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DOUBLE-QUASI-WAVELET NUMERICAL METHOD FOR THE VARIABLE-ORDER TIME FRACTIONAL AND RIESZ SPACE FRACTIONAL REACTION–DIFFUSION EQUATION INVOLVING DERIVATIVES IN CAPUTO–FABRIZIO SENSE

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Abstract

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Our motive in this scientific contribution is to deal with nonlinear reaction–diffusion equation having both space and time variable order. The fractional derivatives which are used are non-singular having exponential kernel. These derivatives are also known as Caputo–Fabrizio derivatives. In our model, time fractional derivative is Caputo type while spatial derivative is variable-order Riesz fractional type. To approximate the variable-order time fractional derivative, we used a difference scheme based upon the Taylor series formula. While approximating the variable order spatial derivatives, we apply the quasi-wavelet-based numerical method. Here, double-quasi-wavelet numerical method is used to investigate this type of model. The discretization of boundary conditions with the help of quasi-wavelet is discussed. We have depicted the efficiency and accuracy of this method by solving the some particular cases of our model. The error tables and graphs clearly show that our method has desired accuracy.

Keywords: Fractional PDE; Diffusion Equation; Caputo–Fabrizio Fractional Derivative; Variable-Order Derivatives; Riesz Derivative; Quasi-Wavelets.

1. INTRODUCTION

Fractional calculus is a new branch of mathematics which originates from the classical one.[1] The integral and derivative in fractional calculus are obtained from the integer order by replacing integer order exponent by real or arbitrary order. In recent time, we are observing that fractional calculus, a branch of mathematics, has emerged as new applicable branch whose application can be found in many areas of physics, chemistry biology, biomathematics and medical sciences. The behavior of diffusion process of contaminants in the groundwater through the medium having pour has been investigated by taking the fractional model of diffusion equation. And the transport of contaminants in groundwater is represented by advection diffusion equation in fractional environment. The development and basics of fractional calculus can be found in the literature.[2,3] We can extend the real order to variable-order in differentiation or integration with the help of theory given in fractional calculus. There are many physical phenomena that cannot be represent by classical derivative so we need the differential equation having fractional order. These fractional differential equations have a lot of applications in control theory, biology, physics and medical science. Initially, only fractional derivative with power law kernel was investigated. These types of fractional derivative include Caputo definition, Riemann Liouville definition, Hadamard and Grunwald–Letnikov definition. The theory of fractional differential equation boosts the application and research in many fields of science and engineering. But the main difficulty was to find out the solution of these FPDEs. By analytical methods such as Laplace transform method, homotopy analysis methods and Fourier transform method, it is too difficult to solve every linear and nonlinear FPDEs. So the researchers started to find out the method to solve these equations numerically. There are many methods available in the literature such as eigen-vector expansion, fractional differential transform method,[4] Adomain decomposition method,[5] predictor–corrector method,[6] homotopy perturbation method[7] and generalized block pulse operational matrix method,[8] etc. A method named as operational matrix method is so popular due to its simplicity and good accuracy. Many numerical methods using the method based upon operational matrices of integration and differentiation with Legendre wavelets,[9] Chebyshev wavelets,[10] sine wavelets, Haar wavelets[11] are given in the literature to derive the numerical solution of fractional PDEs and integro-differential equations. Legendre polynomial,[12] Laguerre polynomial,[13] Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial[14] are commonly used polynomials in deriving the operational matrix and then numerical solution of FPDEs.

Diffusion process is a process in which the fluid flows from higher concentration to low concentration. Diffusion equation is a PDE that describes the physical behaviors of many diffusion phenomena occurring in physics, chemistry, biology and earth science. Reaction–diffusion process is process in which diffusion and reaction both are included. Many phenomena such as diffusion of pollutant in groundwater diffusion process and reaction of pollutant with groundwater are performed simultaneously. In the process of reaction molecules are either created or consumed. The classical form of diffusion equation is represented as

$$\frac{\partial\theta}{\partial t} = D\nabla^2\theta + R(\theta, t). \tag{1}$$

Here, D is diffusive coefficient and the first term on right-hand side represents the diffusion process while the term $R(\theta, t)$ represents the reaction term.

Nowadays, many different fractional operators which are generalization of classical ones are developed. The classical derivative such as Caputo and Riemann–Liouville have power kernel. If we replace this kernel with exponential kernel and Mittag-Leffler kernel, then we get a new generalized class of fractional derivative. The derivative having exponential kernel is known as Caputo–Fabrizio derivative while that having Mittag-Leffler is known as Atangana–Baleanu derivative.

In the previous, few decades fractional differential equation had a lot of application in economics, engineering and science. It is very helpful in the analysis of multi- scale problems having length and wide scale. The beauty of these fractional operator is their nonlocal property which clearly shows that future state not only is a function of the present state but also a function of previous history. It is an excellent instrument for the explanation of fractal and transport walk and characteristic features of anomalous sub-diffusion. The memory effect of many physical process can easily be shown by fractional nonlocal operators.

The characterization of memory property of a system is a tough task in modeling and analysis of complex system. The variable-order fractional derivatives properly describe the memory property of a system which vary with space and time location. We have used the variable-order C-F derivative in our work which is a non-local fractional derivative. These nonlocal derivatives can depict the material heterogeneities and structures with different scales. The classical or local fractional derivatives are not able to describe such behavior. When we study the diffusion process in porous media having variable external field or heterogeneous flow changing with time the constant order diffusion model cannot depict this phenomenon. The study of groundwater flow with variable-order derivatives is given in Ref. 15. The limitation of fixed order fractional calculus is that they cannot characterize the adequate information on complex phenomena such as complex diffusion process occurring in heterogeneous porous medium. To get rid of these difficulties occurring in modeling, the theory of variable-order operators is introduced. The application of variable-order derivatives over the fixed order to a complex system is shown in Ref. 16.

The origin of diffusion equation with Caputo–Fabrizio derivative from statistics. By using continuous time random walk, fractional diffusion equation and Riesz fractional diffusion equation can be derived. Now, we consider the probability density function (PDF) which follows the exponential Debye pattern. Let us consider the following waiting time:

$$\begin{aligned} \phi(t) &= \sigma(1 + \sigma - \delta(t))e^{-\sigma t}, \\ \sigma &= \frac{\gamma}{1-\gamma}, \quad \gamma \in [0, 1] \end{aligned} \tag{2}$$

and taking Gaussian length PDF

$$\mu(x) = (4\pi)^{\frac{-1}{2}} e^{\frac{-x^2}{4}}. \tag{3}$$

The Laplace and Fourier transform of above equations are

$$\begin{aligned} \phi(p) &= 1 - \frac{p}{\gamma + (1-\gamma)p}, \\ \mu(k) &= 1 - k^2 + O(k^4). \end{aligned} \tag{4}$$

After plugging $\phi(p)$ and $\mu(k)$ we find the following specific expression:[17]

$$\begin{aligned} u(k, p) &= \frac{1-\phi(p)}{p} \times \frac{u_0(k)}{1-\mu(k)\phi(p)} \\ &\sim \frac{u_0(k)}{p + [\gamma + (1-\gamma)p]k^2}. \end{aligned} \tag{5}$$

Taking inverse Fourier and Laplace transform we get the following diffusion equation with C-F derivative:

$${}_0^{\mathrm{CFC}}D_t^{\gamma} u(t, x) = \frac{\partial^2 u(t, x)}{\partial x^2} + f(t, x). \tag{6}$$

If we take waiting time as in Eq. (2) and heavy failed jump length PDF

$$\mu(x) = \frac{1}{|x|^{1+\alpha}}, \quad 0 < \alpha < 2, \tag{7}$$

then the same procedure which was applied to derive C-F diffusion equation leads to

$${}_0^{rmCFC}D_t^{\gamma} u(t, x) = K_\alpha \frac{\partial^\alpha u(t, x)}{\partial |x|^\alpha} + f(t, x), \tag{8}$$

where the coefficient K_α depends upon α.

We organize our paper as follows. The definitions of variable-order RL, Caputo and Caputo–Fabrizio is given in Sec. 2. We also discuss about quasi wavelet and quasi- wavelet-based numerical method. In Sec. 3, a difference scheme is developed for the discretization of variable-order time and space fractional derivatives. The error bound and convergence is given in Sec. 5. In Sec. 4, we described the proposed method for solving FPDEs with C-F derivative. In Sec. 6, some numerical examples and results are presents including the variation of different parameters. The conclusion is given in Sec. 6.

2. PRELIMINARY DEFINITIONS

In the last few years, many definitions of fractional integration and differentiation have been come to light. All of them have own special properties and applications. Caputo definition is more reliable as compared to Riemann–Liouville definition from an application point of view. These definitions are with power or singular kernel law. Nowadays, many generalized definitions of fractional derivative with exponential and Mittag–Leffler kernel law have been introduced. We discuss brief definitions and properties of RL, Caputo and recently developed Caputo–Fabrizio derivative.

2.1. Definition of Variable-Order Derivative of Type 1 (V1) and Type 2 (V2)

The definition of Type 1 (V1) variable-order fractional type derivative having order $q-1 < \theta(t,x) \leq q$ of a function $v(t,x)$ with respect to variable t is defined as[18]

$$
{}_0D_t^{\theta(t,x)}v(t,x) = \begin{cases} \dfrac{1}{\Gamma(q-\theta(x,t))} \\ \quad \times \int_0^t (t-s)^{q-\theta(x,t)-1} \frac{\partial^q v(t,x)}{\partial s^q} ds, \\ \qquad q-1 < \theta(t,x) < q, \\ \dfrac{\partial^q v(x,t)}{\partial s^q}, \quad \theta(t,x) = q. \end{cases} \tag{9}
$$

The above definition of fractional derivative is equivalent to the definition in Caputo sense if $\theta(t,x)$ is constant.

The definition of Type 2 (V2) variable-order fractional derivative having order $q-1 < \theta(x,t) \leq q$ of a function $v(x,t)$ is given by[18]

$$
{}_0D_t^{\theta(t,x)}v(t,x) = \begin{cases} \dfrac{1}{\Gamma(q-\theta(t,x))} \dfrac{\partial^q}{\partial t^q} \\ \quad \times \int_0^t (t-s)^{q-\theta(x,t)-1} v(t,x) ds, \\ \qquad q-1 < \theta(t) < q, \\ \dfrac{\partial^q v(x,t)}{\partial s^q}, \quad \theta(x,t) = q. \end{cases} \tag{10}
$$

This definition is equivalent to Riemann–Louisville definition if the order $\vartheta(x,t)$ is constant. These variable-order derivatives are all linear function as they follow the following property:

$$
{}_0D_t^{\vartheta(x,t)}(Af(x,t) + Bg(x,t)) = A_0 D_t^{\vartheta(x,t)} f(x,t) + B_0 D_t^{\vartheta(x,t)} g(x,t). \tag{11}
$$

2.2. Definition of Caputo–Fabrizio Derivative in Caputo Sense

Considering a function $g(t)$ belonging to Sobolev space $H^1(0,1)$ then Caputo–Fabrizio derivative in RL sense of order $\vartheta(x,t)$ is defined by

$$
{}_0^{\mathrm{CFC}}D_t^{\vartheta(x,t)} g(t) = \frac{B(\vartheta(x,t))}{q-\vartheta(x,t)} \times \int_0^t \exp\left[\frac{-\vartheta(x,t)}{q-\vartheta(x,t)}(t-s)\right] \times \frac{\partial^q g(s)}{\partial t^q} ds, \quad q-1 < \vartheta(x,t) \leq q, \tag{12}
$$

where $B(\vartheta(x,t))$ is a normalization function. In all our calculations we have taken $B(\vartheta(x,t)) = 1$.

2.3. Caputo–Fabrizio Derivative in Riemann–Liouville Sense

Considering a function $g(x)$ belonging to Sobolev space $H^1(0,1)$ then the definition of Caputo-Fabrizio derivative in RL sense of order $\vartheta(x,t)$ is given by[19]

$$
{}_0^{\mathrm{CFR}}D_x^{\vartheta(x,t)} g(x) = \frac{B(\vartheta(x,t))}{n-\vartheta(x,t)} \frac{d^q}{dx^q} \times \int_0^x \exp\left[\frac{-\vartheta(x,t)}{n-\vartheta(x,t)}(x-s)\right] \times g(s) ds, \quad q-1 < \vartheta(x,t) \leq q, \tag{13}
$$

where $B(\vartheta(x,t))$ is a normalization function. Here we have taken $B(\vartheta(x,t)) = 1$.

2.4. Riesz Derivative

Riesz fractional derivative of order $q-1 < \vartheta(x,t) \leq q$ is given by[20]

$$\frac{\partial^{\vartheta(x,t)}u(x,t)}{\partial|x|^{\vartheta(x)}} = \frac{-1}{2\cos\frac{\pi\vartheta(x,t)}{2}} \times \left[{}_{a}^{\mathrm{CFR}}D_{x}^{\vartheta(x,t)}u(x,t) + {}^{\mathrm{CFR}}{}_{x}D_{b}^{\vartheta(x,t)}u(x,t)\right], \qquad (14)$$

where ${}_{a}D_{x}^{\vartheta(x,t)}u(x,t)$ and ${}_{x}D_{b}^{\vartheta(x,t)}u(x,t)$ are left and right variable-order Caputo–Fabrizio derivatives in RL sense.

2.5. Why We are Using C-F Derivative?

The operators play an important role in science and the interchange of these operators, which is an important property. Let us consider two operators A and B; we say these two commute if they follow the property $AB = BA$. Many operators arising in physics, biology, statistics and mathematics do not follow the property of commutativity and are called non-commutative operators. We give some examples of non-commutative operators:

(1) Product of two matrices.
(2) Division operator on real numbers as $\frac{3}{4} \neq \frac{4}{3}$.
(3) Linear operators like z and $\frac{d}{dz}$ do not follow the commutative property on wave function $\Psi(y)$ in the case when we formulate the Schrodinger equation in quantum mechanics.
(4) Lie bracket of Lie ring.
(5) Lie bracket of a Lie algebra.

The general form of fractional type derivatives in Caputo and Riemann–Liouville form is defined as:

$$ {}_{0}^{\mathrm{RL}}D_{z}^{\vartheta}g(z) = \frac{d}{dz}\int_{0}^{z}\kappa(z-x)g(z)dz\frac{d}{dz}\kappa * g,$$

$$ {}_{0}^{C}D_{z}^{\vartheta}g(z) = \int_{0}^{z}\kappa(z-x)\frac{d}{dz}g(z)dz = \kappa * \frac{d}{dz}g.$$

In fractional calculus, many forms of kernel are discovered as $\kappa(z-x) = \frac{1}{\Gamma(1-\vartheta)}(z-x)^{-\vartheta}$ and $\kappa(z-x) = \frac{M(\vartheta)}{(1-\vartheta)}\exp\left(\frac{-\vartheta}{1-\vartheta}(z-x)^{-\vartheta}\right)$. The kernel $\kappa(z-x) = \frac{1}{\Gamma(1-\vartheta)}(z-x)^{-\vartheta}$ is known as power kernel law which has been used in classical fractional calculus and the kernel $\kappa(z-x) = \frac{M(\vartheta)}{(1-\vartheta)}\exp\left(\frac{-\vartheta}{1-\vartheta}(z-x)^{-\vartheta}\right)$ is exponential kernel law which is newly discovered. The general derivatives having exponential kernel are known as Caputo–Fabrizio derivatives. In statistics, Pareto distribution describes the fitting of the shape of a large portion of wealth for a small number of portion of the population and the wealth in our society corresponds to power law kernel. The negative exponential distribution is mainly used in statistics as probability distribution. This type of distribution is used to characterize the time between events in Poisson point distribution. The important property of this distribution is that its depicts infinite divisibility and infinite divisible distribution shows an important role in the context of limit theorem and Levy process. This type of derivatives is beneficial when distribution of a waiting time is not dependent upon elapsed time.[21] Here we give some properties of C-F derivative:

(1) The mean square displacement associated with Caputo–Fabrizio fraction derivative is usually a sub-diffusion crossover.
(2) The Caputo–Fabrizio distribution follows the rule from Gaussian to non-Gaussian crossover.
(3) The asymptotic behavior of Caputo–Fabrizio satisfies the power law behavior and connects the theory of fading memory concept with kernels which are non-singular.[22]

Nowadays the derivative with exponential kernel law has become very popular and captured the attention of researchers. This derivative has many applications which can be found in elasticity, Keller–Segel equation, flow of complex rheological medium and flow of ground water in mass-spring damped system.[23]

2.6. Approximation of Function by Quasi-Wavelets

In the literature there are many polynomials and wavelets which are used to approximate an arbitrary function. But the procedure based upon the quasi-wavelets is growing rapidly as spectral collocation method, which is local. It is very useful to solve different types of space-time fractional FPDEs and partial integro-differential equation of different order. We define a mathematical transformation known as the singular discrete convolution in distribution theory

$$\Phi(v) = (F * s)(z) = \int_{-\infty}^{\infty} F(-t+z)s(t)dt, \qquad (15)$$

where $s(t)$ is called a test function and F is recognized as a singular kernel. We can find a family of wavelet by a function which is known as mother wavelet ς using operations of dilation and translation.

$$\varsigma_{\beta,\delta}(z) = \beta^{\frac{-1}{2}}\varsigma\left(\frac{z-\delta}{\beta}\right). \tag{16}$$

The parameter δ represents the translation process while β represents the process of dilation. An orthonormal wavelet base generates any arbitrary subspace by using orthogonal scaling functions. A Shannon's delta sequence kernel is used in our work which is defined as

$$\delta_\alpha(z) = \frac{1}{\pi}\int_0^\pi \cos(zy)dy = \frac{\sin(\alpha z)}{\pi z}, \tag{17}$$

where $\lim_{\alpha\to\alpha_0}\delta_\alpha(z) = \delta(z)$. δ is discussed by Dirac and so known as Dirac delta function.

For a $\alpha > 0$, Shannon's delta sequence kernel generates a basis for the Paley–Wiener reproducing kernel Hilbert space $\mathbf{B}^2_\alpha$ (Ref. 24) which is a subspace of $\mathbf{L}^2(R)$. We can reproduce the function $g(z) \in \mathbf{B}^2_\alpha$ as follows:

$$\begin{aligned} g(z) &= \int_{-\infty}^{\infty} g(z)\delta_\alpha(z-t)dt \\ &= \int_{-\infty}^{\infty} g(z)\frac{\sin((z-t)\alpha)}{(z-t)\pi}dt, \\ &\qquad \forall g(z) \in \mathbf{B}^2_\alpha. \end{aligned} \tag{18}$$

This sampling scaling function can be put in another form in reproducing kernel of Paley–Wiener

$$\delta_{\alpha,k} = \delta_\alpha(z-z_k) = \frac{\sin((z-z_k)\alpha)}{(z-z_k)\pi}, \tag{19}$$

the point $\{x_k\}$ is known as collection of sampling points which is placed around x. We can put all functions $\forall g \in \mathbf{B}^2_\alpha$ in discrete form using Eqs. (11) and (12)

$$g(z) = \sum_{k=-\infty}^{\infty} g(z_k)\delta_\alpha(z-z_k). \tag{20}$$

The Shannon sampling theorem states that samples for uniform spatial discrete in a given band-limited signal in B^2_γ can be depicted as the sampling at the Nyquist frequency γ. We represent Δ by grid size in spatial direction and $\gamma = \frac{\pi}{\Delta}$. So,

$$\begin{aligned} g(z) &= \sum_{k=-\infty}^{\infty} g(z_k)\delta_\alpha(z-z_k) \\ &= \sum_{k=-\infty}^{\infty} g(z_k)\frac{\sin(\frac{\pi(z-z_k)}{\Delta})}{\frac{\pi(z-z_k)}{\Delta}}. \end{aligned} \tag{21}$$

A method for the improvement of kernel Dirichlet's delta type is given by Wan. $R_\sigma(y)$is a regularizer. It is used to increase the regularity

$$\delta_\alpha(z) \to \delta_{\alpha,\sigma} = \delta_\alpha(z)R_\sigma(z). \tag{22}$$

Here R_σ satisfies

$$\lim_{\sigma\to\infty} R_\sigma(z) = 1$$

and

$$\int_{-\infty}^{\infty}\lim_{\sigma\to\infty} R_\sigma(y)\delta_\alpha(y)dy = R_\sigma(0) = 1.$$

Many regularizers satisfy the two conditions as given above. But Gaussian type regularizer is very commonly used

$$R_\sigma(z) = e\left(\frac{-z^2}{2\sigma^2}\right), \quad \sigma > 0. \tag{23}$$

Here σ is width parameter. The relation between Δ and σ is $\sigma = r\times\Delta$, where r is a computation parameter. We can define regularized orthogonal sampling scaling function, which is Gaussian type, as

$$\delta_{\Delta,\sigma}(z) = \frac{\sin(\frac{\pi z}{\Delta})}{\frac{\pi z}{\Delta}}\exp\left(\frac{-z^2}{2\sigma^2}\right). \tag{24}$$

Here

$$\lim_{\sigma\to\infty}\delta_{\Delta,\sigma}(x) = \frac{\sin(\frac{\pi x}{\Delta})}{\frac{\pi x}{\Delta}}.$$

Gaussian regularized sampling scaling function has no property of orthonormal wavelet scaling function so it is called a quasi-scaling function.

By using quasi-scaling function, we can approximate a function $\theta \in \mathbf{B}^2_\alpha$

$$\begin{aligned} \theta(z) &= \sum_{k=-\infty}^{\infty} \theta(z_k)\delta_\alpha(z-z_k) \\ &= \sum_{k=-\infty}^{\infty} \theta(z_k)\delta_\alpha(z-z_k)R_\alpha(z-z_k). \end{aligned} \tag{25}$$

For computation purpose we have to take finite sampling points. The reason behind this is that in computation using infinite sampling point is impossible. We choose $2W+1$ sampling points in our work. All sampling points are chosen close to x. We can rewrite Eq. (18) as

$$\theta(z) = \sum_{k=-W}^{W} \theta(z_k)\delta_{\Delta,\sigma}(z - z_k). \tag{26}$$

The nth order derivatives of a function $\theta(z)$ is

$$\theta^n(z) = \sum_{k=-W}^{W} \theta(z_k)\delta^n_{\Delta,\sigma}(z - z_k), \quad n = 1, 2, \ldots. \tag{27}$$

We have chosen the computational width equal to $2W + 1$. We present the description of formulas of $\delta_{\Delta,\sigma}, \delta^1_{\Delta,\sigma}$ and $\delta^2_{\Delta,\sigma}$ (Ref. 25) which are helpful in calculation as follows:

$$\delta_{\Delta,\sigma}(y) = \begin{cases} \dfrac{\exp\{-\frac{y^2}{2\sigma^2}\}\sin\left(\frac{\pi y}{\Delta}\right)}{\frac{\pi y}{\Delta}}, & y \neq 0, \\ 1 \quad y = 0, \end{cases} \tag{28}$$

$$\delta^1_{\Delta,\sigma}(y) = \begin{cases} \left(-\dfrac{\sin\left(\frac{\pi y}{\Delta}\right)}{\frac{\pi y^2}{\Delta}} - \dfrac{\Delta\sin\left(\frac{\pi y}{\Delta}\right)}{\frac{\pi\sigma^2}{\Delta}} + \dfrac{\cos\left(\frac{\pi y}{\Delta}\right)}{y}\right) \\ \times\exp\left(-\dfrac{y^2}{2\sigma^2}\right), & y \neq 0, \\ 0 \quad y = 0, \end{cases} \tag{29}$$

$$\delta^2_{\Delta,\sigma}(y) = \begin{cases} \left(\dfrac{2\Delta\sin\left(\frac{y\pi}{\Delta}\right)}{\pi y^3} - \dfrac{2\cos\left(\frac{\pi y}{\Delta}\right)}{y^2} \right. \\ \quad + \dfrac{\Delta y\sin\left(\frac{y\pi}{\Delta}\right)}{\pi\sigma^4} + \dfrac{\Delta\sin\left(\frac{\pi y}{\Delta}\right)}{\pi\sigma^2 y} \\ \quad \left. - \dfrac{2\cos\left(\frac{\pi y}{\Delta}\right)}{\sigma^2} - \dfrac{\pi\sin\left(\frac{y\pi}{\Delta}\right)}{y\Delta}\right) \\ \quad \times\exp\left(-\dfrac{y^2}{2\sigma^2}\right), & y \neq 0, \\ 0, \quad y = 0. \end{cases} \tag{30}$$

3. PROPOSED ALGORITHMS

In this section, we will use the finite difference scheme and quasi-wavelet-based numerical method for our Riesz fractional reaction–diffusion model. Here time fractional derivative is of Caputo–Fabrizio type in Caputo sense and space fractional derivative is of Riesz Caputo–Fabrizio type. We consider the following model of Riesz fractional diffusion equation,

$$\begin{aligned} {}_0^{\mathrm{CFC}}D_t^{\alpha(x,t)}u(t,x) = {} & \frac{\partial^{\vartheta(x,t)}u(t,x)}{\partial|x|^{\vartheta(x,t)}} \\ & + au(t,x)(1 - u(t,x)) \\ & + f(t,x), \end{aligned} \tag{31}$$

along with boundary conditions

$$u(0,t) = f_1(t), u(1,t) = f_2(t), \tag{32}$$

and the initial condition

$$u(x,0) = f_3(x), \tag{33}$$

where $0 < \alpha(x,t) \leq 1$, $1 < \vartheta(x,t) \leq 2$ and $f(x,t)$ is the forced term.

3.1. Approximation of Time Fractional Caputo–Fabrizio Derivative

In this section, we develop a new difference scheme in the time direction of FPDE. For discretizing of Caputo–Fabrizio time fractional derivative, we use the Taylor series expansion. We denote the time step by Δt and $t_n = n \times \Delta t$. The notations u^n and f^n are used for the value of $u(x,t)$ and $f(x,t)$ at time $t_n = n \times \Delta t$.

The function $h'(t)$ can be expanded if we use the Taylor series expansion formula in interval (t_n, t_{n+1})

$$\begin{aligned} h'(t) = {} & h'(t_n) + h''(t_n)(t - t_n) \\ & + h'''(t_n)\frac{(t - t_n)^2}{2!} + O((t - t_n)^3). \end{aligned} \tag{34}$$

The following expression can be obtained by the above formula as follows:

$$\begin{aligned} h'(t_n) = {} & \frac{h(t_{n+1}) - h(t_{n-1})}{2\Delta t} \\ & - \frac{h^{(3)}(t_n)}{3!} \times (\Delta t)^2, \end{aligned} \tag{35}$$

and

$$\begin{aligned} h''(t_n) = {} & \frac{h(t_{n+1}) - 2h(t_n) + h(t_{n-1})}{(\Delta t)^2} \\ & - \frac{h^{(4)}(t_n)}{4!} \times (\Delta t)^2 + O(\Delta t)^4. \end{aligned} \tag{36}$$

When the values of $h'(t)$ and $h''(t_n)$ are used in the above equation we get the following expression:

$$h'(t) = \frac{h(t_{n+1}) - h(t_{n-1})}{2\Delta t} + \frac{h(t_{n+1}) - 2h(t_n) + h(t_{n-1})}{(\Delta t)^2}(t - t_n) - \frac{h^{(3)}(t_n)}{3!} \times (\Delta t)^2 - \frac{h^{(4)}(t_n)}{4!} \times (\Delta t)^2 + h'''(t_n)\frac{(t-t_n)^2}{2!} + O((t-t_n)^3). \quad (37)$$

The approximation of C-F derivative of function $u(x,t)$ at point $(t = t_n)$ is derived as follows:

$$\begin{aligned}
{}_0^{\mathrm{CFC}}D_t^{\alpha(x,t)}u(t_n,x) &= \frac{B(\alpha(x,t))}{\Gamma(-\alpha(x,t)+1)} \times \int_0^{t_n} \exp\left[\frac{-\alpha(x,t_n)}{1-\alpha(x,t_n)}(t_n - s)\right] \times \frac{\partial u(x,s)}{\partial s} ds \\
&= \frac{B(\alpha(x,t))}{\Gamma(-\alpha(x,t)+1)} \times \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \left(\frac{u(x,t_{j+1}) - 2u(x,t_j) + u(x,t_{j-1})}{(\Delta t)^2} \times (s - t_n) + \frac{u(x,t_{j+1}) - u(x,t_{j-1})}{2\Delta t} \right) \\
&\quad \times \exp\left[\frac{-\alpha^n}{1-\alpha^n}(t_n - s)\right] ds \\
&= \frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u^{j+1} - u^{j-1}}{2\Delta t} \times \int_{t_j}^{t_{j+1}} \exp\left[\frac{-\alpha^n}{1-\alpha^n}(t_n - s)\right] ds \\
&\quad + \sum_{j=0}^{n-1} \frac{u(x,t_{j+1}) - 2u(x,t_j) + u(x,t_{j-1})}{(\Delta t)^2} \times \int_{t_j}^{t_{j+1}} (s - t_n) \exp\left[\frac{-\alpha^n}{1-\alpha^n}(t_n - s)\right] ds \\
&= \frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u^{j+1} - u^{j-1}}{2\Delta t} \times \left[\frac{\alpha^n - 1}{\alpha^n}\left(\exp\left[\frac{(t_n - t_j)(-\alpha^n)}{-1+\alpha^n}\right] - \exp\left[\frac{(t_n - t_{j+1})(-\alpha^n)}{\alpha^n - 1}\right]\right)\right] \\
&\quad + \frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u^{j+1} - 2u^j + u^{j-1}}{(\Delta t)^2} \times \int_{t_j}^{t_{j+1}} (s - t_n) \exp\left[\frac{-\alpha^n}{1-\alpha^n}(t_n - s)\right] ds.
\end{aligned}$$

Now on simplifying

$$\begin{aligned}
{}_0^{\mathrm{CFC}}D_t^{\alpha(x,t)}u(t_n,x) &= \frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u^{j+1} - u^{j-1}}{2\Delta t} \left[\frac{\alpha^n - 1}{\alpha^n} \times \left(\exp\left[\frac{(t_n - t_j)(\alpha^n)}{-1+\alpha^n}\right] - \exp\left[\frac{(t_n - t_{j+1})(\alpha^n)}{\alpha^n - 1}\right]\right)\right] + \frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \\
&\quad \times \sum_{j=0}^{n-1} \frac{u^{j+1} - 2u^j + u^{j-1}}{(\Delta t)^2} \left[\frac{\alpha^n - 1}{(\alpha^n)^2} \times \left[(-1 + \alpha^n(1 - t_j - t_n)) \exp\left[\frac{(t_n - t_j)(\alpha^n)}{-1+\alpha^n}\right] + (1 + \alpha^n(-1 - t_{j+1} + t_n)) \exp\left[\frac{(t_n - t_{j+1})(\alpha^n)}{\alpha^n - 1}\right]\right]\right], \quad (38)
\end{aligned}$$

a semi-discrete form of our proposed model can be found by using Eq. (38)

$$\begin{aligned}
&\frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u^{j+1} - u^{j-1}}{2\Delta t} \times \left[\frac{\alpha^n - 1}{\alpha^n}\left(\exp\left[\frac{(t_n - t_j)(\alpha^n)}{-1+\alpha^n}\right] - \exp\left[\frac{(t_n - t_{j+1})(\alpha^n)}{\alpha^n - 1}\right]\right)\right] \\
&\quad + \frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u^{j+1} - 2u^j + u^{j-1}}{(\Delta t)^2}
\end{aligned}$$

$$\times\left[\frac{\alpha^n-1}{(\alpha^n)^2}\left[(-1+\alpha^n(1-t_j-t_n))\right.\right.$$
$$\times\exp\left[\frac{(t_n-t_j)(\alpha^n)}{-1+\alpha^n}\right]+(1+\alpha^n(-1-t_{j+1}+t_n))$$
$$\left.\left.\times\exp\left[\frac{(t_n-t_{j+1})(\alpha^n)}{\alpha^n-1}\right]\right]\right]$$
$$=\frac{\partial^{\vartheta(x,t)}u^n(x)}{\partial|x|^{\vartheta(x,t)}}+au^n(x)(1-u^n(x))+f^n. \quad (39)$$

3.2. Discretize the Equation for Spatial Points With the Help of Quasi-Wavelet

We have discussed about the quasi-wavelet-based numerical method in Sec. 2. Now we use it to deal with spatial derivatives and unknown function. We take $\Delta x=\frac{1}{N}$ as spatial step. We assume u_i^n is the approximation of unknown function $u(t,x)$ at point $x=x_i$ and $t=t_j$. In approximation of derivative and unknown function we take 2W neighboring points around. The mth order derivative $u_x^{(m)}(x_i)$ of a function $u(x)$ at the point x_i is approximated by

$$u^m(x_i)=\sum_{s=i-W}^{i+W}u(x_s,t_n)\delta^m_{\Delta,\sigma}(x_i-x_s),\quad n=0,1,2,\ldots,\quad i=0,1,2,\ldots,N-1. \quad (40)$$

Now for the approximation of the term $\frac{\partial^{\vartheta(x,t)}u^n(x)}{\partial|x|^{\vartheta(x,t)}}$, we use the definition of Riesz fractional derivative

$$\frac{\partial^{\vartheta(x,t)}u^n(x)}{\partial|x|^{\vartheta(x,t)}}=\frac{-1}{2\cos\frac{\pi\vartheta\vartheta(x,t)}{2}}\times\left[{}_0D_x^{\vartheta(x,t)}u^n(x)+{}_xD_1^{\vartheta(x,t)}u^n(x)\right]$$
$$=\frac{c(x)}{\Gamma(2-\vartheta^n(x))}\times\left[\frac{d^2}{dx^2}\int_0^x\exp\left[\frac{-\alpha^n}{1-\alpha^n}(x-\eta)\right]u(\eta,t)d\eta+\frac{d^2}{dx^2}\int_x^1\exp\left[\frac{-\alpha^n}{1-\alpha^n}(\eta-x)\right]u(\eta,t)d\eta\right], \quad (41)$$

where $c(x)=\frac{-1}{2\cos\frac{\pi\vartheta\vartheta^{n+\frac{1}{2}}(x)}{2}}$.

For Eq. (41) we assume

$$I_1=\frac{c(x)}{\Gamma(2-\vartheta^n(x))}\frac{d^2}{dx^2}\int_0^x\times\exp\left[\frac{-\alpha^n}{1-\alpha^n}(x-\eta)\right]u(\eta,t)d\eta,$$
$$I_2=\frac{c(x)}{\Gamma(2-\vartheta^n(x))}\frac{d^2}{dx^2}\int_x^1\times\exp\left[\frac{-\alpha^n}{1-\alpha^n}(\eta-x)\right]u(\eta,t)d\eta. \quad (42)$$

Then by using Eq. (40) and considering the value of Eq. (42) at point $x=x_i$, let $x_j=x_s-x_i$ for all $n\geq 0$, $i=0,1,\ldots,N-1$ we have

$$I_1=\frac{c_i^n}{\Gamma(2-\vartheta_i^n)}\sum_{s=i-W}^{i+W}\delta^2_{\Delta,\sigma}(x_i-x_s)\times\int_0^{x_s}\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_s-\eta)\right]u(\eta,t)d\eta$$
$$=\frac{c_i^n}{\Gamma(2-\vartheta_i^n)}\sum_{j=-W}^{W}\delta^2_{\Delta,\sigma}(-j\Delta x)\times\int_0^{x_{i+j}}\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-\eta)\right]u(\eta,t)d\eta$$
$$=,\frac{c_i^n}{\Gamma(2-\vartheta_i^n)}\sum_{j=-W}^{W}\delta^2_{\Delta,\sigma}(-j\Delta x)\times\sum_{l=0}^{i+j-1}\int_{x_l}^{x_{l+1}}\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-\eta)\right]u(\eta,t)d\eta. \quad (43)$$

Let $u(\eta)=\sum_{q=-W_1}^{W_1}\delta_{\Delta,\sigma}(\eta-x_q)u(x_q)$ in Eq. (43) we obtain

$$I_1=\frac{c_i^n}{\Gamma(2-\vartheta_i^n)}\sum_{j=-W}^{W}\delta^2_{\Delta,\sigma}(-j\Delta x)\times\sum_{l=0}^{i+j-1}\int_{x_l}^{x_{l+1}}\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-\eta)\right]\times\sum_{q=-W_1}^{W_1}\delta_{\Delta,\sigma}(\eta-x_q)u^n(x_q)d\eta$$

$$= \frac{c_i^n}{\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta_{\Delta,\sigma}^2(-j\Delta x) \sum_{l=0}^{i+j-1} \sum_{q=-W_1}^{W_1} u^n(x_q) \times \int_{x_l}^{x_{l+1}} \exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-\eta)\right] \times \delta_{\Delta,\sigma}(\eta - x_q) d\eta. \quad (44)$$

Integral term in Eq. (45) is approximated using Simpson $\frac{1}{3}$ rule. After evaluate the integration and using property of gamma function we attain the full discrete form of I_1 for $n \geq 0$, $i = 1, \ldots, N-1$

$$I_1 = \frac{c_i^n \Delta x}{6\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta_{\Delta,\sigma}^2(-j\Delta x) \sum_{l=0}^{i+j-1} \sum_{q=-W_1}^{W_1} u^n(x_q) \times \left[\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-x_l)\right] \times \delta_{\Delta,\sigma}(x_l - x_q) + 4\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-x_{l+\frac{1}{2}})\right] \times \delta_{\Delta,\sigma}(x_{l+\frac{1}{2}} - x_q) + \exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-x_{l+1})\right] \times \delta_{\Delta,\sigma}(x_{l+1} - x_q)\right] + \frac{c_i^n \Delta x}{\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta_{\Delta,\sigma}^2(-j\Delta x) \sum_{q=-W_1}^{W_1} u^n(x_q) \times \left[\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-x_{i+j-1})\right] \times \delta_{\Delta,\sigma}(x_{i+j-1} - x_q)\right]. \quad (45)$$

Similarly, full discrete form of I_2 can be obtained as

$$I_2 = \frac{c_i^n \Delta x}{6\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta_{\Delta,\sigma}^2(-j\Delta x) \times \sum_{p=i+j+1}^{M-1} \sum_{q=-W_1}^{W_1} u^n(x_q) \times \left[\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_p - x_{i+j})\right] \times \delta_{\Delta,\sigma}(x_p - x_q) + 4\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{p+\frac{1}{2}} - x_{i+j})\right] \times \delta_{\Delta,\sigma}(x_{p+\frac{1}{2}} - x_q) + \exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{p+1} - x_{i+j})\right] \times \delta_{\Delta,\sigma}(x_{p+1} - x_q)\right] + \frac{c_i^n \Delta x}{\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta_{\Delta,\sigma}^2(-j\Delta x) \times \sum_{q=-W_1}^{W_1} u^n(x_q)[\delta_{\Delta,\sigma}(x_p - x_q)]. \quad (46)$$

By substituting the value of I_1 and I_2 we attain the full discrete form of Eq. (31)

$$\frac{B(\alpha^n)}{\Gamma(-\alpha^n+1)} \sum_{j=0}^{n-1} \frac{u_i^{j+1} - u_i^{j-1}}{2\Delta t} \times \left[\frac{\alpha^n - 1}{\alpha_i^n}\left(\exp\left[\frac{(t_n - t_j)(\alpha_i^n)}{-1+\alpha_i^n}\right] - \exp\left[\frac{(t_n - t_{j+1})(\alpha_i^n)}{\alpha_i^n - 1}\right]\right)\right] + \frac{B(\alpha_i^n)}{\Gamma(-\alpha_i^n+1)} \sum_{j=0}^{n-1} \frac{u_i^{j+1} - 2u_i^j + u_i^{j-1}}{(\Delta t)^2} \times \left[\frac{\alpha_i^n - 1}{(\alpha_i^n)^2}\left[(-1+\alpha_i^n(1 - t_j - t_n)) \times \exp\left[\frac{(t_n - t_j)(\alpha_i^n)}{-1+\alpha +_i^n}\right] + (1+\alpha_i^n(-1 - t_{j+1} + t_n)) \exp\left[\frac{(t_n - t_{j+1})(\alpha_i^n)}{\alpha^n - 1}\right]\right]\right]$$

$$= \frac{c_i^n \Delta x}{6\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta_{\Delta,\sigma}^2(-j\Delta x) \times \sum_{l=0}^{i+j-1} \sum_{q=-W_1}^{W_1} u_q^n \times \left[\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-x_l)\right] \times \delta_{\Delta,\sigma}(x_l - x_q) + 4\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j}-x_{l+\frac{1}{2}})\right]\right.$$

$$
\begin{aligned}
&\times \delta_{\Delta,\sigma}(x_{l+\frac{1}{2}} - x_q) \\
&+ \exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j} - x_{l+1})\right] \\
&\left.\times \delta_{\Delta,\sigma}(x_{l+1} - x_q)\right] \\
&+ \frac{c_i^n \Delta x}{\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta^2_{\Delta,\sigma}(-j\Delta x) \sum_{q=-W_1}^{W_1} u_q^n \\
&\times \left[\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{i+j} - x_{i+j-1})\right]\right. \\
&\left.\times \delta_{\Delta,\sigma}(x_{i+j-1} - x_q)\right] \\
&\times \sum_{p=i+j+1}^{M-1} \sum_{q=-W_1}^{W_1} u^n(x_q) \left[\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_p - x_{i+j})\right]\right. \\
&\times \delta_{\Delta,\sigma}(x_p - x_q) \\
&+ 4\exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{p+\frac{1}{2}} - x_{i+j})\right] \\
&\times \delta_{\Delta,\sigma}(x_{p+\frac{1}{2}} - x_q) \\
&+ \exp\left[\frac{-\alpha_i^n}{1-\alpha_i^n}(x_{p+1} - x_{i+j})\right] \\
&\left.\times \delta_{\Delta,\sigma}(x_{p+1} - x_q)\right] \\
&+ \frac{c_i^n \Delta x}{\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta^2_{\Delta,\sigma}(-j\Delta x) \\
&\sum_{q=-W_1}^{W_1} u_q^n \times \left[\delta_{\Delta,\sigma}(x_p - x_q)\right] \\
&+ f_i^n + a \times \sum_{j=-W}^{W} \delta^2_{\Delta,\sigma}(-j\Delta x) u_{i+j}^n \\
&\times \left(1 - \sum_{j=-W}^{W} \delta^2_{\Delta,\sigma}(-j\Delta x) u_{i+j}^n\right).
\end{aligned}
$$

This equation gives the full discrete form of variable-order reaction–diffusion equation model.

As $u(x_k)$ is not defined beyond the domain $[0,1]$, so to discretize the boundary conditions, we assume that

$$
u_i^n = \begin{cases} u_0^n, & i < 0, \\ u_N^n, & i > N. \end{cases} \tag{47}
$$

The initial condition can be discretized in the following simple way as follows:

$$
u_i^0 = f_3(x_i), \quad i = 0, 1, \ldots, M. \tag{48}
$$

4. ERROR BOUND AND CONVERGENCE

In this section, we will derive the convergence order for the C-F fractional derivative.

Theorem 1. *Let* $0 < \alpha(x,t) \le 1$, $\varrho(x,t) = \frac{\alpha(x,t)}{1-\alpha(x,t)}$ *and* $u(x,t)$ *be smooth function for* $t \in [0,\infty)$. *Then*

$$
+ \frac{c_i^n \Delta x}{6\Gamma(2-\vartheta_i^n)} \sum_{j=-W}^{W} \delta^2_{\Delta,\sigma}(-j\Delta x)
$$

$$
\begin{aligned}
{}^{\mathrm{CFC}}_{0}D_t^{\alpha(x,t)} u(t_n, x_i) &= \frac{1}{1-\alpha(x,t)} \sum_{j=1}^{n} \frac{u_i^j - u_i^{j-1}}{\varrho_i^j \times \Delta t} \\
&\quad e^{-\varrho(n-j)\Delta t}(1 - e^{-\varrho\Delta t}) + O((\Delta t)^2).
\end{aligned} \tag{49}
$$

Proof. By the definition of Caputo–Fabrizio derivative in the Caputo sense

$$
\begin{aligned}
&{}^{\mathrm{CFC}}_{0}D_t^{\alpha(x,t)} u(t_n, x_i) \\
&= \frac{1}{1-\alpha(x_i,t_n)} \int_0^{t_n} \frac{\partial u(x,s)}{\partial s} e^{-\varrho(t_n - s)} ds \\
&= \frac{1}{1-\alpha_i^n} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{\partial u(x,s)}{\partial s} e^{-\varrho(t_n - s)} ds \\
&= \frac{1}{1-\alpha_i^n} \sum_{j=1}^{n} \frac{u_i^n - u_i^{n-1}}{\Delta t} \\
&\quad \int_{t_{j-1}}^{t_j} e^{-\varrho(t_n - s)} ds + R^n(\Delta t) \\
&= \frac{1}{1-\alpha_i^n} \sum_{j=1}^{n} \frac{u_i^n - u_i^{n-1}}{\Delta t} \\
&\quad \times (1 - e^{-\varrho\Delta t}) e^{-\varrho(n-j)\Delta t} + R^n(\Delta t).
\end{aligned} \tag{50}
$$

The term $R^n(\Delta t)$ can be written as

$$
\begin{aligned}
&R^n(\Delta t) \\
&= -\frac{1}{1-\alpha_i^n} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{\partial^2 u(x,s)}{\partial s^2} \frac{(t_j + t_{j-1} - 2s)}{2} \\
&\quad \times e^{-\varrho(t_n - s)} ds - \frac{1}{1-\alpha_i^n} \\
&\quad \times \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{(t_j - s)^3 u'''(\varsigma_{1j}) - (t_{j-1} - s)^3 u'''(\varsigma_{2j})}{6\Delta t} \\
&\quad \times e^{-\varrho(t_n - s)} ds \\
&= -S_1 - S_2
\end{aligned}
$$

with $\varsigma_{1j}, \varsigma_{2j} \in (t_{j-1}, t_j)$. Now computing the first part S_1 of Eq. (53)

$$S_1 = \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\frac{\partial^2 u(x_i,s)}{\partial s^2} \times \frac{(t_j+t_{j-1}-2s)}{2}e^{-\varrho(t_n-s)}ds$$
$$= \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_{j-\frac{1}{2}}} u''(x_i,s) \times \frac{(t_j+t_{j-1}-2s)}{2}e^{-\varrho(t_n-s)}ds + \frac{1}{1-\alpha_i^n} \times \sum_{j=1}^{n}\int_{t_{j-\frac{1}{2}}}^{t_j} u''(x_i,s)\frac{(t_j+t_{j-1}-2s)}{2} e^{-\varrho(t_n-s)}ds$$
$$= \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\frac{u''(x_i,\zeta_{1j})}{2} \times \int_{t_{j-1}}^{t_{j-\frac{1}{2}}}\frac{(t_j+t_{j-1}-2s)}{2}e^{-\varrho(t_n-s)}ds + \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\frac{u''(x_i,\zeta_{2j})}{2} \times \int_{t_{j-\frac{1}{2}}}^{t_j}\frac{(t_j+t_{j-1}-2s)}{2}e^{-\varrho(t_n-s)}ds$$
$$= \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\frac{u''(x_i,\zeta_{1j})}{2}\left(\frac{-\Delta t}{\varrho}(1+e^{-\varrho\Delta t}) \times + \frac{2}{\varrho^2}(1-e^{-\varrho\Delta t})\right)e^{-\varrho(n-j)\Delta t} + \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\frac{u''(x_i,\zeta_{2j})}{2} \times \left(\frac{-\Delta t}{\varrho}+\frac{2}{\varrho^2}(1-e^{-\frac{1}{2}\varrho\Delta t})\right) \times e^{-\varrho(n-j)\Delta t}, \quad \zeta_{1j},\zeta_{2j}\in(t_{j-1},t_j)$$
$$= \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\frac{u''(x_i,\zeta_{1j})}{2} \times \left(\frac{-\varrho}{6}\Delta t^3 + O((\Delta t)^4)\right)e^{-\varrho(n-j)\Delta t} + \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}(\zeta_{1j}-\zeta_{2j}) \times \frac{u'''(x_i,\zeta_{3j})}{2}\left(\frac{-1}{4}(\Delta t)^2 + O((\Delta t)^3)\right) \times e^{-\varrho(n-j)\Delta t}, \quad \zeta_{1j},\zeta_{2j},\zeta_{3j}\in(t_{j-1},t_j)$$
$$= O\Big((\Delta t)^2\Big). \tag{51}$$

Now we are computing S_2

$$S_2 = \frac{1}{1-\alpha_i^n}\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j} \times \frac{(t_j-s)^3u'''(\varsigma_{1j})-(t_{j-1}-s)^3u'''(\varsigma_{2j})}{6\Delta t} \times e^{-\varrho(t_n-s)}ds$$
$$= \frac{1}{(1-\alpha_i^n)\times 6\Delta t}\sum_{j=1}^{n}u'''(\varsigma_{1j}) \times \int_{t_{j-1}}^{t_j}(t_j-s)^3 e^{-\varrho(t_n-s)}ds - \frac{1}{(1-\alpha_i^n)\times 6\Delta t}\sum_{j=1}^{n}u'''(\varsigma_{2j}) \times \int_{t_{j-1}}^{t_j}(t_{j-1}-s)^3 e^{-\varrho(t_n-s)}ds$$
$$= \frac{1}{(1-\alpha_i^n)\times 6\Delta t}\sum_{j=1}^{n}u'''(\varsigma_{1j})\left[\frac{6}{\varrho^4}(e^{\varrho\Delta t}-1) - \frac{6\Delta t}{\varrho^3} - \frac{3(\Delta t)^2}{\varrho^2} - \frac{(\Delta t)^3}{\varrho}\right]e^{-\varrho(n-j+1)\Delta t} - \frac{1}{(1-\alpha_i^n)\times 6\Delta t}\sum_{j=1}^{n}u'''(\varsigma_{2j})\left[\frac{6}{\varrho^4}(1-e^{-\varrho\Delta t}) - \frac{6\Delta t}{\varrho^3} + \frac{3(\Delta t)^2}{\varrho^2} - \frac{(\Delta t)^3}{\varrho}\right]e^{-\varrho(n-j)\Delta t}$$
$$= e^{-\varrho(n-j+1)\Delta t}\Big(\frac{1}{4}((\Delta t)^4) + O((\Delta t)^5)\Big) + e^{-\varrho(n-j)\Delta t}\Big(\frac{-1}{4}((\Delta t)^4) + O((\Delta t)^5)\Big). \tag{52}$$

Equations (54) and (55) give

$$R^n(\Delta t) \le |S_1| + |S_2| = O\Big((\Delta t)^2\Big). \tag{53}$$

Theorem 2 (Ref. 26). *An approximate relation between the function $f(x)$ and its spatial derivative approximation by quasi wavelet spatial is given by where function is band-limited to $\sigma = \Delta r$, B, $W \in$*

N, $s \in Z^+$, *and* $W \geq \frac{rs}{\sqrt{2}}$

$$\|f^s - \sum_{k=-W}^{W} \delta^s_{\sigma,\Delta}(x - x_k) f(x_k)\| \leq \beta \times \exp\left(\frac{-\gamma^2}{2r^2}\right), \tag{54}$$

where

$$f(x) \in L_\infty \cap L_2(\Omega) \cap C^s(\Omega), \gamma = \min(r^2(\pi - B\Delta), W), \quad \beta = (\sqrt{2B}\|f\|_{L_s(\Omega)} + 2r\|f\|_{L_\infty(\Omega)}) \times \frac{e^\pi (s+1)! r}{\gamma\pi\Delta^s}. \tag{55}$$

5. RESULTS AND DISCUSSION

The performances of our method are shown by solving some numerical examples. We have used Wolfram Mathematica version-11.3. in all numerical computations.

Example 1. If we take $\alpha(x,t) = 1$, $0 \leq x \leq 1$, $0 \leq t \leq 1$ and $\vartheta(x,t) = 2\sin t$, our model reduced to the following equation:

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^{2\sin t} u(t,x)}{\partial |x|^{2\sin t}} + u(t,x)(1 - u(t,x)) + f(t,x). \tag{56}$$

The initial and boundary conditions are taken as follows:

$$u(x,0) = \sin 2\pi x, \quad u(0,t) = 0, \quad u(1,t) = 0. \tag{57}$$

The exact solution of above problem can be taken as $u(x,t) = \frac{\sin(2\pi x)}{\sqrt{t+1}}$.

The plots of numerical and taken exact solution at different times are shown in Figs. 1 and 2. The accuracy of our method can also be seen in Table 1.

Example 2. Considering the following example with $0 \leq x \leq 1$ and $0 \leq t \leq 1$:

$${}_0^{\text{CFC}}D_t^{xt} u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2} + u(1-u) + f(x,t), \tag{58}$$

which, under the prescribed initial and boundary conditions, yields

$$u(0,x) = (1-x)^2 x^2, \quad u(t,0) = 0, \quad u(t,1) = 0. \tag{59}$$

Table 1 **Absolute Error for Different Value of x.**

$x \downarrow$	Absolute Error $t = 0.0001$	Absolute Error $t = 0.1$
$\frac{1}{20}$	5.5×10^{-11}	3.7×10^{-9}
$\frac{3}{20}$	2.7×10^{-10}	7.5×10^{-9}
$\frac{6}{20}$	2.3×10^{-10}	2.5×10^{-10}
$\frac{9}{20}$	1.0×10^{-14}	9.4×10^{-10}
$\frac{12}{20}$	2.3×10^{-10}	1.7×10^{-9}
$\frac{15}{20}$	2.7×10^{-10}	5.6×10^{-9}
$\frac{18}{20}$	5.5×10^{-11}	1.7×10^{-9}

The exact solution of the above problem is given by $u(x,t) = (1-x)^2 x^2 e^t$ with suitable force function $f(x,t)$.

We draw the graph of numerical and exact solution with $M = 20, \Delta t = 0.00001$ and $N = 10$ is depicted by Fig. 3. We also plot the graph between numerical and exact solution for a wide range of time ($t = 0.1$) as shown in Fig. 4. Table 2 represents the variations of absolute error. These results prove the accuracy and validity of our method.

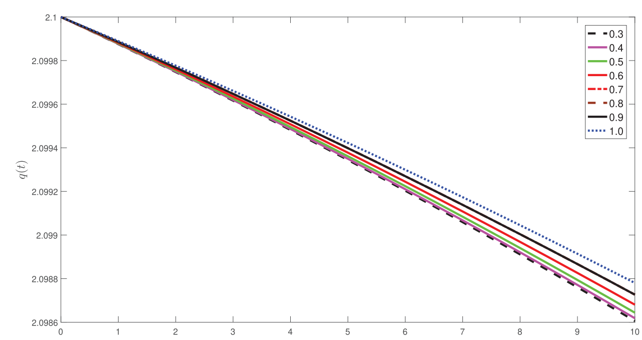

Fig. 1 Plots of $u(t,x)$ versus x for $W = 20$, $\vartheta(x,t) = 2\sin t$, $M = 20$, $r = 3.2$ and $\Delta t = 0.00001$.

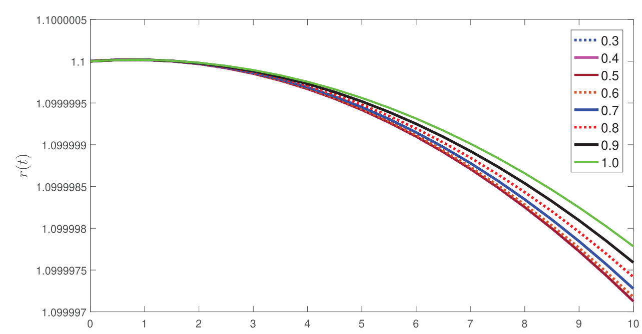

Fig. 2 Plots of $u(x,t)$ for $M = 20$, $\vartheta(x,t) = 2\sin t$, $W = 20$, $t = 0.1$ and $r = 3.2$ in case of numerical and exact solution.

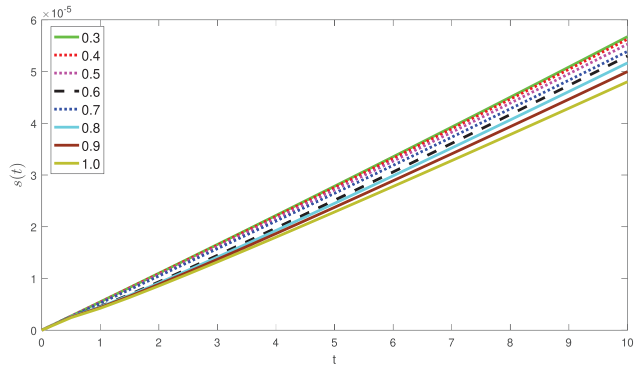

Fig. 3 Behavior of $u(t,x)$ in both numerical and exact solution with parameters $\alpha(x,t) = xt$, $\Delta t = 0.00001$, $M = 20$, $r = 3.2$ and $W = 20$.

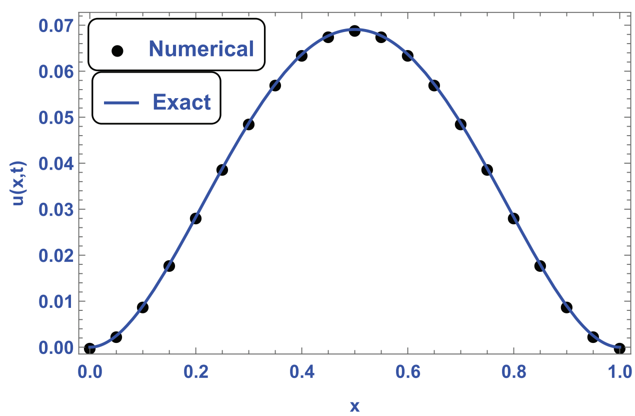

Fig. 4 Plots of $u(t,x)$ for $t = 0.1$, $M = 20$, $\alpha(t,x) = xt$, $r = 3.2$ and $W = 20$ in case of exact and numerical solution.

Table 2 Absolute Error for Different Value of x.

$x \downarrow$	Absolute Error $t = 0.0001$	Absolute Error $t = 0.1$
$\frac{1}{20}$	1.7×10^{-5}	5.6×10^{-5}
$\frac{3}{20}$	1.51×10^{-4}	2.7×10^{-4}
$\frac{6}{20}$	1.55×10^{-4}	6.7×10^{-4}
$\frac{9}{20}$	1.0×10^{-4}	3.6×10^{-4}
$\frac{12}{20}$	1.6×10^{-4}	4.9×10^{-4}
$\frac{15}{20}$	1.7×10^{-4}	9.3×10^{-4}
$\frac{18}{20}$	1.1×10^{-4}	4.2×10^{-4}

Example 3. We consider the time fractional variable-order Riesz space fractional reaction–diffusion equation with $\alpha(x,t) = x$, $0 \leq x \leq 1$, $0 \leq t \leq 1$ and $\vartheta(x,t) = 2t$,

$$ {}_{0}^{\mathrm{CFC}}D_t^x u(t,x) = \frac{\partial^{2t} u(t,x)}{\partial |x|^{2t}} + \quad u(t,x)(1 - u(t,x)) + f(t,x). \tag{60} $$

Equation (63) with initial and boundary conditions

$$ u(x,0) = x^2 \sin \pi x, \;\; u(0,t) = 0, \;\; u(1,t) = e^t \sin \pi \tag{61} $$

gives the exact solution $u(x,t) = e^t x^2 \sin \pi x$ with $f(x,t)$ as force function.

Figures 5 and 6 depict the exact and numerical solution at $\Delta t = 0.00001$ and wide time range $t = 0.1$. The variations of absolute error are shown in Table 3.

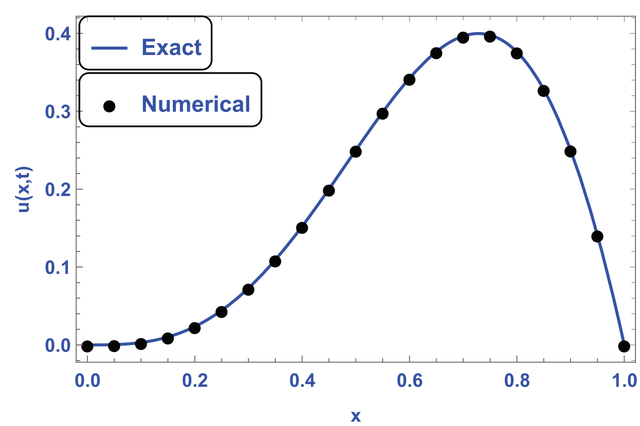

Fig. 5 Behavior of $u(t,x)$ in both numerical and exact solution with parameters $\alpha(x,t) = xt$, $\Delta t = 0.00001$, $M = 20$, $r = 3.2$ and $W = 20$.

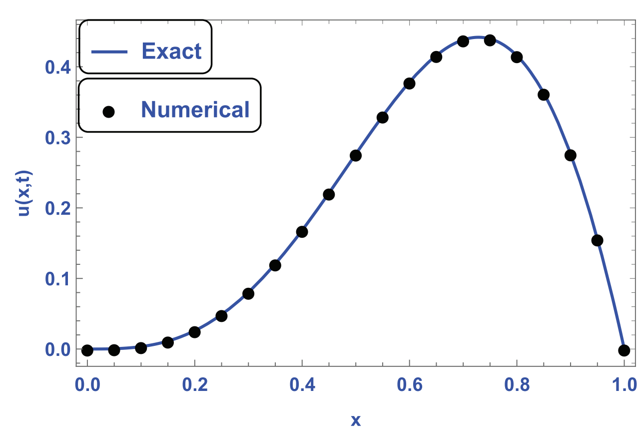

Fig. 6 Behavior of $u(t,x)$ in both numerical and exact solution with parameters $\alpha(x,t) = xt$, $t = 0.1$, $M = 20$, $r = 3.2$ and $W = 20$.

Table 3 Absolute Error for Different Value of x

$x \downarrow$	Absolute Error $t = 0.0001$	Absolute Error $t = 0.1$
$\frac{1}{20}$	3.9×10^{-8}	4.6×10^{-8}
$\frac{3}{20}$	1.4×10^{-7}	3.4×10^{-7}
$\frac{6}{20}$	2.04×10^{-7}	9.4×10^{-7}
$\frac{9}{20}$	1.7×10^{-7}	6.4×10^{-7}
$\frac{12}{20}$	2.1×10^{-7}	1.2×10^{-7}
$\frac{15}{20}$	7.2×10^{-7}	9.7×10^{-7}
$\frac{18}{20}$	1.4×10^{-5}	4.5×10^{-5}

6. CONCLUSION

In this research, we have dealt with a new model of reaction–diffusion equation having variable-order space and time in fractional order derivative with a law named exponential kernel. To solve this type of model, we derived a difference scheme by using Taylor series formula for the discretization of time direction derivative. The discretization of Riesz spatial derivatives and unknown functions is done by using double quasi wavelet method. In the knowledge of authors, double-quasi-wavelet method along with variable-order CFC and Riesz CFR derivative has been presented for the first time. The accuracy and validity of this double-quasi-wavelet method are shown by error tables and graphs.

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