# LYAPUNOV STABILITY THEORY WITH SOME APPLICATIONS

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# LYAPUNOV STABILITY THEORY WITH SOME APPLICATIONS

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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## ABSTRACT

## LYAPUNOV STABILITY THEORY WITH SOME APPLICATIONS

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In this thesis, a detailed overview of Lyapunov stability theorems of linear and nonlinear systems is presented. The Lyapunov first and second methods are investigated and the stability analysis of fractional differential systems is highlighted. A new Lemma for the Caputo fractional derivative is reviewed and a class of fractional-order gene regulatory networks is investigated. Besides the stabilization of continuous-time fractional for positive linear systems is reviewed. An elementary Lemma which estimates the fractional derivatives of Volterra-type Lyapunov functions is also put forward, in order to see how it can satisfy the uniform asymptotic stability of Caputo-type epidemic systems.

**Keywords:** Lyapunov stability, linear and nonlinear systems, Lyapunov function, Lyapunov equation, Riemann-Liouville derivative, Caputo derivative, Mittag-Leffler function.

# ÖZ

## BAZI UYGULAMALARIYLA LYAPUNOV KARARLILIK TEORİSİ

BAGELANY, Ronak
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Bu tezde, doğrusal ve doğrusal olmayan sistemlerin Lyapunov kararlılık teoremleri detaylı bir şekilde gözden gecirilmistir. Birinci ve ikinci Lyapunov metodları incelenmis ve kesirli türevli sistemler için kararlılık analizi vurgulanmıştır. Caputo kesirli türevi için yeni bir Lemma gözden geçirilmiş ve kesirli dereceli gen düzenleyici ağların bir sınıfı incelenmiştir. Ayrıca pozitif doğrusal sistemler için sürekli zaman kesirlerin stabilizasyonu gözden geçirilmiştir. Caputo tipi epidemik sistemlerin düzgün asimptotik kararlılığı nasıl sağladığını görmek için, Volterra-tipi Lyapunov fonksiyonların kesirli türevlerini kestiren bir temel Lemma gözden geçirilmiştir.

**Anahtar Kelimeler:** Lyapunov kararlılık, doğrusal ve doğrusal olmayan sistemler, Lyapunov fonksiyonu, Lyapunov denklemi, Riemann-Liouville türevi, Caputo türevi, Mittag-Leffler fonksiyonu.

## DEDICATION

**To the soul of my mother, that I'm missing her**

**To the soul of my sister, who has been credited to my joining to the school.**

**To my dear father, who did not make me feel for my mother's absence.**

**To my brothers and sisters, who supported and encouraged me.**

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My God bestows health and happiness to all of them.

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## LIST OF SYMBOLS

| | |
|---|---|
| $N$ | The set of the natural numbers $\{1,2,3,\dots\}$ |
| $R$ | The set of real numbers |
| $R^n$ | The vector space of the real vectors of length n |
| $R_+ = [0,+\infty) \subset R$ | The set of all non-negative numbers |
| $R \times R^n$ | The Cartesian product of $R$ and $R^n$ |
| $R^{n\times m}$ | The set of the real $(n\times m)$ matrices |
| $Z$ | Set if integers |
| $\bar{Z}_+$ | Set of non-negative |
| $Z_+$ | Positive |
| $x(t, t_0 x_0)$ | A motion of a syste at $t \in R$ iff $x(t_0 = x_0)$ |
| $A^T$ | The transpose of the matrix $A \in R^{n\times m}$ |
| $\lVert x \rVert$ | The Euclidean norm of $x$ in $R^n$ |
| $\lVert A \rVert$ | = max $\{ \lVert Ax \rVert : \lVert x \rVert = 1 \}$, the induced norm of the matrix $A \in R^{n\times m}$ |
| $\lambda_i(.)$ | The *ith* eigenvalue of a matrix (.) |
| $\lambda_{\max}(.), \lambda_{\min}(.)$ | (Maximal,minimal) eigenvalues of a matrix (.) |
| $B_\varepsilon = \{x \in R^n : \lVert x \rVert < \varepsilon\}$ | Open ball with center at the origin and radius $\varepsilon > 0$ |
| $f : S \to R$ | A vector function mapping from domain $S \subset R^n$ in to $R^n$ |

# CHAPTER 1

# INTRODUCTION

Stability is one of the most important subjects in Mathematics and Engineering. It has an essential role in the system theory as well as in the engineering systems. There are many kinds of stability problems which appear through the study of the dynamic systems, for instance, the stability of equilibrium points, Lyapunov stability, finite time stability, practical stability, technical stability, stability of the periodic orbits and input-output stability [1,2,3].

In 1892, in his doctoral thesis entitled "*A general task about the stability of motion*", a Russian academician, Aleksandr Mikhailovich Lyapunov established the modern stability theory. More than 100 years later this technique helps us to achieve the stability analysis of the equilibrium points or states in any dynamic nonlinear or linear systems [4-10]. Lyapunov not only gave a formal statement of the problem but also proposed the methods serving as the key instruments for treating the stability problems even today, Primarily developed for a family of motions and defined for ordinary differential equations [11,12]. Nowadays Lyapunov stability concept for continuous and discrete time were applied to dynamical systems in more abstract spaces and even to the general motions which are not described by the equations studied in classical analysis [13-18]. Therefore, Lyapunov concepts were adopted to achieve more complicated phenomena in the behavior of dynamical systems such as bifurcation and chaos theory [19,20,21]. The techniques of Lyapunov have been successfully applied in many areas such as examining motion in space, technological devices, automated systems, problems in mechanics, demography, biomedical problems, environmental studies, behavioral science and economics and other fields. Lyapunov concepts of stability are widely used in the other states of equations such that integrals, functional differential equations, nonlinear parabolic equations, difference equations, discrete dynamical systems, fractional calculus and in the fractional dynamic systems [22,23].

Lyapunov methods are classified as the Lyapunov indirect or first method and Lyapunov direct or second method. Lyapunov indirect method gives conditions from a nonlinear system to achieve the local stability near any equilibrium point. Lyapunov direct method is a mathematical directness of the physical properties and the most important method for the analysis of the nonlinear systems and it can be directly applied to a nonlinear system without the need of linearization or solve the system and achieve the global stability. The concept of direct method is if a total energy in a system disappeared, then, the states of this system will reach to the equilibrium point [24,25,26]. In other words, the basic idea behind this method is that if there is a kind of scalar such as energy function and test this (energy) diminishes along the trajectory of the system, then we can determine whether the system is stable or not. Today, Lyapunov linearization method is used to show the theoretical justification of linear system and Lyapunov direct method has become one of the most important methods for nonlinear system analysis [27-31]. By the same methods the stability criteria can be obtained for the discrete-time systems [32,33].

I reviewed in this thesis some basic concepts of the fractional calculus in order to analyze the stability of the nonlinear systems [34,35,36]. The concept of fractional calculus has been known during the stages of the development of the classical calculus founded by Leibnitz and L'Hopital in 1695, when they mentioned half-order derivative [37-42].

The applications of fractional calculus by using Lyapunov methods are very wide nowadays in various branches of applied sciences as well as in engineering, namely, signal and image processing, physics, biology, control theory, chemistry and economics [43-57].

In the last decades, Lyapunov direct method has been a popular technique to study the stability properties of the mathematical models, in which this method is applied in several areas e.g. biological and biomedical science [58-70].

The aim of this thesis is to present a comprehensive review of the uses of Lyapunov theorems and methods for the continuous and discrete time analysis systems in order to achieve the stability properties. On the other hand we have used the fractional calculus to analyze the stability of nonlinear or linear systems. In the applications part of the thesis we presented a new Lemma for the Caputo fractional derivative, a class of fractional-order gene regulatory networks, the stabilization of continuous-time fractional for positive linear systems, an elementary Lemma which estimates the fractional derivatives of Volterra-type Lyapunov functions and the uniform asymptotic stability of Caputo-type epidemic systems.

This thesis contains seven chapters.

**Chapter 1** deals with the concept of stability, Lyapunov theorems together with the methods of stability of nonlinear and linear systems as well as their applications in different fields.

**Chapter 2** includes some preliminaries and definitions utilized in this thesis.

**Chapter 3** introduces the analysis of stability in the sense of Lyapunov by presenting the basic definitions of the direct and indirect methods to determine the stability. We present the way how to construct the Lyapunov function and explaine the Lyapunov theorems to obtain the local or global stability, asymptotic stability or instability for the systems.

**In Chapter 4** the Lyapunov stability theorems for discrete time systems are investigated.

**Chapter 5** presents the stability analysis of fractional systems by using Caputo and Riemann-Liouville fractional derivatives, Bihari's and Bellman-Grönwall's inequality, Mittag-Lefller stability and Lyapunov –Krasovskii theorem are also reviewed.

**Chapter 6** deals with four new and interesting applications of Lyapunov stability theorems.

**Chapter 7** presents the conclusion part.

# CHAPTER 2

# PRELIMINARIES

## 2.1 Definitions

**Definition 1[8]**: A system of the first order of ordinary deferential equation is defined as:

$$\begin{aligned} \frac{dx_1}{dt} &= \dot{x}_1 = g_1(t, x_1, x_2, ...x_n), \\ \frac{dx_1}{dt} &= \dot{x}_2 = g_2(t, x_1, x_2, ...x_n), \\ &\vdots \\ \frac{dx_n}{dt} &= \dot{x}_n = g_n(t, x_1, x_2, ...x_n), \end{aligned} \tag{2.1}$$

where $g_1, g_2, ..., g_n$ are real valued continuous function on an interval $I$.

**Definition 2 [14]**: The system (2.1) is called time-invariant or autonomous system if $g$ does not depend on time $t$, this nonlinear system can be written as

$$\dot{x} = g(x). \tag{2.2}$$

If the function $g$ depends explicitly of $t$ then, the system (2.1) is written as

$$\dot{x} = g(t, x). \tag{2.3}$$

This system is called non-autonomous system.

**Definition 3 [9]**: A differentiable mapping of $g$ of an open set of $S \subset R^n$ to $R^m$ is said to be continuous differentiable in $S$ if $g'$ is such that $\|g'(x_2) - g'(x_1)\| < \varepsilon$ provided that $x_1, x_2 \in S$ and $\|x_2 - x_1\| < \delta$.

**Definition 4 [9]**: A set of $V$ in the plane is a neighborhood of a point $c$ if a small disk around $c$ is contained in $V$ .

**Definition 5 [9]**: A set of $S \subset R^n$ is named as bounded set if there is a real number $L > 0$ leads to $\|x\| < L, \ \forall x \in S$.

**Definition 6 [9]**: Let $B \in R^{n \times n}$ be a square constant matrix, a scalar $\lambda \in R$ is said to be an eigenvalue and $x$ is a nonzero vector called eigenvector of $B$ associated with $\lambda$, if $Bx = \lambda x$.

**Definition 7 [9]**: A function of $g : R^n \to R^m$ is said a locally Lipschitz on $S$ if every point of $S$ has a neighborhood $S_0 \subset S$ in $g$ with domain $S_1$ which satisfies

$$\|g(x_1) - g(x_2)\| \le L\|x_1 - x_2\|, \tag{2.4}$$

it is called as Lipschitz on an open set $S \subset R^n$ if it achieves (2.4) for all $x_1, x_2 \in S$ with the Lipschitz constant $L$. It says to be globally Lipschitz if the condition (2.4) holds on $R^n$.

**Definition 8 [8]**: $x^*$ is said to be the equilibrium point of (2.3), with $x(t_0) = x_0$ if $g(t, x^*) = 0$ for all $t \ge 0$.

**Definition 9 [8]**: If $g(t, x)$ is the trajectories of (2.2) with initial condition $x$ at $t = 0$, the region of attraction to the equilibrium point $x^*$ denoted $R_a$, is defined by

$$R_a = \{x \in S, S \subset R^2, g(t, x) \to x^* \text{ as } t \to \infty\}.$$

**Definition 10 [8]**: A square real matrix $A$ is called Hurwitz if all eigenvalues of $A$ have negative real part.

**Definition 11 [10]:** A family of phase plane trajectories corresponding to various initial conditions is called a phase portrait of (2.3)**.**

**Definition 12 [10]**: A matrix $A$ is called a Schur matrix if ($\max_i |\lambda_i(A)| < 1$).

### 2.2 Comparison functions

In this subsection, we introduce a new class of functions named class ($\mathcal{K}$) and *($\mathcal{KL}$)* [8].

**Definition 13 [10]:** A continuous function $\sigma:[0,a)\to R^{+}$ is said to belong to class $\mathcal{K}$ if

i) $\sigma(0)=0$,

ii) it is strictly increasing.

**Definition 14 [10]:** A continuous function $\sigma:R^{+}\to R^{+}$ is said to belong to class $\mathcal{K}_{\infty}$ if it is achieved in addition to (i),(ii) condition $\sigma(l)\to\infty$ as $l\to\infty$.

**Definition 15 [10]:** A continuous function $\sigma:R^{+}\to R^{+}$ is belongs to a class $\mathcal{L}$ if

i) it is strictly decreasing,

ii) $\sigma(l)\to 0$ when $l\to\infty$

**Definition 16 [10]:** A continuous function $\sigma:[0,a)\times R^{+}\to R^{+}$ is belongs to a class $\mathcal{KL}$ if:

i) for each fixed $s$, the mapping $\sigma(l,s)$ belongs to a class $\mathcal{K}$ according to $l$.

ii) for each fixed $l$, the mapping $\sigma(l,s)$ is decreasing according to $s$.

iii) $\sigma(l,s)\to 0$ as $s\to\infty$ .

### 2.3 Matrices and vector norms

**Definition 17 [21]:** A vector norm on $R^{n}$ is a function $\|.\|$, from $R^{n}\to R$ with the characteristics

i) $\|s\|\geq 0\ \forall s\in R^{n}$ and $\|s\|=0$ if and only if $s=0$

ii) $\|as\|=|a|\|s\|\ \ \forall a\in R$ and $s\in R^{n}$

iii) $\|s+k\|\leq\|s\|+\|k\|\ \forall s,k\in R^{n}$

**Definition 18 [21]:** The definition of the Euclidean norm $l_2$ and the infinity norm $l_\infty$ for the vector $s = (s_1, s_2, ..., s_n)^T$ is

$$\|s\|_2 = \left\{\sum_{i=1}^{n} s_i^2\right\}^{1/2} , \quad \|s\|_\infty = \max_{1 \le i \le n} |s_i| .$$

**Definition 19 [21]:** A matrix norm is a real valued function from $R^{n \times n}$ to $R$. Let $K \in R^{n \times n}$ be a matrix, its norm is symbolize by $\|K\|$ which satisfies a certain number of properties

i) $\|K\| \ge 0$ For all $K \in R^{n \times n}$ and $\|K\| = 0$ if and only if $K = 0$,

ii) $\|aK\| = |a|\|K\|$ For any scalar $a$ and $K \in R^{n \times n}$,

iii) $\|K + L\| \le \|K\| + \|L\|$ for all $K, L \in R^{n \times n}$,

iv) $\|KL\| \le \|K\|\|L\|$ For all $K, L \in R^{n \times n}$.

**Definition 20 [21]:** If $\|.\|$ is a vector norm, the induced matrix norm is given by

$$\|K\| = \max_{\|s\|=1} \|Ks\| = \max_{s \ne 0} \frac{\|Ks\|}{\|s\|} .$$

# CHAPTER 3

# LYAPUNOV STABILITY THEORY OF CONTINUOUS TIME SYSTEMS

There is a simple definition of Lyapunov stability of systems, stating that if the solutions starting around an equilibrium point stay there forever, we can say that the equilibrium point is Lyapunov stable. If the equilibrium point is Lyapunov stable and all the solutions that starting near the equilibrium point converge to it , we say that this equilibrium point is asymptotically stable [9].

Lyapunov work includes two methods for stability analysis Lyapunov direct and indirect methods.

## 3.1 Lyapunov function

Lyapunov function is a stronger and robust method to determine the stability or instability in any equilibrium of nonlinear systems. Lyapunov function means that if we select a positive function $V(.)$ and take its derivative $\dot{V}$ which should be *negative definite* or *negative semi definite*. Then we can say that $V$ is a Lyapunov candidate function [8,9].

**Definition 1 [8]**: Suppose $V : S \to R$ be a continuously differentiable function defined in a domain $S \subset R^n$ and contains the origin, then the derivative of $V$ along the trajectories of (2.2) is denoted by $\dot{V}(x),$ such that

$$\frac{d}{dt}V(x(t)) = \nabla V(x(t)).\dot{x}(t) = \sum_{j=1}^{n} \frac{\partial V}{\partial x_i}(x(t))\dot{x}_j(t). \qquad (3.1)$$

$$\dot{V}(x) = \left[ \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, ..., \frac{\partial V}{\partial x_n} \right] \begin{bmatrix} g_1(x) \\ \vdots \\ g_n(x) \end{bmatrix}. \qquad (3.2)$$

## 3.2 Sign definite functions

We now introduce the notion of *positive definite* functions and the conditions required for the function $V(x)$. In the following $S$ denotes for a connected and open subset of $R^n$ [9].

**Definition 2 [9]**: A scalar function $V : S \rightarrow R$ is named as *positive semi definite* for $S$ when achieves the following conditions

a) $V(0) = 0$ ,

b) $V(x) \geq 0$ , $\forall$ $x$ in $S - \{0\}$.

If we replace the condition (b) with $(V(x) > 0$ in $S - \{0\})$ , then, $V : S \rightarrow R$ is said to be *positive definite* in $S$.

**Definition 3 [9]**: A scalar function $V : S \rightarrow R$ is named as *negative semi definite* in $S$ if it is achieved

a) $V(0) = 0$,

b) $V(x) \leq 0$.

If we replace the condition (b) with $(V(x) < 0$ in $S - \{0\})$ , then, $V : S \rightarrow R$ is called *negative definite* in $S$ .

## 3.3 Definitions of stability

Let us consider [9]

$$\dot{x} = g(x)\,, \tag{3.3}$$

with $g : S \rightarrow R^n$ be a locally Lipschitz map in $S \subset R^n$. Let as assume that $x_e \in S$ is the equilibrium point of (3.3), thus $g(x_e) = 0$. Our aim is to study the stability of $x_e$ , therefore, we state all the definitions and theorems for the case when the equilibrium points are in the origin, meaning that $x_e = 0$

**Definition 4 [10]**: The system (3.3), when $x = 0$, is called Lyapunov stable, if $\varepsilon \succ 0$ , and $\sigma = \sigma(\varepsilon) > 0$ then

$$\|x(0)\| < \sigma \text{, and } \|x(t)\| < \varepsilon\,,\ t \geq 0\,. \tag{3.4}$$

**Definition 5 [10]**: The equilibrium point $x = 0$ of the system (3.3) is called asymptotically stable if it is Lyapunov stable and there is a $\sigma > 0$ such that

$$\|x(0)\| < \sigma \text{, and } \lim_{t \to \infty} x(t) = 0 \text{ .} \tag{3.5}$$

**Definition 6 [10]**: The equilibrium point $x = 0$ of the system (3.3) is called a locally exponentially stable if there are positive real constants $\mu, \lambda$ as well as $\sigma$, such that $\|x(0)\| < \sigma$ then

$$\|x(t)\| \le \mu \|x(0)\| e^{-\lambda t} \text{, } t \ge 0 \text{.} \tag{3.6}$$

**Definition 7 [10]**: The equilibrium point $x = 0$ of the system (3.3) is called a globally exponentially stable, if we have positive real constants $\mu, \lambda$ achieved

$$\|x(t)\| \le \mu \|x(0)\| e^{-\lambda t} \text{, } t \ge 0 \text{. } \forall x(0) \in R^n \text{.} \tag{3.7}$$

**Definition 8 [10]**: The system (3.3), with $x = 0$ is unstable, if it is not Lyapunov stable.

**Definition 9[8]**: Suppose $V : S \to R$ be a continuously differentiable function, then the function $V(x)$ is called radially unbounded if

$$V(x) \to \infty \text{ as } \|x\| \to \infty \text{.}$$

### 3.4 Lyapunov direct method

Lyapunov direct method is a strong method to investigate the stability properties of the equilibrium point for both autonomous and non-autonomous nonlinear systems without the need to solve the nonlinear differential equations. If we construct a scalar function $V$ of a system state $x$, and derive $V$ with respect to time, we see that $V$ is positive everywhere except at the equilibrium point $x = 0$ and $\dot{V} \le 0$ for every $x$, then, we say that the point $x = 0$ is a stable equilibrium point. This result can be extended to provide criteria for determining the equilibrium if it is asymptotically or exponentially stable for global stability analysis [9]. The following theorems deal with Lyapunov direct method and explain the conditions to obtain the types of stability.

**Theorem 1 [8]**: Suppose $x = 0$ be an equilibrium point for (3.3) and $S \subset R^n$ is a domain including the point $x = 0$. Let $V : S \to R$ be a continuous differential function, then

a) $V(0) = 0$,

b) $V(x) > 0$ in a domain $S - \{0\}$,

c) $\dot{V}(x) \le 0$ in a domain $S - \{0\}$.

Thus, $x = 0$ is called stable point.

**Proof [8]**: Let $\varepsilon > 0$, select $l \in (0, \varepsilon]$ such that the closed ball

$$B_l = \{x \in R^n \|x\| \le l\} \subset S$$

is continuous in $S$, Let $\eta = \min_{\|x\|=l} V(x)$. Thus, $\eta \succ 0$ by condition (a) and (b) of Theorem 1, take $\mu \in (0, \eta)$ and let

$$\Psi_\mu = \{x \in B_l | V(x) \le \mu\}.$$

Then, $\Psi_\mu$ is in the interior of $B_l$. The set $\Psi_\mu$ carry the characteristic that each trajectory starts in $\Psi_\mu$ at $t = 0$ remains in $\Psi_\mu$ when $t \ge 0$. This follows from part (c) of the Theorem 1, since

$$\dot{V}(x(t)) \le 0 \to V(x(t)) \le V(x(0)) \le \mu, \text{ for all } t \ge 0.$$

It then follows that any trajectory begins in $\Psi_\mu$ at $t = 0$ remains inside $\Psi_\mu$ for all $t \ge 0$. Thus $V(x)$ is continuous, it follows that $\exists \gamma > 0$ such that

$$\|x\| < \gamma \to V(x) < \mu,$$

then

$$B_\gamma \subset \Psi_\mu \subset B_l,$$

and

$$x(0) \in B_\gamma. \text{This leads to } x(0) \in \Psi_\mu \text{ and } x(t) \in \Psi_\mu, \text{ thus } x(t) \in B_l,$$

so,

$$x(t) < \gamma \to \|x(t)\| < l \le \varepsilon, \ \forall t \ge 0.$$

This means that the equilibrium point is stable at the point $x = 0$.

**Theorem 2 [9]**: In addition to all the conditions in Theorem 1, with $x = 0$, if $V(.)$ is such that

i) $V(0) = 0$,

ii) $V(x) > 0$ in a domain $S - \{0\}$,

iii) $\dot{V}(x) < 0$ in a domain $S - \{0\}$.

Then, we have asymptotically stable at the point $x = 0$.

**Proof [9]**: Under all these statements of the theorem, $V(x)$ actually decreases along the trajectory of $g(x)$. Using the same state utilized in Theorem 1, if each real $a > 0$ can be choosen $b > 0$ thus, $\Psi_b \subset B_a$, whenever the initial condition is inside $\Psi_b$, the solution will remain inside $\Psi_b$. Therefore, to prove the asymptotic stability, all we need to show is that $\Psi_b$ reduces to $0$ in the limit. In other words, $\Psi_b$ shrinks to a single point as $t \to \infty$. However, this is simple, since by assumption $\dot{V}(x) \prec 0$ in $S$. Thus, $V(x)$ tends to steadily reach zero along the solutions of $g(x)$.

**Example 1** [8]:

We consider a simple pendulum with friction, namely [8]

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\frac{g}{l}\sin x - \frac{k}{m}y. \end{aligned} \tag{3.8}$$

The equilibrium points are $(0,0)$ and $(\pi,0)$, in this example it deals with energy, then, the total energy in this system equal to our Lyapunov candidate function $V(x)$ [8], namely

$$E = V(x) = mgl(1-\cos x) + \frac{1}{2}ml^2y^2 > 0.$$

So, if we take the derivative of the function $V(x)$ we get

$$\dot{V}(x) = \left[mgl\sin x, ml^2y\right]\left[y, -\frac{g}{l}\sin x - \frac{k}{m}y\right]^T = -kl^2y^2.$$

Here $\dot{V}(x)$ is not *negative definite*, but *negative semi-definite*, thus, the origin is not asymptotic stable but is stable.

Here Lypunov candidate function fails to identify the asymptotically stability in the origin.

**LaSalle's invariance principle** [9] was developed in 1960 by J.P. LaSalle. It effectively develops a method of obtaining asymptotic stability without the need of the time derivative of the Lyapunov function to be *negative definite* function, but only needs to be *negative semi definite* function. The principle essentially says that if a Lyapunov function exists in a neighborhood of the origin, with a *negative semi definite* time derivative along the trajectories of the system, it can be established that no trajectory can stay identically at the points where $\dot{V}(t) = 0$, except at the origin, therefore, the origin is asymptotically stable. In order to understand better we present a definition and theorems related to the LaSalle's invariance principle [13].

**Definition 10 [9]**: If $S$ is an invariant set belonging to (3.3), then, $x(0) \in S \rightarrow x(t) \in S \quad \forall t \in R^{+}$.

**Theorem 3 [9]**: (LaSalle's Theorem). Suppose $V : S \rightarrow R$ be a differentiable continuous function, such that:

a) $B \subset S$ is a compact set invariant according to the solution of (3.3).

b) $\dot{V}(x) \leq 0$ in $B$.

c) $M : \{x : x \in B, \text{ and } \dot{V} = 0\}$; that is, $M$ is the set of points of $B$ such that $\dot{V} = 0$.

d) $N :$ is the largest invariant set in $M$.

Therefore, each solution starts in $B$ approaches to $N$ as $t \rightarrow \infty$.

**Proof [9]**: Suppose that a solution $x(t)$ in (3.8) starts in $B$. Since $\dot{V}(x) \leq 0 \in B,$ we have a decreasing function $V(x)$ of $t$ and $V(.)$ is a continuous function, it is bounded from below in the compact set $B$. It can be said that $V(x(t))$ has a limit when $t \rightarrow \infty$. Let $\omega$ be a limit set of this trajectory. It follows that $\omega \subset B$ since $B$ is an invariant closed set, for each $s \in \omega$ there is a sequence $t_n$ with $t_n \rightarrow \infty$ and $x(t_n) \rightarrow s$. By continuity of $V(x)$, we conclude

$$V(s) = \lim_{n \to \infty} V(x(t_n)) = a \quad (a \text{ constant}).$$

Note that $V(x) = a$ on $\omega$. Also, by Lemma 3.5 in [9], we consider that $\omega$ be an invariant set, and $\dot{V}(x) = 0$ on $\omega$ because $V(x)$ is constant on $\varepsilon$, then, we obtain

$$\omega \subset N \subset M \subset B$$

Thus, $x(t)$ is boundness by Lemma 3.4 in [9]. This means that $x(t)$ approaches $\omega$ and it is positive limit set as $t \to \infty$. So, $x(t)$ approaches $N$ as $t \to \infty$.

**Remark 1 [9]**:

LaSalle's theorem goes beyond the Lyapunov stability theorems in an important aspect, which is that, $V(.)$ is required to be continuously differentiable and bounded, but it is not required to be *positive definite*.

**Theorem 4 [9]**: The system (3.3), when $x = 0$ is asymptotically stable if

a) $V(x)$ is a *positive definite* function for all $x \in S$, we suppose that $0 \in S$,

b) $\dot{V}(x)$ is a *negative semi definite*,

c) $\dot{V}(x)$ does not vanish identically along the trajectory in $R$, other than the null solution $x = 0$.

**Proof [9]**: By Lyapunov stability theorem, we know that for each $\varepsilon > 0$ there exists $\sigma > 0$, such that

$$\|x_0\| < \sigma \ \to \|x(t)\| < \varepsilon \, .$$

If a solution starting inside the closed ball $B_\sigma$, it will stay within the closed ball $B_\varepsilon$. Hence any solution $x(t, x_0, t_0)$ of (3.8) that starts in $B_\sigma$ is bounded and tends to its limit set of $N$ that is contained in $B_\varepsilon$. Also $V(x)$ can be a continuous function on the set $B_\varepsilon$ and bounded from below in $B_\varepsilon$. It is also non-increasing according to the assumption, thus, it tends to a non-negative limit $L$ as $t \to \infty$. Notice that $V(x) = L$ for all $x$ in the limit set $N$. If $N$ is an invariant set according to (3.3), meaning that if a solution starts in $N$, it will remain there forever. However, along that solution of $\dot{V}(x) = 0$ it can be said that $V(x)$ is a constant $(= L)$ in $N$. Thus, by assumption, $N$ is the origin of the state space. We can conclude that the solution starts in $R \subset B_\sigma$ converges to $x = 0$ as $t \to \infty$.

Let us refer back to example of pendulum, when the origin of the nonlinear system has stable by using Lyapunov direct method. However, the asymptotically stability could not be obtained [9]. Taking in to account that

$$V(x) = mgl(1-\cos x) + \frac{1}{2}ml^2y^2 > 0 \quad (3.9)$$

we obtain

$$\dot{V}(x) = -kl^2y^2. \quad (3.10)$$

Here $\dot{V}(x) = 0$ is *negative semi definite*, for $(x,0)$, if we apply Theorem 4, conditions (a),(b) we obtain that the origin $R = \begin{bmatrix} x \\ y \end{bmatrix}$.

Here $-\pi < x < \pi$, $-a < y < a$, for any $a \in R^+$. By checking the condition (c) if $\dot{V}$ vanish identically along the trajectories trapped in $R$, other than the null solution. Using (3.10) we conclude

$$\dot{V} = 0, \text{ then } -kl^2y^2 = 0 \text{ and } y = 0 \text{ for all } t \text{ and } \dot{y} = 0.$$

By using (3.8) we get $0 = \frac{g}{l}\sin x - \frac{k}{m}y$. Since $y = 0$, this leads to $\sin x = 0$.

Locating $x$ to $x \in (-\pi, \pi)$ condition (c) is satisfied if and only if $x = 0$. So, $\dot{V}(x) = 0$ does not vanish identically along any trajectory other than $x = 0$, therefore, $x = 0$ is an asymptotic stable by Theorem 4 [9].

**Theorem 5 [8]**: Suppose $x = 0$ be an equilibrium point for (3.3) if $V : R^n \to R$ is a continuous differentiable function satisfying

a) $V(0) = 0$,

b) $V(x) > 0$

c) $\|x\| \to \infty \Rightarrow V(x) \to \infty$,

d) $\dot{V}(x) < 0$.

Thus, $x = 0$ is globally asymptotically stable.

**Proof [9]**: the proof follows as in the proof of theorem 2, we just need to prove for given an arbitrary $b > 0$, the condition

$$\Psi_b = \{x \in R^n : V(x) \le b\}.$$

It defines a set which is contained in a ball $B_l = \{x \in R^n : \|x\| \le l\}$, when $l > 0$. To see this, it should noticed that the radial unboundedness of $V(.)$ implies for any $b > 0$, $\exists l > 0$ such that $V(x) > b$ whenever $\|x\| > l$ for some $l > 0$. Thus, $\Psi_b \subset B_a$, which means that $\Psi_b$ is bounded.

**Theorem 6 [8]**: Suppose the equilibrium point $x = 0$ for the non-autonomous system (2.3), and $S \subset R^n$ is a domain including this point, let $V : [0, \infty) \times S \to R$ be a continuously differentiable function such that

$$G_1(x) \le V(t,x) \le G_2(x), \tag{3.11}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} g(t,x) \le -G_3(x). \tag{3.12}$$

For all $t \ge 0$ and for all $x \in S$, where $G_1, G_2$ and $G_3$ are positive continuously differential functions on $S$, then, $x = 0$ is uniformly asymptotically stable. In addition, if $l$ and $k$ are selected such that $B_l = \{\|x\| \le l\} \subset S$ and $k < \min_{\|x\|=l} G_1(x)$, then, each trajectory starting in $\{x \in B_l | G_2(x) \le k\}$ achieves

$$\|x(t)\| \le \mu(\|x(t_0)\|, t - t_0), \text{ for all } t \ge t_0 \ge 0, \tag{3.13}$$

for some class $\mathcal{KL}$ function $\mu$. Finally, if $S = R^n$ and $G_1(x)$ is radially unbounded, then, $x = 0$ is a global uniform asymptotic stable.

**Proof** : See [8].

Before we present the exponential stability theorem, it is better to explain Lemma 1.

**Lemma 1 [9]**: The function $V : S \to R$ is positive differentiable function if and only if there are the functions $\sigma_1, \sigma_2$ in class $\mathcal{K}$ such that

$$\sigma_1(\|x\|) \le V(x) \le \sigma_2(\|x\|), \ \forall x \in B_l \subset S. \tag{3.14}$$

In addition, if $S = R^n$ and $V(.)$ is radially unbounded then, $\sigma_1, \sigma_2$ belong to the class $\mathcal{K}_\infty$.

**Proof [9]**: Given $V(x): S \subset R^n \to R$, we show that $V(x)$ is a positive definite function if and only if there is $\sigma_1 \in \mathcal{K}$ achieves

$$\sigma_1(\|x\|) \leq V(x), \forall x \in S.$$

It is obvious that $\sigma_1$ is a sufficient condition for the positive definiteness of $V$. To prove the necessity condition, we use the definition

$$G(y) = \min_{y \leq \|x\| \leq l} V(x), \text{ for } 0 \leq y \leq l.$$

The function $G(.)$ is well defined since $V(.)$ is continuous and $y \leq \|x\| \leq l$ defines a compact set in $R^n$. Moreover, this function $G(.)$ has the following characteristics:

i) $G \leq V(x)$ $0 \leq \|x\| \leq l$,

ii) It is continuous,

iii) It is a positive definite (since $V(x) > 0$),

iv) It satisfies $G(0) = 0$.

So, $G$ is almost in the class $\mathcal{K}$. It is not, in general, in $\mathcal{K}$ because it is not strictly increasing. We also have that:

$$G(\|x\|) \leq V(x), \text{ for } 0 \leq \|x\| \leq l. \quad (3.15)$$

However $G(x)$ is not in class $\mathcal{K}$ since, in general, it is not strictly increasing. Let $\sigma_1(y)$ be in class $\mathcal{K}$ function such that $\sigma_1(y) \leq kG(y)$ with $0 < k < 1$. Thus, we conclude

$$\sigma_1(\|x\|) \leq G(\|x\|) \leq V(x), \text{ for } \|x\| \leq l. \quad (3.16)$$

The function $\sigma_1$ can be constructed as follows:

$$\sigma_1(r) = \min\left[\frac{r}{y} G(y)\right] \quad r \leq y \leq l. \quad (3.17)$$

This function is strictly increasing. To see this notice that:

$$\frac{\sigma_1}{r} = \min\left[\frac{G(y)}{y}\right] \quad r \leq y \leq l. \quad (3.18)$$

It is positive and also non-decreasing since $r$ increases, the set over which the minimum is computed keeps "shrinking". Thus, from (3.18), $\sigma_1$ is strictly increasing. So, this proves that there is $\sigma_1(.) \in \mathcal{K}$ such that

$$\sigma_1(\|x\|) \leq V(x), \text{ for each } \|x\| \leq l. \quad (3.19)$$

By the same way, the existence of $\sigma_2(.) \in \mathcal{K}$ such that

$$V(x) \le \sigma_2(\|x\|), \text{ when } \|x\| \le l. \tag{3.20}$$

We can be proved similarly.

**Theorem 7 [9]**: Consider that the conditions of Theorem 2 are achieved, moreover, there are positive constants $B_1, B_2, B_3$ and $\sigma$ such that

$$B_1\|x\|^\sigma \le V(t,x) \le B_2\|x\|^\sigma,$$

$$\dot{V}(x) \le -B_3\|x\|^\sigma.$$

So, the origin is exponentially stable, in addition, if the condition occurs globally, then, when $x = 0$ is globally exponentially stable.

**Proof [9]**: According to the suppositions of Theorem 7, also the function $V(x)$ that achieved Lemma 1, when $\sigma_1, \sigma_2$ satisfied some conditions. We have the followings

$$B_1\|x\|^\sigma \le V(t,x) \le B_2\|x\|^\sigma \;,$$

$$\dot{V}(x) \le -B_3\|x\|^\sigma \le -\frac{B_3}{B_2}V(x).$$

Since,

$$\dot{V}(x) \le -\frac{B_3}{B_2}V(x).$$

This leads to

$$V(x) \le V(x_0)e^{-(\frac{B_3}{B_2})t},$$

then

$$\|x\| \le \left[\frac{V(x)}{B_1}\right]^{\frac{1}{\sigma}} \le \left[\frac{V(x_0)e^{-\left(\frac{B_3}{B_2}\right)t}}{B_1}\right]^{\frac{1}{\sigma}},$$

or

$$\|x\| \le \|x_0\|\left[\frac{B_2}{B_1}\right]^{\frac{1}{\sigma}} e^{-\left(\frac{B_3}{\sigma B_2}\right)t}.$$

**Remark 2 [9]**: Exponential stability is the powerful form of stability, because of its precise rate which makes the trajectory converges to the equilibrium point. Moreover, the exponential stability means an asymptotic stability, but the converse is incorrect. In the linear systems, the uniform asymptotic stability and uniform exponential stability is similar.

During our study of Lyapunov direct methods, we observed that some virtues can be mentioned in this section. For example, in this method we can use an energy-like function to study the behavior of the dynamical systems, reflecting in many cases of the physical properties of the system. This method can determines the stability of the nonlinear systems without need to solve the differential equations and it is a powerful method for stability analysis and the control theory by determining the stability, asymptotic stability, exponential stability and instability properties. It is also uses to estimate global stability of the domain of attraction.

On the other hand, there are many flaws of the Lyapunov direct methods, such that, the construction of the Lyapunov function is not easy for general nonlinear systems it is usually a trial-and-error process. Furthermore, there is a lack of systematic methods, and sometimes Lyapunov candidate function fails to identify an asymptotic stable equilibrium, although the system is asymptotic stable. If Lypunov candidate function sometimes fails to achieve the conditions for the stability, this does not means that the equilibrium is not stable or asymptotic stable, but only means that such stability properties cannot be found by using Lyapunov functions [7].

### 3.5 Phase plane diagrams for linear systems

Consider the nonlinear system in (3.3), with the condition $g(0)=0$. If we apply Taylor's theorem of $g$ we obtain that [9]

$$g(x) = A(x) + y(x). \tag{3.21}$$

Here $y(x)$ denotes the sum of higher order terms, $A$ represents the Jacobian matrix of $g$ evaluated at $x=0$, namely

$$A = \left[\frac{\partial g}{\partial x}\right]_{x=0} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}.$$

Neglecting the higher-order terms, then, the linearization of (3.3) about $x=0$ becomes [9]

$$\dot{x} = Ax \,. \tag{3.22}$$

Therefore, the system (3.22) is called the linearization of the system (3.3).

We would like to obtain some information about what happens near the origin for the linear system (3.22). Let us explain some details for a real two dimensional system, namely [22]

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{3.23}$$

The entries of the matrix are:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If we computed the characteristic polynomial we can conclude [22].

$$p(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc = \lambda^2 - tr(A) + \det(A) \,. \tag{3.24}$$

We have $\lambda_1$ and $\lambda_2$ are eigenvalues of $A$ and the characteristic polynomial become

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \,. \tag{3.25}$$

Thus, we can compute that [22]

$$\begin{aligned} \lambda_1 + \lambda_2 &= tr(A) \,, \\ \lambda_1\lambda_2 &= \det(A) \,. \end{aligned} \tag{3.26}$$

If the matrix trace of $A$ is denoted by $T_r = tr(A)$ and the determinant is denoted by $D_{et} = \det(A)$, then, the values of the eigenvalues are [22]

$$\lambda_1, \lambda_2 = \frac{T_r \pm \sqrt{T_r^2 - 4D_{et}}}{2} \,.$$

Let us denotes $\Delta = T_r^2 - 4D_{et}$.

### 3.5.1 Classification of stability

By returning to the system (3.23), we can classify the stability as follows [22].

**1. Stable origin, if [22]**

i) $\Delta > 0$, $D_{et} = 0$, $T_r < 0$,

ii) $\Delta < 0$, $T_r = 0$.

**2. Asymptotic stable origin, if [22]**

i) $\Delta > 0$, $T_r < 0$,

ii) $\Delta = 0$, $T_r < 0$,

iii) $\Delta < 0$, $T_r < 0$.

**3. Unstable origin, if [22]**

i) $\Delta > 0, D_{et} = 0$, $T_r > 0$,

ii) $\Delta = 0$, $T_r > 0$,

iii) $\Delta < 0$, $T_r > 0$.

Now, we classify the behavior of the system (3.23) to various cases for eigenvalues.

### 3.5.2 Classification of the eigenvalues in cases

**Case 1.** Distinct real eigenvalues: $\lambda_1 \neq \lambda_2$ [8]

i) Both eigenvalues are negative $(\lambda_1 < \lambda_2 < 0)$. As $t \to +\infty$, all trajectories flux into the origin, therefore the trajectories are asymptotically stable and in $x = 0$ we have a node.

ii) When eigenvalues are positive $(\lambda_1 > \lambda_2 > 0)$. As $t \to +\infty$, all trajectories flux away from the origin to become arbitrarily large and the equilibrium point $x = 0$ indicates an unstable node.

iii) One negative eigenvalue and one zero eigenvalue $(\lambda_1 < 0), (\lambda_2 = 0)$. The phase portrait is in some sense of decay, and the matrix $A$ has a null space. The points on the $x_1 - axis$ are fixed, then, the origin is stable, but not asymptotic stable.

iv) One positive eigenvalue and one zero eigenvalue $(\lambda_1 > 0), (\lambda_2 = 0)$. Here some points on the $x_1 - axis$ are fixed, and the others ran away to infinity along the vertical line.

v) Both eigenvalues are zero $(\lambda_1 = 0), (\lambda_2 = 0)$. In this case the trajectories start from the equilibrium subspace and move parallel to it.

vi) One negative and one positive eigenvalue $(\lambda_1 < 0 < \lambda_2)$. The points on the $x_1 - axis$ approach the origin, whereas, the points on $x_2 - axis$ go away from the origin.

**Case 2.** Nonzero multiple eigenvalues: $\lambda_1 = \lambda_2 = \lambda \neq 0$ [24].

i) If $\lambda < 0$, then the origin is a stable node.

ii) If $\lambda > 0$, then the origin is unstable, the orbits are like the stable case reversing the time direction.

**Case 3. Complex eigenvalues:** $\lambda_{1,2} = \sigma \pm i\beta$ [8]

i) If $\sigma = 0$, solutions are periodic, and the trajectories are ellipses centered at the origin .

ii) If $\sigma < 0$, the trajectories are spiral converging to the origin as time increase.

iii)If $\sigma > 0$, the trajectories are spiral moving away from the origin as time increases.

### 3.6 Analysis of linear time-invariant systems

Consider an autonomous linear time-invariant system given in the system (3.22) and we can define the Lyapunov function $V(x)$ is a quadratic function candidate, namely [8]

$$V(x) = x^T kx. \tag{3.27}$$

We can choose $k \in R^{n\times n}$ satisfying the conditions:

i) Symmetric, $k = k^T$,

ii) *Positive definite*,

iii) $k$ is constant.

Thus, $V(x)$ is *positive definite*, then, the derivative for the function $V(x)$ along the trajectories of (3.22) gives

$$\dot{V} = \dot{x}^T kx + x^T k\dot{x}.$$

By using (3.22), $\dot{x}^T = x^T A^T$, we report $\dot{V} = x^T A^T kx + x^T kAx = x^T (A^T k + kA)x$

if

$$kA + A^T k = -G. \tag{3.28}$$

Choose $G$ a symmetric matrix, then, we report

$$\dot{V} = -x^T Gx. \tag{3.29}$$

If $G$ is a *positive definite* matrix, then, $\dot{V}(x)$ is *negative definite*, then, as a result the origin is asymptotic stable, to analyze the positive definiteness of the matrices $(k, G)$ we present in the following two steps:

1. Select a symmetric, *positive definite* matrix $G$.
2. Find $k$ achieving the equation (3.28) and must be *positive definite*.

We can consider that $V(x)$ is a Lyapunov function, then, the equation (3.28) is called the matrix of Lyapunov equation. These above steps described as the stability analysis depends on the pair $(k, G)$ which in turn rely on the existence of a unique solution of Lyapunov equation for the matrix $A$. The following theorem explains the stability in Lyapunov equation.

**Theorem 8 [8]**: A matrix $A \in R^{n \times n}$ is called Hurwitz (if $\operatorname{Re}\lambda_i < 0$). For all eigenvalues of $A$, if and only if for every given positive definite symmetric matrix $G$ we find a positive definite symmetric matrix $k$ that achieves Lyapunov equation (3.28). In addition, if $A$ is Hurwitz, then, $k$ is a unique solution of (3.28).

**Proof [8]**: Suppose that $G \geq 0$ then, there exists $k > 0$ satisfies (3.28). Since $V(x) = x^T kx > 0$ with $\dot{V}(x) = -x^T Gx < 0$, the asymptotic stability follows Theorem 2. For the converse, assume that $\operatorname{Re}\lambda_i < 0$ and given $G$, define $k$ as follows:

$$k = \int_0^\infty e^{A^T t} G e^{At} dt \ . \tag{3.30}$$

This $k$ is well defined, given the assumptions on the eigenvalues of $A$. The matrix $k$ is also symmetric, since $(e^{A^T t})^T = e^{At}$. We assume that $k$ is positive definite. To satisfy this case, we reason by contradiction and assume that the opposite is true, that is, $\exists\ x \neq 0$ such that $x^T kx = 0$, but if $x^T kx = 0$ it leads to [8]

$$\int_0^\infty x^T e^{A^T t} G e^{At} x dt = 0 \ .$$

Then, we have

$$\int_0^\infty y^T G y dt = 0 \text{, if } y = e^{At} x \text{,}$$

where

$$y = e^{At}x = 0 \text{ for all } t \geq 0, \text{ then}, x = 0.$$

$e^{at}$ is nonsingular $\forall t$. This contradicts the assumption, and we think that $k$ is indeed a positive definite. Now, we show that $k$ achieves the Lyapunov equation [8]

$$kA + A^T k = \int_0^\infty e^{A^T t} G e^{At} A dt + \int_0^\infty A^T e^{A^T t} G e^{At} dt$$

$$= \int_0^\infty \frac{d}{dt}(e^{A^T t} G e^{At}) dt = e^{A^T t} G e^{At} \Big|_0^\infty = -G,$$

proving that, $k$ is indeed a solution of Lyapunov equation. To finalize this, it remains to display that this $k$ is unique. To understand this, assume that there is another solution $\tilde{k} = k$, such that

$$(k - \tilde{k})A + A^T(k - \tilde{k}) = 0,$$

then, we conclude that

$$e^{A^T t}\left[(k - \tilde{k})A + A^T(k - \tilde{k})\right]e^{At} = 0.$$

Thus, we obtain

$$\frac{d}{dt}\left[e^{A^T t}(k - \tilde{k})e^{At}\right] = 0,$$

meaning that $e^{A^T t}(k - \tilde{k})e^{At}$ is a constant $\forall t$. This can be the case if and only if $k - \tilde{k} = 0$, or $k = \tilde{k}$.

### 3.7 Lyapunov indirect method

Lyapunov indirect method gives conditions from a nonlinear system to achieve the local stability near any equilibrium point by examining the equilibrium point of the linearized system [10]. The following theorem achieves this property.

**Theorem 9 [8]**: Let $x = 0$ be an equilibrium point of (3.3), where $g : S \rightarrow R^n$ is differential continuous and $S$ is a neighborhood when $x = 0$, then

$$B = \frac{\partial g}{\partial x}(x)\Big|_{x=0}.$$

Thus we have:

i) Asymptotically stable origin if $\operatorname{Re}\lambda_i < 0$ for all eigenvalues of matrix $B$.

ii) Unstable origin if $\operatorname{Re}\lambda_i > 0$ for all eigenvalues of $B$.

**Proof [8]**: First, we prove (i). Suppose that $B$ is a Hurwitz matrix, by Theorem 7, since for every symmetric positive definite matrix $O$, the solution $k$ of Lyapunov equation (3.28) is a positive definite. If we suppose that $V(x) = x^T kx$ is a Lyapunov candidate function, then, to derive this function along the trajectories of the system, we conclude

$$\begin{aligned}\dot{V}(x) &= x^T kg(x) + g^T(x)kx \\ &= x^T k[Bx + y(x)] + [x^T B^T + y^T(x)]kx \\ &= x^T(kB + B^T k)x + 2x^T ky(x) \\ &= -x^T Ox + 2x^T ky(x).\end{aligned}$$

As it can be seen, the first term $(-x^T Ox)$ is *negative definite*, but the other term $(2x^T ky(x)$ is an indefinite sign, so the function $y(x)$ satisfies [8]

$$\frac{\|y(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0.$$

So, for each $\sigma > 0$, there is $l \succ 0$ then

$$\|y(x)\|_2 < \sigma\|x\|_2, \ \forall \ \|x\|_2 < l.$$

Hence, it implies that

$$\dot{V}(x) < -x^T Ox + 2\sigma\|k\|_2\|x\|_2^2, \ \forall \ \|x\|_2 < l,$$

but

$$x^T Ox \geq \lambda_{\min}(O)\|x\|_2^2.$$

Here $\lambda_{\min}(.)$ refers to the minimum eigenvalue of $B$, and $\lambda_{\min}(O)$ is positive and real, since $O$ is a symmetric and *posative definite* , we have [8]

$$\dot{V}(x) < -[\lambda_{\min}(O) - 2\sigma\|k\|_2]\|x\|_2^2, \text{ for all } \|x\|_2 < l.$$

Selecting $\sigma < (1/2)\lambda_{\min}(O)/\|k\|_2$ note that $\dot{V}(x)$ is a *negative definite*. By Theorems 1 and 2, we achieved that the origin is asymptotically stable.

There are some flaws of the Lyapunov indirect methods. For example, it can estimate just the local stability and determine the asymptotic stability properties only [8].

### 3.8 Instability theorems

**Theorem 10 [12]**: If we have the system (3.3), let

$$J = \left[\frac{\partial g}{\partial x}\right]_{x=0}.$$

It is the system which Jacobian has estimated at $x = 0$. Then, if at least one of the eigenvalues of the matrix $J$ carries a positive real part, here we can say that $x = 0$ is unstable.

**Theorem 11 [9]**: Suppose $x = 0$ is an equilibrium of (3.3), let $V : S \to R$ be a continuous differentiable function and $V(0) = 0,\ V(x_0) > 0$ for any $x_0$ with the arbitrarily small $\|x_0\|$, if we define the set $H$ such that

$$H = \{x \in B_l \,|\, V(x) > 0\},$$

and

$$B_l = \{x \in R^n \,\|\, x\| \le l\}.$$

It can be considered that $\dot{V}(x) > 0$ in $H$, then, the system (3.3) is unstable at the equilibrium point $x = 0$.

**Proof [8]**: Suppose the point $x_0$ is inside $H$ and $V(x_0) = b > 0$, then the trajectory $x(t)$ which starts at $x(0) = x_0$ should leave the set $H$, since $x(t)$ is inside $H$ and $V(0) > 0$ in $H$, moreover $V(x(t)) \ge b$, let

$$\sigma = \min\{\dot{V}(x) | x \in H \text{ and } V(x) \ge b\},$$

which exists as long as $\dot{V}(x)$ has a minimum value over the compact set $\{x \in H$ and $V(x) \ge b\}$=$\{x \in B_l$ and $V(x) \ge b\}$, so, $\sigma \succ 0$ and [8]

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \ge b + \int_0^t \sigma ds = b + \sigma t.$$

This shows that $x(t)$ does not stay for a long time in $H$ because $V(x)$ is bounded on $H$, and $x(t)$ cannot leave $H$ through the surface $V(x)=0$. As we know $V(x) \geq b$, it must leave $H$ through the sphere $\|x\| = l$, because this occurs for the arbitrarily small $\|x_0\|$. Therefore, the origin is unstable [8].

# CHAPTER 4

# LYAPUNOV STABILITY FOR DISCRETE TIME SYSTEMS

## 4.1 Nonlinear systems

Suppose the nonlinear discrete time system [10] is namely

$$x_{r+1} = g(x_r)\,. \tag{4.1}$$

If $x(k) \in S \subseteq R^n$, and $r \in \overline{Z}_+$, and $g: R^n \to R^n$ be a differentiable continuous in a neighborhood of the origin, an equilibrium point $x_\varepsilon \in R^n$ satisfies $g(x_\varepsilon) = x_\varepsilon$. The equilibrium point of the system (4.1) can be defined bellow [10].

**Definition 1 [14]**: If $g(x_r^*) = 0$ in the nonlinear discrete time system (4.1), then, the point $x_r^*$ is an equilibrium point. Consider the origin $x_r = 0$ as the equilibrium point. In the next definition, we define the types of stability of the equilibrium point.

**Definition 2 [10]**: If the point $x_r = 0$ is the equilibrium point of (4.1), then, it is

1- Stable, for every $\varepsilon > 0$ there is $\delta(\varepsilon) \succ 0$ such that

$$\|x_0\| < \delta \;\to \|x_r\| < \varepsilon\,,\; r \geq 0$$

2- Locally asymptotic stable, if it achieves the condition of stability and

$$\|x_0\| < \delta \;\to \lim_{r\to\infty} x_r = 0\,.$$

3- Globally asymptotic stable, if it is asymptotic stable for all $x_0 \in R^n$.

4- Unstable, when it is not Lyapunov stable.

**Theorem 1 [12]**: Let $x_r = 0$ be an equilibrium point for the system (4.1), then, the system is asymptotic stable near the zero equilibrium point. If there is a function $V(x_r)$ defined in a domain $S$ and continuous in $x_r$ and satisfy

a) $V(0) = 0$,

b) $V(x_r) > 0 \ \forall \ x_r \neq 0$ in $S$,

c) $\Delta V(x(r)) = V(x(r+1)) - V(x(r))$ For all $x(r) \in S$.

So, $x_r = 0$ is asymptotically stable, if further

d) $V \to \infty$ as $\|x_r\| \to \infty$.

Then, the point $x_r = 0$ is globally asymptotically stable.

### 4.2 Linear systems

Suppose a discrete time linear system [8], namely

$$x(r+1) = Bx(r). \tag{4.2}$$

Here $B \in R^{n\times n}$, is said to be symmetric if $B = B^T$ and the asymptotic stability of such a system is determined by the eigenvalues found directly in the interior the unit circle in the complex plane.

**Definition 3 [8]**: A matrix $B$ is called asymptotically stable, for all $|\lambda_i| < 1$, and $\forall i = 1,2,...,n$. Here $\lambda_i$'s are refer to the eigenvalues of the matrix $B$, moreover it called a Schur matrix, if all its eigenvalues lay inside the unit circle in the complex plane. For the discrete time linear system (4.1) the Lyapunov function has a quadratic form [8]

$$V(x(r)) = x^T(r)kx(r) > 0,$$

$$\Delta V(x(r)) = V(x(r+1) - V(x(r)) \leq 0.$$

We have

$$V(x(r+1)) - V(x(r)) = x^T(r+1)kx(r+1) - x^T(r)kx(r)$$

$$= x^T(r)(B^T kB - k)x(r) \leq 0. \tag{4.3}$$

We should note that the relation between the continuous time argument and discrete-time referring to (4.3) as the algebraic Lyapunov equation is similar. We consider the following equation

$$B^T kB - k = -G. \qquad (4.4)$$

The equation (4.4) is named Lyapunov matrix equation for the discrete time system (4.1), and the matrix $G$ is *positive definite*. Thus, the system is asymptotic stable, if we can pick that $G = I$, then, the identity matrix can solve

$$B^T kB - k = -I. \qquad (4.5)$$

Related to $k$, then, we see that if $k$ is a *positive definite*.

**Theorem 2 [8]**: Suppose the linear discrete time system in (4.1), the conditions are equivalent

1- The matrix $B$ is asymptotically stable.

2- Given any matrix $G = G^T > 0$ there is a positive matrix $k = k^T$ achieving the discrete-time of Lyapunov matrix equation:

$$B^T kB - k = -G.$$

**Proof [8]**: Firstly show that $1 \to 2$. Let $B$ be asymptotically stable and choose any matrix $G = G^T > 0$. Take the matrix

$$k = \sum_{i=0}^{\infty} (B^T)^i GB^i. \qquad (4.6)$$

It is well defined by the asymptotically stability of $B$, and $k = k^T \succ 0$ by definition. Now, let us substitute $k$ in the Lyapunov matrix equation (4.4)**,** we obtain

$$B^T kB - k = B^T \left( \sum_{i=0}^{\infty} \left(B^T\right)^i GB^i \right) B - \sum_{i=0}^{\infty} \left(B^T\right)^i GB^i$$

$$= \sum_{i=0}^{\infty} \left(B^T\right)^i GB^i - \sum_{i=0}^{\infty} \left(B^T\right)^i GB^i = -G$$

In order to show the uniqueness, suppose that there is another matrix $\tilde{k}$ that satisfies the Lyapunov equation. After some steps, we can show that

$$\left(B^T\right)^M (k - \tilde{k}) B^M = k - \tilde{k}$$

Letting $M \to \infty$ gives the desired result.

In order to show that $2 \to 1$, suppose the Lyapunov function $V(x) = x^T kx$, fix $x(0)$ to be an initial state. We obtain

$$V(x(M)) - V(x(0)) \le -\sum_{i=0}^{\infty} x(i)^T Gx(i) \le \lambda_{\min}(G) \sum_{i=0}^{\infty} \|x(i)\|_2^2. \qquad (4.7)$$

Therefore, the sequence $[V(x(r))]_{r\in M}$ is strictly decreasing and bounded from below, therefore, it attains a non-negative limit. Moreover, we can show by contradiction that this limit is actually 0, or equivalently $\lim_{i\to\infty}\|x(i)\| = 0$, since this holds for any choice of $x(0)$, so, $B$ is asymptotically stable.

**Theorem 3 [12]**: Suppose that $x^* = 0$ is an equilibrium point of the system (4.1), where $g: S \to R^n$ is continuous differentiable near a neighborhood of the origin, $S \subseteq R^n$, consider

$$J = \left[\frac{\partial g}{\partial x_r}\right]_{x_r = x^* = 0},$$

be the Jacobian of the above system at the equilibrium point $x^* = 0$. If all the eigenvalues of the Jacobian matrix $J$ are less than one in absolute value, therefore the system (4.1) is named asymptotically stable near this zero equilibrium point.

**Proof**: See [12].

# CHAPTER 5

## STABILITY ANALYSIS OF FRACTIONAL ORDER SYSTEMS

Fractional calculus is related to the calculus of integrals and derivatives of orders that may be real or complex. Nowadays the applications of fractional calculus are very wide in various fields of sciences and engineering, such as signal and image processing, chemistry, physics, biology, economics, chaos theory and control theory [37,41]. One of the features of using fractional-order derivatives instead of integer-order derivatives for solving a system, is that some times when we solve a system which is of the kind of integer-order derivative, this system turned to be unstable. But if we solved this system by fractional-order derivative, the system turned out to be stable [46].

### 5.1 Preliminaries

The formula of a fractional integral with $\sigma \in (0,1)$ is given bellow [38]

$$ {}_a\mathcal{D}_t^{-\sigma} g(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \frac{g(\tau)}{(t-\tau)^{1-\sigma}} \, d\tau \, , \tag{5.1} $$

where $g(t)$ is an arbitrary integrable function. ${}_a\mathcal{D}_t^{-\sigma}$ represents the fractional integral of order $\sigma$ on $[a,t]$, and $\Gamma(.)$ is the Gamma function which is defined as: $\Gamma(z) = \int_0^\infty t^{z-1} e^t dt$ .

The Riemann-Liouville derivative of fractional order $\sigma$ can be defined by [39]:

$$ {}_a^R D_t^\sigma g(t) = \frac{1}{\Gamma(n-\sigma)} \frac{d^n}{dt^n} \int_a^t \frac{g(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau \ , \quad (5.2) $$

for $(n-1 < \sigma < n)$.

The Grunwald–Letnikov derivative is defined by [39]:

$$ {}_a^G D_t^\sigma g(t) = \lim_{h\to 0} g_h^{(\sigma)}(t) = \lim_{h\to 0} h^{-\sigma} \sum_{i=0}^{\left[\frac{(t-\sigma)}{h}\right]} (-1)^i \binom{\sigma}{i} g(t-ih) , \quad (5.3) $$

in which $[.]$ refers to the integer part.

The Caputo fractional derivative has the form [39]:

$$ {}_a^C D_t^\sigma g(t) = \frac{1}{\Gamma(n-\sigma)} \int_a^t \frac{g^n(\tau)}{(t-\tau)^{\sigma-n+1}} d\tau, \ n-1 < \sigma < n. \quad (5.4) $$

The Mittag-Leffler function has the form [36]:

$$ E_\sigma(H) = \sum_{k=0}^{\infty} \frac{H^k}{\Gamma(k\sigma+1)} , \quad (5.5) $$

where $\sigma > 0$, $H \in R$. The definition of Mittag-Leffler function of two–parameters is [36]:

$$ E_{\sigma,\beta}(H) = \sum_{k=0}^{\infty} \frac{H^k}{\Gamma(k\sigma+\beta)} , $$

(5.6)

where $\sigma > 0$, $\beta \in R$, $H \in R$.

The Laplace transform of $g(t)$ can be defined as [48]:

$$ G(s) = \mathcal{L}\{g(t); s\} = \int_0^\infty s^{-st} g(t) dt \ . \quad (5.7) $$

$\mathcal{L}\{.\}$ stands for the Laplace transform.

The Laplace transform for Mittag-Leffler function has the form [36]:

$$ \mathcal{L}\{t^{\beta-1} E_{\sigma,\beta}(\pm\lambda t^\sigma)\} = \frac{s^{\sigma-\beta}}{s^\sigma \mp \lambda}, \ (R(s) > |\lambda|^{\frac{1}{\sigma}}) . \quad (5.8) $$

Here, $s$ is the variable in Laplace domain, $R(s)$ is the real part of $s$, $\lambda \in R$.

The Laplace transform of the Caputo derivative is written as [48]:

$$ \mathcal{L}\{{}_0^C D_t^\sigma g(t)\} = s^\sigma G(s) - \sum_{k=0}^{n-1} s^{(\sigma-k-1)} g^{(k)}(0), \quad n-1 < \sigma < n . \quad (5.9) $$

## 5.2 Stability analysis by Caputo fractional system

Consider the fractional system [38], namely

$$ {}_{t_0}^{C}D_t^{\sigma} x(t) = g(t, x). \tag{5.10}$$

$x(t_0)$ denoting the initial condition, where $\sigma \in (0,1)$, $g : [t_0, \infty] \times \Omega \rightarrow R^n$ is a locally Lipschitz in $x$, it is continuous in $t$ on $[t_0, \infty] \times \Omega$, and $\Omega \in R^n$ is a domain containing the equilibrium point $x = 0$ [38].

**Definition 1 [36]**: If $x_0$ is the equilibrium point of the Caputo fractional system (5.10), then, it is true if and only if $g(t, x_0) = 0$.

**Definition 2 [36]**: If $\sigma : [0, t) \rightarrow [0, \infty)$ is a continuous function is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$.

**Lemma 1 [36]**: (Fractional Comparison Principle): Suppose ${}_0^C D_t^{\beta} g(t) \geq {}_0^C D_t^{\beta} h(t)$ and $g(0) = h(0)$, where $\beta \in (0,1)$, then, $g(t) \geq h(t)$.

**Proof [36]**: From ${}_0^C D_t^{\beta} g(t) \geq {}_0^C D_t^{\beta} h(t)$, there is a nonnegative function $w(t)$ achieves

$$ {}_0^C D_t^{\beta} g(t) \geq w(t) + {}_0^C D_t^{\beta} h(t). \tag{5.11}$$

Taking the Laplace transform of equation (5.11), we get

$$ s^{\beta} G(s) - s^{\beta - 1} g(0) = w(s) + s^{\beta} H(s) - s^{\beta - 1} h(0) .$$

It is followed by $g(0) = h(0)$ that

$$ G(s) = s^{-\beta} w(s) + H(s) . \tag{5.12}$$

Using the inverse transform to (5.12), it gives us that [36]:

$$ g(t) = {}_0 D_t^{-\beta} w(t) + h(t) .$$

At last, from $w(t) \geq 0$ and (5.1), we conclude that $g(t) \geq h(t)$.

**Theorem 1 [38]**: Suppose that $x = 0$ is the equilibrium point for the system (5.10), consider a Lyapunov function $V(t, x(t))$ with functions $\sigma_1, \sigma_2, \sigma_3$ belong to class $\mathcal{K}$ satisfying

$$ \sigma_1 \|x\| \leq V(t, x) \leq \sigma_2 \|x\| , \tag{5.13}$$

$$ {}_0^C D_t^{\mu} V(t, x(t)) \le -\sigma_3 \|x\|. \tag{5.14} $$

Where $\mu \in (0,1)$, so, the system (5.11)is asymptotic stable .

**Proof [38]:** From (5.13) and (5.14), we obtain that

$$ {}_0^C D_t^{\mu} V(t, x(t)) \le -\frac{\sigma_3}{\sigma_2}(V). $$

If we look at Lemma 1, we can see that $V(t, x(t))$ is bounded by the unique nonnegative solution of the scalar differential equation

$$ {}_0^C D_t^{\mu} g(t) \le -\frac{\sigma_3}{\sigma_2}(g(t)), \quad g(0) = V(0, x(0)), \tag{5.15} $$

from definition 1, $g(t) = 0$ for $t \ge 0$, $g(0) = 0$ because $\frac{\sigma_3}{\sigma_2}$ is a class $\mathcal{K}$ function, or $g(t) \ge 0$ on $t \in [0, \infty)$, then from (5.15) we get ${}_0^C D_t^{\mu} g(t) \le 0$. By Lemma 1, we get $g(t) \le g(0)$, then for $t \in [0, \infty)$ we prove that (5.15) is asymptotic stable by contradiction. The second part of the proof can be found in [38].

**Example 1 [40]**: Consider the system

$$ {}^C D_t^{\sigma} x(t) = \sin x + gx. \tag{5.16} $$

The aim is to find the values of parameter $g$ to satisfy the system that will be asymptotic stable of the system (5.16).

Firstly, we choose $V(x) = \frac{1}{2}x^2$ to be Lyapunov candidate function, then, we get

$$ {}^C D_t^{\sigma} V(t) = {}^C D_t^{\sigma} \frac{1}{2}x^2 = \frac{1}{2} {}^C D_t^{\sigma} x^2 < x\, {}^C D_t^{\sigma} x(t) $$

$$ = x \sin x + gx^2 \approx x^2 + gx^2 = (g+1)x^2. \tag{5.17} $$

Here we considered that $\sin x = x$ for every $x \in (-\frac{\pi}{36}, \frac{\pi}{36})$. Suppose that $x$ is in the given interval, we have for $g < -1$ and it is corrected when $(g+1)x^2 < 0$ . So, the origin is locally asymptotically stable for $g < -1$.

### 5.3 Stability analysis by Riemann–Liouville fractional systems

At first we review several properties of Riemann–Liouville and Caputo derivatives.

**Property 1 [42]**: If $0 < \sigma < 1$, we get

$$ {}_{t_0}^{C}D_t^{\sigma} x(t) = {}_{t_0}D_t^{\sigma} x(t) - \frac{x(t_0)}{\Gamma(1-\sigma)}(t-t_0)^{-\sigma}. \tag{5.18}$$

If $x(t_0) = 0$, we conclude that

$$ {}_{t_0}^{C}D_t^{\sigma} x(t) = {}_{t_0}D_t^{\sigma} x(t). \tag{5.19}$$

**Property 2 [42]**: When $u > -1$, we have

$$ {}_{t_0}D_t^{\sigma}(t-t_0)^u = \frac{\Gamma(1+u)}{\Gamma(1+u-\sigma)}(t-t_0)^{u-\sigma}. \tag{5.20}$$

If $0 < \sigma < 1$ and $x(t) = (t-t_0)^u$, then, from Property 1, we get

$$ {}_{t_0}^{C}D_t^{\sigma}(t-t_0)^u = \frac{\Gamma(1+u)}{\Gamma(1+u-\sigma)}(t-t_0)^{u-\sigma}. \tag{5.21}$$

**Property 3 [42]**:

$$ {}_{t_0}^{C}D_t^{\sigma}(ax(t)+by(t)) = a\,{}_{t_0}^{C}D_t^{\sigma} x(t) + b\,{}_{t_0}^{C}D_t^{\sigma} y(t). \tag{5.22}$$

**Property 4 [42]**: From (5.4), if $0 < \sigma < 1$ we conclude

$$ I_t^{\sigma}\ {}_{t_0}^{C}D_t^{\sigma} x(t) = x(t) - x(t_0), \tag{5.23}$$

and

$$ (I_t^{\sigma}\,\mathrm{g})(\mathrm{t}) = \frac{1}{\Gamma(\sigma)}\int_a^t \frac{g(s)ds}{(t-s)^{1-\sigma}},\ t \succ t_0. $$

**Property 5 [48]**: If $\sigma, \beta \in R$, $n \in Z$ and $-1 \le \beta < n$.

$$ {}_{t_0}D_t^{\sigma}({}_{t_0}D_t^{\beta} g(t)) = {}_{t_0}D_t^{\sigma+\beta} g(t) - \sum_{j=1}^{n}\left[{}_{t_0}D_t^{\beta-j} g(t)\right]_{t-t_0} \frac{(t-t_0)^{-\sigma-j}}{\Gamma(1-\sigma-j)}. \tag{5.24}$$

**Remark 1 [51]**: The derivative for a constant by Caputo is zero, whereas the derivative for a constant by Riemann–Liouville is not zero. However it is equal to $D^{\beta}C = \dfrac{C(t-t_0)^{-\beta}}{\Gamma(1-\beta)}$.

**Lemma 2 [36]**: Let $\beta \in (0,1)$ and $W(t)$ be a non-negative arbitrary function on $t \in [0,\infty)$, then

$$ {}_0^C D_t^{\beta} W(t) \le {}_0 D_t^{\beta} W(t) . \tag{5.25} $$

Here ${}^C D$ and $D$ are the fractional operators of Caputo and Riemann-Liouville types.

**Proof [36]**: If we use (5.24), we can conclude that

$$ {}_0^C D_t^{\beta} W(t) = {}_0 D_t^{\beta-1} \frac{d}{dt} W(t) = {}_0 D_t^{\beta} W(t) - \frac{W(0) t^{-\beta}}{\Gamma(1-\beta)} . \tag{5.26} $$

Because $\beta \in (0,1)$ and $W(t) \ge 0$, we have

$$ {}_0^C D_t^{\beta} W(t) \le {}_0 D_t^{\beta} W(t) \; . $$

**Theorem 2 [38]**: If the suppositions in Theorem 1 are achieved when we replace ${}_0^C D_t^{\beta}$ by ${}_0 D_t^{\beta}$, so we have $\lim_{t\to\infty} x(t) = 0$.

**Proof [38]**: By Lemma 2, with $V(t,x) \ge 0$ we have

$$ {}_0^C D_t^{\beta} V(t, x(t)) \le {}_0 D_t^{\beta} V(t, x(t)), $$

meaning

$$ {}_0^C D_t^{\beta} V(t, x(t)) \le {}_0 D_t^{\beta} V(t, x(t)) \le -\sigma_3(\|x\|) . $$

If we follow the same proof is in Theorem 1, we get

$$ \lim_{t\to\infty} x(t) = 0 . $$

### 5.4 Stability analysis of Bihari's and Bellman-Grönwall's inequality

In this section we use Bellman-Grönwall's and Bihari's inequality to study some theorems that explain the stability of fractional order systems by using Lyapunov second method.

**Theorem 3 [43]**: (Bihari's Inequality). Suppose $v$ and $g$ be non-negative continuous functions on $[0,\infty)$, Let $k$ be a continuous increasing function on $[0,\infty)$, where $k(t) > 0$ in $(0,\infty)$, if there is a positive constant $\alpha \ge 0$ , such that $v$ satisfy

$$ v(t) \le \alpha + \int_0^t g(s) k(v(s)) ds , \; t \ge 0. \tag{5.27} $$

Then,

$$v(t) \le H^{-1}\left[H(\alpha) + \int_0^t g(s)ds\right],\ 0 \le t \le T. \qquad (5.28)$$

Here, $H$ is defined by $H(x) = \int_{x_0}^{x} \frac{dy}{k(y)},\ x, x_0 > 0.$ Whereas $H^{-1}$ is inverse of $H$, when $T$ is chosen such that

$$H(\alpha) + \int_0^t g(s)ds \in \text{Domain } H^{-1} \text{ for } 0 \le t \le T.$$

**Proof [43]:** Let

$$y(t) \le \alpha + \int_o^t g(s)k(v(s))ds,$$

Since $g, v$ and $k$ are continuous functions, then

$$y'(t) \le g(t)k(v(t)).$$

Since

$$0 \le v(t) \le \alpha + \int_0^t g(s)k(v(s))ds = y(t),$$

and $k$ is increasing function, then

$$k(v(t)) \le k(y(t)).$$

Since, $g(t) \ge 0$, then we obtain

$$y'(t) = g(t)k(v(t)) \le g(t)k(y(t)).$$

Then

$$\frac{y'(t)}{k(y(t))} \le g(t).$$

If $g(0) = \alpha,$ then

$$\int_\alpha^{y(t)} \frac{1}{k(r)} dr = \int_0^t \frac{y'(s)}{k(y(s))} d(s) \le \int_0^t g(s)ds. \text{ Suppose } r = y(s)$$

Then, we have

$$H(v(t)) - H(\alpha) = \int_{x_0}^{v(t)} \frac{1}{k(r)} dr - \int_{x_0}^{\alpha} \frac{1}{k(r)} dr = \int_{\alpha}^{v(t)} \frac{1}{k(r)} dr$$

$$\leq \int_{\alpha}^{y(t)} \frac{1}{k(r)} dr \text{, since } v(t) \leq y(t)$$

$$\leq \int_{0}^{t} g(s) ds.$$

That is

$$H(v(t)) \leq H(\alpha) + \int_{0}^{t} g(s) ds.$$

Since $H$ is increasing function, then $H^{-1}$ in increasing function. So,

$$v(t) \leq H^{-1}\left[ H(\alpha) + \int_{0}^{t} g(s) ds \right], \quad 0 \leq t \leq T.$$

**Theorem 4 [43]**: (Bellman-Grönwall integral inequality). Suppose $g(t)$ fulfills

$$g(t) \leq \int_{0}^{t} h(\tau) g(\tau) d\tau + k(t), \tag{5.29}$$

with $h(t)$ and $k(t)$ that are real functions, then, we can conclude that

$$g(t) \leq \int_{0}^{t} h(\tau) k(\tau) \exp\left[ \int_{\tau}^{t} h(r) dr \right] d\tau + k(t).$$

(5.30)

If $k(t)$ is differentiable, then

$$g(t) \leq k(0) \exp\left[ \int_{0}^{t} h(\tau) d\tau \right] + \int_{0}^{t} k(\tau) \exp\left[ \int_{\tau}^{t} h(r) dr \right] d\tau. \tag{5.31}$$

Particularly, if $k(t)$ is a constant, then, we have

$$g(t) \leq k(0) \exp\left[ \int_{0}^{t} h(\tau) d\tau \right]. \tag{5.32}$$

**Proof [18]:** To prove this Theorem we want to define a new variable and transform the integral inequality into a differential equation. Suppose that

$$V(t) = \int_{0}^{t} h(\tau) g(\tau) d\tau. \tag{5.33}$$

Now if we take the derivative of $V$ and using (5.29), we get

$$\dot{V} = h(t) g(t) \leq h(t) V(t) + h(t) k(t).$$

Suppose that

$$s(t) = h(t)g(t) - h(t)V(t) - h(t)k(t)$$

Which is non-positive function, then $V(t)$ achieves

$$\dot{V}(t) - h(t)V(t) = h(t)k(t) + s(t).$$

To solve this equation with the initial condition $V(0) = 0,$ we get

$$V(t) = \int_0^t \exp\left[\int_\tau^t h(r)dr\right][h(\tau)k(\tau) + s(\tau)]d\tau. \tag{5.34}$$

Since $s(t)$ is non-positive function, we conclude

$$V(t) \le \int_0^t \exp\left[\int_\tau^t h(r)dr\right]h(\tau)k(\tau)d\tau.$$

From the definition of $V$ in (5.32) and (5.29) we conclude

$$g(t) \le \int_0^t \exp\left[\int_\tau^t h(r)dr\right]h(\tau)k(\tau)d\tau + k(\tau).$$

If $k(\tau) = k(t)$ and $0 \le \tau \le t$, then

$$g(t) \le k(t)\left(1 + \int_0^t h(\tau)\exp\left[\int_\tau^t h(r)dr\right]d\tau\right) = k(t)\left(1 - \int_0^t \frac{d\exp\left[\int_\tau^t h(r)dr\right]}{d\tau}\right)$$

$$= k(t)\left(1 - \exp\left[\int_\tau^t h(r)dr\right]_0^t\right) = k(t)\left(1 - 1 + \exp\left[\int_0^t h(r)dr\right]\right)$$

$$= k(t)\exp\left[\int_0^t h(r)dr\right].$$

In the following Theorem, we extend the Lyapunov second method for Caputo type by using Bellman-Grönwall and Bihari's inequality.

**Theorem 5 [43]**: Suppose that we have the equilibrium point $x = 0$ if

$${}^{C}D^{\sigma}x(t) = g(t,x),\ S \subset R^n,$$

is a domain containing $x = 0$. Suppose $V(t,x):[0,\infty)\times S \to R$ be a continuously differentiable function such as

$$M_1(x) \le V(t,x) \le M_2(x),$$
$$^C D^\sigma V(t,x) \le -M_3(x).$$

(5.35)

For all $t \ge 0$ and for all $x \in S$, $0 < \sigma < 1$, where $M_1(x), M_2(x)$ and $M_3(x)$ are continuous positive definite functions on $S$, then, $x = 0$ is uniform asymptotic stable.

**Proof**: See [43].

**Example 2 [43]**: Suppose a fractional order derivative system as given bellow

$$D^\sigma(H(t)) = -c\,\mathrm{sgn}(H(t)),\ 0 < \sigma < 1,\ c > 0. \tag{5.36}$$

Then, we select the function

$$V(t) = \frac{1}{2}(H(t))^2, \tag{5.37}$$

to be a Lyapunov candidate function, we have

$$\dot{x}(t) > 0 \rightarrow {}^C D^\sigma x(t) > 0.$$

So,

$$\dot{x}(t) \le 0 \rightarrow {}^C D^\sigma x(t) \le 0. \tag{5.38}$$

Therefore, we get

$$\dot{V}(t) = H(t)\dot{H}(t). \tag{5.39}$$

From (5.36) it is implied that

$$\dot{H}(t) = D^{1-\sigma}(-c\,\mathrm{sgn}(H(t)),\ 0 < \sigma < 1, \tag{5.40}$$

If we use the signum function definition, namely

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases} \tag{5.41}$$

Then, we have

$$\mathrm{sgn}(D^{1-\sigma}(-c\,\mathrm{sgn}(H(t)) = -\mathrm{sgn}(H(t)),\ c > 0. \tag{5.42}$$

Thus, we conclude that

$$\dot{V}(t) = H(t)\dot{H}(t) = H(t)D^{1-\sigma}(-c\,\mathrm{sgn}(H(t)). \tag{5.43}$$

Using the signum function,

$$\operatorname{sgn}(\dot{V}(t)) = \operatorname{sgn}(H(t)\dot{H}(t)) = \operatorname{sgn}(H(t))\operatorname{sgn}(D^{1-\sigma}(-c\operatorname{sgn}(H(t)))\,,$$
$$= -\operatorname{sgn}(H(t)\operatorname{sgn}(H(t)) = -1, \tag{5.44}$$

we prove that

$$\dot{V}(t) \le 0\text{, then, } D^{\sigma}V(t) \le 0.$$

If we use the Theorem 4, we conclude

$$V(t) = \frac{1}{2}(H(t))^2 \ge 0\text{, which implies that } D^{\sigma}V(t) \le 0. \tag{5.45}$$

Therefore, the system (5.36) is stable.

### 5.5 Mittag-Leffler stability

**Definition 3 [38]**: The solution of (5.10) is Mittag-Leffler stable if

$$\|x(t)\| \le \{w[x(t_0)]E_{\sigma}(-\lambda(t-t_0)^{\sigma}\}^{c}. \tag{5.46}$$

Whereas $t_0$ is the initial time, $\sigma \in (0,1)$, $\lambda \ge 0$, $c \succ 0$, $w(0) = 0$, $w(x) \ge 0$ and $w(x)$ is locally Lipschitz on $x \in B \in R^n$ with Lipschitz constant $w_0$.

**Definition 4 [38]**: The solution of (5.10) is generalized Mittag-Leffler stable if

$$\|x(t)\| \le \{w[x(t_0)](t-t_0)^{-\delta} E_{\sigma,1-\delta}(-\lambda(t-t_0)^{\sigma}\}^{c}, \tag{5.47}$$

where $t_0$ is the initial time, $\sigma \in (0,1)$, $-\sigma < \delta < 1-\sigma$, $\lambda \ge 0$, $c > 0$, $w(0) = 0$, $w(x) \ge 0$ and $w(x)$ is locally Lipschitz on $x \in B \in R^n$ with Lipschitz constant $w_0$.

**Theorem 6 [38]**: If $x = 0$ is an equilibrium point for (5.10), $S \subset R^n$ is a domain contains the origin. Suppose that $V(t, x(t)) : [0,\infty) \times S \to R$ is a locally Lipschitz and continuous differentiable function depends on $x$ such that

$$\delta_1\|x\|^{h} \le V(t, x(t)) \le \delta_2\|x\|^{hr}, \tag{5.48}$$

$${}_{0}^{C}D_{t}^{\sigma}V(t, x(t)) \le -\delta_3\|x\|^{hr}. \tag{5.49}$$

For all $t \ge 0$ and $x \in S$, $\sigma \in (0,1)$, where $\delta_1, \delta_2, \delta_3$, $h$ and $r$ are arbitrary positive constants, therefore $x = 0$ is Mittag-Leffler stable. If it is happen globally on $R^n$, then, $x = 0$ is globally Mittag-Leffler stable.

**Proof [38]**. With the equation (5.48) and (5.49) we obtain

$$ {}_{0}^{C}D_t^{\sigma}V(t,x(t)) \le -\frac{\delta_3}{\delta_2}V(t,x(t)). $$

If there is a nonnegative function $K(t)$ achieves

$$ {}_{0}^{C}D_t^{\sigma}V(t,x(t)) + K(t) = -\frac{\delta_3}{\delta_2}V(t,x(t)). \tag{5.50} $$

If we take the Laplace transform of (5.50), we get

$$ s^{\sigma}V(s) - V(0)s^{\sigma-1} + K(t) = -\frac{\delta_3}{\delta_2}V(s), \tag{5.51} $$

since the nonnegative constant $V(0)=V(0,x(0))$ and $V(s)=\mathcal{L}\{V(t,x(t))\}$. We can understand that [38]

$$ V(s) = \frac{V(0)s^{\sigma-1} - K(s)}{s^{\sigma} + \frac{\delta_3}{\delta_2}}. $$

If $x(0)=0$ leads to $V(0)=0,$ then, the solution of (5.10) is $x=0.$ If $x(0)\neq 0$ leads to $V(0)>0,$ because $V(t,x)$ is locally Lipschitz according to $x,$ from (Existence and uniqueness Theorem [38]), and taking the inverse Laplace transform of (5.51), we get

$$ V(t) = V(0)E_{\sigma}\left(-\frac{\delta_3}{\delta_2}t^{\sigma}\right) - K(t) * \left[t^{\sigma-1}E_{\sigma,\sigma}\left(-\frac{\delta_3}{\delta_2}t^{\sigma}\right)\right]. $$

Since $t^{\sigma-1}$ and $E_{\sigma,\sigma}\left(\frac{\delta_3}{\delta_2}t^{\sigma}\right)$ are nonnegative functions, then

$$ V(t) \le V(0)E_{\sigma}\left(-\frac{\delta_3}{\delta_2}t^{\sigma}\right). \tag{5.52} $$

If we substitute (5.52) in to (5.48), we get

$$ \|x(t)\| \le \left[\frac{V(0)}{\delta_1}E_{\sigma}\left(-\frac{\delta_3}{\delta_2}t^{\sigma}\right)\right]^{\frac{1}{h}}, $$

for $x(0)\neq 0$, then, $\frac{V(0)}{\delta_1} > 0$.

Suppose that $k = \frac{V(0)}{\delta_1} = \frac{V(0,x(0))}{\delta_1} \ge 0$, then, we obtain

$$\|x(t)\| \leq \left[ kE_{\sigma}\left( -\frac{\delta_3}{\delta_2} t^{\sigma} \right) \right]^{\frac{1}{h}}.$$

Here it can said that $k = 0$ holds if and only if $x(0) = 0.$ Because $V(t, x)$ is locally Lipschitz according to $x$ and $V(0, x(0)) = 0$ if and only if $x(0) = 0.$ Therefore $k = \frac{V(0, x(0))}{\delta_1}$ is also Lipschitz according to $x(0)$ and $k(0) = 0,$ meaning that the system (5.10) is Mittag-Leffler stable.

**Example 3 [38]**: Suppose we have the following system

$$_0D_t^{\sigma}|g(t)| = -|g(t)|, \tag{5.53}$$

in which $\sigma \in (0,1)$. Let us consider the function candidate Lipschitz $V(t, g) = |g|$. By Lemma 2 we have

$${}_0^C D_t^{\sigma} V = {}_0^C D_t^{\sigma}|g| \leq {}_0D_t^{\sigma}|g| = {}_0D_t^{\sigma} V \leq -|g|. \tag{5.54}$$

Let $\sigma_1 = \sigma_2 = 1$ and $\sigma_3 = -1$, if we apply the Theorem 6, we obtain

$$|g(t)| \leq |g(0)| E_{\sigma}(-t^{\sigma}).$$

This implies that the system (5.53) is Mittag-Leffler stable.

### 5.6 Lyapunov–Krasovskii stability theory with time-delay

Through this part we study the stability of fractional order time-delay nonlinear systems by using the Lyapunov–Krasovskii theory. The definition of time delay can be as time interval of an event starting in one point to another point in the output within the system, which can occur in several areas, especially in chemical, biological, physical and economic systems, in addition in the processes of computation and measurement [52]. The existence of a Lyapunov–Krasovskii functional is a necessary as well as sufficient condition for the globally exponential stability and the uniform globally asymptotically stability of autonomous systems [52].

#### 5.6.1 Nonlinear time-delay systems

Let us consider a Caputo fractional time-delay nonlinear system [42], namely

$$ {}_{t_0}^{C}D_t^{\sigma} x(t) = g(t, x_t). \tag{5.55} $$

Whereas $x(t) \in R^n$, $0 < \sigma < 1$ and $g : R \times U \to R^n$, so, to achieve the evolution of the state, we can determine the initial case variables $x(t)$ in the interval of a time of length $l$, starting from $t_0 - l$ to $t_0$, such that [42]

$$ x_{t_0} = \varphi. \tag{5.56} $$

Here $\varphi \in U$, and $x(t_0 + \theta) = \varphi(\theta)$, $-l \le \theta \le 0$.

To explain the next definitions, let $U([h,k], R^n)$ be the set of continuous functions mapping the interval $[h,k]$ to $R^n$. If we want to identify $l$ as a maximum time delay of a system, we can define the set of continuous closed interval function mapping $[-l,0]$ to $R^n$. Let $U = U([-l,0], R^n)$. For every $B \succ 0$ and a continuous function $\Psi \in U([t_0 - l, t_0 + B], R^n)$, for $t_0 \le t \le t_0 + B$, let $\Psi_t \in U$ be a part of function $\Psi$ and the definition $\Psi_t(\theta) = \Psi_t(t + \theta)$, $-l \le \theta \le 0$.

**Definition 5 [52]**: Let $\varphi \in U([h,k], R^n)$, the uniform norm of $\varphi$ can be defined as

$$ \|\varphi\|_U = \max_{h \le \theta \le k} \|\varphi(\theta)\|. \tag{5.57} $$

**Definition 6 [52]**: Suppose the time-delay system (5.55), therefore the solution at $x(0) = 0$ can be:

1. Stable, if for every $t_0 \ge 0$, for each $\varepsilon > 0$, there is $\delta = \delta(t_0, \varepsilon) > 0$ that achieves [52]

$$ \|x(t_0)\|_c \le \delta \to \|x(t)\| \le \varepsilon, \text{ if } t \ge t_0. \tag{5.58} $$

2. Attractive, if for every $t_0 \ge 0$ and any $\varepsilon > 0$, there is $\delta_h = \delta_h(t_0, \varepsilon) > 0$ that achieves the property [52]

$$ \|x(t_0)\|_U \le \delta_h \to \lim_{t \to \infty} \|x(t)\| = 0. $$

(5.59)

3. Asymptotic stable, since it is stable as well as attractive [52].
4. Uniformly stable, in addition, it is stable, and $\delta = \delta(\varepsilon) > 0$ can be selected in an independent form of $t_0$ [52].

5. Uniformly asymptotic stable, in addition to it is uniform stability, there is a $\delta_0 > 0$ and function $\delta(\varepsilon)$, $T(\varepsilon)$ as $\|x_{t_0}\| < \delta_0$ and $t \geq t_0 + T(\varepsilon)$ for $\|x(t)\| < \varepsilon$ [52].
6. Globally uniformly asymptotic stable, in addition to the fact that it is uniformly asymptotically stable, so, $\delta_0$ can be finite number and arbitrary large [52].
7. Exponentially stable, if there are $\delta, \sigma > 0$ and $\mu \geq 1$ such that [52]

$$\|x(t_0)\|_U \leq \delta \rightarrow \|x(t)\| \leq \mu e^{-\sigma t}\|x_0\|. \tag{5.60}$$

### 5.6.2 Lyapunov-Krasovskii stability theorem

Before we begin to prove the theorem, it is better to mention that if $V(t,\varphi)$ is a differentiable function, we can suppose that $x_t(\tau,\varphi)$ is the solution of (5.55) at the time $t$ with initial condition $x_t = \varphi$. We can define the Caputo derivative of the function $V(t,x_t)$ according to $t$ and evaluate it at $t = \tau$, for $0 \prec \sigma \prec 1$, then, we get [42]

$$ {}_{t_0}^{C}D_t^{\sigma} V(\tau,\varphi) = {}_{t_0}^{C}D_t^{\sigma} V(t, x_t(\tau,\varphi))\Big|_{t=\tau, x_t=\varphi} = \frac{1}{\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{V'(s,x_s)}{(t-s)^{\sigma}} ds \Big|_{t=\tau, x_t=\varphi}. \tag{5.61}$$

The requirement for the time-delay system of the state at time $t$ can be the value of $x(t)$ in the interval $[t-l, t]$, and the Lyapunov function $V(t,x_t)$ depending on $x_t$ [42].

**Theorem 7 [42]**: Let $g: R \times U \rightarrow R$ in (5.19) maps $R \times$ (bounded sets in $U$) to be bounded sets in $R^n$, and $\rho_1, \rho_2, \rho_3 : \overline{R}_+ \rightarrow \overline{R}_+$ are continuous non-decreasing functions, where additionally $\rho_1(s), \rho_2(s)$ are positive for $s > 0$ and $\rho_1(0) = \rho_2(0) = 0$, there is a continuous differentiable function $V : R \times S_\alpha \rightarrow R$, where $S_\alpha = \{\varphi \in U : \|\varphi\|_U < \alpha\}$, then,

$$\rho_1(\|\varphi(0)\|) \leq V(t,\varphi) \leq \rho_2(\|\varphi\|_U), \tag{5.62}$$

$$ {}_{t_0}^{C}D_t^{\sigma} V(t,\varphi) \leq -\rho_3(\|\varphi(0)\|). \quad 0 < \sigma < 1. \tag{5.63}$$

Therefore, the system (5.55) is uniformly stable, whereas if $\rho_3(s) > 0$ for $(s) > 0$ then, it is uniformly asymptotic stable. Moreover, if $\lim_{s\rightarrow\infty} \rho_1(s) = \infty$, therefore, it is globally uniformly asymptotic stable.

**Proof [42]:** Since $\rho_2$ is continuous and $\rho_2(0)=0$. Let us say that $\varepsilon \succ 0$ we can find a sufficiently small $\delta=\delta(\varepsilon)>0$, such that $\rho_2(\delta)<\rho_1(\varepsilon)$. It is important to note that for any initial time $t_0$ and any initial condition $x_{t_0}=\varphi$ with $\|\varphi\|_h<\delta,$ we have ${}_{t_0}^{C}D_t^{\sigma}V(t,x_t)\le 0$ and from property 4, $V(t,x_t)\le V(t_0,\varphi),$ for eny $t\ge t_0$. This means that

$$\rho_1(\|x(t)\|)\le V(t,x_t)\le V(t_0,\varphi)\le \rho_2(\|\varphi\|_h)\le \rho_2(\delta)<\rho_1(\varepsilon), \tag{5.64}$$

where $\|x(t)\|<\varepsilon$ for $t\ge t_0$. Then the system (5.55) is uniform stable. To prove that the system being uniform asymptotic stable, suppose that $0<\varepsilon<\alpha$ and $\delta=\delta(\varepsilon)>0$ correspond to uniform stability, select $\varepsilon_0\le\alpha$ and appointed by $\delta_0=\delta(\varepsilon_0)>0$ and fixed $\varepsilon_0$. Now, we choose $\|x_{t_0}\|\le\delta_0$ and $T(\varepsilon)=\left[\frac{\rho_2(\delta_0)}{\rho_3(\delta(\varepsilon))}\Gamma(1+\sigma)\right]^{\frac{1}{\sigma}}$, thus $\delta(\varepsilon)$ corresponds to uniform stability, consider

$\|x_{t_0}\|\le\delta_0$ , if we have $\|x(t)\|\ge\delta(\varepsilon)$ for all $t\ge t_0$, then

$$-\rho_3(\|x(t)\|)\le -\rho_3(\delta(\varepsilon)). \tag{5.65}$$

In addition we have

$$ {}_{t_0}^{C}D_t^{\sigma}V(t,\varphi)\le -\rho_3(\delta(\varepsilon)), \text{ for } t\ge t_0 \ . \tag{5.66}$$

Then, by properties 2 and 3 we get

$$ {}_{t_0}^{C}D_t^{\sigma}\left(V(t,x_t)+\rho_3(\delta(\varepsilon))\frac{(t-t_0)^{\sigma}}{\Gamma(1+\sigma)}\right)\le 0. \tag{5.67}$$

If we use the property 4, we conclude

$$V(t,x_t)+\rho_3(\delta(\varepsilon))\frac{(t-t_0)^{\sigma}}{\Gamma(1+\sigma)}\le V(t_0,\varphi). \tag{5.68}$$

So, we have

$$\begin{aligned} V(t,x_t) &\le V(t_0,\varphi)-\rho_3(\delta(\varepsilon))\frac{(t-t_0)^{\sigma}}{\Gamma(1+\sigma)} \\ &\le \rho_2(\|\varphi\|_h)-\rho_3(\delta(\varepsilon))\frac{(t-t_0)^{\sigma}}{\Gamma(1+\sigma)} \\ &\le \rho_2(\delta_0)-\rho_3(\delta(\varepsilon))\frac{(t-t_0)^{\sigma}}{\Gamma(1+\sigma)}. \end{aligned} \tag{5.69}$$

If for any $t = t_0 + T(\varepsilon)$, we get

$$0 \prec \rho_1(\delta(\varepsilon)) \leq V(t_0 + T, x_{t_0+T}) \leq \rho_2(\delta_0) - \frac{\rho_3(\delta(\varepsilon))}{\Gamma(1+\sigma)} T^{\sigma} = 0. \tag{5.70}$$

This contradiction proves that there is a $t \in [t_0, t_0 + T(\varepsilon)]$, $\|x(t_1)\| < \delta(\varepsilon)$. We have $\|x(t)\| < \varepsilon$ and $t \geq t_0 + T(\varepsilon)$, when $\|x_{t_0}\| < \delta_0$,so, the system (5.55) is uniform asymptotic stable.

To prove globally, let us suppose $\lim_{s \to \infty} \rho_1(s) = \infty$, then, $\delta_0$ that we selected is arbitrary large, so, we choose $\varepsilon$ after $\delta_0$ that can satisfy $\rho_2(\delta_0) < \rho_1(\varepsilon)$. Thus we can conclude that the system is globally uniform asymptotic stable.

# CHAPTER 6

# APPLICATIONS OF LYAPUNOV STABILITY THEORY

This chapter reviews the Lyapunov direct method and shows the stability properties of mathematical models in biology using the fractional order systems.

## 6.1 Finding the Lyapunov candidate

In this section we show a new Lemma to achieve the stability of fractional derivatives by quadratic Lyapunov functions in the sense of Caputo when $0 < \sigma < 1$. This new Lemma helps to prove the stability of the diseases equilibrium in fractional-order, for example, epidemic systems [41,60].

**Lemma 1 [41]**: Let us consider $x(t) \in R$ is a continuously differentiable function. If for every time instant $t \geq t_0$ we conclude that

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\sigma} x^2(t) \leq x(t)\,{}_{t_0}^{C}D_t^{\sigma} x(t), \quad \forall \sigma \in (0,1). \tag{6.1}$$

**Proof [41]**: Suppose that expression (6.1) is true, so, it is equivalent to prove that,

$$x(t)\,{}_{t_0}^{C}D_t^{\sigma} x(t) - \frac{1}{2}\,{}_{t_0}^{C}D_t^{\sigma} x^2(t) \geq 0, \quad \forall \sigma \in (0,1). \tag{6.2}$$

Using the definition of Caputo fractional derivative (5.4)

$${}_{t_0}^{C}D_t^{\sigma} g(t) = \frac{1}{\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{\dot{x}(\tau)}{(t-\tau)^{\sigma}}\, d\tau, \tag{6.3}$$

and in the same way we conclude

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\sigma} x^2(t) = \frac{1}{\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{x(\tau)\dot{x}(\tau)}{(t-\tau)^{\sigma}}\, d\tau. \tag{6.4}$$

So, the expression (6.4) is defined as

$$\frac{1}{\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{[x(t)-x(\tau)]\dot{x}(\tau)}{(t-\tau)^{\sigma}}\, d\tau \geq 0. \tag{6.5}$$

If we define the auxiliary variable $g(\tau) = x(t) - x(\tau),$ meaning that

$$g'(\tau) = \frac{dg(\tau)}{d\tau} = -\frac{dx(\tau)}{d\tau},$$

then, the expression (6.5) becomes

$$\frac{1}{\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{g(\tau) g'(\tau)}{(t-\tau)^{\sigma}} d\tau \le 0 . \qquad (6.6)$$

Here we apply the integration by parts on expression (6.6). So, we get [41]

$du = g(\tau) g'(\tau) d\tau$ , $\quad u = \frac{1}{2} g^2,$

$v = \frac{1}{\Gamma(1-\sigma)} (t-\tau)^{-\sigma},$ $dv = \frac{\sigma}{\Gamma(1-\sigma)} (t-\tau)^{-\sigma-1}$, respectively

In that way, the expression (6.6) is defined as:

$$-\left[ \frac{g^2(\tau)}{2\Gamma(1-\sigma)(t-\tau)^{\sigma}} \right]\Bigg|_{\tau=t} + \left[ \frac{g_0^2}{2\Gamma(1-\sigma)(t-t_0)^{\sigma}} \right] + \frac{\sigma}{2\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{g^2(\tau)}{(t-\tau)^{\sigma+1}} d\tau \ge 0 . \qquad (6.7)$$

If we check the first term of expression (6.7), which has an indetermination at $\tau = t$. Let us analyze the corresponding limit, namely

$$\lim_{\tau \to t} \frac{g^2(\tau)}{2\Gamma(1-\sigma)(t-\tau)^{\sigma}} = \frac{1}{2\Gamma(1-\sigma)} \lim_{\tau \to t} \frac{[x(t) - x(\tau)]^2}{(t-\tau)^{\sigma}}$$

$$= \frac{1}{2\Gamma(1-\sigma)} \lim_{\tau \to t} \frac{[x^2(t) - 2x(t)x(\tau) + x^2(\tau)]}{(t-\tau)^{\sigma}} . \qquad (6.8)$$

Given that the function is derivable, L'Hopital's rule can be applied, then [41]:

$$\frac{1}{2\Gamma(1-\sigma)} \lim_{\tau \to t} \frac{[x^2(t) - 2x(t)x(\tau) + x^2(\tau)]}{(t-\tau)^{\sigma}}$$

$$= \frac{1}{2\Gamma(1-\sigma)} \lim_{\tau \to t} \frac{[-2x(t)\dot{x}(\tau) + 2x(\tau) + \dot{x}(\tau)]}{-\sigma(t-\tau)^{\sigma-1}}$$

$$= \frac{1}{2\Gamma(1-\sigma)} \lim_{\tau \to t} \frac{[2x(t)\dot{x}(\tau) - 2x(\tau)\dot{x}(\tau)](t-\tau)^{1-\sigma}}{\sigma} = 0 . \qquad (6.9)$$

So, the expression (6.7) is reduced to [41]

$$\frac{g_0^2}{2\Gamma(1-\sigma)(t-t_0)^{\sigma}} + \frac{\sigma}{2\Gamma(1-\sigma)} \int_{t_0}^{t} \frac{g^2(\tau)}{(t-\tau)^{\sigma+1}} d\tau \ge 0 .$$

(6.10)

Expression (6.10) is clearly true, and this concludes the proof [41].

**Example 1 [41]**: Suppose the fractional order nonlinear system with $0 < \sigma < 1$,

$$\begin{aligned} {}_0^C D_t^\sigma z(t) &= -z(t) + y^3(t), \\ {}_0^C D_t^\sigma y(t) &= -z(t) - y(t). \end{aligned} \tag{6.11}$$

We choose the Lyapunov candidate function as

$$V(z(t), y(t)) = \frac{1}{2} z^2(t) + \frac{1}{4} y^4(t). \tag{6.12}$$

So, applying the Lemma 1, we conclude that

$$\begin{aligned} {}_0^C D_t^\sigma V(z(t), y(t)) &= \frac{1}{2} {}_0^C D_t^\sigma z^2(t) + \frac{1}{4} {}_0^C D_t^\sigma y^4(t) \\ &\le z(t) {}_0^C D_t^\sigma z(t) + \frac{1}{2} y^2(t) {}_0^C D_t^\sigma y^2(t) \\ &\le z(t) {}_0^C D_t^\sigma z(t) + y^3(t) {}_0^C D_t^\sigma y(t) = -z^2(t) - y^4(t) \prec 0. \end{aligned} \tag{6.13}$$

If we notice (6.13) we can prove that the fractional derivative of Lyapunov function is *negative definite* function. Then, the system (6.11) is asymptotically stable.

## 6.2 Fractional–order gene regulatory networks

In this section we present the fractional–order gene regulatory networks and we check the global Mittag-Lefller stability as well as the global generalized Mittag-Lefller stability. The mechanism improved to regulate the expression of genes is called the gene regulatory networks [58].

A gene regulatory network includes a number of genes that can regulate the expression of every gene by proteins. Changes of these genes are governed by the translational processes and stimulation of proteins in transcriptional [58].

We consider fractional–order gene regulatory networks [58]

$$\begin{cases} D^\sigma y_i(t) = -e_i y_i(t) + \sum_{j=1}^{n} k_{ij} g_i(p_j(t)) + A_i \\ D^\sigma p_i(t) = -r_i p_i(t) + l_i y_i(t), i = 1,2,...,n. \end{cases} \tag{6.14}$$

If $\sigma \in (0,1)$, $y_i(t), p_i(t) \in R$ are the concentrations of messenger regulatory network acid (mRNA : is a transcription of prokaryotic protein-coding genes making messenger RNA ready to moves into protein), where the protein of the *ith* is node, and $e_i, r_i$ are the vanish rates of mRNA and the protein, when $l_i > 0$ is the translation rate, such that

$$g_i(x) = \left(\frac{x}{\mu_j}\right)^{G_j} \Bigg/ \left[1 + \left(\frac{x}{\mu_j}\right)^{G_j}\right],$$

This function can be monotonically increasing function and $G_j$ is the Hill coefficients, $\mu_j$ is the positive constant $A_i = \sum_{j \in I_i} a_{ij}$ , $a_{ij}$ are bounded constant, and $I_i$ can be the set of all $j$ that is a repressor of gene $i$. We have the matrix $K = (k_{ij}) \in R^{n \times n}$ which is the coupling matrix of the gene network, defined as [58]

$$k_{ij} = \begin{cases} a_{ij}, & \text{if reproduction factor } j \text{ is an animator of gene } i, \\ -a_{ij}, & \text{if reproduction factor } j \text{ is an repressor of gene } i, \\ 0, & \text{if there is no link from node } j \text{ to } i. \end{cases}$$

One gene or mRNA $y_i$ is generally activated by multiple proteins $p = (p_1, p_2, ..., p_n)^T$ in the transcription process for (6.14) [58].

**Definition 1 [58]**: The vectors $y^* = (y_1^*, ..., y_n^*)^T$ , $p^* = (p_1^*, ..., p_n^*)^T$ is an equilibrium point of fractional gene regulatory networks (FGRNs) if and only if [58]:

$$\begin{cases} -e_i y_i^* + \sum_{j=1}^{n} k_{ij} g_i(p_j^*) + A_i = 0, \\ -r_i p_i^* + l_i y_i(t) = 0, i = 1,2,..., n. \end{cases} \quad (6.15)$$

To prove the theorems of global Mittag-Lefller stability and global generalized Mittag-Lefller stability of an equilibrium point for (6.14), we should present the next Lemma.

**Lemma 2 [58]**: Suppose $V(t)$ be a continuous function on $[0, \infty)$ and satisfies

$$D^{\sigma} V(t) \le -\rho V(t).$$

Let $\sigma \in (0,1)$ and $\rho$ is a constant, then

$$V(t) = V(0)E_{\sigma}(-\rho t^{\sigma}),\ t \geq 0 .$$

**Definition 2 [58]**: If there are positive constants $\mu_i (i = 1,2,...,2n),$ then, we have

$$\begin{cases} \delta_i = e_i - \dfrac{\mu_{n+i}}{\mu_i} l_i > 0, \\ \delta_{n+i} = r_i - \displaystyle\sum_{j=1}^{n} \dfrac{\mu_i}{\mu_{n+i}} |w_{ji}| k_i > 0. \end{cases} \quad i = 1,2,..., n. \tag{6.16}$$

Here we shift the equilibrium point $(y^{*T}, p^{*T})^T$ of fractional gene regulatory networks in (6.14) to the origin, by using [58]:

$u_i(t) = y_i(t) - y_i^*, \qquad v_i(t) = p_i(t) - p_i^*,\ i = 1,2,..., n.$

Then, fractional gene regulatory networks in (6.14) can be transformed as

$$\begin{cases} D^{\sigma} u_i(t) = -e_i u_i(t) + \displaystyle\sum_{j=1}^{n} k_{ij} g_j(v_j(t)) \\ D^{\sigma} v_i(t) = -r_i v_i(t) + l_i u_i(t), i = 1,2,..., n. \end{cases} \tag{6.17}$$

Here $g_j(v_j(t)) = f_j(v_j(t) + p_j^*) - f_i(p_j^*).$

The above Lemma is useful to prove the next Theorem

**Theorem 1 [58]**: Assume that definition 2 holds, then, the fractional gene regulatory networks of (6.14) is globally Mittag-Leffler stable.

**Proof [58]**: Ler construct the function $V(t)$ as

$$V(t) = \sum_{i=1}^{n} \mu_i |u_i(t)| + \sum_{i=1}^{n} \mu_{n+i} |v_i(t)| .$$

From [66] we had $D^{\sigma}|u_i(t)| = \mathrm{sgn}(u_i(t)) D^{\sigma} u_i(t).$

Taking the fractional order derivative of $V(t)$ along the solution of (6.17), we get

$$D^{\sigma}V(t)=\sum_{i=1}^{n}\mu_i D^{\sigma}|u_i(t)|+\sum_{i=1}^{n}\mu_{n+i}D^{\sigma}|v_i(t)|$$

$$=\sum_{i=1}^{n}\mu_i\ \mathrm{sgn}(u_i(t))\left\{-e_iu_i(t)+\sum_{j=1}^{n}k_{ij}f_j(v_j(t))\right\}+\sum_{i=1}^{n}\mu_{n+i}\ \mathrm{sgn}(v_i(t))\{-r_i(v_i(t))+l_iu_i(t)\}$$

$$\leq\sum_{i=1}^{n}\mu_i\left\{-e_i|u_i(t)|+\sum_{j=1}^{n}|k_{ij}||w_j||(v_j(t))|\right\}+\sum_{i=1}^{n}\mu_{n+i}\{-r_i|v_i(t)|+l_i|u_i(t)|\}$$

$$=-\sum_{i=1}^{n}\{\mu_ie_i|u_i(t)|-\mu_{n+i}l_i|u_i(t)|\}-\sum_{i=1}^{n}\left\{\mu_{n+i}r_i|v_i(t)|-\sum_{j=1}^{n}|k_{ji}||w_i||v_i(t)|\right\}$$

$$=-\sum_{i=1}^{n}\mu_i\left(e_i-\frac{\mu_{n+i}}{\mu_i}l_i\right)|u_i(t)|-\sum_{i=1}^{n}\mu_{n+i}\left(r_i-\sum_{j=1}^{n}\frac{\mu_j}{\mu_{n+i}}|k_{ji}||w_i|\right)|v_i(t)|$$

$$\leq-\rho\left[\sum_{i=1}^{n}\mu_i|u_i(t)|+\sum_{i=1}^{n}\mu_{n+i}|v_i(t)|\right]$$

$$\leq-\rho V(t).$$

According to Lemma 2, where $\rho=\min_{1\leq i\leq n}\{\rho_i,\rho_{n+i}\}$, $D^{\sigma}V(t)\leq-\rho V(t)$. Which means that

$$\sum_{i=1}^{n}|y_i(t)-y_i^*|+\sum_{i=1}^{n}|p_i(t)-p_i^*|\leq\frac{\mu_{\max}}{\mu_{\min}}\left(\sum_{i=1}^{n}|y_i(0)-y_i^*|+\sum_{i=1}^{n}|p_i(0)-p_i^*|\right)E_{\sigma}(-\rho t^{\sigma}),$$

therefore,

$$\|y(t)-y^*\|+\|p(t)-p^*\|\leq Y(\|y_0-y^*\|+\|p_0-p^*\|)E_{\sigma}(-\rho t^{\sigma}),\ \text{for}\ t\geq 0.$$

Here $Y=\frac{\mu_{\max}}{\mu_{\min}}$. So, the fractional gene regulatory networks of (6.14) is globally Mittag-Leffler stable.

Before proving the generalized Mittag-Leffler stability of the system (6.14), we should define the next Lemma.

**Lemma 3 [58]**: Suppose $V(t)$ is a continuous function on $[0,\infty)$ satisfying

$$D^{\sigma}V(t)\leq\theta V(t),$$

and $0<\sigma<1$, where $\theta$ is a constant, then, there exist constant $t_1$ and $\gamma$ achieve

$$V(t)\leq V(0)t^{-\gamma}E_{\sigma,1-\gamma}(\theta t^{\sigma}),\ t\geq t_1.$$

**Proof [58]**: We find a nonnegative function $Y(t)$ such that

$$D^{\sigma}V(t)+Y(t)\le \theta V(t). \tag{6.18}$$

Here we take the Laplace transform of (6.18) and we have

$$s^{\sigma}V(s)-V(0)s^{\sigma-1}+Y(s)=\theta V(s).$$

Since $V(s)=\mathcal{L}\{V(t)\}$ and $Y(s)=\mathcal{L}\{Y(t)\}$,

let

$\mathcal{L}\{\tilde{Y}(t)\}=\tilde{y}(s)=Y(s)-V(0)s^{\sigma-1}+V(0)s^{\sigma-\tilde{\sigma}}$, where $\tilde{\sigma}\in[\sigma,1+\sigma)$. We obtain that

$$V(s)=\frac{V(0)s^{\sigma-1}-Y(s)}{s^{\sigma-\theta}}=\frac{V(0)s^{\sigma-\tilde{\sigma}}}{s^{\sigma}-\theta}$$

If we use the inverse Laplace transform, then, the solution of (6.18) become

$$V(t)=V(0)t^{\tilde{\sigma}-1}E_{\sigma,\tilde{\sigma}}(\theta t^{\sigma})-\tilde{Y}(t)*\left[t^{\sigma-1}E_{\sigma,\sigma}(\theta t^{\sigma})\right],$$

where * is a convolution operation and [58]

$$\tilde{Y}(t)=\mathcal{L}^{-1}(\tilde{Y}(s)=Y(t)-V(0)\frac{t^{-\sigma}}{\Gamma(1-\sigma)}+V(0)\frac{t^{\tilde{\sigma}-\sigma-1}}{\Gamma(\tilde{\sigma}-\sigma)}.$$

Since $t^{\sigma-1}$ and $E_{\sigma,\sigma}(\theta t^{\sigma})$ are nonnegative functions [67], it follows from $\sigma\in(0,1)$ when $\varepsilon>0$ and $t_1>0$ that [58]

$$\left[\frac{t^{\tilde{\sigma}-\sigma-1}}{\Gamma(\tilde{\sigma}-\sigma)}-\frac{t^{-\sigma}}{\Gamma(1-\sigma)}\right]*\left[t^{\sigma-1}E_{\sigma,\sigma}(\theta t^{\sigma})\right]\ge 0, \tag{6.19}$$

for all $t\ge t_1$ and $\tilde{\sigma}\in(1+\sigma-\varepsilon,1+\sigma)$. Therefore,

$$\tilde{Y}(t)*\left[t^{\sigma-1}E_{\sigma,\sigma}(\theta t^{\sigma})\right]\ge 0,$$

for all $t\ge t_1$ and $\tilde{\sigma}\in(1+\sigma-\varepsilon,1+\sigma)$, since

$$V(t)=V(0)t^{\tilde{\sigma}-1}E_{\sigma,\tilde{\sigma}}(\theta t^{\sigma})-\tilde{Y}(t)*t^{\sigma-1}E_{\sigma,\sigma}(\theta t^{\sigma}).$$

Here, we obtain

$$V(t)\le V(0)t^{\tilde{\sigma}-1}E_{\sigma,\tilde{\sigma}}(\theta t^{\sigma})$$

for all $t\ge t_1$ and $\tilde{\sigma}\in(1+\sigma-\varepsilon,1+\sigma)$, let $\tilde{\sigma}-1=-\gamma$, then, we get [58]

$$V(t)\le V(0)t^{-\gamma}E_{\sigma,1-\gamma}(\theta t^{\sigma}).$$

**Theorem 2 [58]**: Assume that the Definition 2 holding, the view that there is a constant $t_1$ such that fractional gene regulatory networks (6.14) is globally generalized Mittag-Leffler stable, for $t\ge t_1$.

**Proof**: See [58].

### 6.3 Stabilization of continuous-time fractional systems

Through this section we explain the stabilization of continuous-time fractional for positive linear systems.

**1- Stability**

Let us consider a linear fractional order continuous time system [59], namely

$$D^{\sigma} g(t) = Bg(t). \tag{6.20}$$

Let $0 < \sigma \le 1,\ g_0 \ge 0,$ suppose $B$ is Metzler matrix.

**Definition 3 [59]**: A matrix $B \in R^{n \times n}$ is named the Metzler if all of its off-diagonal entries are nonnegative i.e. $B = [bij] \in R^{n \times n}, bij \ge 0, i \ne j$.

**Lemma 4 [59]**: We can say that the continuous-time fractional system (6.20) is positive if and only if $B$ is a Metzler matrix.

**Proposition 1 [59]**: The function $V(g(t)) = \rho^T g(t)$, $\rho > 0$ is a Lyapunov function for the positive system (6.20) if and only if

$$\rho^T B < 0. \tag{6.21}$$

**Proof [59]**:

**Necessity**: $B$ is a Metzler matrix, then, by Lemma 4, the system is positive, it means that $g(t) \ge 0$, to prove the necessity condition. Suppose that $V(g(t)) = \rho^T g(t),$ $\rho > 0$, is a Lyapunov function for the system (6.20), $D^{\sigma} V(g(t))$ can be *negative definite*. Taking the Riemann-Liouville fractional derivative with respect to (6.20), then we get

$$D^{\sigma} V(g(t)) = {}^{C}D^{\sigma} V(t)) + \frac{V(0)}{\Gamma(1-\sigma) t^{\sigma}}$$

$$= \rho^T ({}^{C}D^{\sigma}) g(t) + \frac{\rho^T g_0}{\Gamma(1-\sigma) t^{\sigma}}$$

$$= \rho^T \left[ ({}^C D^\sigma) g(t) + \frac{g_0}{\Gamma(1-\sigma)t^\sigma} \right] = \rho^T D^\sigma g(t) = \rho^T Bg(t) < 0 .$$

This means that $\rho^T B < 0$.

**Sufficiency**: We assume that the condition (6.21) is true. Suppose now the function $V(g(t)) = \rho^T g(t)$ is *positive definite*, we compute its fractional derivative by the same way in the necessity part of the proof, since $D^\sigma V(g(t)) = \rho^T Bg(t)$. It follows that $D^\sigma V(g(t)) < 0$ ,then, by Theorem 1 in chapter 5, the function $V(g(t)) = \rho^T g(t)$ is the Lyapunov function of the system (6.20).

**2- Stabilizability**

Suppose a linear fractional order continuous time system

$$D^\sigma g(t) = Hg(t) + Bu(t). \tag{6.22}$$

Let $0 < \sigma \le 1$, $g \in R^n$, $g(0) = g_0 \ge 0$, $u \in R^m$ and $B$ is Metzler matrix. If we use a state feedback control $u(t) = Kg(t)$, then, we get the closed-loop system as follows

$$D^\sigma g(t) = (H + BK)g(t), \tag{6.23}$$

where $0 < \sigma \le 1$, $g(0) = g_0 \ge 0.$

**Theorem 3 [59]:** If we find a positive vector $\rho \in R^n$ and vectors $x_1, x_2, ..., x_n \in R^m$, then

$$G\rho + B\sum_{i=1}^{n} x_i < 0, \tag{6.24}$$

$$h_{ij}\rho_j + b_i x_j \ge 0. \quad i \ne j \tag{6.25}$$

Whereas $h_{ij}$ denotes the element ( $ij$ ) of the matrix $G$ and $b_i$ are the raw vectors of $B$, then (6.23) is asymptotic stable by remaining the state non negative $\forall g_0 \ge 0$.

**Proof**: See [59].

**Example 2 [59]**: Consider a fractional continuous time system

$$D^\sigma g(t) = Gg(t) + Hy(t), \tag{6.26}$$

with $0 < \sigma \le 1$, $g(0) = g_0 \ge 0$ and $\sigma = 0.5$. If we have the following system matrices:

$$G = \begin{bmatrix} -1 & -0.5 \\ -0.3 & -0.5 \end{bmatrix}, \qquad H = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix},$$

we can see in the matrix $G$, the open-loop system is not positive, because there are off-diagonal negative elements in $G$. We can design a case feedback controller $u = Kg$, to stabilized the system and make the closed-loop cases nonnegative. We take as [59]

$$K = \begin{bmatrix} 1.5707 & 1.5459 \end{bmatrix}.$$

Then, by multiply $H$ with $K$, we get the closed-loop for the new matrix

$$G + HK = G_C = \begin{bmatrix} -0.3717 & 0.1184 \\ 0.0141 & -0.1908 \end{bmatrix}.$$

Thus, the matrix $G_C$ is Metzler, and the eigenvalues of the matrix $G_C$ are $\{-0.3805 \quad -0.1820\}$. By Theorem 3, the system (6.26) is asymptotic stable [59].

### 6.4 The epidemic systems

In this section we present a Lemma which estimates the fractional derivatives of Volterra-type Lyapunov functions and study the uniform asymptotic stability in the Caputo's sense if $\sigma \in (0,1)$. this result is used in Caputo-type epidemic systems. The epidemic systems are the Susceptible Infected Recovered (SIR), Susceptible Infected Susceptible (SIS), Susceptible Infected Recovered-Susceptible (SIRS) and Ross Macdonald models for vector-borne diseases; consequently, if the basic reproductive number is greater than one, then, we can say that the unique endemic equilibrium is uniformly asymptotically stable [60].

**Lemma 5 [60]**: Suppose that $y(t) \in R^{+}$ be a derivable and continuous function. So, for each time instant $t \ge t_0$

$$ {}_{t_0}^{C}D_t^{\sigma}\left[ y(t) - y^* - y^* \ln \frac{y(t)}{y^*} \right] \le \left( 1 - \frac{y^*}{y(t)} \right) {}_{t_0}^{C}D_t^{\sigma} y(t), \; y^* \in R^{+}, \forall \sigma \in (0,1). \qquad (6.27)$$

**Proof [60]**: By direct calculation we conclude

$$ {}_{t_0}^{C}D_t^{\sigma} y(t) - {}_{t_0}^{C}D_t^{\sigma} y^* - y^* {}_{t_0}^{C}D_t^{\sigma}\left[\ln\frac{y(t)}{y^*}\right] \le \left(\frac{y(t)-y^*}{y(t)}\right) {}_{t_0}^{C}D_t^{\sigma} y(t) . \tag{6.28} $$

Here we apply the property 3 in (5.22), then, we conclude

$$ y(t) {}_{t_0}^{C}D_t^{\sigma} y(t) - y^* y(t) {}_{t_0}^{C}D_t^{\sigma}\left[\ln\frac{y(t)}{y^*}\right] \le \left(y(t)-y^*\right) {}_{t_0}^{C}D_t^{\sigma} y(t) . \tag{6.29} $$

Rewriting the inequality (6.29), we conclude

$$ {}_{t_0}^{C}D_t^{\sigma} y(t) - y(t) {}_{t_0}^{C}D_t^{\sigma}\left[\ln\frac{y(t)}{y^*}\right] \le 0 . \tag{6.30} $$

Using the Caputo fractional derivative in (5.4), so, we can write that as [60]

$$ {}_{t_0}^{C}D_t^{\sigma} y(t) = \frac{1}{\Gamma(1-\sigma)}\int_{t_0}^{t}\frac{\dot{y}(\mu)}{(t-\mu)^{\sigma}}\,d\mu . \tag{6.31} $$

In the same way we conclude [60]

$$ {}_{t_0}^{C}D_t^{\sigma}\left[\ln\frac{y(t)}{y^*}\right] = \frac{1}{\Gamma(1-\sigma)}\int_{t_0}^{t}\frac{1}{y(\mu)}\frac{\dot{y}(\mu)}{(t-\mu)^{\sigma}}\,d\mu . \tag{6.32} $$

So, we can write the inequality (6.30), namely

$$ \frac{1}{\Gamma(1-\sigma)}\int_{t_0}^{t}\left[\frac{y(\mu)-y(t)}{y(\mu)}\right]\frac{\dot{y}(\mu)}{(t-\mu)^{\sigma}}\,d\mu \le 0 . \tag{6.33} $$

Now, define the auxiliary variable $W(\mu) = \dfrac{y(\mu)-y(t)}{y(t)}$, which means that $\dot{W}(\mu) = \dfrac{\dot{y}(\mu)}{y(t)}$. In this way, the inequality (6.33) becomes [60]

$$ \frac{1}{\Gamma(1-\sigma)}\int_{t_0}^{t} y(t)\left(1-\frac{1}{W(\mu)+1}\right)\frac{\dot{W}(\mu)}{(t-\mu)^{\sigma}}\,d\mu \le 0 . $$

If we integrate the last integral by parts, we can get [60]

$$ v = \frac{1}{\Gamma(1-\sigma)}(t-\mu)^{-\sigma} , \qquad dv = \frac{\sigma}{\Gamma(1-\sigma)}(t-\mu)^{-\sigma-1} , $$

$$ du = y(t)\left(1-\frac{1}{W(\mu)+1}\right)\dot{W}(\mu)d\mu , \qquad u = y(t)(W(\mu)-\ln(W(\mu)+1)) . $$

We have the followings [60]

$$\frac{1}{\Gamma(1-\sigma)}\int_{t_0}^{t} y(t)\left(1-\frac{1}{W(\mu)}\right)\frac{\dot{W}(\mu)}{(t-\mu)^{\sigma}}\,d\mu =$$

$$\left[\frac{y(t)(W(\mu)-\ln(W(\mu)+1))}{\Gamma(1-\sigma)(t-\mu)^{\sigma}}\right]_{\mu=t} - \left[\frac{y(t)(W(t_0)-\ln(W(t_0)+1))}{\Gamma(1-\sigma)(t-t_0)^{\sigma}}\right]$$

$$-\frac{\sigma}{\Gamma(1-\sigma)}\int_{t_0}^{t}\frac{y(t)(W(\mu)-\ln(W(\mu)+1))}{(t-\mu)^{\sigma+1}}d\mu \le 0. \qquad (6.34)$$

We notice that the first part of (6.34) is an indetermination at $\mu = t$.

Now we analyze the corresponding limit [60]

$$\lim_{\mu\to t}\frac{y(t)(W(\mu)-\ln(W(\mu)+1))}{\Gamma(1-\sigma)(t-\mu)^{\sigma}}$$

$$=\frac{1}{\Gamma(1-\sigma)}\lim_{\mu\to t}\frac{y(t)(W(\mu)-\ln(W(\mu)+1))}{(t-\mu)^{\sigma}}$$

$$=\frac{1}{\Gamma(1-\sigma)}\lim_{\mu\to t}\frac{\left[y(\mu)-y(t)-y(t)\ln\frac{y(\mu)}{y(t)}\right]}{(t-\mu)^{\sigma}}.$$

Now, let us use L'Hopital's rule for the limit, differentiating both the numerator and the denomonator, we can obtain [60]

$$\frac{1}{\Gamma(1-\sigma)}\lim_{\mu\to t}\frac{\left[y(\mu)-y(t)-y(t)\ln\frac{y(\mu)}{y(t)}\right]}{(t-\mu)^{\sigma}}$$

$$=\frac{1}{\Gamma(1-\sigma)}\lim_{\mu\to t}\frac{\left[1-\frac{y(t)}{y(\mu)}\right]\dot{y}(\mu)}{\sigma(t-\mu)^{\sigma-1}}$$

$$=\frac{1}{\Gamma(1-\sigma)}\lim_{\mu\to t}\left[\frac{(t-\mu)^{1-\sigma}}{\sigma}\left(1-\frac{y(t)}{y(\mu)}\right)\dot{y}(\mu)\right]=0.$$

So, the inequality (6.34) is reduced to [60]

$$-\left[\frac{y(t)(W(t_0)-\ln(W(t_0)+1))}{\Gamma(1-\sigma)(t-t_0)^{\sigma}}\right]-\frac{\sigma}{\Gamma(1-\sigma)}\int_{t_0}^{t}\frac{y(t)(W(\mu)-\ln(W(\mu)+1))}{(t-\mu)^{\sigma+1}}d\mu \le 0,$$

or equivalenty

$$-\left[\frac{y(t_0)-y(t)-y(t)\ln\frac{y(t_0)}{y(t)}}{\Gamma(1-\sigma)(t-\mu)^{\sigma}}\right]-\frac{\sigma}{\Gamma(1-\sigma)}\int_{t_0}^{t}\frac{y(\mu)-y(t)-y(t)\ln\frac{y(\mu)}{y(t)}}{(t-\mu)^{\sigma+1}}d\mu\leq 0. \quad (6.35)$$

It is easy to see that the inequality (6.35) is true, and this concludes the proof.

Now, we review an example using this Lemma to investigate the stability of some fractional-order differential equation models of infectious diseases.

**Example 3 [60]**: The differential equations for Susceptible-Infected-Susceptible model are

$$\begin{aligned}\frac{dx}{dt}&=\Lambda-\frac{\sigma xy}{x+y}-\beta x+\delta y,\\ \frac{dy}{dt}&=\frac{\sigma xy}{x+y}-(\alpha+\beta+\delta)y.\end{aligned} \quad (6.36)$$

Let $x$ be the number of susceptible individuals and $y$ be the number of infected individuals. Since the parameters and the initial conditions are positive values. If we refer to the feasible region of (6.36) by

$$\Omega=\left\{(x,y)\in R_{0+}^{2}:x\geq 0,y\geq 0,x+y\leq\frac{\Lambda}{\beta}\right\},$$

and

$$R_{0+}^{2}=\{(x,y)\in R^{2}:x\geq 0,y\geq 0\}.$$

If the system (6.36) has a basic reproductive number given by [60]

$$R_0=\frac{\sigma}{\alpha+\beta+\delta}.$$

Consequently, the system (6.36) has a disease-free (non-negative boundary) equilibrium $(\frac{\Lambda}{\beta},0)$, and an endemic equilibrium is $(x^*,y^*)$ when $R_0>0$. Therefore [60],

$$x^*=\frac{\Lambda}{\beta+(\alpha+\beta)(R_0-1)},\qquad y^*=\frac{(R_0-1)\Lambda}{\beta+(\alpha+\beta)(R_0-1)}. \quad (6.37)$$

So, the integer order system (6.36) is asymptotic stable in the interior of the feasible region $\Omega$, when the basic reproductive number $R_0>1$, then, we have a unique endemic equilibrium $(x^*,y^*)$. Note that if $(x^*,y^*)$ is asymptotic stable it is also uniformly asymptotic stable. In addition, if we apply fractional order model on the system (6.36) by using Caputo derivative, we get [60]

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\rho} x(t) &= \Lambda - \frac{\sigma xy}{x+y} - \beta x + \delta y, \\
{}_{t_0}^{C}D_t^{\rho} y(t) &= \frac{\sigma xy}{x+y} - (\alpha + \beta + \delta) y.
\end{aligned}
\tag{6.38}
$$

With the same equilibrium points found in (6.36), to achieve the uniformly asymptotically stable of the endemic equilibrium $(x^*, y^*)$, first we suppose Lyapunov function $L = \{(x, y) \in \Omega : x > 0, y > 0\} \to R$ and define as [60]

$$
\begin{aligned}
L(x, y) = &\left[ x + y - (x^* + y^*) - (x^* + y^*) \ln \frac{(x+y)}{x^* + y^*)} \right] \\
&+ \frac{(\alpha + 2\beta)(x^* + y^*)}{\sigma y^*} \left( y - y^* - y^* \ln \frac{y}{y^*} \right).
\end{aligned}
$$

If $L(x, y)$ is positive definite, continuous function for all $x > 0, y > 0$, since $(x^*, y^*)$ is an endemic equilibrium point of (6.38), then [60]

$$
\Lambda = \beta(x^* + y^*) + \alpha y^* \ , \quad (\alpha + \beta + \delta) = \frac{\sigma x^*}{x^* + y^*} . \tag{6.39}
$$

Using Lemma 5, we get

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\rho} L(x, y) &\le \left[ 1 - \frac{(x^* + y^*)}{x+y} \right] {}_{t_0}^{C}D_t^{\rho} (x+y) + \frac{(\alpha + 2\beta)(x^* + y^*)}{\sigma y} \left( 1 - \frac{y^*}{y} \right) {}_{t_0}^{C}D_t^{\rho} y \ , \\
&\le \frac{\left[ (x - x^*) + (y - y^*) \right]}{x+y} \left( \Lambda - \beta(x+y) - \alpha y \right) + \frac{(\alpha + 2\beta)(x^* + y^*)}{\sigma y} \ \frac{(y - y^*)}{y} \\
&\quad \left( \sigma \frac{xy}{x+y} - (\delta + \beta + \alpha) y \right).
\end{aligned}
$$

Now, we use (6.39) and we conclude

$$
\begin{aligned}
{}_{t_0}^{C}D_t^{\rho} L(x, y) \le &\frac{\left[ (x - x^*) + (y - y^*) \right]}{x+y} \left( -\beta(x - x^*) - (\alpha + \beta)(y - y^*) \right) \\
&+ \frac{(\alpha + 2\beta)(x^* + y^*)}{y^*} \quad (y - y^*) \left( \frac{x}{x+y} - \frac{x^*}{x^* + y^*} \right) .
\end{aligned}
$$

Then, we have

$$
{}_{t_0}^{C}D_t^{\rho} L(x, y) \le \frac{\left[ (x - x^*) + (y - y^*) \right]}{x+y} \left( -\beta(x - x^*) - (\alpha + \beta)(y - y^*) \right)
$$

$$+\frac{(\alpha+2\beta)(x^*+y^*)}{y^*}\ (y-y^*)\left(\frac{y^*(x-x^*)-x^*(y-y^*)}{(x^*+y^*)(x+y)}\right),$$

$$\leq -\beta\frac{(x+x^*)^2}{x+y}-\left(\alpha+\beta+(\alpha+2\beta)\frac{x^*}{y^*}\right)\frac{(y+y^*)^2}{x+y}.$$

We know that ${}_{t_0}^{C}D_t^{\rho}L(x,y)$ is a *negative definite* if $\rho\in(0,1)$ (by theorem 5 in chapter 5), then, the system (6.38) is uniform asymptotic stable inside of $\Omega$, with the coordinates in (6.37) [60].

# CHAPTER 7

# CONCLUSION

In this thesis we have presented the concept of Lyapunov stability theory and some of its applications in a detailed overview.

At the beginning of the thesis we recall some preliminaries and definitions that were useful for the context of the study. After that we have defined the Lyapunov function, methods and theorems to determine the stability properties of the dynamical systems. In the example of pendulum, we have seen that the origin is stable, but not asymptotic stable. In this case we have applied the Lasalle's invariance principle to prove the asymptotic stability. In the indirect method we have showed that the local stability of a system is studied through the Jacobian matrix. If the real parts of its eigenvalues are all strictly negative, the equilibrium point, then, is locally stable, but if at least one is strictly positive, then, it is unstable. The next topic was about a review of the Lyapunov stability for discrete time systems. Also to review the Lyapunov fractional stability, we have presented the stability analysis in the Caputo and Riemann-Liouville senses. We have reviewed Bihari's and Bellman-Grönwall's inequality, Mittag-Leffler stability and Lyapunov-Krasovskii theorem with time delay.

Finally, we have presented some applications of Lyapunov stability theory. In the first application we have recalled a new Lemma that helps to satisfy the stability of fractional derivatives by quadratic Lyapunov functions in the sense of Caputo when $0 < \sigma < 1$. In the second one a class of fractional order gene regulatory networks has reviewed. Some criteria of the Mittag–Leffler stability and generalized Mittag–Leffler stability have been shown by utilising the fractional Lyapunov method for these networks. In the third application, the stabilization problem for continuous-time fractional linear systems with the additional condition of non negativity of the states has been discussed. Finally, the Volterra-type Lyapunov functions has been used to prove the stability of equilibrium points in integral order epidemic systems, to estimate the uniform asymptotic stability of the Caputo-type epidemic systems.

I hope that my thesis can be considered as a review about the Lyapunov function and some of its applications, will help the young researchers in their studies about the fascinating area of stability of the dynamics of complex systems.

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# APPENDICES

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## TEZ VERİ GİRİŞİ VE YAYIMLAMA İZİN FORMU

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| Üniversite | Çankaya Üniversitesi |
| Enstitü / Hastane | Fen Bilimleri Enstitüsü |
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