

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/345734767>

Exact analytical solutions of fractional order telegraph equations via triple Laplace transform

Article in *Discrete & Continuous Dynamical Systems - S* - January 2018

DOI: 10.3934/dcdss.2020427

CITATIONS

0

READS

61

3 authors:



Rahmat Ali Khan

University of Malakand

188 PUBLICATIONS 3,105 CITATIONS

SEE PROFILE



Yongjin Li

Sun Yat-Sen University

200 PUBLICATIONS 1,573 CITATIONS

SEE PROFILE



Fahd Jarad

Cankaya University

244 PUBLICATIONS 4,777 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



Coherent control of high photon and plasmon polariton sensing [View project](#)



Fuzzy sets theory [View project](#)

EXACT ANALYTICAL SOLUTIONS OF FRACTIONAL ORDER TELEGRAPH EQUATIONS VIA TRIPLE LAPLACE TRANSFORM

RAHMAT ALI KHAN

Department of Mathematics, University of Malakand, Chakadara Dir(L), Khyber
Pakhtunkhwa, Pakistan

YONGJIN LI

Department of Mathematics, Sun Yat-Sen University
Guangzhou, China

FAHD JARAD*

Department of Mathematics, Çankaya University
06790 Etimesgut, Ankara, Turkey

ABSTRACT. In this paper, we study initial/boundary value problems for $1 + 1$ dimensional and $1 + 2$ dimensional fractional order telegraph equations. We develop the technique of double and triple Laplace transforms and obtain exact analytical solutions of these problems. The techniques we develop are new and are not limited to only telegraph equations but can be used for exact solutions of large class of linear fractional order partial differential equations

1. Introduction. The fractional calculus has become one of the most notable branches of mathematics for the outstanding findings obtained when scientists working on various areas of science and engineering [13, 4, 16, 18, 25, 27]. The most conventional fractional derivatives having singular kernels are defined as follows:

Definition 1.1. [25, 16, 27] The Riemann- Liouville fractional order integral operator of order $p > 0$ of function ϕ starting at 0 is given as:

$$\begin{aligned} I^p \psi(v) &= \frac{1}{\Gamma(p)} \int_0^v (t - \tau)^{p-1} \phi(\tau) d\tau, \\ I^0 \phi(t) &= \phi(t). \end{aligned} \tag{1}$$

Definition 1.2. [25, 16, 27] The Caputo derivative of a function ψ of order $p \geq 0$ is defined as

$$\begin{aligned} {}^c D^p \phi(v) &= I^{k-p} D^k \phi(v) \\ &= \frac{1}{\Gamma(k-p)} \int_0^v (v - \tau)^{k-p-1} \psi^{(k)}(\tau) d\tau, \end{aligned} \tag{2}$$

for $k - 1 < p \leq k$, $k \in \mathbb{N}$.

2010 *Mathematics Subject Classification.* Primary: 26A33; Secondary: 34A08, 35R11.

Key words and phrases. Fractional telegraph equations, initial/ boundary value problems, double and triple Laplace transform, Exact solutions.

* Corresponding author.

On the other hand, integral transforms are very effective tools to solve analytically ordinary and partial differential equation with positive or fractional orders. For such integral transforms, we refer the readers to [14, 2] and the works cited in these references.

Meanwhile, The initial/boundary value problems for telegraph equation have a rich history and have a lot of applications. This equation was studied and modified by numerous researchers. We refer to [24, 5, 6, 26, 7, 8, 19, 20, 21, 22, 23, 11, 9, 10, 1, 3] for different techniques for obtaining the approximate solutions of telegraph equations. To the best of our knowledge, exact analytic solutions to two-dimensional telegraph equations has never been studied previously. In this paper, we aim to study initial/boundary value problems for fractional order telegraph equations via multiple Laplace transform method and obtain exact analytical solutions. we develop a technique to find the exact analytical solutions of 1 + 1 dimensional and 1 + 2 dimensional fractional order transient hyperbolic telegraph equation with initial conditions(IC) and boundary conditions(BC). The initial/ boundary value problems for fractional order transient hyperbolic telegraph equation are given as below

$$\begin{aligned} D_t^p u + 2\alpha D_t^q u + \beta^2 u &= D_x^p u + f(x, t), \quad 1 < p \leq 2, 0 < q \leq 1 \\ \text{IC: } u(x, 0) &= \phi_1(x), \quad \frac{\partial u}{\partial t}(x, 0) = \phi_2(x), \quad x \in \mathbb{R}_+, \\ \text{BC: } u(0, t) &= \psi_1(t), \quad \frac{\partial u}{\partial x}(0, t) = \psi_2(t), \quad t \in \mathbb{R}_+, \end{aligned} \quad (3)$$

where $(x, t) \in \mathbb{R}_+^2$, D_t^r is fractional partial derivative of order r with respect to t in the sense of Caputo, $u \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, $\phi_1, \phi_2, \psi_1, \psi_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ and f is a continuous source term. The two dimensional transient hyperbolic telegraph is of the form

$$\begin{aligned} D_t^p v + 2\alpha D_t^q v + \beta^2 v &= D_x^p v + D_y^p v + F(x, y, t), \quad 1 < p \leq 2, 0 < q \leq 1 \\ \text{IC: } v(x, y, 0) &= \rho_1(x, y), \quad \frac{\partial v}{\partial t}(x, y, 0) = \rho_2(x, y), \quad (x, y) \in \mathbb{R}_+^2, \\ \text{BC: } v(0, y, t) &= \phi_1(y, t), \quad \frac{\partial v}{\partial x}(0, y, t) = \phi_2(y, t), \quad (y, t) \in \mathbb{R}_+^2 \\ v(x, 0, t) &= \psi_1(x, t), \quad \frac{\partial v}{\partial y}(x, 0, t) = \psi_2(x, t), \quad (x, t) \in \mathbb{R}_+^2, \end{aligned} \quad (4)$$

where $v \in C(\mathbb{R}_+^3, \mathbb{R}_+)$, $\rho_1, \rho_2, \phi_1, \phi_2, \psi_1, \psi_2 \in C(\mathbb{R}_+^2, \mathbb{R}_+)$, F is a continuous source term and α, β are parameters.

2. Double and Tripple laplace transform. The double Laplace transform of a function $u(x, t)$ defined in the positive quadrant of the xt -plane is defined by the equation

$$L_x L_t \{u(x, t)\}(s_1, s_2) = \int_0^\infty \int_0^\infty e^{-(s_1 x + s_2 t)} u(x, t) dx dt,$$

provided that the integral exist, while the triple laplace transform of a function $v(x, y, t)$ is defined by the following equation

$$L_x L_y L_t \{v(x, y, t)\}(s_1, s_2, s_3) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s_1 x + s_2 y + s_3 t)} v(x, y, t) dx dy dt,$$

provided that the integrals exist. We need the following results for existence of double laplace transform for partial derivatives [2].

Theorem 2.1. [2] *Let $\alpha, \beta > 0$, $m - 1 < \alpha \leq m$, $n - 1 < \beta \leq n$, $n, m \in \mathbb{N}$. Let $l = \max\{m, n\}$, $u \in L_l((0, a) \times (0, b))$ for any $a, b > 0$, $|u(x, t)| < \lambda e^{x\tau_1 + t\tau_2}$ for $x > a > 0$, $t > b > 0$, the laplace transform of u and $\frac{\partial^{i+j}u(x,t)}{\partial x^i \partial t^j}$, $i = 0, 1, \dots, m$, $j = 0, 1, \dots, n$ exist. Then the following hold:*

$$\begin{aligned}
 L_x L_t \{D_x^\alpha u(x, t)\} &= s_1^\alpha L_x L_t \{u(x, t)\} - \sum_{i=0}^{m-1} s_1^{\alpha-1-i} L_t \left\{ \frac{\partial^i u}{\partial x^i}(0, t) \right\}, \\
 L_x L_t \{D_t^\beta u(x, t)\} &= s_2^\beta L_x L_t \{u(x, t)\} - \sum_{j=0}^{n-1} s_2^{\beta-1-j} L_x \left\{ \frac{\partial^j u}{\partial t^j}(x, 0) \right\}, \\
 L_x L_t \{D_x^\alpha D_t^\beta u(x, t)\} &= s_1^\alpha s_2^\beta L_x L_t \{u(x, t)\} - \sum_{i=0}^{m-1} s_1^{\alpha-1-i} L_x \left\{ \frac{\partial^i u}{\partial t^i}(x, 0) \right\} \\
 &\quad - \sum_{j=0}^{n-1} s_2^{\beta-1-j} L_t \left\{ \frac{\partial^j u}{\partial x^j}(0, t) \right\} + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} s_1^{\alpha-1-i} s_2^{\beta-1-j} \left\{ \frac{\partial^{i+j} u}{\partial x^i \partial t^j}(0, 0) \right\}
 \end{aligned} \tag{5}$$

The extension of Theorem (2.1) to triple Laplace transform is provided in [15] as the following :

Theorem 2.2. *For $\alpha, \beta, \gamma > 0$, $m - 1 < \alpha \leq m$, $n - 1 < \beta \leq n$, $r - 1 < \gamma \leq r$, $v \in C^l(\mathbb{R}_+^3, \mathbb{R}_+)$, $l = \max\{m, n, r\}$, there exist $\tau_1, \tau_2, \tau_3 > 0$ such that $|\frac{\partial^{i+j+k}v(x,y,t)}{\partial x^i \partial y^j \partial t^k}| < \lambda e^{x\tau_1 + y\tau_2 + t\tau_3}$, $i = 0, 1, \dots, m$, $j = 0, 1, \dots, n$, $k = 0, 1, \dots, r$, then the following hold:*

$$\begin{aligned}
 L_x L_y L_t \{D_x^\alpha v(x, y, t)\} &= s_1^\alpha L_x L_y L_t \{v(x, y, t)\} - \sum_{i=0}^{m-1} s_1^{\alpha-1-i} L_y L_t \left\{ \frac{\partial^i v}{\partial x^i}(0, y, t) \right\}, \\
 L_x L_y L_t \{D_y^\beta v(x, y, t)\} &= s_2^\beta L_x L_y L_t \{v(x, y, t)\} - \sum_{j=0}^{n-1} s_2^{\beta-1-j} L_x L_t \left\{ \frac{\partial^j v}{\partial y^j}(x, 0, t) \right\}, \\
 L_x L_y L_t \{D_t^\gamma v(x, y, t)\} &= s_3^\gamma L_x L_y L_t \{v(x, y, t)\} - \sum_{k=0}^{r-1} s_3^{\gamma-1-k} L_x L_y \left\{ \frac{\partial^k v}{\partial t^k}(x, y, 0) \right\} \\
 L_x L_y L_t \{D_x^\alpha D_y^\beta v(x, y, t)\} &= s_1^\alpha s_2^\beta L_x L_y L_t \{v(x, y, t)\} \\
 &\quad - \sum_{i=0}^{m-1} s_1^{\alpha-1-i} L_y L_t \left\{ \frac{\partial^i v}{\partial x^i}(0, y, t) \right\} - \sum_{j=0}^{n-1} s_2^{\beta-1-j} L_x L_t \left\{ \frac{\partial^j v}{\partial y^j}(x, 0, t) \right\} \\
 &\quad + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} s_1^{\alpha-1-i} s_2^{\beta-1-j} L_t \left\{ \frac{\partial^{i+j} v}{\partial x^i \partial y^j}(0, 0, t) \right\}.
 \end{aligned} \tag{6}$$

3. Solution of One dimensional telegraph equation. We now consider the initial/boundary value problem for one dimensional fractional order hyperbolic telegraph equation (3). Applying the double Laplace transform on the differential equation in (3) and using (5), we obtain

$$\begin{aligned}
 (s_2^p + 2\alpha s_2^q + \beta^2 - s_1^p) L_x L_t \{u\} &= (s_2^{p-1} + 2\alpha s_2^{q-1}) L_x \{u(x, 0)\} + s_2^{p-2} L_x \left\{ \frac{\partial u}{\partial t}(x, 0) \right\} \\
 &\quad - s_1^{p-1} L_t \{u(0, t)\} - s_1^{p-2} L_t \left\{ \frac{\partial u}{\partial x}(0, t) \right\} + L_x L_t \{f(x, t)\}.
 \end{aligned} \tag{7}$$

Using the initial and boundary conditions, we obtain

$$\begin{aligned} L_x L_t \{u(x, t)\} &= \frac{1}{(s_2^p + 2\alpha s_2^q + \beta^2 - s_1^p)} \{ (s_2^{p-1} + 2\alpha s_2^{q-1}) L_x \{ \phi_1(x) \} \\ &+ s_2^{p-2} L_x \{ \phi_2(x) \} - s_1^{p-1} L_t \{ \psi_1(t) \} - s_1^{p-2} L_t \{ \psi_2(t) \} + L_x L_t \{ f(x, t) \} \}. \end{aligned} \quad (8)$$

As a particular case, we choose the source term $f(x, t) = \beta^2 \exp(-t) \sin x$, the initial and boundary conditions as $\phi_1(x) = \sin(x)$, $\phi_2(x) = -\sin(x)$, $\psi_1(t) = 0$, $\psi_2(t) = \exp(-t)$, then, the initial/boundary value problem for the telegraph equation (3) takes the form

$$\begin{aligned} D_t^p u + 2\alpha D_t^q u + \beta^2 u &= D_x^p u + \beta^2 \exp(-t) \sin x, \quad (x, t) \in \mathbb{R}_+^2, \\ \text{IC: } u(x, 0) &= \sin(x), \quad \frac{\partial u}{\partial t}(x, 0) = -\sin(x), \quad x \in \mathbb{R}_+, \\ \text{BC: } u(0, t) &= 0, \quad \frac{\partial u}{\partial x}(0, t) = \exp(-t), \quad t \in \mathbb{R}_+. \end{aligned} \quad (9)$$

Hence, the solution (8) of (9) reduces to

$$\begin{aligned} u(x, t) &= L_x^{-1} L_t^{-1} \left\{ \frac{1}{(s_2 + 1)(s_1^2 + 1)} \right\} \\ &+ L_x^{-1} L_t^{-1} \left\{ \frac{2\alpha s_2^{q-1} - s_2^{p-2} - s_1^{p-2}}{(s_2 + 1)(s_1^2 + 1)(s_2^p + 2\alpha s_2^q + \beta^2 - s_1^p)} \right\} \\ &= e^{-t} \sin x + L_x^{-1} L_t^{-1} u_1 \{ (x, t) \}, \end{aligned} \quad (10)$$

where $u_1(s_1, s_2) = \frac{2\alpha s_2^{q-1} - s_2^{p-2} - s_1^{p-2}}{(s_2 + 1)(s_1^2 + 1)(s_2^p + 2\alpha s_2^q + \beta^2 - s_1^p)}$. Using the series expansion

$$\frac{1}{s_2 + 1} = \frac{1}{s_2} \left(1 + \frac{1}{s_2}\right)^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{s_2^{k+1}}, \quad (11)$$

we obtain

$$u_1(s_1, s_2) = \sum_{k=0}^{\infty} (-1)^k \frac{2\alpha s_2^{q-k-2} - s_2^{p-k-3} - s_1^{p-2} s_2^{-k-1}}{(s_1^2 + 1)(s_2^p + 2\alpha s_2^q + \beta^2 - s_1^p)}. \quad (12)$$

Applying the inverse transform L_t^{-1} on (12) and using the formula [12]

$$L_t^{-1} \left\{ \frac{s^{\text{Rho}-1}}{(s^\alpha + a s^\beta + b)}, t \right\} = t^{\alpha-\rho} \sum_{r=0}^{\infty} (-a)^r t^{(\alpha-\beta)r} E_{\alpha, \alpha+(\alpha-\beta)r-\rho+1}^{r+1}(-bt^\alpha), \quad (13)$$

we obtain

$$\begin{aligned} L_t^{-1} \{u_1(s_1, s_2)\} &= \\ &\sum_{k=0}^{\infty} (-1)^k \frac{1}{(s_1^2 + 1)} \left[2\alpha t^{p-(q-k-1)} \sum_{r=0}^{\infty} (-2\alpha)^r t^{(p-q)r} E_{p, \delta_1}^{r+1}(\theta(s_1)t^p) - \right. \\ &t^{p-(p-k-2)} \sum_{r=0}^{\infty} (-2\alpha)^r t^{(p-q)r} E_{p, \delta_2}^{r+1}(\theta(s_1)t^p) - \\ &\left. s_1^{p-2} t^{p-k} \sum_{r=0}^{\infty} (-2\alpha)^r t^{(p-q)r} E_{p, \delta_3}^{r+1}(\theta(s_1)t^p) \right], \end{aligned}$$

which can be rewritten as

$$L_t^{-1}\{u_1(s_1, s_2)\} = \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} (-2\alpha)^r t^{(p-q)r} \frac{t^{(p-q)r}}{(s_1^2 + 1)} \left[2\alpha t^{p-q+k+1} E_{p, \delta_1}^{r+1}(\theta(s_1)t^p) - t^{(k+2)} E_{p, \delta_2}^{r+1}(\theta(s_1)t^p) - s_1^{p-2} t^{p-k} E_{p, \delta_3}^{r+1}(\theta(s_1)t^p) \right], \quad (14)$$

where, $\delta_1 = p + (p-q)r - q + k + 2$, $\delta_2 = p + (p-q)r - p + k + 3$, $\delta_3 = p + (p-q)r + k + 1$ and $\theta(s_1) = s_1^p - \beta^2$. Now, we use the formula

$$E_{\alpha, \beta}^r(z) = \frac{1}{\Gamma(r)} \sum_{v=0}^{\infty} \frac{\Gamma(r+v)}{v! \Gamma(\alpha v + \beta)} (z^v), \quad (15)$$

and obtain

$$\frac{1}{s_1^2 + 1} E_{p, \delta_i}^{r+1}(\theta(s_1)t^p) = \sum_{v=0}^{\infty} \frac{1}{\Gamma(r+1)} \frac{\Gamma(r+v+1)}{v! \Gamma(pv + \delta_i) (s_1^2 + 1)} t^{pv} (s_1^p - \beta^2)^v, \quad i = 1, 2, 3$$

which in view of (11) takes the form

$$\begin{aligned} \frac{1}{s_1^2 + 1} E_{p, \delta_i}^{r+1}(\theta(s_1)t^p) &= \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^\rho \frac{t^{pv}}{\Gamma(r+1)} \frac{\Gamma(r+v+1)}{v! \Gamma(pv + \delta_i)} s_1^{-2(\rho+1)} (s_1^p - \beta^2)^v \\ &= \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^\rho \frac{t^{pv}}{\Gamma(r+1)} \frac{\Gamma(r+v+1)}{v! \Gamma(pv + \delta_i)} s_1^{-(2\rho+2-pv)} [1 + \beta^2 s_1^{-p}]^v. \end{aligned} \quad (16)$$

Using the linearity of the inverse Laplace transform L_x^{-1} formula [12], and the following formula

$$L_x \{x^{\beta-1} E_{\alpha, \beta}^r(ax^\alpha), s\} = s^{-\beta} [1 - as^{-\alpha}]^{-r}, \quad (17)$$

the equation (16) yields

$$\begin{aligned} L_x^{-1} \left\{ \frac{E_{p, \delta_i}^{r+1}(\theta(s_1)t^p)}{s_1^2 + 1} \right\} &= \\ \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^\rho \frac{t^{pv} x^{2(\rho+1)-pv-1}}{\Gamma(r+1)} \frac{\Gamma(r+v+1)}{v! \Gamma(pv + \delta_i)} E_{p, 2(\rho+1)-pv}^{-v}(\beta^2 x^p), \end{aligned} \quad (18)$$

where

$$E_{\alpha, \beta}^{-v}(z) = \sum_{l=0}^v (-1)^l \binom{v}{l} \frac{z^l}{\Gamma(\alpha l + \beta)}, \quad l \in \mathbb{N}, \text{ is a polynomial.} \quad (19)$$

Using (18) in (14), we obtain

$$L_x^{-1} L_t^{-1} \{u(s_1, s_2)\} = \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} (-2\alpha)^r t^{(p-q)r} [2\alpha t^{p-q+k+1} I_1 - t^{k+2} I_2 - t^{p-k} I_3], \quad (20)$$

where

$$\begin{aligned}
I_1 &= \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^{\rho} \frac{t^{pv} x^{2\rho-pv+1} \Gamma(r+v+1)}{\Gamma(r+1) v! \Gamma(pv+\delta_1)} E_{p,2\rho-pv+2}^{-v}(\beta^2 x^p), \\
I_2 &= \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^{\rho} \frac{t^{pv} x^{2\rho-pv+1} \Gamma(r+v+1)}{\Gamma(r+1) v! \Gamma(pv+\delta_2)} E_{p,2\rho-pv+2}^{-v}(\beta^2 x^p) \\
I_3 &= \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^{\rho} \frac{t^{pv} x^{2\rho-p(v+1)+3} \Gamma(r+v+1)}{\Gamma(r+1) v! \Gamma(pv+\delta_3)} E_{p,2(\rho+2)-p(v+1)}^{-v}(\beta^2 x^p).
\end{aligned} \tag{21}$$

Hence the exact solution of the one dimensional telegraph equation is given by

$$u(x, t) = e^{-t} \sin x + \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} (-2\alpha)^r t^{(p-q)r} [2\alpha t^{p-q+k+1} I_1 - t^{k+2} I_2 - t^{p-k} I_3]. \tag{22}$$

4. Analytical solutions of two dimensional telegraph equations using triple Laplace transform. Now, we obtain exact analytic solution of 1 + 2 dimensional telegraph equations (4) with initial and boundary conditions via triple Laplace transform. Apply triple transform on the differential equation in (4), using the linearity property, the results (6) and the initial/ boundary conditions, we obtain

$$\begin{aligned}
(s_3^p + 2\alpha s_3^q + \beta^2 - s_1^p - s_2^p) L_x L_y L_t \{v(x, y, t)\} &= (s_3^{p-1} + 2\alpha s_3^{q-1}) L_x L_y \{\rho_1(x, y)\} \\
&+ s_3^{p-2} L_x L_y \{\rho_2(x, y)\} - s_1^{p-1} L_y L_t \{\phi_1(y, t)\} - s_1^{p-2} L_y L_t \{\phi_2(y, t)\} \\
&- s_2^{p-1} L_x L_t \{\psi_1(x, t)\} - s_2^{p-2} L_x L_t \{\psi_2(x, t)\} + L_x L_y L_t \{F(x, y, t)\},
\end{aligned} \tag{23}$$

Hence, it follows that

$$\begin{aligned}
v(x, y, t) &= L_x^{-1} L_y^{-1} L_t^{-1} \frac{1}{(s_3^p + 2\alpha s_3^q + \beta^2 - s_1^p - s_2^p)} \left[(s_3^{p-1} + 2\alpha s_3^{q-1}) L_x L_y \{\rho_1(x, y)\} \right. \\
&+ s_3^{p-2} L_x L_y \{\rho_2(x, y)\} - s_1^{p-1} L_y L_t \{\phi_1(y, t)\} - s_1^{p-2} L_y L_t \{\phi_2(y, t)\} \\
&\left. - s_2^{p-1} L_x L_t \{\psi_1(x, t)\} - s_2^{p-2} L_x L_t \{\psi_2(x, t)\} + L_x L_y L_t \{F(x, y, t)\} \right]
\end{aligned} \tag{24}$$

As a special case, choose $F(x, y, t) = \beta^2 e^{-t} \sin x \sin y$, the initial conditions as $\rho_1(x, y) = \sin x \sin y$, $\rho_2(x, y) = -\sin x \sin y$ and the boundary conditions as $\phi_1(y, t) = 0$, $\phi_2(y, t) = e^{-t} \sin y$, $\psi_1(x, t) = 0$, $\psi_2(x, t) = e^{-t} \sin x$, then, the problem (4) takes the form

$$\begin{aligned}
D_t^p v + 2\alpha D_t^q v + \beta^2 v &= D_x^p v + D_y^p v + \beta^2 e^{-t} \sin x \sin y, (x, y, t) \in \mathbb{R}_+^3, \\
\text{IC: } v(x, y, 0) &= \sin x \sin y, \frac{\partial v}{\partial t}(x, y, 0) = e^{-t} \sin y, (x, y) \in \mathbb{R}_+^2, \\
\text{BC: } v(0, y, t) &= 0, \frac{\partial v}{\partial x}(0, y, t) = e^{-t} \sin y, (y, t) \in \mathbb{R}_+^2 \\
v(x, 0, t) &= 0, \frac{\partial v}{\partial y}(x, 0, t) = e^{-t} \sin x, (x, t) \in \mathbb{R}_+^2.
\end{aligned} \tag{25}$$

The solution (24) of (25) takes the form

$$v(x, y, t) = L_x^{-1} L_y^{-1} L_t^{-1} \left\{ \frac{1}{(s_1^2 + 1)(s_2^2 + 1)(s_3 + 1)(s_3^p + 2\alpha s_3^q + \beta^2 - s_1^p - s_2^p)} \left[(s_3^{p-1} \right. \right.$$

$$\begin{aligned}
 & + 2\alpha s_3^{q-1} + \beta^2 - s_1^p - s_2^p) + 2\alpha s_3^{q-1} - s_3^{p-2} - s_2^{p-2} - s_1^{p-2}] \} \\
 & = L_x^{-1} L_y^{-1} L_t^{-1} \left\{ \frac{1}{(s_1^2 + 1)(s_2^2 + 1)(s_3 + 1)} + \right. \\
 & \quad \left. \frac{2\alpha s_3^{q-1} - s_3^{p-2} - s_2^{p-2} - s_1^{p-2}}{(s_1^2 + 1)(s_2^2 + 1)(s_3 + 1)(s_3^p + 2\alpha s_3^q + \beta^2 - s_1^p - s_2^p)} \right\} \\
 & = e^{-t} \sin x \sin y + L_x^{-1} L_y^{-1} L_t^{-1} \{v_1(s_1, s_2, s_3)\},
 \end{aligned} \tag{26}$$

where $v_1(s_1, s_2, s_3) = \frac{2\alpha s_3^{q-1} - s_3^{p-2} - s_2^{p-2} - s_1^{p-2}}{(s_1^2 + 1)(s_2^2 + 1)(s_3 + 1)(s_3^p + 2\alpha s_3^q + \beta^2 - s_1^p - s_2^p)}$.

Now, using the relation (11), we obtain

$$v_1(s_1, s_2, s_3) = \sum_{k=0}^{\infty} (-1)^k \frac{2\alpha s_3^{q-k-2} - s_3^{p-k-3} - (s_1^{p-2} + s_2^{p-2})s_3^{-k-1}}{(s_1^2 + 1)(s_2^2 + 1)(s_3^p + 2\alpha s_3^q + \beta^2 - (s_1^p + s_2^p))}. \tag{27}$$

Applying L_t^{-1} and using (13), we obtain

$$\begin{aligned}
 & L_t^{-1} \{v_1(s_1, s_2, s_3)\} = \\
 & \sum_{k=0}^{\infty} (-1)^k \sum_{r=0}^{\infty} (-2\alpha)^r \frac{t^{(p-q)r}}{(s_1^2 + 1)(s_2^2 + 1)} \left[2\alpha t^{p-q+k+1} E_{p,\delta_1}^{r+1}(\theta(s)t^p) \right. \\
 & \quad \left. - t^{(k+2)} E_{p,\delta_2}^{r+1}(\theta(s)t^p) - (s_1^{p-2} + s_2^{p-2})t^{p-k} E_{p,\delta_3}^{r+1}(\theta(s)t^p) \right],
 \end{aligned} \tag{28}$$

where, $\delta_1 = p + (p-q)r - q + k + 2$, $\delta_2 = p + (p-q)r - p + k + 3$, $\delta_3 = p + (p-q)r + k + 1$ and $\theta(s) = s_1^p + s_2^p - \beta^2$. Using (15), we obtain

$$\begin{aligned}
 & \frac{1}{(s_1^2 + 1)(s_2^2 + 1)} E_{p,\delta_i}^{r+1}(\theta(s)t^p) = \\
 & \sum_{v=0}^{\infty} \frac{1}{\Gamma(r+1)} \frac{t^{pv} \Gamma(r+v+1)}{v! \Gamma(pv + \delta_i)(s_1^2 + 1)(s_2^2 + 1)} (s_1^p + s_2^p - \beta^2)^v
 \end{aligned} \tag{29}$$

which in view of (11) can be written as

$$\begin{aligned}
 & \frac{1}{(s_1^2 + 1)(s_2^2 + 1)} E_{p,\delta_i}^{r+1}(\theta(s)t^p) = \\
 & \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^\rho \frac{t^{pv} \Gamma(r+v+1) s_2^{pv-2(\rho+1)}}{\Gamma(r+1) v! \Gamma(pv + \delta_i)(s_1^2 + 1)} [1 - (\beta^2 - s_1^p) s_2^{-p}]^v
 \end{aligned} \tag{30}$$

Applying L_y^{-1} and using (17) we obtain

$$\begin{aligned}
 & L_y^{-1} \left\{ \frac{E_{p,\delta_i}^{r+1}(\theta(s)t^p)}{(s_1^2 + 1)(s_2^2 + 1)} \right\} = \\
 & \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} (-1)^\rho \frac{t^{pv} y^{2\rho-pv+1} \Gamma(r+v+1)}{\Gamma(r+1) v! \Gamma(pv + \delta_i)(s_1^2 + 1)} E_{p,2(\rho+1)-pv}^{-v}((\beta^2 - s_1^p) y^p)
 \end{aligned} \tag{31}$$

where $E_{\alpha,\beta}^{-v}(z) = \sum_{l=0}^v (-1)^l \binom{v}{l} \frac{z^l}{\Gamma(\alpha l + \beta)}$, $l \in \mathbb{N}$ is a polynomial. Hence it follows that

$$L_y^{-1} \left\{ \frac{E_{p,\delta_i}^{r+1}(\theta(s)t^p)}{(s_1^2 + 1)(s_2^2 + 1)} \right\} = \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} \sum_{l=0}^v (-1)^{\rho+l} \bigwedge_{v\rho l} \frac{t^{pv} y^{pl+2\rho-pv+1} (\beta^2 - s_1^p)^l}{\Gamma(pv + \delta_i)(s_1^2 + 1)}, \tag{32}$$

where $\Lambda_{v\rho l} = \binom{v}{l} \frac{\Gamma(r+v+1)}{\Gamma(r+1)v!\Gamma(pl+2\rho-pv+2)}$. Using (11), we obtain

$$\begin{aligned} L_y^{-1} \left\{ \frac{E_{p,\delta_i}^{r+1}(\theta(s)t^p)}{(s_1^2+1)(s_2^2+1)} \right\} &= \\ \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} \sum_{l=0}^v \sum_{m=0}^{\infty} (-1)^{\rho+l+m} \bigwedge_{v\rho l} \frac{t^{pv} y^{pl+2\rho-pv+1} s_1^{-2(m+1)} (\beta^2 - s_1^p)^l}{\Gamma(pv + \delta_i)} \end{aligned} \quad (33)$$

Applying L_x^{-1} and using (13), we obtain

$$\begin{aligned} L_y^{-1} L_x^{-1} \left\{ \frac{E_{p,\delta_i}^{r+1}(\theta(s)t^p)}{(s_1^2+1)(s_2^2+1)} \right\} &= \\ \sum_{v=0}^{\infty} \sum_{\rho=0}^{\infty} \sum_{l=0}^v \sum_{m=0}^{\infty} (-1)^{\rho+m} \bigwedge_{v\rho l} \frac{t^{pv} y^{pl+2\rho-pv+1} x^{2m-pl+1} E_{p,\tau}^{-l}(\beta^2 x^p)}{\Gamma(pv + \delta_i)} \end{aligned} \quad (34)$$

where $\tau = 2(m+1) - pl$.

Now, using (11), we have

$$\frac{s_2^{p-2} + s_1^{p-2}}{(s_1^2+1)(s_2^2+1)} E_{p,\delta_3}^{r+1}(\theta(s)t^p) = \sum_{\rho=0}^{\infty} (-1)^\rho \left(\frac{s_1^{p-2} s_2^{-2(\rho+1)}}{s_1^2+1} + \frac{s_2^{p-2(\rho+2)}}{s_1^2+1} \right) E_{p,\delta_3}^{r+1}(\theta(s)t^p), \quad (35)$$

which in view of (15) implies that

$$\begin{aligned} \frac{s_2^{p-2} + s_1^{p-2}}{(s_1^2+1)(s_2^2+1)} E_{p,\delta_3}^{r+1}(\theta(s)t^p) &= \\ \sum_{\rho=0}^{\infty} \sum_{v=0}^{\infty} (-1)^\rho \Lambda_v t^{pv} \left\{ \frac{s_1^{p-2}}{s_1^2+1} s_2^{-(2(\rho+1)-pv)} [1 - (\beta^2 - s_1^p) s_2^{-p}]^v \right. \\ \left. + \frac{1}{s_1^2+1} s_2^{-(2(\rho+2)-p(v+1))} [1 - (\beta^2 - s_1^p) s_2^{-p}]^v \right\}, \end{aligned} \quad (36)$$

where $\Lambda_v = \frac{\Gamma(r+v+1)}{\Gamma(r+1)v!\Gamma(pv+\delta_3)}$. Using (17), we obtain

$$\begin{aligned} L_y^{-1} \left\{ \frac{s_2^{p-2} + s_1^{p-2}}{(s_1^2+1)(s_2^2+1)} E_{p,\delta_3}^{r+1}(\theta(s)t^p) \right\} &= \\ \sum_{\rho=0}^{\infty} \sum_{v=0}^{\infty} (-1)^\rho \Lambda_v t^{pv} y^{2\rho-pv+1} \left\{ \frac{s_1^{p-2}}{s_1^2+1} E_{p,2(\rho+1)-pv}^{-v}(\beta^2 - s_1^p) y^p \right. \\ \left. + \frac{y^{(2-p)}}{s_1^2+1} E_{p,2\rho-pv-p+3}^{-v}(\beta^2 - s_1^p) y^p \right\}, \end{aligned} \quad (37)$$

which in view of (19) implies that

$$\begin{aligned} L_y^{-1} \left\{ \frac{s_2^{p-2} + s_1^{p-2}}{(s_1^2+1)(s_2^2+1)} E_{p,\delta_3}^{r+1}(\theta(s)t^p) \right\} &= \sum_{\rho=0}^{\infty} \sum_{v=0}^{\infty} \sum_{l=0}^v (-1)^{\rho+l} \Lambda_v \binom{v}{l} t^{pv} y^{pl+2\rho-pv+1} \\ &\quad \left\{ \frac{s_1^{pl+p-2} (\beta^2 s_1^{-p} - 1)^l}{(s_1^2+1)\Gamma(pl+\tau_1)} + \frac{y^{(2-p)} s_1^{pl} (\beta^2 s_1^{-p} - 1)^l}{(s_1^2+1)\Gamma(pl+\tau_2)} \right\} \end{aligned} \quad (38)$$

where $\tau_1 = 2(\rho + 1) - pv$, $\tau_2 = 2\rho - p(v + 1) + 3$. Using (11), we obtain

$$L_y^{-1} \left\{ \frac{s_2^{p-2} + s_1^{p-2}}{(s_1^2 + 1)(s_2^2 + 1)} E_{p,\delta_3}^{r+1}(\theta(s)t^p) \right\} = \sum_{\rho=0}^{\infty} \sum_{v=0}^{\infty} \sum_{l=0}^v \sum_{m=0}^{\infty} (-1)^{\rho+m} \Lambda_v \binom{v}{l} t^{pv} y^{pl+2\rho-pv+1} \left\{ \frac{s_1^{-(2(m+2)-p(l+1))} (1 - \beta^2 s_1^{-p})^l}{\Gamma(pl + \tau_1)} + \frac{y^{(2-p)} s_1^{-(2(m+1)-pl)} (1 - \beta^2 s_1^{-p})^l}{\Gamma(pl + \tau_2)} \right\}. \quad (39)$$

Taking the inverse transform L_x^{-1} and using (17), we obtain

$$L_x^{-1} L_y^{-1} \left\{ \frac{s_2^{p-2} + s_1^{p-2}}{(s_1^2 + 1)(s_2^2 + 1)} E_{p,\delta_3}^{r+1}(\theta(s)t^p) \right\} = \sum_{\rho=0}^{\infty} \sum_{v=0}^{\infty} \sum_{l=0}^v \sum_{m=0}^{\infty} (-1)^{\rho+m} \Lambda_v \binom{v}{l} t^{pv} y^{pl+2\rho-pv+1} x^{2m-pl+1} \left\{ \frac{x^{2-p} E_{p,\delta_4}^{-l}(\beta^2 x^p)}{\Gamma(pl + \tau_1)} + \frac{y^{(2-p)} E_{p,\delta_5}^{-l}(\beta^2 x^p)}{\Gamma(pl + \tau_2)} \right\}, \quad (40)$$

where $\delta_4 = 2m - p(l + 1) + 3$, $\delta_5 = 2m - pl + 2$. Applying $L_x^{-1} L_y^{-1}$ on (41), using (34) and (40), we obtain

$$L_x^{-1} L_y^{-1} L_t^{-1} \{v_1(s_1, s_2, s_3)\} = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} (-1)^k (-2\alpha)^r t^{(p-q)r} \sum_{\rho=0}^{\infty} \sum_{v=0}^{\infty} \sum_{l=0}^v \sum_{m=0}^{\infty} (-1)^{\rho+m} \bigwedge_{v\rho l} t^{pv} y^{pl+2\rho-pv+1} x^{2m-pl+1} \left\{ \frac{2\alpha t^{(p-q+k+1)}}{\Gamma(pv + \delta_1)} - \frac{t^{-(k+2)}}{\Gamma(pv + \delta_1)} E_{p,\tau}^{-l}(\beta^2 x^p) - \frac{t^{p-k} \Gamma(pl + 2\rho - pv + 2)}{\Gamma(pv + \delta_3)} \left(\frac{x^{2-p} E_{p,\delta_4}^{-l}(\beta^2 x^p)}{\Gamma(pl + \tau_1)} + \frac{y^{(2-p)} E_{p,\delta_5}^{-l}(\beta^2 x^p)}{\Gamma(pl + \tau_2)} \right) \right\}. \quad (41)$$

Hence the exact solution is obtained by putting the values in (26).

5. Conclusion. We developed triple Laplace transform technique to obtain exact solutions of 1 + 1 and 1 + 2 dimensional telegraph equations. We conclude that the proposed techniques are also appropriate to further types of fractional order partial differential equations.

Acknowledgments. This research has been supported by the National Natural Science Foundation of China (11571378). The research visit to Sun Yet Sen University was supported by National Natural Science Foundation of China.

REFERENCES

- [1] M. A. Abdou, [Adomian decomposition method for solving telegraph equation in charged particle transport](#), *J. Quant. Spectrosc. Ra.*, **95** (2005), 407-414.
- [2] A. M. O. Anwar, F Jarad, D Baleanu and F Ayaz, Fractional Caputo heat equation within the double Laplace transform, *Romanian J. Phys.*, **58** (2013), 15-22.

- [3] J. Biazar and M. Eslami, [Analytic solution for telegraph equation by differential transform method](#), *Phys. Lett. A*, **374** (2010), 2904-2906.
- [4] L. Debnath, [Recent applications of fractional calculus to science and engineering](#), *Int. J. Math. Math. Sci.*, (**2003**) (2003), 3413-3442.
- [5] M. Dehghan and A. Shokri, [A numerical method for solving the hyperbolic telegraph equation](#), *Numer. Meth. Part. D. E.*, **24** (2008), 1080-1093.
- [6] M. Dehghan and M. Lakestani, [The use of Chebyshev cardinal functions for solution of the second order one-dimensional telegraph equation](#), *Numer. Meth. for Part. D. E.*, **25** (2009), 931-938.
- [7] M. Dehghan and A. Ghesmati, [Solution of the second-order one-dimensional hyperbolic telegraph equation by using the dual reciprocity boundary integral equation method](#), *Eng. Anal. Bound. Elem.*, **34** (2010), 51-59.
- [8] M. Dehghan and A. Mohebbi, [High order implicit collocation method for the solution of two-dimensional linear hyperbolic equation](#), *Numer. Meth. for Part. D. E.*, **25** (2009), 232-243.
- [9] M. Dehghan and A. Ghesmati, [Combination of meshless local weak and strong forms to solve the two dimensional hyperbolic telegraph equation](#), *Eng. Anal. Bound. Elem.*, **34** (2010), 324-336.
- [10] M. Dehghan and A. Shokri, [A meshless method for numerical solution of a linear hyperbolic equation with variable coefficients in two space dimensions](#), *Numer. Meth. Part. D. E.*, **25** (2009), 494-506.
- [11] H. Ding and Y. Zhang, [A new fourth-order compact finite difference scheme for the two-dimensional second-order hyperbolic equation](#), *J. Comput. Appl. Math.*, **230** (2009), 626-632.
- [12] H. J. Haubold, A. M. Mathai and R. K. Saxena, [Mittag-Leffler Functions and Their Applications](#), *J. Appl. Math.*, **2011** (2011), Art. ID 298628, 51 pp.
- [13] R. Hilfer, [Applications of Fractional Calculus in Physics](#), Word Scientific, Publishing Co., Inc., River Edge, NJ, 2000.
- [14] F. Jarad and K. Tas, [Application of Sumudu and double Sumudu transforms to Caputo-Fractional differential equations](#), *J. Comput. Anal. Appl.*, **14** (2012), 475-483.
- [15] T. Khan, K. Shah, A. Khan and R. A. Khan, [Solution of fractional order heat equation via triple Laplace transform in two dimensions](#), *Math Meth Appl Sci.*, **41** (2018), 818-825.
- [16] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, [Theory and Application of Fractional Differential Equations](#), North Holland Mathematics Studies 204, Elsevier, Amsterdam, 2006.
- [17] M. Lakestani and B. N. Saray, [Numerical solution of telegraph equation using interpolating scalling function](#), *Comp. Math. Appl.*, **60** (2010), 1964-1972
- [18] R. L. Magin, [Fractional Calculus in Bioengineering](#) Begell House Publishers, Redding, 2006.
- [19] R. K. Mohanty and M. K. Jain, [An unconditionally stable alternating direction implicit scheme for the two space dimensional linear hyperbolic equation](#), *Numer. Meth. for Part. D. E.*, **17** (2001), 684-688.
- [20] R. K. Mohanty, [An operator splitting method for an unconditionally stable difference scheme for a linear hyperbolic equation with variable coefficients in two space dimensions](#), *Appl. Math. Comput.*, **152** (2004), 799-806.
- [21] R. K. Mohanty, M. K. Jain and K. George, [On the use of high order difference methods for the system of one space second order nonlinear hyperbolic equations with variable coefficients](#), *J. Comp. Appl. Math.*, **72** (1996), 421-431.
- [22] R. K. Mohanty, [New unconditionally stable difference schemes for the solution of multi-dimensional telegraphic equations](#), *Int. J. Comp. Math.*, **86** (2009), 2061-2071
- [23] R. K. Mohanty, M. K. Jain and U. Arora, [An unconditionally stable ADI method for the linear hyperbolic equation in three space dimensions](#), *Int. J. Comp. Math.*, **79** (2002), 133-142.
- [24] A. Mohebbi and M. Dehghan, [High order compact solution of the one-space-dimensional linear hyperbolic equation](#), *Numer. Meth. Part. D. E.*, **24** (2008), 1222-1235.
- [25] I. Podlubny, [Fractional Differential Equations](#), Mathematics in Science and Engineering, Academic Press, New York, 1999.
- [26] A. Saadatmandi and M. Dehghan, [Numerical solution of hyperbolic telegraph equation using the Chebyshev tau method](#), *Numer. Meth. for Part. D. E.*, **26** (2010), 239-252.
- [27] S. G. Samko, A. A. Kilbas and O. I. Marichev, [Fractional Integrals and Derivatives: Theory and Applications](#), Gordon and Breach, Yverdon, (1993).

- [28] C. Shu, Q. Yao and K. S. Yeo, [Block-marching in time with DQ discretization: An efficient method for time-dependent problems](#), *Comput. Methods Appl. Mech. Engrg*, **191** (2002), 4587-4597.

Received for publication May 2019.

E-mail address: rahmat.alipk@yahoo.com

E-mail address: stslyj@mail.sysu.edu.cn

E-mail address: fahd@cankaya.edu.tr