44] researched generalized synchronization on integer order CNNs with mismatched dimension nodes, which ensured that pinning synchronization can be acquired by means of some inequality techniques and Lyapunov theory. Regrettably, there are few results focused on synchronization of complex coupled neural networks with fractional order derivatives, for instance [32, 34, 46].

In reality, the presence of external disturbances, environmental noises and model errors in many practical circumstances, the correct estimations of parameters in fractional order CNNs are usually

cannot be acquired. Therefore, it is necessary and significant to research the fractional order CNNs with uncertain parameters and very few consequences on studies were paid. For example, by means of Kronecker product, Mittag-Leffler function and Lyapunov stability theory, Shuxue et al. [34] considered several synchronization conditions which can ensure the asymptotical synchronization of complex structure on fractional order CNNs with uncertain parameters and without time delays. On the other hand, time delays are not omitted into the dynamical behavior of neural networks, which can lead to the system oscillation, instability behaviors and divergence owing to the finite switching speed of amplifiers. Therefore, it is much more important to consider time delays in studying in investigating synchronization of fractional order CNNs and some remarkable results on this topic have been paid in existing literature. For example, Zhang et al. [46] researched synchronization stability of fractional order complex CNNs with coupling delays and several conditions to ensure the synchronization stability of complex CNNs were established based on Riemann-Liouville fractional order derivative properties, LMI approach, and Lyapunov theory.

Cohen-Grossberg neural networks (CGNNs) is a standout amongst the most renowned type and its special case of Hopfield type neural networks, which became first of all originated by means of Cohen and Grossberg in 1983 [3]. In recent years, CGNNs have received growing attention due to their widespread application in different areas, such as secure communications, nonlinear optimization problems, image processing, and parallel computation. As a type of FNN, fractional order CGNNs dynamical behavior has been extensively investigated by many researchers and some excellent results have been devoted to fractional order CGNNs, see Ref [30, 38]. On the other hand, the result of complex coupled Cohen-Grossberg neural networks is more complicated and unpredictable dynamical behaviors than different forms of CNNs. Owing to the complex structure of CGNNs, there is few works published on synchronization analysis of coupled CGNNs [35,45]. The authors of [35] proposed some criteria to ensure the synchronization criteria in finite time issues of integer order CGNNs with linear coupling delays and nonlinear coupling delays. In [45], the authors derived several criteria which can guarantee the synchronization criteria in fixed time issues for integer order CGNNs with delayed couplings. To the best of our knowledge, nevertheless, asymptotical synchronization stability of fractional order coupled Cohen-Grossberg neural networks with and without parameter uncertainty has not yet been investigated.

Sparked by the above reason and discussion, the main aim is to study the synchronization stability analysis of fractional order coupled complex interconnected Cohen-Grossberg neural networks under with and without parameter uncertainties under linear coupling delays. The main contributions of this work are indexed as pursues:

1). The complex interconnected fractional order coupled Cohen-Grossberg neural networks model with and without parameter uncertainties are presented in the first time.

2). By using fractional-order stability theory, a new fractional-order comparison theorem for multiple delayed fractional order linear system is established and it is improved those in the existing works literature.

3). Several sufficient criteria in voice LMI techniques for synchronization stability and robust synchronization stability are established theoretically via proposed fractional order comparison theorem.

4). Our proposed synchronization stability results are enhancing the present fractional order Cohen-Grossberg time delayed neural networks and integer-order coupled neural networks.

5). Moreover, the presented results in this paper are also still valid for the synchronization stability of delay-coupled integer order complex Cohen-Grossberg neural networks with and without parameter uncertainties, and these results do not discuss in the previous works of literature.

Notations. In this paper, $\mathbb{N}$ signify the space of natural numbers from 1 to n, $\mathbb{R}^n$ stands for the space of n-D Euclidean space, respectively, and $\mathbb{R}^{n \times n}$ stands for a set of all $n \times n$ real matrices. $\otimes$ means the Kronecker product of two matrices. $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ denote the maximum and minimum eigenvalues of the corresponding matrix. I_n represent the identity matrix with n dimensions. $C([-\delta, 0], \mathbb{R}^n)$ signifies the set of all continuous functions from $[-\delta, 0]$ to $\mathbb{R}^n$, where $\delta > 0$. For $m(t) = (m_1(t), ..., m_n(t))^T \in \mathbb{R}^n$, we denote

$$\|m(t)\|_2 = \sqrt{\sum_{l=1}^{n} m_l^2(t)}.$$

2. Preliminaries and problem model formulation

In this part, some basic knowledge of fractional order calculus, some useful lemma's, problem statement and some necessary assumptions will be given.

Definition 2.1 [29] The Riemann-Liouville fractional integral order $\alpha \in (0, 1)$ for a function $M(t)$ is defined as

$$D_{t_0,t}^{-\alpha} M(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \theta)^{\alpha-1} M(\theta)\, d\theta,\ \alpha > 0,\ t \geq t_0.$$

Definition 2.2 [29] The Caputo type fractional-order derivative with order α for a function $M(t)$ implies:

$$D_{t_0,t}^{\alpha} M(t) = \frac{1}{\Gamma(n - \alpha)} \int_{t_0}^{t} \frac{M^{(n)}(\theta)}{(t - \theta)^{\theta-n+1}}\, d\theta,$$

where $t \geq t_0$ and $n - 1 < \alpha < n \in \mathbb{Z}^+$.

Preposition 1 [10] For $n - 1 < \alpha < n$, the Laplace transform of the Caputo type fractional-order derivative $M(t)$ implies:

$$L[D^{\alpha} M(t)] = s^{\alpha} M(s) - \sum_{\rho=0}^{n-1} s^{\alpha-\rho-1} M^{(\rho)}(0).$$

When $M^{(\rho)}(0) = 0$, $\rho = 1, 2, ..., n$, we have

$$L[D^{\alpha} M(t)] = s^{\alpha} M(s).$$

Preposition 2 [19] The linearity of Caputo type fractional-order derivative is defined by

$$D_{t_0,t}^{\alpha}\Big[\tau_1 M_1(t) + \tau_2 M_2(t)\Big] = \tau_1 D_{t_0,t}^{\alpha} M_1(t) + \tau_2 D_{t_0,t}^{\alpha} M_2(t).$$

Preposition 3 [19] For $n - 1 < \alpha < n$, we have

$$D_{t_0,t}^{-\alpha}[D_{t_0,t}^{\alpha} M(t)] = M(t) - \sum_{\rho=0}^{n-1} \frac{(t - t_0)^{\rho}}{\rho!} M^{(\rho)}(t_0), \ \alpha \geq 0.$$

Especially, $0 < \alpha < 1$, one has

$$D_{t_0,t}^{-\alpha}[D^{\alpha} M(t)] = M(t) - M(t_0).$$

In the following lines D^{α} is an the abbreviation of $D_{t_0,t}^{\alpha}$.

Consider the fractional order linear systems with time delays as follows:

$$D^{\alpha} M(t) = EM(t) + M(t_{\delta_1}) + M(t_{\delta_2}), \ 0 < \alpha < 1 \tag{2.1}$$

where $E = (e_{kl})_{n \times n} \in \mathbb{R}^{n \times n}$, $M(t_{\delta_1}) = \left(\sum_{l=1}^{n} p_{1l} m_l(t - \delta_{1l}^1), ..., \sum_{l=1}^{n} p_{nl} m_l(t - \delta_{nl}^1) \right)^T$,

$M(t_{\delta_2}) = \left(\sum_{l=1}^{n} q_{1l} m_l(t - \delta_{1l}^2), ..., \sum_{l=1}^{n} q_{nl} m_l(t - \delta_{nl}^2) \right)^T$, $M(t) = (M_1(t), ..., M_n(t))^T \in \mathbb{R}^n$.

Particularly, if $\delta_{k1}^1 = \delta_{1l}$, $\delta_{k1}^2 = \delta_{2l}$, $k, l = 1, 2, ..., n$, $P = (p_{kl})_{n \times n} \in \mathbb{R}^{n \times n}$, $Q = (q_{kl})_{n \times n} \in \mathbb{R}^{n \times n}$, fractional order linear system (2.1) can be written as,

$$D^{\alpha} M(t) = EM(t) + PM(t - \delta_1) + QM(t - \delta_2), \ 0 < \alpha < 1, \tag{2.2}$$

where $M(t - \delta_1) = (m_1(t - \delta_{11}), m_2(t - \delta_{12}), ..., m_n(t - \delta_{1n}))^T$ and $M(t - \delta_2) = (m_1(t - \delta_{21}), m_2(t - \delta_{22}), ..., m_n(t - \delta_{2n}))^T$.

Making Laplace transform of (2.2) on both sides, we get

$$
\begin{cases}
\begin{cases}
s^{\alpha}G_1(s) - s^{\alpha-1}\xi_1(0) = e_{11}G_1(s) + p_{11}e^{-s\delta_{11}^1}\Big[G_1(s) + \int_{-\delta_{11}^1}^0 e^{-st}\xi_1(t)\mathrm{d}t\Big] + q_{11}e^{-s\delta_{11}^2} \\
\quad \times\Big[G_1(s) + \int_{-\delta_{11}^2}^0 e^{-st}\xi_1(t)\mathrm{d}t\Big] + e_{12}G_2(s) + p_{12}e^{-s\delta_{12}^1}\Big[G_2(s) + \int_{-\delta_{12}^1}^0 e^{-st}\xi_2(t)\mathrm{d}t\Big] + q_{12}e^{-s\delta_{12}^2} \\
\quad \times\Big[G_2(s) + \int_{-\delta_{12}^2}^0 e^{-st}\xi_2(t)\mathrm{d}t\Big] + \ldots\ldots + e_{1n}G_n(s) + p_{1n}e^{-s\delta_{1n}^1}\Big[G_n(s) + \int_{-\delta_{1n}^1}^0 e^{-st}\xi_n(t)\mathrm{d}t\Big] \\
\quad +q_{1n}e^{-s\delta_{1n}^2}\Big[G_n(s) + \int_{-\delta_{1n}^2}^0 e^{-st}\xi_n(t)\mathrm{d}t\Big]
\end{cases} \\
\\
\begin{cases}
s^{\alpha}G_2(s) - s^{\alpha-1}\xi_2(0) = e_{21}G_1(s) + p_{21}e^{-s\delta_{21}^1}\Big[G_1(s) + \int_{-\delta_{21}^1}^0 e^{-st}\xi_1(t)\mathrm{d}t\Big] + q_{21}e^{-s\delta_{21}^2} \\
\quad \times\Big[G_1(s) + \int_{-\delta_{21}^2}^0 e^{-st}\xi_1(t)\mathrm{d}t\Big] + e_{22}G_2(s) + p_{22}e^{-s\delta_{22}^1}\Big[G_2(s) + \int_{-\delta_{22}^1}^0 e^{-st}\xi_2(t)\mathrm{d}t\Big] + q_{22}e^{-s\delta_{22}^2} \\
\quad \times\Big[G_2(s) + \int_{-\delta_{22}^2}^0 e^{-st}\xi_2(t)\mathrm{d}t\Big] + \ldots\ldots + e_{2n}G_n(s) + p_{2n}e^{-s\delta_{1n}^1}\Big[G_n(s) + \int_{-\delta_{2n}^1}^0 e^{-st}\xi_n(t)\mathrm{d}t\Big] \\
\quad +q_{2n}e^{-s\delta_{2n}^2}\Big[G_n(s) + \int_{-\delta_{2n}^2}^0 e^{-st}\xi_n(t)\mathrm{d}t\Big]
\end{cases} \\
\\
\qquad\qquad\qquad\qquad\qquad . \\
\qquad\qquad\qquad\qquad\qquad . \\
\qquad\qquad\qquad\qquad\qquad . \\
\\
\begin{cases}
s^{\alpha}G_n(s) - s^{\alpha-1}\xi_n(0) = e_{n1}G_1(s) + p_{n1}e^{-s\delta_{n1}^1}\Big[G_1(s) + \int_{-\delta_{n1}^1}^0 e^{-st}\xi_1(t)\mathrm{d}t\Big] + q_{n1}e^{-s\delta_{n1}^2} \\
\quad \times\Big[G_1(s) + \int_{-\delta_{n1}^2}^0 e^{-st}\xi_1(t)\mathrm{d}t\Big] + e_{n2}G_2(s) + p_{n2}e^{-s\delta_{n2}^1}\Big[G_2(s) + \int_{-\delta_{n2}^1}^0 e^{-st}\xi_2(t)\mathrm{d}t\Big] + q_{n2}e^{-s\delta_{n2}^2} \\
\quad \times\Big[G_2(s) + \int_{-\delta_{n2}^2}^0 e^{-st}\xi_2(t)\mathrm{d}t\Big] + \ldots\ldots + e_{nn}G_n(s) + p_{nn}e^{-s\delta_{nn}^1}\Big[G_n(s) + \int_{-\delta_{nn}^1}^0 e^{-st}\xi_n(t)\mathrm{d}t\Big] \\
\quad +q_{nn}e^{-s\delta_{nn}^2}\Big[G_n(s) + \int_{-\delta_{nn}^2}^0 e^{-st}\xi_n(t)\mathrm{d}t\Big]
\end{cases}
\end{cases}
$$

$$(2.3)$$

where $G_l(s)$ is the Laplace transform of $M_l(t)$ with $G_l(s) = L[M_l(t)]$ and $\xi_l(t)\big(0 \le l \le n, \ t \in [-\hat{\delta} = -\max\{\delta_1, \delta_2\}, 0]\big)$ are the initial conditions of $M_l(t)$.

Equation (2.3) can be written as follows:

$$
\Delta(s)\begin{pmatrix} G_1(s) \\ G_2(s) \\ . \\ . \\ . \\ G_n(s) \end{pmatrix} = \begin{pmatrix} h_1(s) \\ h_2(s) \\ . \\ . \\ . \\ h_n(s) \end{pmatrix}
$$

in which

$$
\begin{cases}
\begin{cases}
h_1(s) = s^{\alpha-1}\xi_1(0) + p_{11}e^{-s\delta_{11}^1}\int_{-\delta_{11}^1}^{0} e^{-st}\xi_1(t)\mathrm{d}t + q_{11}e^{-s\delta_{11}^2}\int_{-\delta_{11}^2}^{0} e^{-st}\xi_1(t)\mathrm{d}t \\
\qquad + p_{12}e^{-s\delta_{12}^1}\int_{-\delta_{12}^1}^{0} e^{-st}\xi_2(t)\mathrm{d}t + q_{12}e^{-s\delta_{12}^2}\int_{-\delta_{12}^2}^{0} e^{-st}\xi_2(t)\mathrm{d}t \\
\qquad + \cdots + p_{1n}e^{-s\delta_{1n}^1}\int_{-\delta_{1n}^1}^{0} e^{-st}\xi_n(t)\mathrm{d}t + q_{1n}e^{-s\delta_{1n}^2}\int_{-\delta_{1n}^2}^{0} e^{-st}\xi_n(t)\mathrm{d}t
\end{cases} \\[2mm]
\begin{cases}
h_2(s) = s^{\alpha-1}\xi_2(0) + p_{21}e^{-s\delta_{21}^1}\int_{-\delta_{21}^1}^{0} e^{-st}\xi_1(t)\mathrm{d}t + q_{21}e^{-s\delta_{21}^2}\int_{-\delta_{21}^2}^{0} e^{-st}\xi_1(t)\mathrm{d}t \\
\qquad + p_{22}e^{-s\delta_{22}^1}\int_{-\delta_{22}^1}^{0} e^{-st}\xi_2(t)\mathrm{d}t + q_{22}e^{-s\delta_{22}^2}\int_{-\delta_{22}^2}^{0} e^{-st}\xi_2(t)\mathrm{d}t \\
\qquad + \cdots + p_{2n}e^{-s\delta_{2n}^1}\int_{-\delta_{2n}^1}^{0} e^{-st}\xi_n(t)\mathrm{d}t + q_{2n}e^{-s\delta_{2n}^2}\int_{-\delta_{2n}^2}^{0} e^{-st}\xi_n(t)\mathrm{d}t
\end{cases} \\[2mm]
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\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\
\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\[2mm]
\begin{cases}
h_n(s) = s^{\alpha-1}\xi_n(0) + p_{n1}e^{-s\delta_{n1}^1}\int_{-\delta_{n1}^1}^{0} e^{-st}\xi_1(t)\mathrm{d}t + q_{n1}e^{-s\delta_{n1}^2}\int_{-\delta_{n1}^2}^{0} e^{-st}\xi_1(t)\mathrm{d}t \\
\qquad + p_{n2}e^{-s\delta_{n2}^1}\int_{-\delta_{n2}^1}^{0} e^{-st}\xi_2(t)\mathrm{d}t + q_{n2}e^{-s\delta_{n2}^2}\int_{-\delta_{n2}^2}^{0} e^{-st}\xi_2(t)\mathrm{d}t \\
\qquad + \cdots + p_{nn}e^{-s\delta_{nn}^1}\int_{-\delta_{nn}^1}^{0} e^{-st}\xi_n(t)\mathrm{d}t + q_{nn}e^{-s\delta_{nn}^2}\int_{-\delta_{nn}^2}^{0} e^{-st}\xi_n(t)\mathrm{d}t,
\end{cases}
\end{cases}
$$

$$
\Delta(s) = \begin{pmatrix}
s^{\alpha} + \Upsilon_{11} & \Upsilon_{12} & \cdots & \Upsilon_{1n} \\
\Upsilon_{21} & s^{\alpha} + \Upsilon_{22} & \cdots & \Upsilon_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Upsilon_{n1} & \Upsilon_{n2} & \vdots & s^{\alpha} + \Upsilon_{nn}
\end{pmatrix},
$$

where

$$
\begin{aligned}
\Upsilon_{11} &= -e_{11} - p_{11}e^{-s\delta_{11}^1} - q_{11}e^{-s\delta_{11}^2}, \quad \Upsilon_{12} = -e_{12} - p_{12}e^{-s\delta_{12}^1} - q_{12}e^{-s\delta_{12}^2}, \\
\Upsilon_{1n} &= -e_{1n} - p_{1n}e^{-s\delta_{1n}^1} - q_{1n}e^{-s\delta_{1n}^2}, \quad \Upsilon_{21} = -e_{21} - p_{21}e^{-s\delta_{21}^1} - q_{21}e^{-s\delta_{21}^2}, \\
\Upsilon_{22} &= -e_{22} - p_{22}e^{-s\delta_{22}^1} - q_{22}e^{-s\delta_{22}^2}, \quad \Upsilon_{2n} = -e_{2n} - p_{2n}e^{-s\delta_{2n}^1} - q_{2n}e^{-s\delta_{2n}^2}, \\
\Upsilon_{n1} &= -e_{n1} - p_{n1}e^{-s\delta_{n1}^1} - q_{n1}e^{-s\delta_{n1}^2}, \quad \Upsilon_{n2} = -e_{n2} - p_{n2}e^{-s\delta_{n2}^1} - q_{n2}e^{-s\delta_{n2}^2}, \\
\Upsilon_{nn} &= -e_{nn} - p_{nn}e^{-s\delta_{nn}^1} - q_{nn}e^{-s\delta_{nn}^2}.
\end{aligned}
$$

$\Delta(s)$ represent the characteristic matrix of system (2.2) and $det(\Delta(s))$ stands for the characteristic polynomial of $\Delta(s)$. It's obvious that the stability of system (2.2) is completely determined by the distribution of eigenvalues of $\Delta(s)$.

Remark 2.3 If $\delta_{kl}^2 = 0$, system (2.1) is equivalent to the following expression:

$$
D^{\alpha}M(t) = \tilde{E}M(t) + M(t_{\delta_1}), \ 0 < \alpha < 1 \tag{2.4}
$$

where $\tilde{E} = E + Q = (\tilde{e}_{kl})_{n \times n}$. The characteristic matrix of system (2.4) is denoted by:

$$\tilde{\Delta}(s) = \begin{pmatrix} s^\alpha - e_{11} - p_{11}e^{-s\delta_{11}^1} & -e_{1n} - p_{1n}e^{-s\delta_{1n}^1} & \cdots & -e_{1n} - p_{1n}e^{-s\delta_{1n}^1} \\ -e_{21} - p_{21}e^{-s\delta_{21}^1} & s^\alpha - e_{22} - p_{22}e^{-s\delta_{22}^1} & \cdots & -e_{2n} - p_{2n}e^{-s\delta_{2n}^1} \\ \vdots & \vdots & \ddots & \vdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -e_{n1} - p_{n1}e^{-s\delta_{n1}^1} & -e_{n2} - p_{n2}e^{-s\delta_{n2}^1} & \vdots & -e_{nn} - p_{nn}e^{-s\delta_{nn}^1} \end{pmatrix}.$$

It is obviously, stability of system (2.4) is completely determined by the distribution values of eigenvalues of $\tilde{\Delta}(s)$.

Remark 2.4 If $\delta_{kl}^1 = \delta_{kl}^2 = 0$, system (2.1) is equivalent to the following expression:

$$D^\alpha M(t) = \hat{E}M(t), \; 0 < \alpha < 1 \tag{2.5}$$

where $\hat{E} = E + P + Q = (\hat{e}_{kl})_{n \times n}$. The characteristic matrix of system (2.5) is denoted by:

$$\hat{E} = \begin{pmatrix} e_{11} + p_{11} + q_{11} & e_{12} + p_{12} + q_{12} & \cdots & e_{1n} + p_{1n} + q_{1n} \\ e_{21} + p_{21} + q_{21} & e_{22} + p_{22} + q_{22} & \cdots & e_{2n} + p_{2n} + q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_{n1} + p_{n1} + q_{n1} & e_{n2} + p_{n2} + q_{n2} & \cdots & e_{nn} + p_{nn} + q_{nn} \end{pmatrix}.$$

It is obviously, stability of system (2.4) is completely determined by the distribution values of eigenvalues of $\Delta(s)$.

Theorem 2.5 If $0 < \alpha < 1$, all the roots of characteristic equation $det(\Delta(s)) = 0$ have negative real parts, then the zero solution of system (2.1) is Lyapunov globally asymptotically stable.

Proof. The proof of theorem is almost the similar as those of Theorem 3.1 in [13], so we omit it here.

Theorem 2.6 If a matrix $\hat{E}$ is stable, i.e., all the eigenvalues $\lambda's$ of $\hat{E}$ satisfy $|arg(\lambda)| > \frac{\pi}{2}$ and the characteristic equation $det(\Delta(s)) = 0$ has no pure imaginary roots for any $\delta_1, \; \delta_2 > 0$, then the zero solution of system (2.1) is Lyapunov globally asymptotically stable.

Proof. The proof of theorem is almost the similar as those of Theorem 3.2 in [13], so we omit it here.

Lemma 2.7 Consider the following delayed fractional order differential inequality

$$\begin{cases} D^\alpha m(t) \leq -Em(t) + Fm(t - \delta_1) + Gm(t - \delta_2), \; t > t_0, \; 0 < \alpha \leq 1, \\ m(s) = \hat{m}(s), \; s \in [-\hat{\delta} = -\max\{\delta_1, \delta_2\}, 0], \end{cases} \tag{2.6}$$

and delayed fractional order linear system

$$\begin{cases} D^\alpha \tilde{m}(t) = -E\tilde{m}(t) + F\tilde{m}(t - \delta_1) + G\tilde{m}(t - \delta_2), \; t > t_0, \; 0 < \alpha \leq 1, \\ \tilde{m}(s) = \hat{m}(s), \; s \in [-\hat{\delta} = -\max\{\delta_1, \delta_2\}, 0], \end{cases} \tag{2.7}$$

where $m(t)$ and $\tilde{m}(t)$ are continuous and non negative in $[0, +\infty)$, and $\hat{m}(t) \geq 0$, $t \in [-\hat{\delta}, 0]$. If E, F, $G > 0$ are scalars, then $m(t) \leq \tilde{m}(t)$ for all $t \in [0, +\infty)$.

Proof. From (2.6), there exists a non negative function $r(t)$ such that

$$
\begin{cases}
D^\alpha m(t) = -Em(t) + Fm(t - \delta_1) + Gm(t - \delta_2) - r(t), & 0 < \alpha \leq 1, \\
m(s) = \hat{m}(s), & s \in [-\hat{\delta}, 0].
\end{cases} \tag{2.8}
$$

Let $\check{\delta} = \min\{\delta_1, \delta_2\}$ and $\mu_1 = [\frac{t_1}{\check{\delta}}] + 1$, where $[\frac{t_1}{\check{\delta}}]$ is the greatest integer smaller than $\frac{t_1}{\check{\delta}}$. Obviously, $[0, t_1) \subseteq [0, \mu_1 \check{\delta}]$, and from [27], the unique solution of equation (2.8) has expressed by $m(t) = m_{l\check{\delta}}$, and

$$
m_{l\check{\delta}} = k_{l\check{\delta}} \mathbb{E}_{\alpha,1}(-Et^\alpha) + \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(-E(t-s)^\alpha) \Psi_{l\check{\delta}} ds, \ 0 < \alpha \leq 1, \ t \in [(l-1)\check{\delta}, l\check{\delta}], \tag{2.9}
$$

where $k_{l\check{\delta}}$, $l = 1, 2, .., \mu_1$ is constant, $\mathbb{E}_{\alpha,1}$ is one parameter Mittag-Leffler function, $m_{0\check{\delta}} = \hat{m}(s)$ and $\Psi_{l\check{\delta}}$ is represented as

$$
\Psi_{l\check{\delta}} = \begin{cases}
Fm_{0\check{\delta}}(t - \delta_1) + Gm_{0\check{\delta}}(t - \delta_2) - r(t), & 0 < t \leq \check{\delta} \\
Fm_{1\check{\delta}}(t - \delta_1) + Gm_{1\check{\delta}}(t - \delta_2) - r(t), & \check{\delta} < t \leq 2\check{\delta} \\
\quad \cdot & \\
\quad \cdot & \\
\quad \cdot & \\
Fm_{(\mu_1-1)\check{\delta}}(t - \delta_1) + Gm_{(\mu_1-1)\check{\delta}}(t - \delta_2) - r(t), & (\mu_1 - 1)\check{\delta} < t \leq \mu_1 \check{\delta}.
\end{cases} \tag{2.10}
$$

Since $t^{\alpha-1}$, $\mathbb{E}_{\alpha,\alpha}$ and $r(t)$ are non negative functions. From (2.9) and (2.10), we have

$$
\begin{aligned}
m_{l\check{\delta}} \leq{}& k_{l\check{\delta}} \mathbb{E}_{\alpha,1}(-Et^\alpha) + \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(-E(t-s)^\alpha) Fm_{l\check{\delta}}(s - \delta_1) ds \\
&+ \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(-E(t-s)^\alpha) Gm_{l\check{\delta}}(s - \delta_2) ds, \ t \in [(\mu_1 - 1)\check{\delta}, \mu_1 \check{\delta}].
\end{aligned} \tag{2.11}
$$

Similarly, the unique solution of system (2.7) is expressed by

$$
\begin{aligned}
\tilde{m}_{l\check{\delta}} ={}& k_{l\check{\delta}} \mathbb{E}_{\alpha,1}(-Et^\alpha) + \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(-E(t-s)^\alpha) F\tilde{m}_{l\check{\delta}}(s - \delta_1) ds \\
&+ \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(-E(t-s)^\alpha) G\tilde{m}_{l\check{\delta}}(s - \delta_2) ds, \ t \in [(\mu_1 - 1)\check{\delta}, \mu_1 \check{\delta}].
\end{aligned} \tag{2.12}
$$

Next, we will show that $m(t) \leq \tilde{m}(t)$, $t \in [(\mu_1 - 1)\check{\delta}, \mu_1 \check{\delta}]$, $l = 1, 2, ..., \mu_1$. Now we will use the method of induction on μ_1.

Firstly, we will show that $m(t) \leq \tilde{m}(t)$ for $\mu_1 = 1$. If $t \in (0, \check{\delta}]$, then $t - \delta_j \in [-\hat{\delta}, 0]$ and $m(t - \delta_j) = \tilde{m}(t - \delta_j) = \hat{m}(t - \delta_j)$ for $j = 1, 2$. From (2.11) and (2.12), we have

$$
m_{\check{\delta}}(t) \leq k_{l\check{\delta}} \mathbb{E}_{\alpha,1}(-Et^\alpha) + \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}(-E(t-s)^\alpha) F\hat{m}_{l\check{\delta}}(s - \delta_1) ds
$$

$$+ \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G \hat{m}_{l\breve{\delta}}(s-\delta_2) ds = \tilde{m}_{\breve{\delta}}(t). \tag{2.13}$$

Note that the initial values of system (2.7) and (2.8) into collection, one has $k_{\breve{\delta}} = 0$. Hence $m(t) \leq \tilde{m}(t)$ for $\mu_1 = 1$. Next, we will assume that $m(t) \leq \tilde{m}(t)$ true for $\mu_1 = 1$, that is for $t \in [(\mu_1 - 1)\breve{\delta}, \mu_1\breve{\delta}]$, we have $m_{l\breve{\delta}}(t) \leq \tilde{m}_{l\breve{\delta}}(t)$, $l = 1, 2, ..., \mu_1$. Then, we will prove that $m(t) \leq \tilde{m}(t)$ for $\mu_1 + 1$.

For $t \in [\mu_1\breve{\delta}, (\mu_1 + 1)\breve{\delta}]$, by virtue of inequality (2.9), one has

$$
\begin{aligned}
m(t) &= m_{(\mu_1+1)\breve{\delta}} \\
&\leq k_{(\mu_1+1)\breve{\delta}} \mathbb{E}_{\alpha,1}(-Et^{\alpha}) + \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F m_{(\mu_1+1)\breve{\delta}}(s-\delta_1) ds \\
&\quad + \int_0^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G m_{(\mu_1+1)\breve{\delta}}(s-\delta_2) ds \\
&= k_{(\mu_1+1)\breve{\delta}} \mathbb{E}_{\alpha,1}(-Et^{\alpha}) + \int_0^{\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F m_{0\breve{\delta}}(s-\delta_1) ds \\
&\quad + \sum_{l=2}^{\mu_1} \int_{(l-1)\breve{\delta}}^{l\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F m_{l\breve{\delta}}(s-\delta_1) ds \\
&\quad + \int_0^{\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G m_{0\breve{\delta}}(s-\delta_2) ds \\
&\quad + \sum_{l=2}^{\mu_1} \int_{(l-1)\breve{\delta}}^{l\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G m_{l\breve{\delta}}(s-\delta_2) ds \\
&\quad + \int_{\mu_1\breve{\delta}}^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F m_{(\mu_1+1)\breve{\delta}}(s-\delta_1) ds \\
&\quad + \int_{\mu_1\breve{\delta}}^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G m_{(\mu_1+1)\breve{\delta}}(s-\delta_2) ds \\
&\leq k_{(\mu_1+1)\breve{\delta}} \mathbb{E}_{\alpha,1}(-Et^{\alpha}) + \int_0^{\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F \tilde{m}_{0\breve{\delta}}(s-\delta_1) ds \\
&\quad + \sum_{l=2}^{\mu_1} \int_{(l-1)\breve{\delta}}^{l\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F \tilde{m}_{l\breve{\delta}}(s-\delta_1) ds \\
&\quad + \int_0^{\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G \tilde{m}_{0\breve{\delta}}(s-\delta_2) ds \\
&\quad + \sum_{l=2}^{\mu_1} \int_{(l-1)\breve{\delta}}^{l\breve{\delta}} (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G \tilde{m}_{l\breve{\delta}}(s-\delta_2) ds \\
&\quad + \int_{\mu_1\breve{\delta}}^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) F \tilde{m}_{(\mu_1+1)\breve{\delta}}(s-\delta_1) ds \\
&\quad + \int_{\mu_1\breve{\delta}}^t (t-s)^{\alpha-1} \mathbb{E}_{\alpha,\alpha}\big(-E(t-s)^{\alpha}\big) G \tilde{m}_{(\mu_1+1)\breve{\delta}}(s-\delta_2) ds \\
&= \tilde{m}_{(\mu_1+1)\breve{\delta}} = \tilde{m}(t). \tag{2.14}
\end{aligned}
$$

Set $\mu_2 = [\frac{t_2}{\delta}] + 1$. Then, obviously, $[t_1, t_2] \subseteq [t_1, \mu_2 \check{\delta}]$. The initial values of (2.7) and (2.8) is represented by:

$$
\begin{aligned}
m(s) &= \hat{m}(s), \ s \in [t_1 - \hat{\delta}, t_1) \\
\tilde{m}(s) &= \hat{m}(s), \ s \in [t_1 - \hat{\delta}, t_1).
\end{aligned}
$$

The proof is almost the similar for $t \in [0, t_1)$, we get that $m(t) \leq \tilde{m}(t), \ t \in [t_1, t_2)$. So splitting the union of all subsets $[0, t_1) \cup [t_1, t_2) \cup ....$, we show that $m(t) \leq \tilde{m}(t)$ for all $t \in [0, +\infty)$. This completes the proof of Lemma 2.7.

Lemma 2.8 [8] Let $m(t) \in \mathbb{R}^n$ be a differentiable vector valued function and $M \in \mathbb{R}^{n \times n}$ is constant, symmetric and positive definite matrix. Then the following relationship is holds:

$$
D^\alpha [m^T(t) M m(t)] \leq 2m^T(t) M D^\alpha m(t), \ \forall \ 0 < \alpha < 1.
$$

Lemma 2.9 [13] Consider the following delayed fractional order differential inequality

$$
\begin{cases}
D^\alpha m(t) \leq -E m(t) + F m(t - \delta_1), \ t > t_0, \ 0 < \alpha \leq 1, \\
m(s) = \hat{m}(s), \ s \in [-\delta_1, 0],
\end{cases}
$$

and delayed fractional order linear system

$$
\begin{cases}
D^\alpha \tilde{m}(t) = -E \tilde{m}(t) + F \tilde{m}(t - \delta_1), \ t > t_0, \ 0 < \alpha \leq 1, \\
\tilde{m}(s) = \hat{m}(s), \ s \in [-\delta_1, 0],
\end{cases}
$$

where $m(t)$ and $\tilde{m}(t)$ are continuous and non negative in $[0, +\infty)$, and $\hat{m}(t) \geq 0, \ t \in [-\delta_1, 0]$. If $E, \ F > 0$ are scalars, then $m(t) \leq \tilde{m}(t)$ for all $t \in [0, +\infty)$.

Lemma 2.10 [13] If $0 < \alpha < 1$, all the roots of characteristic equation $det(\tilde{\Delta}(s)) = 0$ have negative real parts, then the zero solution of system (2.4) is Lyapunov globally asymptotically stable.

Lemma 2.11 [13] If $0 < \alpha < 1$, all the eigenvalues of $\hat{E}$ satisfy $|arg(\lambda) > \frac{\pi}{2}|$ and the characteristic equation $det(\tilde{\Delta}(s)) = 0$ has no pure imaginary roots for any $\delta_1 > 0$, and $-E + F < 0$, then the zero solution of system (2.4) is Lyapunov globally asymptotically stable.

Lemma 2.12 [20] Let $\gamma \in \mathbb{R}, \ U, \ V, \ W, \ Z$ be matrices with suitable dimensions. Then the Kronecker product has the following properties:

(1). $(\gamma U) \otimes V = U \otimes (\gamma V)$;
(2). $(U + V) \otimes W = (U \otimes W) + (V \otimes W)$;
(3). $(U \otimes V)^T = (U^T \otimes V^T)$;
(4). $(U \otimes V)(W \otimes Z) = (UW \otimes VZ)$;

Lemma 2.13 [25] Let $m = 0$ be the equilibrium point of fractional order differential system $D^\alpha m(t) = h(t, m(t))$. Assume that there exists a Lyapunov functional $H(t, m(t))$ and k-class function θ_l, $(l = 1, 2, 3)$, satisfying:

$$\theta_1(\|m\|) \leq H(t, m(t)) \leq \theta_2(\|m\|), \ D^\alpha H(t, m(t)) \leq \theta_3(\|m\|), \ 0 < \alpha < 1.$$

Then the fractional order differential system is asymptotically stable.

Lemma 2.14 [41] For any vectors β_1, $\beta_2 \in \mathbb{R}^m$ and any matrix $0 < G \in \mathbb{R}^{n \times n}$, then the following relationship holds:

$$2\beta_1^T \beta_2 \leq \beta_1^T G \beta_1 + \beta_2^T G^{-1} \beta_2.$$

In this article, we consider an array of linear coupled fractional order Cohen-Grossberg neural networks (FCCGNNs) consisting of N identical nodes with each isolated node network being an n-dimensional dynamical system, which is presented by:

$$
\begin{aligned}
D^\alpha z_l(t) &= -D(z_l(t)) \left[U(z_l(t)) - Vh(z_l(t)) - Wh(z_l(t - \delta_1)) - J \right] \\
&\quad + \beta_1 \sum_{k=1}^{N} A_{lk} \Phi z_k(t) + \beta_2 \sum_{k=1}^{N} A_{lk} \Psi z_k(t - \delta_2),
\end{aligned}
\tag{2.15}
$$

with the singe delayed isolated node networks

$$
D^\alpha z_l(t) = -D(z_l(t)) \left[U(z_l(t)) - Vh(z_l(t)) - Wh(z_l(t - \delta_1)) - J \right]
\tag{2.16}
$$

in which $l = 1, 2, ..., N$, N is the total number of nodes in the networks, $z_l(t) = \left(z_{l1}(t), ..., z_{ln}(t) \right)^T$ signifies the state of the neuron at time t; $D(\cdot)$ signifies an amplification function; $U(\cdot)$ signifies an an appropriately behaved function; $h(z_l(t)) = \left(h_1(z_{l1}(t)), ..., h_n(z_{ln}(t)) \right)^T$ signifies the activation function of the neurons at time t; $V = (v_{lk})_{n \times n}$ and $W = (w_{lk})_{n \times n}$ represents the connection weights of the k-th neuron on l-th neuron; $\delta_1 > 0$ and $\delta_2 > 0$ represents the positive and constant delays, respectively; $J = (J_1, ..., J_n)$ is the constant external input of the network; $\beta_1 > 0$ and $\beta_2 > 0$ represents the strengths of constant and delayed coupling weights, respectively. $\Phi = diag\{\phi_1, ..., \phi_n\} > 0$ and $\Psi = diag\{\psi_1, ..., \psi_n\} > 0$ denotes the inertial coupling between two nodes; $A = (A_{lk})_{N \times N}$ is the topological structure of the network and coupling strengths, where A_{lk} satisfies the following conditions: for $k \neq l$, $A_{lk} = A_{kl} > 0$ if there is link between node l and k, otherwise $A_{lk} = A_{kl} = 0$; for $l = k$, the diagonal elements are

$$
A_{ll} = - \sum_{k=1, k \neq l}^{N} A_{lk}, \ l = 1, 2, ..., N.
\tag{2.17}
$$

The initial values of system (2.15) are presented by

$$z_l(s) = \eta_l(s), \ s \in [-\hat{\delta}, 0],$$

where $\eta_l(s) \in C([-\hat{\delta}, 0], \mathbb{R}^n)$.

In order to prove our main results, we need the following fundamental assumptions.

Assumption [$\mathcal{H}_1$]. For every $l = 1, 2, ..., n$, the amplification function $d_l(\tau)$ is continuous functions and there exist positive constants $0 < \underline{d}_l \in \mathbb{R}$ and $0 < \overline{d}_l \in \mathbb{R}$ such that

$$0 < \underline{d}_l \le d_l(\tau) \le \overline{d}_l < +\infty, \text{ for any } \tau \in \mathbb{R}.$$

Assumption [$\mathcal{H}_2$]. For every $l = 1, 2, ..., n$, there exist a non-negative constant $u_l \in \mathbb{R}$ for arbitrary $\chi_1, \chi_2 \in \mathbb{R}$ and $\chi_1 \ne \chi_2$ such that

$$\frac{u_l(\chi_l) - u_l(\chi_2)}{\chi_1 - \chi_2} \ge u_l > 0.$$

Assumption [$\mathcal{H}_3$]. The activation function $h(\cdot)$ satisfy the global Lipschitz condition, that is, there exist a positive constant $L_k > 0$, $k = 1, 2, ..., n$ such that

$$|h_k(\chi_1) - h_k(\chi_2)| \le L_k|\chi_1 - \chi_2|,$$

for any $\chi_1, \chi_2 \in \mathbb{R}$ and $\chi_1 \ne \chi_2$.

Supposing that the constant vector $\check{z} = (\check{z}_1, ..., \check{z}_N)^T$ satisfies

$$0 = -D(\check{z}_l)\left[U(\check{z}_l) - Vh(\check{z}_l) - Wh(\check{z}_l)) - J\right] + \beta_1 \sum_{k=1}^{N} A_{lk}\Phi\check{z}_k + \beta_2 \sum_{k=1}^{N} A_{lk}\Psi\check{z}_k,$$

then it is said to be an arbitrary desired equilibrium solution for FCCGNNs (2.15), where $\check{z}_l = (\check{z}_{l1}, ..., \check{z}_{ln})^T$, $l = 1, 2, ..., N$.

Definition 2.15 The FCCGNNs with linear coupling delays (2.15) achieves global asymptotically stable if

$$\lim_{t \to +\infty} \|z_l(t) - \check{z}_l\|_2 = 0$$

holds, where $l = 1, 2, ..., N$.

Take $m_l(t) = z_l(t) - \check{z}_l$, $l = 1, 2, ..., N$, then

$$D^\alpha m_l(t) = -D(m_l(t) + \check{z}_l)\left[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t - \delta_1))\right]$$

$$+\beta_1 \sum_{k=1}^{N} A_{lk}\Phi m_k(t) + \beta_2 \sum_{k=1}^{N} A_{lk}\Psi m_k(t - \delta_2), \tag{2.18}$$

where $\tilde{U}(m_l(t)) = U(m_l(t) + \check{z}_l) - U(\check{z}_l)$ and $H(m_l(t)) = h(m_l(t) + \check{z}_l) - h(\check{z}_l)$.

3. Main results

In this section, several synchronization stability results are derived to ensure that FCCGNNs with and without parameter uncertainties is globally asymptotical synchronization stability depending on comparison theorem and LMI techniques, respectively.

3.1. Asymptotical synchronization stability of FCCGNNs

Theorem 3.1 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (2.15) is globally asymptotically stable, if the following conditions hold:

$$\theta = \lambda_m \Big[2\underline{D}U - VV^T - WW^T - 4\beta_1\varphi\Phi - 2\beta_2\varphi\Psi - \Lambda \Big] > 0$$

$$\kappa = \lambda_M(\Lambda) > 0, \ \varsigma = 2\beta_2\varphi\lambda_M(\Psi) > 0 \ \text{and} \ 0 < (\kappa + \varsigma) < \theta\sin(\frac{\alpha\pi}{2}),$$

where $\underline{D} = diag\{\underline{d}_1, ..., \underline{d}_n\}$, $U = diag\{u_1, ..., u_n\}$, $\Lambda = diag\{\overline{d}_1^2 L_1^2, ..., \overline{d}_n^2 L_n^2\}$, $\varphi = \max_{1 \le l \le N}\{|A_{ll}|\}$.

Proof. For the FCCGNNs error system (2.18), construct the following Lyapunov functional:

$$H(t) = m^T(t)m(t) = \sum_{l=1}^{N} m_l^T(t)m_l(t) \tag{3.1}$$

Taking the fractional order time derivative of $H(t)$ along the trajectories of (2.18) and, based on Lemma 2.8, one can get

$$
\begin{aligned}
D^\alpha H(t) &\le 2\sum_{l=1}^{N} m_l^T(t)D^\alpha m_l(t) \\
&= 2\sum_{l=1}^{N} m_l^T(t)\Big\{ -D(m_l(t) + \check{z}_l)\Big[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t - \delta_1))\Big] \\
&\quad + \beta_1 \sum_{k=1}^{N} A_{lk}\Phi m_k(t) + \beta_2 \sum_{k=1}^{N} A_{lk}\Psi m_k(t - \delta_2) \Big\} \\
&= -2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)\Big[U(m_l(t) + \check{z}_l) - U(\check{z}_l)\Big] \\
&\quad + 2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)VH(m_l(t)) + 2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)WH(m_l(t - \delta_1)) \\
&\quad + 2\beta_1 \sum_{l=1}^{N}\sum_{k=1}^{N} m_l^T(t)A_{lk}\Phi m_k(t) + 2\beta_2 \sum_{l=1}^{N}\sum_{k=1}^{N} m_l^T(t)A_{lk}\Psi m_k(t - \delta_2) \tag{3.2}
\end{aligned}
$$

By virtue of Assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ and Lemma 2.14, one gets

$$2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)VH(m_l(t)) \le 2\sum_{l=1}^{N} m_l^T(t)V\overline{d}_l H(m_l(t))$$

$$\leq \quad \sum_{l=1}^{N} m_l^T(t)VV^T m_l(t) + \sum_{l=1}^{N} m_l^T(t)\Lambda m_l(t) \qquad (3.3)$$

$$2\sum_{l=1}^{N} m_l^T(t)D(m_l(t)+\check{z}_l)WH(m_l(t-\delta_1)) \quad \leq \quad 2\sum_{l=1}^{N} m_l^T(t)W\bar{d}_l H(m_l(t-\delta_1))$$

$$\leq \quad \sum_{l=1}^{N} m_l^T(t)WW^T m_l(t)$$

$$+ \sum_{l=1}^{N} m_l^T(t-\delta_1)\Lambda m_l(t-\delta_1) \qquad (3.4)$$

and

$$-2\sum_{l=1}^{N} m_l^T(t)D(m_l(t)+\check{z}_l)\Big[U(m_l(t)+\check{z}_l)-U(\check{z}_l)\Big] \quad \leq \quad -2\sum_{l=1}^{N} m_l^T(t)\bar{d}_l u_l m_l(t)$$

$$\leq \quad -2m^T(t)\underline{D}U m_l(t) \qquad (3.5)$$

By application of positive the diagonal property of matrix, positive definiteness and based on Lemma 2.14, and from (2.17), it is deduced to

$$2\beta_1 \sum_{l=1}^{N}\sum_{k=1}^{N} m_l^T(t)A_{lk}\Phi m_k(t) \quad \leq \quad 2\beta_1 \sum_{l=1}^{N}\sum_{k=1}^{N} |A_{lk}|m_l^T(t)\Phi^{\frac{1}{2}}\Phi^{\frac{1}{2}} m_k(t)$$

$$\leq \quad \beta_1 \sum_{l=1}^{N}\sum_{k=1}^{N} |A_{lk}|m_l^T(t)\Phi m_l(t) + \beta_1 \sum_{l=1}^{N}\sum_{k=1}^{N} |A_{lk}|m_k^T(t)\Phi m_k(t)$$

$$= \quad 2\beta_1 \sum_{l=1}^{N} |A_{ll}|m_l^T(t)\Phi m_l(t) + 2\beta_1 \sum_{k=1}^{N} |A_{kk}|m_k^T(t)\Phi m_k(t)$$

$$\leq \quad 4\beta_1\varphi \sum_{l=1}^{N} m_l^T(t)\Phi m_l(t). \qquad (3.6)$$

Similarly,

$$2\beta_2 \sum_{l=1}^{N}\sum_{k=1}^{N} m_l^T(t)A_{lk}\Psi m_k(t-\delta_2) \quad \leq \quad 2\beta_2 \sum_{l=1}^{N} |A_{ll}|m_l^T(t)\Psi m_l(t)$$

$$+2\beta_2 \sum_{k=1}^{N} |A_{kk}|m_k^T(t-\delta_2)\Psi m_k(t-\delta_2)$$

$$= \quad 2\beta_2\varphi \sum_{l=1}^{N} m_l^T(t)\Psi m_l(t)$$

$$+2\beta_2\varphi \sum_{l=1}^{N} m_l^T(t-\delta_2)\Psi m_l(t-\delta_2) \qquad (3.7)$$

Substituting (3.3)–(3.7) to (3.2), it yields

$$
\begin{aligned}
D^\alpha H(t) \ \leq\ & -\sum_{l=1}^{N} m_l^T(t)\Big[2\underline{D}U - VV^T - WW^T - 4\beta_1\varphi\Phi - 2\beta_2\varphi\Psi - \Lambda\Big]m_l(t) \\
& + \sum_{l=1}^{N} m_l^T(t-\delta_1)\Lambda m_l(t-\delta_1) + 2\beta_2\varphi\sum_{l=1}^{N} m_l^T(t-\delta_2)\Psi m_l(t-\delta_2) \\
\leq\ & -\lambda_m(\Upsilon)\sum_{l=1}^{N} m_l^T(t)m_l(t) + \lambda_M(\Lambda)\sum_{l=1}^{N} m_l^T(t-\delta_1)m_l(t-\delta_1) \\
& + 2\beta_2\varphi\lambda_M(\Psi)\sum_{l=1}^{N} m_l^T(t-\delta_1)m_l(t-\delta_1) \\
=\ & -\theta H(t) + \kappa H(t-\delta_1) + \varsigma H(t-\delta_2)
\end{aligned}
\tag{3.8}
$$

Consider the following fractional order linear system:

$$
D^\alpha Q(t) \ =\ -\theta Q(t) + \kappa Q(t-\delta_1) + \varsigma Q(t-\delta_2)
\tag{3.9}
$$

where $Q(t) \geq 0$ ($Q(t) \in \mathbb{R}$), and take the similar initial values with $H(t)$. Then, by application of Lemma 2.7, one has $0 \leq H(t) \leq Q(t)$, $\forall\ t \in [0,+\infty)$. It point out that, there exists a unique equilibrium point in (3.9).

When $\delta_1 \neq 0$, $\delta_2 \neq 0$, assume that $s = vi = |v|\big(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\big)$ is purely imaginary root of characteristic equation $det(\Delta(s)) = 0$, where v is real number. Therefore, we have

$$
det(\Delta(s)) = s^\alpha + \theta - \kappa e^{-s\delta_1} - \varsigma e^{-s\delta_2} = 0
\tag{3.10}
$$

with $e^{-s\delta_k} = \cos v\delta_k - i\sin v\delta_k$, $k = 1,2$. Then, by substituting $s = vi = |v|\big(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\big)$ into (3.10), we have

$$
|v|^\alpha\big(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\big)^\alpha + \theta - \kappa\big(\cos v\delta_1 - i\sin v\delta_1\big) - \varsigma\big(\cos v\delta_2 - i\sin v\delta_2\big) = 0
$$

which implies that

$$
|v|^\alpha\cos(\frac{\alpha\pi}{2}) + \theta - \kappa\cos v\delta_1 - \varsigma\cos v\delta_2 + i\Big[|v|^\alpha\sin(\frac{\alpha\pi}{2}) + \kappa\sin v\delta_1 + \varsigma\sin v\delta_2\Big] = 0
\tag{3.11}
$$

Separating real and imaginary part of (3.11), one can get

$$
\begin{cases}
|v|^\alpha\cos(\frac{\alpha\pi}{2}) + \theta = \kappa\cos v\delta_1 + \varsigma\cos v\delta_2 \\
|v|^\alpha\sin(\frac{\alpha\pi}{2}) = -\kappa\sin v\delta_1 - \varsigma\sin v\delta_2.
\end{cases}
\tag{3.12}
$$

which is equivalent to

$$
|v|^{2\alpha} + 2\theta|v|^\alpha\cos(\frac{\alpha\pi}{2}) + \theta^2 - (\kappa^2 + \varsigma^2) - 2\kappa\varsigma\cos v(\delta_1 - \delta_2) = 0.
\tag{3.13}
$$

When $(\kappa + \varsigma)^2 < \theta^2 \sin(\frac{\alpha\pi}{2})$, because $\kappa, \ \varsigma > 0$ holds, we have

$$
\begin{aligned}
&|v|^{2\alpha} + 2\theta|v|^\alpha \cos(\tfrac{\alpha\pi}{2}) + \theta^2 - (\kappa^2 + \varsigma^2) - 2\kappa\varsigma \cos v(\delta_1 - \delta_2) \\
&\quad = \ |v|^{2\alpha} + 2\theta|v|^\alpha \cos(\frac{\alpha\pi}{2}) + \theta^2 - (\kappa + \varsigma)^2 + 2\kappa\varsigma\big(1 - \cos v(\delta_1 - \delta_2)\big). \\
&\quad > \ \Big(|v|^\alpha + \theta\cos(\frac{\alpha\pi}{2})\Big)^2 + 2\kappa\varsigma\big(1 - \cos v(\delta_1 - \delta_2)\big). \\
&\quad \geq \ 0.
\end{aligned}
\tag{3.14}
$$

Which is contradiction. Obviously, when $\kappa + \varsigma < \theta \sin(\frac{\alpha\pi}{2})$ and $0 < \alpha \leq 1$, $det(\Delta(s)) = 0$ has no purely imaginary roots for any $\delta_k \neq 0$, $k = 1, 2$ if $\theta > \kappa + \varsigma$, which means the zero solution of the system (3.9) is globally Lyapunov asymptotically stable based on Lemma 2.6. According to Lemma 2.7, we have $0 < H(t) \leq Q(t)$ and depending on the aforementioned analysis, $H(t)$ is globally asymptotically stable. i.e., $H(t) \to 0$, $[0, +\infty)$. As $H(t) = m^T(t)m(t) = \sum_{l=1}^{N} m_l^T(t)m_l(t) \to 0$, then $m_l(t) \to 0$ as $t \to +\infty$. Hence we declare that, the zero solution of (2.15) realizes globally asymptotically stable. The proof is ended.

Theorem 3.2 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (2.15) is globally asymptotically stable, if there exists a positive constant γ such that the following conditions hold:

$$
\tilde{\theta} = -\lambda_m\left[-\Big(I_N \otimes \big(VV^T + WW^T - 2\underline{D}U + \Lambda + \gamma^{-1}\beta_2\Psi^2\big) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)\Big)\right] > 0
$$

$$
\tilde{\kappa} = \lambda_M(\Lambda) > 0, \ \tilde{\varsigma} = \beta_2\gamma\lambda_M(AA^T) > 0 \ \text{ and } \ 0 < (\tilde{\kappa} + \tilde{\varsigma}) < \tilde{\theta}\sin(\frac{\alpha\pi}{2}),
$$

where $\underline{D} = diag\{\underline{d}_1, ..., \underline{d}_n\}$, $U = diag\{u_1, ..., u_n\}$, $\Lambda = diag\{\overline{d}_1^2 L_1^2, ..., \overline{d}_n^2 L_n^2\}$.

Proof. For the FCCGNNs (2.18), construct the following Lyapunov functional:

$$
H(t) = m^T(t)m(t) = \sum_{l=1}^{N} m_l^T(t)m_l(t)
\tag{3.15}
$$

Taking the fractional order time derivative of $H(t)$ along the trajectories of (2.18) and, based on Lemma 2.8, one can get

$$
\begin{aligned}
D^\alpha H(t) &\leq 2\sum_{l=1}^{N} m_l^T(t)D^\alpha m_l(t) \\
&= 2\sum_{l=1}^{N} m_l^T(t)\bigg\{ -D(m_l(t) + \check{z}_l)\Big[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t - \delta_1))\Big] \\
&\qquad + \beta_1\sum_{k=1}^{N} A_{lk}\Phi m_k(t) + \beta_2\sum_{k=1}^{N} A_{lk}\Psi m_k(t - \delta_2)\bigg\}
\end{aligned}
\tag{3.16}
$$

By virtue of Assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ and Lemma 2.14, one gets

$$
2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)VH(m_l(t)) \leq m^T(t)\Big[I_N \otimes (VV^T + \Lambda)\Big]m(t)
\tag{3.17}
$$

$$2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)WH(m_l(t - \delta_1) \leq m^T(t)\Big[I_N \otimes WW^T\Big]m(t)$$

$$+m^T(t - \delta_1)\Big[I_N \otimes \Lambda\Big]m(t - \delta_1) \qquad (3.18)$$

Combined with (3.16)-(3.18), we have

$$
\begin{aligned}
D^\alpha H(t) &\leq m^T(t)\Big[I_N \otimes (VV^T + WW^T - 2\underline{D}U\Lambda)\Big]m(t) + m^T(t - \delta_1)\Big[I_N \otimes \Lambda\Big]m(t - \delta_1) \\
&\quad +2\beta_1 m^T(t)(A \otimes \Phi)m(t) + 2\beta_2 m^T(t)[A \otimes \Psi]m(t - \delta_2) \\
&\leq m^T(t)\Big[(I_N \otimes \Omega) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)\Big]m(t) \\
&\quad +m^T(t - \delta_1)\Big[I_N \otimes \Lambda\Big]m(t - \delta_1) + \beta_2\gamma m^T(t - \delta_2)[AA^T \otimes I_n]m(t - \delta_2) \\
&\leq \lambda_M(\Delta)m^T(t)m(t) + \lambda_M(\Lambda)m^T(t - \delta_1)m(t - \delta_1) + \beta_2\lambda_M(AA^T)m^T(t - \delta_2)m(t - \delta_2) \\
&= -\lambda_m(-\Delta)H(t) + \lambda_M(\Lambda)H(t - \delta_1) + \beta_2\gamma\lambda_M(AA^T)H(t - \delta_2) \\
&= -\tilde{\theta}H(t) + \tilde{\kappa}H(t - \delta_1) + \tilde{\varsigma}H(t - \delta_2) \qquad (3.19)
\end{aligned}
$$

where $\Omega = VV^T + WW^T - 2\underline{D}U\Lambda + \beta_2\gamma^{-1}\Psi^2$. The rest of the proof for $\lim_{t\to+\infty}\|m(t)\|_2 = 0$ similar as in Theorem 3.1. Therefore, the FCCGNNs (2.15) realizes globally asymptotically stable.

If there is no coupling delays in FCCGNNs (2.18), then the result is given as follows.

Corollary 3.3 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (2.15) is globally asymptotically stable, if the following conditions hold:

$$
\begin{aligned}
\tilde{\theta}_1 &= \lambda_m\Big[-(I_N \otimes (VV^T + WW^T - 2\underline{D}U + \Lambda) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T))\Big] > 0 \\
\tilde{\kappa} &= \lambda_M(\Lambda) > 0, \quad \text{and} \quad 0 < \tilde{\kappa} < \tilde{\theta}_1\sin(\frac{\alpha\pi}{2}),
\end{aligned}
$$

where $\underline{D}$, U, Λ are same definitions in Theorem 3.2.

Proof. For the error system (2.18) with no coupling delays, take the same Lyapunov functional (3.15) as in Theorem 3.2:

$$
\begin{aligned}
D^\alpha H(t) &\leq 2\sum_{l=1}^{N} m_l^T(t)D^\alpha m_l(t) \\
&= 2\sum_{l=1}^{N} m_l^T(t)\Big\{ -D(m_l(t) + \check{z}_l)\Big[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t - \delta_1))\Big] \\
&\quad +\beta_1\sum_{k=1}^{N} A_{lk}\Phi m_k(t)\Big\} \\
&\leq m^T(t)\Big[I_N \otimes (VV^T + WW^T - 2\underline{D}U + \Lambda)\Big]m(t) + m^T(t - \delta_1)\Big[I_N \otimes \Lambda\Big]m(t - \delta_1) \\
&\quad +2\beta_1 m^T(t)(A \otimes \Phi)m(t) \\
&\leq m^T(t)\Big[(I_N \otimes \tilde{\Omega}) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)\Big]m(t) + m^T(t - \delta_1)\Big[I_N \otimes \Lambda\Big]m(t - \delta_1)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad \lambda_M(\Delta_1) m^T(t) m(t) + \lambda_M(\Lambda) m^T(t - \delta_1) m(t - \delta_1) \\
&= \quad -\lambda_m(-\Delta_1) H(t) + \lambda_M(\Lambda) H(t - \delta_1) \\
&= \quad -\tilde{\theta}_1 H(t) + \tilde{\kappa} H(t - \delta_1)
\end{aligned}
\tag{3.20}
$$

where $\tilde{\Omega} = VV^T + WW^T - 2\underline{D}U\Lambda$ and $\Delta_1 = (I_N \otimes \tilde{\Omega}) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)$.

Consider the following fractional order linear system:

$$
D^\alpha Q(t) \quad = \quad -\tilde{\theta}_1 Q(t) + \tilde{\kappa} Q(t - \delta_1)
\tag{3.21}
$$

where $Q(t) \geq 0$ ($Q(t) \in \mathbb{R}$), and take the same initial values with $H(t)$. By virtue of Remark 2.3 and Lemma 2.9, the zero solution of (3.21) is globally asymptotically Lyapunov stable and the obtained characteristic equation is

$$
\Delta(s) = s^\alpha + \tilde{\theta}_1 - \tilde{\kappa} e^{-s\delta_1} = 0
\tag{3.22}
$$

for any $\delta_1 > 0$ and $\tilde{\theta}_1 - \tilde{\kappa} > 0$. In the following we have to prove that system (39) has no purely imaginary roots. Suppose that $s = vi = |v|\left(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right)$ is purely imaginary roots of (40), where v is real number. With $e^{-s\delta_1} = \cos v\delta_1 - i\sin v\delta_1$ and substituting $s = vi = |v|\left(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right)$ into (40), we get

$$
|v|^\alpha\left(\cos(\frac{\alpha\pi}{2}) + i\sin(\frac{\alpha\pi}{2})\right) + \tilde{\theta}_1 - \tilde{\kappa}(\cos v\delta_1 - i\sin v\delta_1) = 0.
$$

Separating real part and imaginary part of (41), we obtain

$$
|v|^\alpha \cos(\frac{\alpha\pi}{2}) + \tilde{\theta}_1 \quad = \quad \tilde{\kappa}\cos v\delta_1
\tag{3.23}
$$

$$
|v|^\alpha \sin(\frac{\alpha\pi}{2}) \quad = \quad -\tilde{\kappa}\sin v\delta_1
\tag{3.24}
$$

From (3.23) and (3.24), it follows that

$$
\left[|v|^\alpha \cos(\frac{\alpha\pi}{2}) + \tilde{\theta}_1\right]^2 + \left[|v|^\alpha \sin(\frac{\alpha\pi}{2})\right]^2 - \tilde{\kappa}^2 = 0
\tag{3.25}
$$

which is equivalent to

$$
|v|^{2\alpha} + \tilde{\theta}_1^2 + 2\tilde{\theta}_1|v|^\alpha \cos(\frac{\alpha\pi}{2}) - \tilde{\kappa}^2 = 0
\tag{3.26}
$$

Since $|v|^\alpha > 0$, $\cos(\frac{\alpha\pi}{2}) > 0$ and $\tilde{\kappa} > 0$, thus if $\tilde{\theta}_1 - \tilde{\kappa} > 0$. Eq (3.26) has no real roots. Hence the characteristic equation of (3.26) has no purely imaginary roots for any $\delta_1 > 0$ if $\tilde{\theta}_1 - \tilde{\kappa} > 0$, which means the zero solution of the system (3.21) is globally Lyapunov asymptotically stable according to Lemma 2.10 and Lemma 2.11. Again by using Lemma 2.9, we have $0 < H(t) \leq Q(t)$ and depending on the aforementioned analysis, $H(t)$ is globally asymptotically synchronize. i.e., $H(t) \to 0$, $[0, +\infty)$. As $H(t) = m^T(t)m(t) = \sum_{l=1}^N m_l^T(t)m_l(t) \to 0$, then $\lim_{t \to +\infty} \|m(t)\|_2 = 0$. Hence we conclude that, the zero solution of system (2.18) with no coupling delays realizes globally asymptotically stable. The proof is completed.

If there is no coupling delays and time delays in FCCGNNs (2.15), then the result is given as follows.

Corollary 3.4 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (2.15) is globally asymptotically stable, if the following conditions hold:

$$\tilde{\theta} = -\lambda_m\left[-\left(I_N \otimes \left(VV^T - 2\underline{D}U + \Lambda\right) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)\right)\right] > 0$$

where $\underline{D}$, U, Λ are same definitions in Theorem 3.2.

Proof. For the error system (2.18) without coupling delays and time delays, take the same Lyapunov functional (3.15) as in Theorem 3.2:

$$
\begin{aligned}
D^\alpha H(t) &\leq 2\sum_{l=1}^{N} m_l^T(t)D^\alpha m_l(t) \\
&= 2\sum_{l=1}^{N} m_l^T(t)\left\{-D(m_l(t) + \check{z}_l)\left[\tilde{U}(m_l(t)) - VH(m_l(t))\right] + \beta_1\sum_{k=1}^{N} A_{lk}\Phi m_k(t)\right\} \\
&\leq m^T(t)\left[I_N \otimes (VV^T - 2\underline{D}U + \Lambda)\right]m(t) + 2\beta_1 m^T(t)(A \otimes \Phi)m(t) \\
&\leq m^T(t)\left[(I_N \otimes (VV^T - 2\underline{D}U + \Lambda) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)\right]m(t) \\
&= -\lambda_m\left[-\left(I_N \otimes (VV^T - 2\underline{D}U + \Lambda) + \beta_1(A \otimes \Phi) + \beta_1(A^T \otimes \Phi^T)\right)\right]H(t) \\
&= -\tilde{\theta}_1\|m(t)\|_2^2. \tag{3.27}
\end{aligned}
$$

The rest of the proof is similar to the proof of Theorem 4.1 in Ref [34], the FCCGNNs (2.15) with no coupling delays and time delays realizes globally asymptotically stable, thus the proof is ended.

3.2. Robust asymptotical synchronization stability of FCCGNNs

In fact, the existence of external disturbances and model errors are unavoidable in many situations. Therefore in this subsection we introduce the FCCGNNs with uncertainty consisting N identical nodes described by:

$$
\begin{aligned}
D^\alpha z_l(t) &= -D(z_l(t))\left[U(z_l(t)) - Vh(z_l(t)) - Wh(z_l(t - \delta_1)) - J\right] \\
&\quad + \beta_1\sum_{k=1}^{N} A_{lk}\Phi z_k(t) + \beta_2\sum_{k=1}^{N} A_{lk}\Psi z_k(t - \delta_2), \tag{3.28}
\end{aligned}
$$

where $z_l(t)$, $D(z_l(t))$, $U(z_l(t))$, $h(z_l(t))$ and J defined same as in (2.15). The parameters β_1, β_2, V, W, Φ, Ψ, A change in some given precision, which is intervalized as follows:

$$
\begin{aligned}
\beta_1^I &:= \left\{0 < \beta_1^- \leq \beta_1 \leq \beta_1^+, \; \forall \, \beta_1 \in \beta_1^I\right\}; \\
\beta_2^I &:= \left\{0 < \beta_2^- \leq \beta_2 \leq \beta_2^+, \; \forall \, \beta_2 \in \beta_2^I\right\}; \\
\Phi^I &:= \left\{\Phi = diag(\phi_l) : \Phi^- \leq \Phi \leq \Phi^+, \; 0 < \phi^- \leq \phi \leq \phi^+, \; l = 1, 2, ..., n \; \forall \, \Phi \in \Phi^I\right\}; \\
\Psi^I &:= \left\{\Psi = diag(\psi_l) : \Psi^- \leq \Psi \leq \Psi^+, \; 0 < \psi^- \leq \psi \leq \psi^+, \; l = 1, 2, ..., n \; \forall \, \Psi \in \Psi^I\right\};
\end{aligned}
$$

$$
\begin{aligned}
V^I &:= \Big\{ V = (v_{lk})_{n \times n} : V^- \le V \le V^+, \ 0 < v_{lk}^- \le v_{lk} \le v_{lk}^+, \ l = 1, 2, ..., n, \\
&\quad k = 1, 2, ..., n \ \forall \ V \in V^I \Big\}; \\
W^I &:= \Big\{ W = (w_{lk})_{n \times n} : W^- \le W \le W^+, \ 0 < w_{lk}^- \le w_{lk} \le w_{lk}^+, \ l = 1, 2, ..., n, \\
&\quad k = 1, 2, ..., n \ \forall \ W \in W^I \Big\}; \\
A^I &:= \Big\{ A = (A_{lk})_{N \times N} : A_{lk}^- \le A_{lk} \le A_{lk}^+, l \ne k, \ l = 1, 2, ..., N, \ k = 1, 2, ..., N \ \forall \ A \in A_1^I \Big\};
\end{aligned}
\tag{3.29}
$$

In order to convenience, we define

$$
\begin{aligned}
\tilde{v}_{lk} &= \max\{|v_{lk}^-|, |v_{lk}^+|\}, \ l = 1, 2, ..., n, \ k = 1, 2, ..., n \\
\tilde{w}_{lk} &= \max\{|w_{lk}^-|, |w_{lk}^+|\}, \ l = 1, 2, ..., n, \ k = 1, 2, ..., n \\
\tilde{A}_{ll} &= \sum_{k=1, l \ne k}^{N} A_{lk}^+, \ \tilde{A}_{lk}(l \ne k) = A_{lk}^+, \ l = 1, 2, ..., N, \ k = 1, 2, ..., N
\end{aligned}
$$

Supposing that the constant vector $\check{z} = (\check{z}_1, ..., \check{z}_N)^T$ satisfies

$$
0 = -D(\check{z}_l)\Big[U(\check{z}_l) - Vh(\check{z}_l) - Wh(\check{z}_l)) - J \Big] + \beta_1 \sum_{k=1}^{N} A_{lk} \Phi \check{z}_k + \beta_2 \sum_{k=1}^{N} A_{lk} \Psi \check{z}_k,
$$

then it is said to be an equilibrium solution of (3.28). The error $m_l(t) = z_l(t) - \check{z}_l, \ l = 1, 2, ..., N$ is given as follows:

$$
\begin{aligned}
D^\alpha m_l(t) &= -D(m_l(t) + \check{z}_l)\Big[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t - \delta_1)) \Big] \\
&\quad + \beta_1 \sum_{k=1}^{N} A_{lk} \Phi m_k(t) + \beta_2 \sum_{k=1}^{N} A_{lk} \Psi m_k(t - \delta_2),
\end{aligned}
\tag{3.30}
$$

where $\tilde{U}(m_l(t))$ and $H(m_l(t))$ have similar meanings as in error system (2.18).

Theorem 3.5 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (3.28) is globally robust asymptotically synchronize, if the following conditions hold:

$$
\begin{aligned}
\zeta &= \lambda_m\Big[-(-2\underline{D}U + (\varepsilon_1 + \varepsilon_2)L + \beta_1^+(\gamma_a + \gamma_\phi)I_n + \beta_2^+ \gamma_\psi I_n + \varepsilon_1^{-1} \gamma_v \overline{D}^2) \Big] > 0 \\
\mu &= \varepsilon_2^{-1} \gamma_w \lambda_M(\overline{D}^2) > 0, \ \varpi = \beta_2^+ \gamma_\psi > 0 \ \text{and} \ 0 < (\mu + \varpi) < \zeta \sin(\frac{\alpha\pi}{2}),
\end{aligned}
$$

where $\underline{D} = diag\{\underline{d}_1, ..., \underline{d}_n\}$, $U = diag\{u_1, ..., u_n\}$, $L = diag\{L_1^2, ..., L_n^2\}$, $\gamma_a = \sum_{l=1}^{N} \sum_{k=1}^{N} \tilde{A}_{lk}^2$, $\gamma_v = \sum_{l=1}^{n} \sum_{k=1}^{n} \tilde{v}_{lk}^2$, $\gamma_w = \sum_{l=1}^{n} \sum_{k=1}^{n} \tilde{w}_{lk}^2$, $\gamma_\phi = \sum_{l=1}^{n} \phi_l^{+2}$, $\gamma_\psi = \sum_{l=1}^{n} \psi_l^{+2}$, $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are arbitrary constants.

Proof. For the error system (3.30), choose the same Lyapunov functional in (3.15), then one has

$$
D^\alpha H(t) \le 2 \sum_{l=1}^{N} m_l^T(t) D^\alpha m_l(t)
$$

$$= 2\sum_{l=1}^{N} m_l^T(t)\Big\{ -D(m_l(t) + \check{z}_l)\Big[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t - \delta_1))\Big]$$

$$+\beta_1 \sum_{k=1}^{N} A_{lk}\Phi m_k(t) + \beta_2 \sum_{k=1}^{N} A_{lk}\Psi m_k(t - \delta_2)\Big\} \tag{3.31}$$

By some relevant inequality techniques, we can get

$$\begin{aligned}
2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)VH(m_l(t)) &\leq 2\sum_{l=1}^{N} m_l^T(t)\overline{D}VH(m_l(t)) \\
&\leq \sum_{l=1}^{N} \varepsilon_1 m_l^T(t)Lm_l(t) + \sum_{l=1}^{N} \varepsilon_1^{-1} m_l^T(t)\overline{D}^2 VV^T m_l(t) \\
&= m^T(t)(I_N \otimes \varepsilon_1 L)m(t) + m^T(t)(I_N \otimes \varepsilon_1^{-1}\gamma_v \overline{D}^2)m(t)
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
2\sum_{l=1}^{N} m_l^T(t)D(m_l(t) + \check{z}_l)WH(m_l(t - \delta_1)) &\leq 2\sum_{l=1}^{N} m_l^T(t)\overline{D}WH(m_l(t - \delta_1)) \\
&\leq \sum_{l=1}^{N} m_l^T(t)\varepsilon_2 Lm_l(t) \\
&\quad + \sum_{l=1}^{N} m_l^T(t - \delta_1)\varepsilon_2^{-1}\overline{D}^2 WW^T m_l(t - \delta_1) \\
&= m^T(t)(I_N \otimes \varepsilon_2 L)m(t) \\
&\quad + m^T(t - \delta_1)\Big(I_N \otimes \varepsilon_2^{-1}\gamma_w \overline{D}^2\Big)m(t - \delta_1)
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
2\beta_1 \sum_{k=1}^{N}\sum_{k=1}^{N} A_{lk}\Phi m_k(t) &= 2\beta_1 m^T(t)\Big[(A \otimes I_n)(I_N \otimes \Phi)\Big]m(t) \\
&\leq \beta_1 m^T(t)\Big[AA^T \otimes I_n\Big]m(t) \\
&\quad + \beta_1 m^T(t)\Big[(I_N \otimes \Phi^2)\Big]m(t) \\
&= m^T(t)\Big[I_N \otimes \beta_1^+(\gamma_a + \gamma_\phi)I_n\Big]m(t)
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
2\beta_2 \sum_{k=1}^{N}\sum_{k=1}^{N} A_{lk}G\Psi m_k(t - \delta_2) &\leq m^T(t)\Big[I_N \otimes (\beta_2^+\gamma_a I_n)\Big]m(t) \\
&\quad + m^T(t - \delta_2)\Big[I_N \otimes (\beta_2^+\gamma_\psi I_n)\Big]m(t - \delta_2)
\end{aligned} \tag{3.35}$$

Combined with (3.31)–(3.35), we have

$$D^\alpha H(t) \leq m^T(t)\Big[I_N \otimes \big(-2\underline{D}U + (\varepsilon_1 + \varepsilon_2)L + \beta_1^+(\gamma_a + \gamma_\phi)I_n + \beta_2^+\gamma_\psi I_n\big) + \varepsilon_1^{-1}\gamma_v \overline{D}^2\Big]m(t)$$

$$+m^T(t-\delta_1)\Big(I_N \otimes \varepsilon_2^{-1}\gamma_w\overline{D}^2\Big)m(t-\delta_1) + m^T(t-\delta_2)\Big[I_N \otimes (\beta_2^+\gamma_\psi I_n)\Big]m(t-\delta_2)$$

$$\le \lambda_M(\Theta)m^T(t)m(t) + \varepsilon_2^{-1}\gamma_w\lambda_M(\overline{D}^2)m^T(t-\delta_1)m(t-\delta_1) + \beta_2^+\gamma_\psi m^T(t-\delta_2)m(t-\delta_2)$$

$$= -\lambda_m(-\Theta)H(t) + \varepsilon_2^{-1}\gamma_w\lambda_M(\overline{D}^2)H(t-\delta_1) + \beta_2^+\gamma_\psi H(t-\delta_2)$$

$$= -\zeta H(t) + \mu H(t-\delta_1) + \varpi H(t-\delta_2) \tag{3.36}$$

The rest of the proof for $\lim_{t \to +\infty} \|m(t)\|_2 = 0$ similar as in Theorem 3.1. Therefore, the FCCGNNs (3.28) achieves globally robust asymptotically stable.

If there are no coupling delays in FCCGNNs (3.28) with the ranges of parameters given by (3.29), then the result is given as follows.

Corollary 3.6 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (3.28) with the ranges of parameters given by (3.29) achieves globally robust asymptotically stable, if the following conditions hold:

$$\zeta = \lambda_m\Big[-\big(-2\underline{D}U + (\varepsilon_1 + \varepsilon_2)L + \beta_1^+(\gamma_a + \gamma_\phi)I_n + \varepsilon_1^{-1}\gamma_v\overline{D}^2\big)\Big] > 0$$

$$\mu = \varepsilon_2^{-1}\gamma_w\lambda_M(\overline{D}^2) > 0 \text{ and } 0 < \mu < \zeta\sin(\frac{\alpha\pi}{2}),$$

where $\underline{D}$, U, L, γ_a, γ_v, γ_w, γ_ϕ, ε_1 and ε_2 are already defined in Theorem 3.5.

Proof. For the error system (3.30) without coupling delays, choose the same Lyapunov functional in (3.15), then one has

$$D^\alpha H(t) \le 2\sum_{l=1}^{N} m_l^T(t)D^\alpha m_l(t)$$

$$= 2\sum_{l=1}^{N} m_l^T(t)\Big\{-D(m_l(t)+\check{z}_l)\Big[\tilde{U}(m_l(t)) - VH(m_l(t)) - WH(m_l(t-\delta_1))\Big]$$

$$+\beta_1\sum_{k=1}^{N} A_{lk}\Phi m_k(t)\Big\}$$

$$\le m^T(t)\Big[I_N \otimes \big(-2\underline{D}U + (\varepsilon_1 + \varepsilon_2)L + \beta_1^+(\gamma_a + \gamma_\phi)I_n\big) + \varepsilon_1^{-1}\gamma_v\overline{D}^2\Big]m(t)$$

$$+m^T(t-\delta_1)\Big(I_N \otimes \varepsilon_2^{-1}\gamma_w\overline{D}^2\Big)m(t-\delta_1)$$

$$\le -\zeta H(t) + \mu H(t-\delta_1) \tag{3.37}$$

The rest of the proof for $\lim_{t \to +\infty} \|m(t)\|_2 = 0$ similar as in Corollary 3.3. Therefore, the FCCGNNs (3.28) with no coupling delays is globally robust asymptotically stable.

If there is no coupling delays and time delays in FCCGNNs (3.28) with the ranges of parameters given by (3.29), then the result is given as follows.

Corollary 3.7 If the assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ hold, then the FCCGNNs (3.28) with the ranges of parameters given by (3.29) achieves globally robust asymptotically stable, if the following conditions hold:

$$\zeta = \lambda_m\left[-\left(-2\underline{D}U + \varepsilon_1 L + \beta_1^+(\gamma_a + \gamma_\phi)I_n + \varepsilon_1^{-1}\gamma_v\overline{D}^2\right)\right] > 0$$

where $\underline{D}$, U, L, γ_a, γ_v, γ_ϕ, ε_1 and ε_2 are already defined in Theorem 3.5.

Proof. For the error system (3.30) with no coupling delays and no time delays, choose the same Lyapunov functional in (3.15), then one has

$$
\begin{aligned}
D^\alpha H(t) &\leq 2\sum_{l=1}^{N} m_l^T(t)D^\alpha m_l(t) \\
&= 2\sum_{l=1}^{N} m_l^T(t)\left\{-D(m_l(t) + \check{z}_l)\left[\tilde{U}(m_l(t)) - VH(m_l(t))\right] + \beta_1 \sum_{k=1}^{N} A_{lk}\Phi m_k(t)\right\} \\
&\leq m^T(t)\left[I_N \otimes \left(-2\underline{D}U + \varepsilon_1 L + \beta_1^+(\gamma_a + \gamma_\phi)I_n\right) + \varepsilon_1^{-1}\gamma_v\overline{D}^2\right]m(t) \\
&\leq -\zeta\|m(t)\|_2^2
\end{aligned}
\tag{3.38}
$$

The rest of the proof for $\lim_{t\to+\infty}\|m(t)\|_2 = 0$ similar as in Corollary 3.4. Therefore, the FCCGNNs (3.30) with no coupling delays and no time delays achieves globally robust asymptotically stable.

Remark 3.8 The author of [46] presented the synchronization stability conditions of Riemann Liouville sense fractional-order complex coupled neural networks under coupling delays. By using, Riemann Liouville derivative properties and some inequality techniques, several algebraic sufficient conditions are derived to verify the global asymptotic synchronization stability conditions of the proposed model. While in this paper, Caputo derivative properties, Cohen-Grossberg neural networks type models, Kronecker product and uncertain parameter are taken into consideration. Moreover, our obtained corollaries are new and purely different from those existing works.

4. Numerical examples

To verify the advantage of the above analytical results, two numerical simulations are performed in the following few lines.

Example 4.1 A FCCGNNs consisting of six identical two dimensional fractional order Cohen-Grossberg neural network is considered in the following:

$$
\begin{aligned}
D^{0.95}z_l(t) &= -D(z_l(t))\left[U(z_l(t)) - Vh(z_l(t)) - Wh(z_l(t - 0.2)) - J\right] \\
&\quad + \beta_1\sum_{k=1}^{6} A_{lk}\Phi z_k(t) + \beta_2\sum_{k=1}^{6} A_{lk}\Psi z_k(t - 0.1),
\end{aligned}
\tag{4.1}
$$

where $l = 1, 2, ..., 6$, $D(\tau) = diag\{0.1,\ 0.1\}$, $U_1(\tau) = U_2(\tau) = 20\tau$, $h_1(\tau) = h_2(\tau) = 0.05 * \tanh(\tau)$, $\beta_1 = 0.25, \beta_2 = 0.45$, $J = (0\ \ 0)^T$, the matrices $V,\ W,\ A,\ \Phi,\ \Psi$ are selected as, respectively

$$V = \begin{bmatrix} 2 & 3 \\ 1.3 & 1.2 \end{bmatrix},\ W = \begin{bmatrix} 2.5 & 1 \\ 1.5 & 2 \end{bmatrix},\ \Phi = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.95 \end{bmatrix},\ \Psi = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.9 \end{bmatrix}.$$

The topology structure of (4.1) is described by

$$A = \begin{bmatrix} -1 & 0 & 0.7 & 0.7 & 0 & 0.45 \\ 0 & -2 & 0 & 0.6 & 0.12 & 0.4 \\ 0.7 & 0 & -2 & 0.3 & 0.67 & 0.25 \\ 0.7 & 0.6 & 0.3 & -1.4 & 0.6 & 0.97 \\ 0 & 0.12 & 0.67 & 0.6 & -2.5 & 0 \\ 0.45 & 0.4 & 0.25 & 0.97 & 0 & -2 \end{bmatrix}$$

Apparently, $(0\ \ 0)^T \in \mathbb{R}^2$ is an equilibrium point of FCCGNNs (4.1). It is clear that, the activation function $h_k(\tau)$, $k = 1, 2$ satisfy the Lipschitz condition with $L_k = 1.5$, and the function $D(\cdot)$ and $U(\cdot)$ meets the assumptions $[\mathcal{H}_2] - [\mathcal{H}_3]$ with $\underline{d}_l = 0.05$, $\overline{d}_l = 0.5$, $u_l = 1$. Employing the MATLAB LMI contol Toolbox, it is simply to calculate $\theta = 28.758$, $\kappa = 0.5625$ and $\varsigma = 1.6009$ of Theorem 3.2 with $\gamma = 1$, and it satisfy all the conditions in proposed Theorem 3.2. Therefore, the FCCGNNs (4.1) is globally asymptotically stable from Theorem 3.2. The numerical simulations are depicted in Figures 1 and 2.
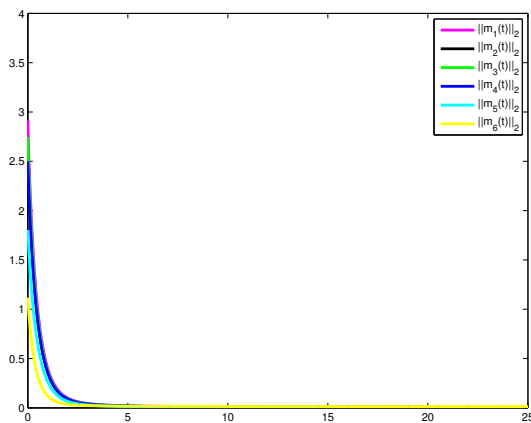
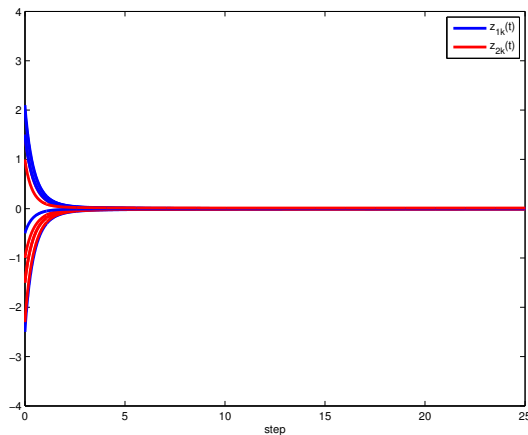**Figure 1.** The change processes of $z_{l1}(t),\ z_{l2}(t),\ l = 1, 2..., 6$.

Figure 2. The change processes of $\|m_l(t)\|_2,\ l = 1, 2..., 6$.

Example 4.2 The model of FCCGNNs with uncertainty consisting five identical nodes can be described as follows:

$$D^{0.988} z_l(t) = -D(z_l(t)) \left[U(z_l(t)) - Vh(z_l(t)) - Wh(z_l(t - 0.5)) - J \right]$$
$$+ \beta_1 \sum_{k=1}^{N} A_{lk} \Phi z_k(t) + \beta_2 \sum_{k=1}^{N} A_{lk} \Psi z_k(t - 0.5), \tag{4.2}$$

where $l = 1, 2, ..., 5$, $D(\tau) = diag\{0.9, \ 0.9\}$, $U_1(\tau) = U_2(\tau) = 14\tau$, $h_1(\tau) = h_2(\tau) = 0.08 * \tanh(\tau)$, $J = (0 \ \ 0)^T$.

The parameters β_1, β_2, V, W, Φ, Ψ, A in the networks (4.2) change in some given precision, which is intervalized as follows:

$$\beta_1^I \ := \ \left\{ 0 < 0.03 \leq \beta_1 \leq 0.3, \ \forall \ \beta_1 \in \beta_1^I \right\};$$

$$\beta_2^I \ := \ \left\{ 0 < 0.02 \leq \beta_2 \leq 0.2, \ \forall \ \beta_2 \in \beta_2^I \right\};$$

$$\Phi^I \ := \ \left\{ \Phi = diag(\phi_l) : \Phi^- \leq \Phi \leq \Phi^+, \ 0 < \frac{1}{l+1} + 0.01 \leq \phi \leq \frac{1}{l+1} + 0.4, \ l = 1, 2 \ \forall \ \Phi \in \Phi^I \right\};$$

$$\Psi^I \ := \ \left\{ \Psi = diag(\psi_l) : \Psi^- \leq \Psi \leq \Psi^+, \ 0 < \frac{1}{l+1} + 0.02 \leq \psi \leq \frac{1}{l+1} + 0.3, \ l = 1, 2 \ \forall \ \Psi \in \Psi^I \right\};$$

$$V^I \ := \ \left\{ V = (v_{lk})_{2 \times 2} : \ 0 < \frac{1}{l+k} + 0.04 \leq v_{lk} \leq 0 < \frac{1}{l+k} + 0.06, \ l, k = 1, 2, \ \forall \ V \in V^I \right\};$$

$$W^I \ := \ \left\{ W = (w_{lk})_{2 \times 2} : \ 0 < \frac{1}{l+k} + 0.02 \leq w_{lk} \leq \frac{1}{l+k} + 0.08, \ l, k = 1, 2, \ \forall \ W \in W^I \right\};$$

$$A^I \ := \ \left\{ A = (A_{lk})_{5 \times 5} : \frac{1}{2l+3k} + 0.01 \leq A_{lk} \leq \frac{1}{2l+3k} + 0.03, l \neq k, \ l, k = 1, 2, .., 5 \ \forall \ A \in A^I \right\}.$$

It is easy to obtain that $D_l(\cdot)$, $U_l(\cdot)$ and $h_k(\cdot)$ satisfy assumptions $[\mathcal{H}_1] - [\mathcal{H}_3]$ with $\underline{d}_l = 0.2$, $\overline{d}_l = 0.4$, $u_l = 7$, $L_k = 0.5$ for $l, k = 1, 2$ and $\check{z} = (0 \ \ 0)^T \in \mathbb{R}^2$ is an equilibrium point of (4.2). By simple calculation, it is easy to get $\zeta = 0.541$, $\mu = 0.223$ and $\varpi = 0.104$ with $\varepsilon_1 = 0.5$ and $\varepsilon_2 = 1$, and the conditions of Theorem 3.5 holds. Hence the FCCGNNs (4.2) achieve globally asymptotically stable from Theorem 3.5. Figures 3 and 4 demonstrate the numerical simulation results, which confirms the obtained theoretical findings.
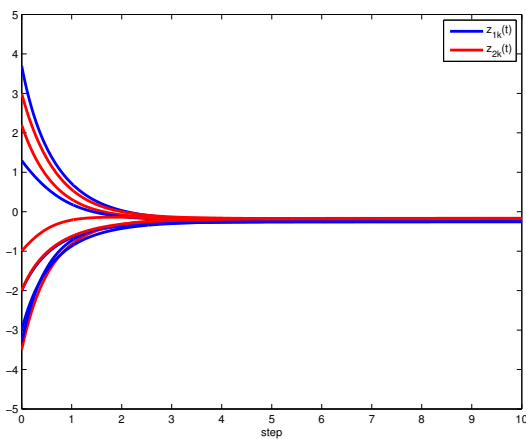

Figure 3. The change processes of $z_{l1}(t)$, $z_{l2}(t)$, $l = 1, 2..., 5$.
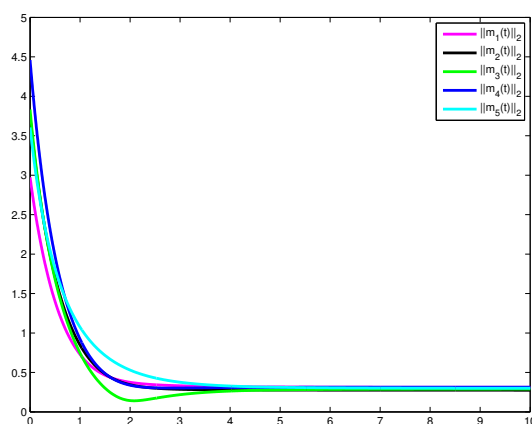

Figure 4. The change processes of $\|m_l(t)\|_2$, $l = 1, 2..., 5$.

5. Conclusions

In this paper, we have investigated the global asymptotical synchronization stability and global robust asymptotical synchronization stability topic for FCCGNNs under coupling delays. On the one

side, via Lyapunov method, LMI technique, and proposed comparison principle theorem, we have derived the several asymptotical synchronization stability results for the considered complex network without parameter uncertainties. On the other side, thanks to some inequality techniques and robust analysis skills, the author's concerns the issues of global robust asymptotical synchronization stability for the considered complex network with parameter uncertainties. In the end, we provide two computer simulations to demonstrate the validity of the proposed analytical methods. Our future work will be extended to stability, stabilization and synchronization of Riemann-Liouville sense delay-coupled fractional order memristive Cohen-Grossberg BAM neural networks with time varying discrete and distributed delays.

Acknowledgments

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