EXACT SOLUTIONS OF THE CUBIC BOUSSINESQ AND THE COUPLED HIGGS SYSTEM

by

Mahmoud A. E. ABDELRAHMAN[a,b], Hanan A. ALKHIDHR[c], Dumitru BALEANU[d,e,f*], and Mustafa INC[g]

[a] Department of Mathematics, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia
[b] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt
[c] Department of Mathematics, Qassim University, Buraidah, Saudi Arabia
[d] Department of Mathematics, Cankaya University, Balgat, Ankara, Turkey
[f] Institute of Space Sciences, Bucharest, Romania
[e] Department of Medical Research, China Medical University, Taichung, Taiwan
[g] Department of Mathematics, Firat University, Elazig, Turkey

Original scientific paper
https://doi.org/10.2298/TSCI20S1333A

We present explicit exact solutions of some evolution equations including cubic Boussinesq and coupled Higgs system by the unified method. The explicit solutions are expressed in terms of some elementary functions including trigonometric, exponential, and polynomial. The method is applied to a number of special test problems to test the strength of the method and computational results indicate the power and efficiency of the method.

Key words: *unified solver, solitons, elliptic functions, exact solutions, physical applications*

Introduction

Non-linear evolution equations (NLEE) are significant type of PDE having applications in many different branches of science and engineering including quantum mechanics, optical fibres, relativity, plasma, nuclear industry, heat flow, biology, statistical mechanics *etc*., [1-11]. Different types of traveling wave solutions including exponential, rational, hyperbolic, trigonometric, dark, bright, complex, elliptic, and Jacobi elliptic, functions model many phenomena in science. Recently, physicist and mathematician have made many efforts in finding the analytic solutions to a number of NLEE. Methods such as F-expansion [12], extended tanh [13], generalized Kudryashov [14], tanh-sech [15], homogeneous balance [16], exp-function [17], multiple exp-function [18], Jacobi elliptic function [19], sine-cosine [20], expansion [19] and Riccati-Bernoulli sub-ODE [21, 22] are highly useful techniques for solving NLEE.

For $\theta(x, t)$, consider:

$$R\left(\theta,\theta_x,\theta_t,\theta_{xx},\theta_{xt},\theta_{tt},\ldots\right)=0 \tag{1}$$

For $c > 0$:

$$\theta\left(x,t\right)=\theta\left(\zeta\right),\ \ \zeta=kx-ct \tag{2}$$

* Corresponding author, e-mail: dumitru@cankaya.edu.tr

Equation (1) is converted to the following ODE:

$$G\left(\theta,\frac{\mathrm{d}\theta}{\mathrm{d}\zeta},\frac{\mathrm{d}^2\theta}{\mathrm{d}\zeta^2},\frac{\mathrm{d}^3\theta}{\mathrm{d}\zeta^3},...\right)=0 \tag{3}$$

Most standard methods for solving NLEE based on transfer these equations to ODE, using appropriate transformation and then solve it, which give travelling wave solutions and consequently give the solutions of the original NLEE. We observed that there are so many classes of NLEE arising in physics, fluid mechanics and engineering fields transferred to the following ODE:

$$\alpha\theta''+\beta\theta^3+\gamma\theta=0 \tag{4}$$

see [9, 10, 19, 22-29, 31] and so on. This equation is often referred to the pseudoptential or sagdeev potential [30].

The Jacobi elliptic function expansion method

The JEFEM [19, 31] expresses eq. (3):

$$\theta(\zeta)=a_0+\sum_{j=1}^{N}\theta_i^{j-1}(\zeta)\left[a_j\theta_i(\zeta)+b_j\tilde{\theta}_i(\zeta)\right],\ i=1,2,3,... \tag{5}$$

with

$$\begin{gathered}\theta_1(\zeta)=sn\zeta,\ \tilde{\theta}_1(\zeta)=cn\zeta,\ \theta_2(\zeta)=sn\zeta,\ \tilde{\theta}_2(\zeta)=dn\zeta\\ \theta_3(\zeta)=ns\zeta,\ \tilde{\theta}_3(\zeta)=cs\zeta,\ \theta_4(\zeta)=ns\zeta,\ \tilde{\theta}_4(\zeta)=ds\zeta\\ \theta_5(\zeta)=sc\zeta,\ \tilde{\theta}_5(\zeta)=nc\ \zeta,\theta_6(\zeta)=sd\zeta,\ \tilde{\theta}_6(\zeta)=nd\zeta\end{gathered} \tag{6}$$

in which *snζ, cnζ*, and *dnζ* are Jacobian elliptic sine, cosine, and third kind functions, respectively. The Glaishers symbols:

$$\begin{gathered}ns\zeta=\frac{1}{sn\ \zeta},\ nc\zeta=\frac{1}{cn\ \zeta},\ nd\zeta=\frac{1}{dn\ \zeta},\ sc\zeta=\frac{sn\zeta}{cn\zeta},\\ cs\zeta=\frac{cn\zeta}{sn\zeta},\ ds\zeta=\frac{dn\zeta}{sn\zeta},\ sd\zeta=\frac{sn\zeta}{dn\zeta}\end{gathered} \tag{7}$$

These functions obey:

$$\begin{gathered}sn^2\zeta+cn^2\zeta=1,\ dn^2\zeta+m^2sn^2\zeta=1,\ ns^2\zeta=1+cs^2\zeta,\\ ns^2\zeta=m^2+ds^2\zeta,\ sc^2\zeta+1=nc^2\zeta,\ m^2sd^2+1=nd^2\zeta\end{gathered} \tag{8}$$

in which $m \in (0, 1)$ is a modulus:

$$sn'\zeta=cn\zeta dn\zeta,\ cn'\zeta=-sn\zeta dn\zeta,\ dn'^2sn\zeta cn\zeta \tag{9}$$

$$ns'\zeta=-ds\zeta cs\zeta,\ ds'\zeta=-cs\zeta ns\zeta,\ cs'\zeta=-ns\zeta ds\zeta \tag{10}$$

$$sc'\zeta=nc\zeta dc\zeta,\ nc'\zeta=sc\zeta dc\zeta,\ cd'\zeta=cd\zeta nd\zeta,\ nd'^2sd\,cd\zeta \tag{11}$$

consider

$$D\left[\frac{\mathrm{d}^q\theta}{\mathrm{d}\zeta^q}\right]=n+q,\ D\left[\theta^p\left(\frac{\mathrm{d}^q\theta}{\mathrm{d}\zeta^q}\right)^s\right]=np+s(n+q) \tag{12}$$

Hence, we get:

$$\theta(\zeta) = a_0 + \sum_{j=1}^{N} \tanh^{j-1}(\zeta)\left[a_j \tanh(\zeta) + b_j \operatorname{sech}(\zeta)\right] \tag{13}$$

$$\theta(\zeta) = a_0 + \sum_{j=1}^{N} \coth h^{j-1}(\zeta)\left[a_j \coth(\zeta) + b_j \operatorname{csch}(\zeta)\right] \tag{14}$$

$$\theta(\zeta) = a_0 + \sum_{j=1}^{N} \tan^{j-1}(\zeta)\left[a_j \tan(\zeta) + b_j \sec(\zeta)\right] \tag{15}$$

$$\theta(\zeta) = a + \sum \cot (\zeta)\left[a_j \cot(\zeta) + b_j \csc(\zeta)\right] \tag{16}$$

Unified solver

Now we give the unified solver for equation:

$$\alpha\theta'' + \beta\theta^3 + \gamma\theta = 0 \tag{17}$$

Balancing θ'' and θ^3, gives $m = 1$. Thus, the solution of eq. (17) takes the form [19, 31]:

$$\theta = a_0 + a_1 sn(\zeta) + b_1 cn(\zeta) \tag{18}$$

in which a_0, a_1, and b_1 are constants. From eq. (18) we get:

$$\theta' = a_1 cn(\zeta) dn(\zeta) - b_1 sn(\zeta) dn(\zeta) \tag{19}$$

$$\theta'' = -m^2 sn(\zeta) a_1 + 2 a_1 sn(\zeta)^3 m^2 + 2m^2 sn(\zeta)^2 cn(\zeta) b_1 - a_1 sn(\zeta) - b_1 cn(\zeta) \tag{20}$$

Writing eqs. (18)-(20) in (17) and equating all coefficients of sn^3, sn^2cn, sn^2, $sncn$, sn, cn, sn^0 to 0, we get:

$$2\alpha m^2 a_1 + \beta\left(a_1^3 - 3a_1 b_1^2\right) = 0 \tag{21}$$

$$2\alpha m^2 b_1 + \beta\left(3a_1^2 b_1 - b_1^3\right) = 0 \tag{22}$$

$$a_0\left(a_1^2 - b_1^2\right) = 0 \tag{23}$$

$$a_0 a_1 b_1 = 0 \tag{24}$$

$$\alpha a_1\left(1 + m^2\right) - \beta\left(3a_0^2 a_1 + 3a_1 b_1^2\right) - \gamma a_1 = 0 \tag{25}$$

$$\alpha b_1 - \beta\left(3a_0^2 b_1 + b_1^3\right) - \gamma b_1 = 0 \tag{26}$$

$$\beta\left(a_0^3 + 3a_0 b_1^2\right) + \gamma a_0 = 0 \tag{27}$$

Solving the system, we obtain:

Case 1.

$$a_0 = 0, \; a_1 = \pm\sqrt{\qquad} m, \; b_1 = 0, \; \gamma = \alpha\left(1+m \quad\right)$$

thus

$$\theta_1(x,t) = \pm\sqrt{\frac{-2\alpha}{\beta}} msn(\zeta) \tag{28}$$

For $m \to 1$, eq. (39) is expressed:

$$\theta_1(x,t) = \pm\sqrt{\frac{-2\alpha}{\beta}} \tanh(\zeta) \tag{29}$$

Case 2.

$$a_0 = 0, \quad a_1 = \pm\sqrt{\frac{-\alpha}{2\beta}} m, \quad b_1 = -\sqrt{\frac{\alpha}{2\beta}} m, \quad \gamma = \frac{1}{2}\alpha\left(2-m^2\right)$$

thus

$$\theta_2(x,t) = \pm\sqrt{\frac{-\alpha}{2\beta}} m\, sn(\zeta) - \sqrt{\frac{\alpha}{2\beta}} m\, cn(\zeta) \tag{30}$$

For $m \to 1$, eq. (41) is written:

$$\theta_2(x,t) = \pm\sqrt{\frac{-\alpha}{2\beta}}\, tanh(\zeta) - \sqrt{\frac{\alpha}{2\beta}}\, sech(\zeta) \tag{31}$$

Case 3.

$$a_0 = 0, \quad a_1 = \pm\sqrt{\frac{-\alpha}{2\beta}} m, \quad b_1 = \sqrt{\frac{\alpha}{2\beta}} m, \quad \gamma = \frac{1}{2}\alpha\left(2-m^2\right)$$

then

$$\theta_3(x,t) = \pm\sqrt{\frac{-\alpha}{2\beta}} m\, sn(\zeta) + \sqrt{\frac{\alpha}{2\beta}} m\, cn(\zeta) \tag{32}$$

For $m \to 1$, eq. (43):

$$\theta_3(x,t) = \pm\sqrt{\frac{-\alpha}{2\beta}}\, tanh(\zeta) + \sqrt{\frac{\alpha}{2\beta}}\, sech(\zeta) \tag{33}$$

Case 4.

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = \pm\sqrt{\frac{2\alpha}{\beta}}\, m, \quad \gamma = \alpha\left(1-2m^2\right)$$

then

$$\theta_4(x,t) = \pm\sqrt{\frac{2\alpha}{\beta}}\, m\, cn(\zeta) \tag{34}$$

For $m \to 1$, eq. (62):

$$\theta_4(x,t) = \pm\sqrt{\frac{2\alpha}{\beta}}\, \text{sech}(\zeta) \tag{35}$$

Test cases for equation $\alpha\theta'' + \beta\theta^3 + \gamma\theta = 0$

Test *Case 1*: Consider the following cubic Boussinesq eq. [24]:

$$\chi_{tt} - \chi_{xx} - \chi_{xxxx} + 2(\chi^3)_{xx} = 0 \tag{36}$$

This equation has physical applications including vibrations in a non-linear string, non-linear lattice waves and iron sound waves in plasma [32, 33]. Using the transformation

$$\chi(x,t) = \chi(\zeta),\ \zeta = k(x - wt) \tag{37}$$

transform eq. (36) into ODE:

$$-k^2\chi'' + 2\chi^3 + (w^2 - 1)\chi = 0 \tag{38}$$

Comparing it with (17) results that $\alpha = -k^2$, $\beta = 2$, and $\gamma = w^2 - 1$. Hence, we have:

Case 1. The first family of solutions:

$$\chi_1(x,t) = \pm kmsn\left\{k\left[x - \sqrt{1-k^2(1+m^2)}\,t\right]\right\} \tag{39}$$

As long as $m \to 1$, eq. (39) becomes:

$$\chi_1(x,t) = \pm k\tanh\left[k\left(x - \sqrt{1-2k^2}\,t\right)\right] \tag{40}$$

where k is arbitrary constant. This solution is depicted in fig. 1.

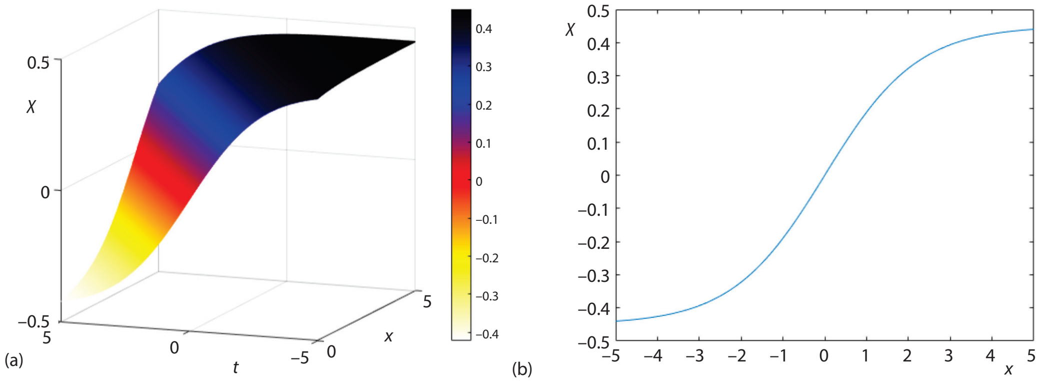

Figure 1. Graph of solution of eq. (40) with k = 0.45

Case 2. The second family of solutions:

$$\chi_2(x,t) = \pm\frac{k}{2}msn\left\{k\left[x - \sqrt{1-\frac{1}{2}k^2(2-m^2)}\,t\right]\right\} - i\frac{k}{2}mcn\left\{k\left[x - \sqrt{1-\frac{1}{2}k^2(2-m^2)}\,t\right]\right\} \tag{41}$$

As long as $m \to 1$, eq. (41) becomes:

$$\chi_2(x,t) = \pm\frac{k}{2}\tanh\left\{k\left[x - \sqrt{1-\frac{1}{2}k^2}\,t\right]\right\} - i\frac{k}{2}\operatorname{sech}\left\{k\left[x - \sqrt{1-\frac{1}{2}k^2}\,t\right]\right\} \tag{42}$$

Case 3. The second family of solutions:

$$\chi_3(x,t) = \pm\frac{k}{2}msn\left\{k\left[x - \sqrt{1-\frac{1}{2}k^2(2-m^2)}\,t\right]\right\} + i\frac{k}{2}mcn\left\{k\left[\left(x - \sqrt{1-\frac{1}{2}k^2(2-m^2)}\,t\right)\right]\right\} \tag{43}$$

As long as $m \to 1$, eq. (43) becomes:

$$\chi_3(x,t)=\pm\frac{k}{2}\tanh\left\{k\left[x-\sqrt{1-\frac{1}{2}k^2}\,t\right]\right\}+i\frac{k}{2}\operatorname{sech}\left\{\left[k\left(x-\sqrt{1-\frac{1}{2}k^2}\,t\right)\right]\right\} \tag{44}$$

where k is arbitrary constant.

Case 4. The fourth family of solutions:

$$\chi_4(x,t)=\pm ik\,mcn\left\{k\left[x-\sqrt{1-k^2\left(1-2m^2\right)}\,t\right]\right\} \tag{45}$$

As long as $m \to 1$, eq. (45) becomes:

$$\chi_4(x,t)=\pm ik\operatorname{sech}\left\{k\left[x-\sqrt{1+k^2}\,t\right]\right\} \tag{46}$$

where k is arbitrary constant.

Test Case 2

The second test case is the coupled Higgs system:

$$u_{tt}-u_{xx}+|u|^2u-2u\phi=0,\ \phi_{tt}+\phi_{xx}-(|q|^2)_{xx}=0 \tag{47}$$

in which $u(x,\,t)$ and $\phi(x,\,t)$ are the complex scalar nucleon field and the real scalar meson field, respectively [34]. Using the traveling wave transformation:

$$u(x,t)=\mathrm{e}^{i\eta(x,t)}q(\zeta),\ \phi(x,t)=\mathrm{e}^{i\eta(x,t)}\psi(\zeta),\ \eta(x,t)=px+rt,\ \zeta=x+\mu t \tag{48}$$

where p, r , and μ are constants. Writing eq. (48) into eq. (47) yields:

$$\left(\mu^2-1\right)q''+\left(p^2-r^2\right)q-2q\psi+q^3=0 \tag{49}$$

$$\left(\mu^2+1\right)q''-2(q')^2-2qq''=0 \tag{50}$$

Integrating eq. (49) and neglecting the constant of integration:

$$\left(\mu^2+1\right)\psi=q^2 \tag{51}$$

Inserting eq. (51) into eq. (50), we obtain:

$$\left(\mu^4-1\right)q''+\left(\mu^2-1\right)q^3+\left(\mu^2+1\right)\left(p^2-r^2\right)q=0 \tag{52}$$

Comparing it with (17) gives $\alpha=(\mu^4-1)$, $\beta=(\mu^2-1)$, and $\gamma=(\mu^2+1)(p^2-r^2)$:
Case 1.

$$q_1(x,t)=\pm i\sqrt{2\left(\frac{p^2-r^2}{m^2+1}+2\right)}\,msn\left(x+\sqrt{\frac{p^2-r^2}{m^2+1}+1t}\right) \tag{53}$$

For $m \to 1$, eq. (53) becomes:

$$q_1(x,t)=\pm i\sqrt{p^2-r^2+4}\,\tanh\left(x+\sqrt{\frac{p^2-r^2}{2}+1t}\right) \tag{54}$$

Thus the solution of eq. (47):

$$u_1(x,t) = \pm i\, e^{i(px+rt)} \sqrt{p^2 - r^2 + 4}\, \tanh\left(x + \sqrt{\frac{p^2 - r^2}{2} + 1t}\right) \tag{55}$$

where p, r are arbitrary constants.

Case 2.

$$\begin{aligned} q_2(x,t) = \pm i \sqrt{\frac{p^2 - r^2}{2 - m^2} + 1}\, msn\left[x + \sqrt{\frac{2\left(p^2 - r^2\right)}{2 - m^2} + 1t}\right] + \\ + \sqrt{\frac{p^2 - r^2}{2 - m^2} + 1}\, mcn\left[x + \sqrt{\frac{2\left(p^2 - r^2\right)}{2 - m^2} + 1t}\right] \end{aligned} \tag{56}$$

For $m \to 1$, eq. (56) becomes:

$$\begin{aligned} q_2(x,t) = \pm i \sqrt{p^2 - r^2 + 1}\, \tanh\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right] + \\ + \sqrt{p^2 - r^2 + 1}\, \text{sech}\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right] \end{aligned} \tag{57}$$

Thus the solution of eq. (47):

$$\begin{aligned} u_2(x,t) = \pm e^{i(px+rt)}\Bigg(\pm i \sqrt{p^2 - r^2 + 1} \tanh\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right] + \\ + \sqrt{p^2 - r^2 + 1} \text{sech}\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right]\Bigg) \end{aligned} \tag{58}$$

where p, r are arbitrary constants.

Case 3. The third family of equation:

$$\begin{aligned} q_3(x,t) = \pm i \sqrt{\frac{p^2 - r^2}{2 - m^2} + 1}\, msn\left[x + \sqrt{\frac{2\left(p^2 - r^2\right)}{2 - m^2} + 1t}\right] - \\ - \sqrt{\frac{p^2 - r^2}{2 - m^2} + 1}\, mcn\left[x + \sqrt{\frac{2\left(p^2 - r^2\right)}{2 - m^2} + 1t}\right] \end{aligned} \tag{59}$$

For $m \to 1$, eq. (56) becomes:

$$q_3(x,t) = \pm i \sqrt{p^2 - r^2 + 1}\, \tanh\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right] - \sqrt{p^2 - r^2 + 1}\, sech\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right] \tag{60}$$

Thus the solution of eq. (47):

$$\begin{aligned} u_3(x,t) = \pm e^{i(px+rt)}\Bigg(\pm i \sqrt{p^2 - r^2 + 1} \tanh\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right] + \\ + \sqrt{p^2 + r^2 + 1} \text{sech}\left[x + \sqrt{2\left(p^2 - r^2\right) + 1t}\right]\Bigg) \end{aligned} \tag{61}$$

where p, r are arbitrary constants.

Case 4.

$$q_4\left(x,t\right)=\pm\sqrt{2\left(\frac{p^2-r^2}{1-2m^2}+2\right)}\,m\,cn\left(x+\sqrt{\frac{p^2-r^2}{1-2m^2}+1}t\right) \tag{62}$$

As long as $m\longrightarrow 1$, eq. (62) becomes:

$$q_4\left(x,t\right)=\pm\sqrt{2\left(r^2-p^2+2\right)}\ \operatorname{sech}\left[x+\sqrt{r^2-p^2+1}t\right] \tag{63}$$

Thus the solution of eq. (47):

$$u_4\left(x,t\right)=\pm e^{i\left(px+rt\right)}\sqrt{2\left(r^2-p^2+2\right)}\operatorname{sech}\left[x+\sqrt{r^2-p^2+1}t\right] \tag{64}$$

where p, r are arbitrary constants. Figures 2 and 3 illustrate the solutions.

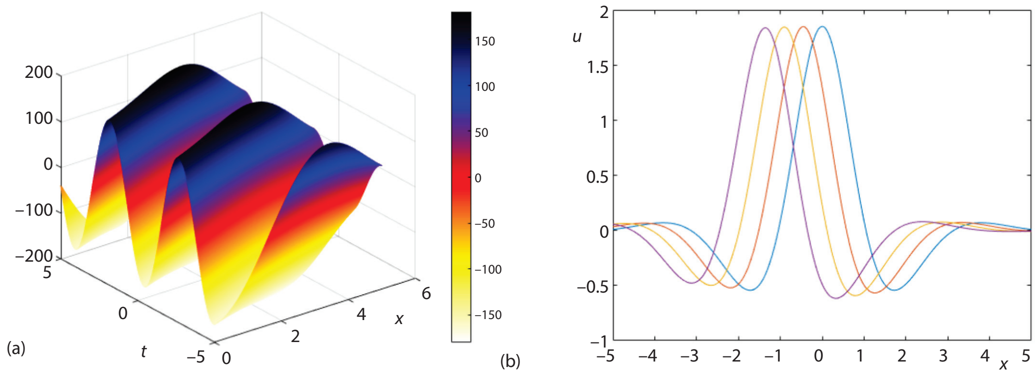

Figure 2. Graph of real part of solution of eq. (64) with p = 1.5 and r = 1.4

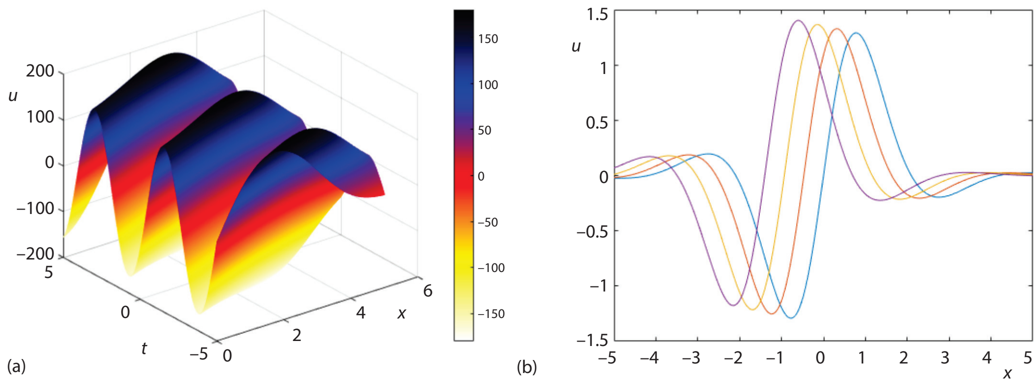

Figure 3. Graph of imaginary part of solution of eq. (64) with p = 1.5 and r = 1.4

Conclusion

We obtained exact solutions of cubic Boussinesq and coupled Higgs system by the unified method. The explicit solutions are expressed in terms of some elementary functions including trigonometric, exponential, and polynomial. Tests functions indicated the strength of the method. As a future expansion of the work, we will apply the method to some other NLEE having a noise term.

References

[1] Abdelrahman, M. A. E., Kunik, M., The Ultra-Relativistic Euler Equations, *Math. Meth. Appl. Sci*., *12* (2015), 1, pp. 1247-1264
[2] Abdelrahman, M. A. E., Alkhidhr, H., A Robust and Accurate Solver for Some Non-Linear Partial Differential Equations and Tow Applications, *Physica Scripta*, *95* (2020), 2, 065212
[3] Abdelrahman, M. A. E., *et al*., The Coupled Non-Linear Schrodinger-Type Equations, *Modern Physics Letters B*, *34* (2020), 6, 2050078
[4] Malik, R., *et al*., Theoretical Investigations on Propagating and Growing Modes in a Pair Plasma Havin-Dust Grains, *Phys. Plasmas*, *19* (2012), 032107
[5] M. Kaplan, *et al*., A Generalized Kudryashov Method to Some Non-Linear Evolution Equations in Mathematical Physics, *Non-Linear Dyn*., *85* (2016), June, pp. 2843-2850
[6] Wazwaz, A. M., Bright and Dark Optical Solitons for (2+1)-Dimensional Schrodinger (NLS) equations in the Anomalous Dispersion Regimes and the Normal Dispersive Regimes, *Optik*, *192* (2019), 162948
[7] Bhrawy, A. H., An Efficient Jacobi Pseudospectral Approximation for Non-Linear Complex Generalized Zakharov System, *Appl. Math. Comput*., *247* (2014), Nov., pp. 30-46
[8] Hassan, S. Z., Abdelrahman, M. A. E., Solitary Wave Solutions for Some Non-Linear Time Fractional Partial Differential Equations, *Pramana J. Phys*., *91* (2018), 5, pp. 91-67
[9] Abdelrahman, M. A. E., Sohaly, M. A., On the New Wave Solutions to the MCH Equation, *Indian Journal of Physics*, *93* (2019), Dec., pp. 903-911
[10] Abdelrahman, M. A. E., Sohaly, M. A., Solitary Waves for the Non-Linear Schrodinger Problem With the Probability Distribution Function in Stochastic Input Case, *Eur. Phys. J. Plus*, *132* (2017), 339
[11] Abdelrahman, M. A. E., A Note on Riccati-Bernoulli sub-ODE Method Combined with Complex Transform Method Applied to Fractional Differential Equations, *Non-Linear Engineering Modelling and Application*, *7* (2018), 4, pp. 279-285
[12] Zhang, J. L., *et al.,* The Improved F-Expansion Method and Its Applications, *Phys. Lett. A*, *350* (2006), 1-2, pp. 103-109
[13] Wazwaz, A. M. The Extended Tanh Method for Abundant Solitary Wave Solutions of Non-Linear Wave Equations, *Appl. Math. Comput.*, *187* (2007), 2, pp. 1131-1142
[14] Mirzazadeh, M., *et al.*, 1-Soliton Solution of KdV Equation, *Non-Linear Dyn*., *80* (2015), 1-2, pp. 387-396
[15] Wazwaz, A. M., The Tanh Method for Travelling Wave Solutions of Non-Linear Equations, *Appl. Math. Comput*., *154* (2004), 3, pp. 714-723
[16] Fan, E., Zhang, H., A Note on the Homogeneous Balance Method, *Phys. Lett. A*, *246* (1998), 5, pp. 403-406
[17] Aminikhad, H., *et al.*, Exact Solutions for Non-Linear Partial Differential Equations Via Exp-Function Method, *Numer. Methods Partial Differ. Equations*, *26* (2009), 6, pp. 1427-1433
[18] Ma, W.-X., *et al.*, A Multiple Exp-Function Method for Non-Linear Differential Equations and Its Application, *Physica Scripta*, *82* (2010), 065003
[19] Dai. C. Q., Zhang, J. F., Jacobian Elliptic Function Method for Non-Linear Differential Difference Equtions, *Chaos Solutions Fractals*, *27* (2006), 4, pp. 1042-1049
[20] Wazwaz, A. M., Exact Solutions to the Double Sinh-Gordon Equation by the Tanh Method and a Variable Separated ODE Method, *Comput. Math. Appl*., *50* (2005), 10-12, pp. 1685-1696
[21] Yang, X. F., *et al.*, A Riccati-Bernoulli sub-ODE Method for Non-Linear Partial Differential Equations and Its Application, *Adv. Diff. Equa*., *1* (2015), Dec., pp. 117-133
[22] Abdelrahman, M. A. E., Sohaly, M. A., The Development of the Deterministic Non-Linear PDE in Particle Physics to Stochastic Case, *Results in Physics*, *9* (2018), June, pp. 344-350
[23] Wazwaz, A. M., A Sine-Cosine Method for Handling Non-Linear Wave Equations, *Math. Comput. Modelling*, *40* (2004), 5-6, pp. 499-508
[24] Baskonus, H. M., Bulut, H., New Wave Behaviors of the System of Equations for the Ion Sound and Langmuir Waves, *Waves Random Complex Media*, *26* (2016), 4, pp. 613-625
[25] Liu, C., Exact Solutions for the Higher-Order Non-Linear Schrodinger Equation in Non-Linear Optical Fibres, *Chaos, Solitons and Fractals, 23* (2005), 3, pp. 949-955
[26] Zhang, S., Exp-Function Method for Solving Maccari's System, *Phys. Lett. A*., *371* (2007), 1-2, pp. 65-71
[27] El Achab, A., Amine, A., A Construction of New Exact Periodic Wave and Solitary Wave Solutions for the 2-D Ginzburg-Landau Equation, *Non-Linear Dyn*., *91* (2017), 2, pp. 995-999
[28] Hosseini, K., *et al.*, New Exact Traveling Wave Solutions of the Unstable Non-Linear Schrodinger Equations, *Commun. Theor. Phys*., *68* (2017), 6, pp. 761-767

[29] Bulut, H., *et al.*, Optical Solitons to the Resonant Non-Linear Schrodinger Equation with Both Spatio-Temporal and Inter-Modal Dispersions under Kerr Law Non-Linearity, *Optik*, *163* (2018), June, pp. 49-55
[30] Akbari-Moghanjoughi, M., Energy Spectrum of Oscillations in Generalized Sagdeev Potential, *Physics of Plasmas*, *24* (2017), 072107
[31] Wanga, Q., *et al.*, An Extended Jacobi Elliptic Function Rational Expansion Method and Its Application (2+1)-Dimensional Dispersive Long Wave Equation, *Phys. Lett. A*, *289* (2005), 5-6, pp. 411-426
[32] Daripa, P., Hua, W., A Numerical Study of an Ill-Posed Boussinesq Equation Arising in Water Waves and Non-Linear Lattices: Filtering and Regularization Techniques, *Appl. Math. Comput*., *101* (1999), 2-3, pp. 159-207
[33] Nagasawa, T., Nishida, Y., Mechanism of Resonant Interaction of Plane Ion-Acoustic Solitons, *Phys. Rev. A*, (1992), Sept., pp. 3446-3471
[34] Hafez, M. G., *et al.*,Traveling Wave Solutions for Some Important Coupled Non-Linear Physical Models Via the Coupled Higgs Equation and the Maccari System, *Journal of King Saud University Science*, *27* (2015), pp. 105-112

Paper submitted: April 5, 2020
Paper revised: June 29, 2020
Paper accepted: July 10, 2020