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Existence of solutions for the Caputo-Hadamard fractional differential equations and inclusions

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Abstract. In this article, we investigate the existence results, with multi-point and integral boundary conditions, for Caputo-Hadamard fractional differential equations (CHFDEs) and inclusions. To get the desired results, which are clearly illustrated by examples, we use standard fixed point theorems for single-valued and multi-valued maps.

1. Introduction

As of late years, fractional-order differential equations(FDEs) have expanded attention from both the conceptual and the applied perspectives. There are numerous applications in several fields, such as chemical mechanics, signal processing, aerodynamics, fluid flow, electrical systems, etc. Instead of integer-order differential and integral operators, differential fractional-order operators are non-local and have the means to examine the inherited properties of a few materials and procedures. The monographs [15, 16, 20, 22] typically applied to the theory of fractional derivatives and integrals and applications of FDEs. See [4, 5, 10, 19, 23–29, 31] for more points of interest and templates, and the references therein. In either case, it has shown that the bulk of the work on the subject is concerned with the FDE of Riemann-Liouville or Caputo form. Other than these fractional derivatives, another type of fractional derivatives defined in the literature is the fractional derivative known to Hadamard in 1892 [13], varying from the aforementioned derivatives in the sense that the integral kernel in the Hadamard derivative description contains an arbitrary exponent's logarithmic function. A point-by-point overview of the integral and Hadamard derivatives found in [1–3, 6, 7, 11, 14, 18, 21, 30, 32, 34, 35]. Ahmad et.al [38] recently examined sequential fractional-order neutral functional differential equations with the Caputo-Hadamard fractional derivative (CHFD). Similarly, with three-point boundary conditions, Boutiara et.al [36] studied the Caputo-Hadamard fractional boundary value problem (BVP). Recently, Tariboon et.al [37] investigated the existence of solutions of CHFDEs for separated BVPs. In this paper, we investigate a new BVP of CHFDEs and inclusions:

$$^{CH}\mathcal{D}^{\varrho}z(\tau) = h(\tau, z(\tau)), \quad \tau \in \mathcal{E} := [1, T], \tag{1}$$

$$^{CH}\mathcal{D}^{\varrho}z(\tau) \in \mathcal{H}(\tau, z(\tau)), \quad \tau \in \mathcal{E} := [1, T], \tag{2}$$

$$z(1) = 0, \quad z'(1) = 0, \quad z(T) = \xi^{H}\mathcal{I}^{\varsigma} z(\upsilon) + \zeta \sum_{i=1}^{m-2} \nu_i z(\vartheta_i), \tag{3}$$

where ${}^{CH}\mathcal{D}^{\varrho}$ is the CHFDs of order $2 < \varsigma \leq 3$, $1 < \vartheta < T$, ${}^{H}\mathcal{I}^{\varsigma}$ is the Hadamard fractional integral of order $1 < \varsigma < 2$, and h: $\mathcal{E} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\mathcal{H} : \mathcal{E} \to \mathcal{S}(\mathbb{R})$ is a multi-valued map, $\mathcal{S}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and ξ, ζ are positive real constants. The article is carried out as follows. In Section 2, We'll present valuable preliminaries and lemmas. Section 3 deals with the existence and uniqueness result for problem (1) and (3) established through fixed point theorems Krasnoselskii and Banach. In Section 4, we discuss the solutions of existence for the problem (2) and (3) using the alternative of Leray-Schauder and fixed-point theorem due to Covitz. We address two examples to explain our main results.

2. Preliminaries

We start with some fundamental definitions, semigroup properties, and lemmas with results [15, 20].

Definition 2.1. *Let $0 \leq b \leq c \leq \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The HFIs of order $\varrho \in \mathbb{C}$ are defined by*

$$\begin{aligned} (\mathcal{I}^{\varrho}_{b+} h)(\tau) &= \frac{1}{\Gamma(\varrho)} \int_b^{\tau} \Big(\log \frac{\tau}{\theta}\Big)^{\varrho-1} h(\theta) \frac{d\theta}{\theta}, \quad b < \tau < c, \quad \textit{and} \\ (\mathcal{I}^{\varrho}_{c-} h)(\tau) &= \frac{1}{\Gamma(\varrho)} \int_{\tau}^{c} \Big(\log \frac{\theta}{\tau}\Big)^{\varrho-1} h(\theta) \frac{d\theta}{\theta}, \quad b < \tau < c. \end{aligned}$$

Definition 2.2. *The left and right-sided Hadamard fractional derivatives of order $\varrho \in \mathbb{C}$ with $\mathbb{R}(\varrho) \geq 0$ on (b, c) and $b < \tau < c$ are defined by*

$$\begin{aligned} (\mathcal{D}^{\varrho}_{b+} h)(\tau) &= \Big(\tau \frac{d}{d\tau}\Big)^n \frac{1}{\Gamma(n-\varrho)} \int_b^{\tau} \Big(\log \frac{\tau}{\theta}\Big)^{n-\varrho-1} h(\theta) \frac{d\theta}{\theta}, \quad \textit{and} \\ (\mathcal{D}^{\varrho}_{c-} h)(\tau) &= \Big(-\tau \frac{d}{d\tau}\Big)^n \frac{1}{\Gamma(n-\varrho)} \int_{\tau}^{c} \Big(\log \frac{\theta}{\tau}\Big)^{n-\varrho-1} h(\theta) \frac{d\theta}{\theta}, \end{aligned}$$

where $n = [\mathbb{R}(\varrho)] + 1$.

Lemma 2.3. *If $\mathbb{R}(\varrho) > 0$, $\mathbb{R}(\varsigma) > 0$ and $0 < b < c < \infty$, then we have*

$$\begin{aligned} \Big(\mathcal{I}^{\varrho}_{b+} \Big(\log \frac{\theta}{b}\Big)^{\varsigma-1}\Big)(\tau) &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)} \Big(\log \frac{\tau}{b}\Big)^{\varsigma+\varrho-1}, \\ \Big(\mathcal{I}^{\varrho}_{c-} \Big(\log \frac{c}{\theta}\Big)^{\varsigma-1}\Big)(\tau) &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)} \Big(\log \frac{c}{\tau}\Big)^{\varsigma+\varrho-1}. \end{aligned}$$

Definition 2.4. *Let $0 < b < c < \infty$, $\mathbb{R}(\varrho) \geq 0$, $n = [\mathbb{R}(\varrho) + 1]$. The left and right CHFDs of order ϱ are respectively defined by*

$$({}^{C}\mathcal{D}^{\varrho}_{b+} h)(\tau) = \mathcal{D}^{\varrho}_{b+} \left[h(\theta) - \sum_{k=0}^{n-1} \frac{\delta^k h(b)}{k!} \Big(\log \frac{\theta}{b}\Big)^k \right](\tau),$$

and

$$({}^{C}\mathcal{D}^{\varrho}_{c-} h)(\tau) = \mathcal{D}^{\varrho}_{c-} \left[h(\theta) - \sum_{k=0}^{n-1} \frac{(-1)^k \delta^k h(c)}{k!} \Big(\log \frac{c}{\theta}\Big)^k \right](\tau).$$

We define space $\mathcal{W} = \mathcal{C}(\mathcal{E}, \mathbb{R})$ the Banach space of all continuous functions from $\mathcal{E} \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|z\| = \sup\{|z(\tau)|, \tau \in \mathcal{E}\}$. Let $\mathcal{AC}[1, T]$ be the space functions that are absolutely continuous on $[1, T]$. Let us introduce the space $\mathcal{AC}^n_\delta[1, T]$, which consists of those function h by

$$\mathcal{AC}^n_\delta[1,T] = \Big\{h : [1,T] \to \mathbb{C}, \delta^{n-1}h(\tau) \in \mathcal{AC}[1,T], \delta = \tau\frac{d}{d\tau}\Big\}.$$

Lemma 2.5. *If $\mathcal{T} : \mathcal{X} \to \mathcal{S}_{cld}(\mathcal{Y})$ is upper semi-continuous (USC), then $\mathcal{G}r(T)$ is a closed subset of $\mathcal{X} \times \mathcal{Y}$; i.e., for every sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ and $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{Y}$, if $u_n \to u_*$ and $v_n \to v_*$, then $u_n \to \mathcal{T}v_*$. Conversely, if $\mathcal{T}$ is completely continuous and has a closed graph, then it is USC.*

Definition 2.6. *A function $z \in C^3(\mathcal{E}, \mathbb{R})$ is called a solution of problem (2) and (3) if $\exists$ a function $\alpha(\tau) \in \mathcal{L}^1(\mathcal{E}, \mathbb{R})$ with $\alpha(\tau) \in \mathcal{H}(\tau, z(\tau))$ such that*

$$\begin{aligned} &{}^{CH}\mathcal{D}^{\varrho} z(\tau) = \alpha(\tau), \quad 2 < \varsigma \leq 3, \quad \forall \ \ \tau \in \mathcal{E}, \\ &z(1) = 0, \quad z'(1) = 0, \quad z(T) = \xi\, {}^{H}\mathcal{I}^{\varsigma} z(\upsilon) + \zeta \sum_{i=1}^{m-2} \nu_i z(\vartheta_i), \end{aligned}$$

where $1 < \upsilon < \vartheta_1 < \vartheta_2 < \cdots < \vartheta_{m-2} < T$.

Lemma 2.7. *For any $\hat{h} \in \mathcal{C}(\mathcal{E}, \mathbb{R})$, $z \in \mathcal{C}^3(\mathcal{E}, \mathbb{R})$, the function z is the solution of the problem*

$$\begin{aligned} &{}^{CH}\mathcal{D}^{\varrho} z(\tau) = \hat{h}(\tau), \quad \tau \in \mathcal{E}, \\ &z(1) = 0, \quad z'(1) = 0, \quad z(T) = \xi\, {}^{H}\mathcal{I}^{\varsigma} z(\upsilon) + \zeta \sum_{i=1}^{m-2} \nu_i z(\vartheta_i), \end{aligned} \tag{4}$$

if and only if

$$\begin{aligned} z(\tau) = \ & \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1} \hat{h}(\theta)\frac{d\theta}{\theta} + \frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\hat{h}(\theta)\frac{d\theta}{\theta} \\ & +\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\hat{h}(\theta)\frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\hat{h}(\theta)\frac{d\theta}{\theta}\Bigg], \end{aligned} \tag{5}$$

where

$$\Delta = (\log T)^2 - \frac{2\xi(\log\upsilon)^{2+\varsigma}}{\Gamma(3+\varsigma)} - \zeta\sum_{i=1}^{m-2}\nu_i(\log\vartheta_i)^2. \tag{6}$$

3. Single-valued case for the problem (1) and (3)

We define $\Pi : \mathcal{W} \to \mathcal{W}$ as

$$\begin{aligned} \Pi(z)(\tau) = & \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}h(\theta, z(\theta))\frac{d\theta}{\theta} + \frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}h(\theta, z(\theta))\frac{d\theta}{\theta} \\ & + \zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}h(\theta, z(\theta))\frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}h(\theta, z(\theta))\frac{d\theta}{\theta}\Bigg], \end{aligned} \tag{7}$$

in view of Lemma 2.7. Suitable for computation, we represent:

$$\Omega = \frac{1}{\Delta\Gamma(\varrho+1)}\Bigg(\Delta(\log T)^{\varrho} + (\log T)^2\Big(\zeta\sum_{i=1}^{m-2}\nu_i(\log\vartheta_i)^{\varrho}\Big)\Bigg) + \frac{(\log T)^2}{\Delta}\Bigg(\frac{\xi(\log\upsilon)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)} + \frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}\Bigg). \tag{8}$$

Let a continuous function be $h : \mathcal{E} \times \mathbb{R} \to \mathbb{R}$. We need the following premises in order to prove the existence and uniqueness results.

$(\mathcal{K}_1)$ $|h(\tau,z_1)-h(\tau,z_2)|\leq \mathcal{P}|z_1-z_2|,\ \forall\ \tau\in\mathcal{E}, z_1,z_2\in\mathbb{R},\mathcal{P}>0.$

$(\mathcal{K}_2)$ $|h(\tau,z(\tau))|\leq\delta(\tau)$ for $(\tau,z)\in\mathcal{E}\times\mathbb{R}$, and $\delta\in\mathcal{C}(\mathcal{E},\mathbb{R}^+)$ with $\|\delta\|=\max_{\tau\in\mathcal{E}}|\delta(\tau)|$.

$(\mathcal{K}_3)$ $\mathcal{H}:\mathcal{E}\times\mathbb{R}\to\mathcal{S}(\mathbb{R})$ has non-empty compact and convex values and is Caratheodory.

$(\mathcal{K}_4)$ $\exists$ a non-decreasing continuous function $\phi:[0,\infty]\to[0,\infty]$and a function $\kappa\in C(\mathcal{E},\mathbb{R}^+)$ such that $\|\mathcal{H}(\tau,z)\|_{\mathcal{S}}=\sup\{|w|:w\in\mathcal{H}(\tau,z)\}\leq\kappa(\tau)\phi(\|z\|)$ for each $(\tau,z)\in\mathcal{E}\times\mathbb{R}$.

$(\mathcal{K}_5)$ The $\mathcal{Q}$ constant exists such that $\dfrac{\mathcal{Q}}{\phi(\mathcal{Q})\|\kappa\|\Omega}>1$, where Ω is set by (8).

$(\mathcal{K}_6)$ $\mathcal{H}:\mathcal{E}\times\mathbb{R}\to\mathcal{S}_{cpt}(\mathbb{R})$ is such that $\mathcal{H}(\cdot,z):\mathcal{E}\to\mathcal{S}_{cpt}(\mathbb{R})$ is measurable for each $z\in\mathbb{R}$.

$(\mathcal{K}_7)$ $\mathcal{F}_g(\mathcal{H}(\tau,z),\mathcal{H}(\tau,\overline{z}))\leq\rho(\tau)|z-\overline{z}|\ \forall\ \mathcal{E}$ and $z,\overline{z}\in\mathbb{R}$ with $\rho\in\mathcal{L}^1(\mathcal{E},\mathbb{R}^+)$ and $g(0,\mathcal{H}(\tau,0))\leq\rho(\tau)\ \forall\ \tau\in\mathcal{E}$.

Theorem 3.1. *Suppose $(\mathcal{K}_1)$, $(\mathcal{K}_2)$ holds. If*

$$\left\{\frac{\mathcal{P}(\log T)^2}{\Delta}\left(\frac{\xi(\log\upsilon)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)}+\zeta\sum_{i=1}^{m-2}\nu_i\frac{(\log\vartheta_i)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}\right)\right\}<1. \tag{9}$$

Then there is at least one solution to the problem (1) and (3).

Proof. Determining $\mathcal{B}_\varepsilon=\{z\in\mathcal{W}:\|z\|\leq\varepsilon\}$, where $\varepsilon\geq\|\delta\|\Omega$. To prove the hypothesis of Theorem (see [17]), we divide the Π operator given by (7) as $\Pi=\Pi_1+\Pi_2$ to $\mathcal{B}_\varepsilon$, where

$$\begin{aligned}(\Pi_1 z)(\tau)=&\frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}h(\theta,z(\theta))\frac{d\theta}{\theta},\\(\Pi_2 z)(\tau)=&\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^\upsilon\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}h(\theta,z(\theta))\frac{d\theta}{\theta}\\&+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}h(\theta,z(\theta))\frac{d\theta}{\theta}-\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}h(\theta,z(\theta))\frac{d\theta}{\theta}\Bigg].\end{aligned}$$

For $\hat{z}_1,\hat{z}_2\in\mathcal{B}_\varepsilon$,

$$\begin{aligned}|(\Pi_1\hat{z}_1)(\tau)+(\Pi_2\hat{z}_2)(\tau)|\leq&\sup_{\tau\in\mathcal{E}}\Bigg\{\frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}|h(\theta,z(\theta))|\frac{d\theta}{\theta}\\&+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^\upsilon\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}|h(\theta,z(\theta))|\frac{d\theta}{\theta}\\&+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}|h(\theta,z(\theta))|\frac{d\theta}{\theta}\\&+\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|h(\theta,z(\theta))|\frac{d\theta}{\theta}\Bigg]\Bigg\}\\\leq&\|\delta\|\left\{\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^2}{\Delta}\left(\frac{\xi(\log\upsilon)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)}+\zeta\sum_{i=1}^{m-2}\nu_i\frac{(\log\vartheta_i)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}\right)\right\}\\\leq&\|\delta\|\Omega\leq\varepsilon,\end{aligned}$$

That implies $\Pi_1\hat{z}_1 + \Pi_2\hat{z}_2 \in \mathcal{B}_\varepsilon$. Now, we have to prove that Π_2 is a contractive. Let $z_1, z_2 \in \mathbb{R}$, and $\tau \in \mathcal{E}$. Then, along with (9) with the $(\mathcal{K}_1)$ assumption, we get

$$\|\Pi_2 z_1 - \Pi_2 z_2\| \leq \frac{\mathcal{P}(\log T)^2}{\Delta}\left(\frac{\xi(\log \upsilon)^{\varrho+\varsigma}}{\Gamma(\varrho+\varsigma+1)}+\zeta\sum_{i=1}^{m-2}\nu_i\frac{(\log\vartheta_i)^{\varrho}}{\Gamma(\varrho+1)}+\frac{(\log T)^{\varrho}}{\Gamma(\varrho+1)}\right)\|z_1 - z_2\|.$$

The operator Π_2, as specified in statement $(\mathcal{K}_1)$, is a contraction. Next, we'll show Π_1's compact and continuous. h-continuity implies continuous operator Π_1. Π_1 is bounded uniformly as $\mathcal{B}_\varepsilon$,

$$\|\Pi_1 z\| \leq \frac{\|\delta\|(\log T)^{\varrho}}{\Gamma(\varrho+1)}.$$

In addition, with $\sup_{(\tau,z)\in\mathcal{E}\times\mathcal{B}_\varepsilon} |h(\tau, z)| = \hat{h} < \infty$ and $\tau_1 < \tau_2$, $\tau_1, \tau_2 \in \mathcal{E}$, we have

$$\begin{aligned}|(\Pi_1 z)(\tau_2) - (\Pi_1 z)(\tau_1)| =&\left|\frac{1}{\Gamma(\varrho)}\int_1^{\tau_2}\left(\log\frac{\tau_2}{\theta}\right)^{\varrho-1}h(\theta, z(\theta))\frac{d\theta}{\theta}\right.\\ &\left.-\frac{1}{\Gamma(\varrho)}\int_1^{\tau_1}\left(\log\frac{\tau_1}{\theta}\right)^{\varrho-1}h(\theta, z(\theta))\frac{d\theta}{\theta}\right|\\ \leq&\frac{\hat{h}}{\Gamma(\varrho)}\left|\int_0^{\tau_1}\left[\left(\log\frac{\tau_2}{\theta}\right)^{\varrho-1}-\left(\log\frac{\tau_1}{\theta}\right)^{\varrho-1}\right]\frac{d\theta}{\theta}+\int_{\tau_1}^{\tau_2}\left(\log\frac{\tau_2}{\theta}\right)^{\varrho-1}\frac{d\theta}{\theta}\right|.\end{aligned} \tag{10}$$

As $\tau_1 \to \tau_2$, the RHS of the above inequality tends to zero. So Π_1 on $\mathcal{B}_\varepsilon$ is relatively compact. Then Π_1 is compact on $\mathcal{B}_\varepsilon$ by Theorem (see Lemma 1.2 [33]). Therefore, all of Theorem's assumptions (see [17]) are fulfilled. Hence, there is at least one solution for the problem (1) and (3) on $\mathcal{E}$. □

Theorem 3.2. *Suppose $(\mathcal{K}_1)$ hold. Additionally, it assumes $\mathcal{P}\Omega < 1$, where Ω is specified in (8). Then, there exists an unique solution for the problem (1) and (3) on $\mathcal{E}$.*

Proof. Define $\sup_{\tau\in\mathcal{E}}|h(\tau, 0)| = \mathcal{V} < \infty$. Choosing $\varepsilon \geq \frac{\mathcal{V}\Omega}{1-\mathcal{P}\Omega}$, we show that $\Pi\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$, where $\mathcal{B}_\varepsilon = \{z \in \mathcal{W} : \|z\| \leq \varepsilon\}$. For $z \in \mathcal{B}_\varepsilon$, we have

$$\begin{aligned}|(\Pi z)(\tau)| \leq& \sup_{\tau\in\mathcal{E}}\left\{\frac{1}{\Gamma(\varrho)}\int_1^{\tau}\left(\log\frac{\tau}{\theta}\right)^{\varrho-1}|h(\theta, z(\theta))|\frac{d\theta}{\theta}\right.\\ &+\frac{(\log\tau)^2}{\Delta}\left[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\left(\log\frac{\upsilon}{\theta}\right)^{\varrho+\varsigma-1}|h(\theta, z(\theta))|\frac{d\theta}{\theta}\right.\\ &+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\left(\log\frac{\vartheta_i}{\theta}\right)^{\varrho-1}|h(\theta, z(\theta))|\frac{d\theta}{\theta}\\ &\left.\left.+\frac{1}{\Gamma(\varrho)}\int_1^{T}\left(\log\frac{T}{\theta}\right)^{\varrho-1}|h(\theta, z(\theta))|\frac{d\theta}{\theta}\right]\right\}\\ \leq&(\mathcal{P}\varepsilon+\mathcal{V})\sup_{\tau\in\mathcal{E}}\left\{\frac{1}{\Gamma(\varrho)}\int_1^{\tau}\left(\log\frac{\tau}{\theta}\right)^{\varrho-1}\frac{d\theta}{\theta}\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\frac{d\theta}{\theta}\\
&+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\\
&+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\Bigg]\Bigg\}\le(\mathcal{P}\varepsilon+\mathcal{V})\Omega.
\end{aligned}\tag{11}
$$

Thus, $\|(\Pi z)\|\le\varepsilon$ is derived from (11). Now, for $z,\hat{z}\in\mathcal{W}$, we're getting

$$
\begin{aligned}
|\Pi z(\tau)-\Pi\hat{z}(\tau)|\le&\sup_{\tau\in\mathcal{E}}\Bigg\{\frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}|h(\theta,z(\theta))-h(\theta,\hat{z}(\theta))|\frac{d\theta}{\theta}\\
&+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}|h(\theta,z(\theta))-h(\theta,\hat{z}(\theta))|\frac{d\theta}{\theta}\\
&+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}|h(\theta,z(\theta))-h(\theta,\hat{z}(\theta))|\frac{d\theta}{\theta}\\
&+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|h(\theta,z(\theta))-h(\theta,\hat{z}(\theta))|\frac{d\theta}{\theta}\Bigg]\Bigg\}\\
\le&\mathcal{P}\|z-\hat{z}\|\Bigg\{\frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\\
&+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\frac{d\theta}{\theta}\\
&+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\\
&+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\Bigg]\Bigg\}=\mathcal{P}\Omega\|z-\hat{z}\|.
\end{aligned}
$$

Therefore,

$$\|\Pi z-\Pi\hat{z}\|\le\mathcal{P}\Omega\|z-\hat{z}\|.$$

Since by definition $\mathcal{P}\Omega<1$, Π represents a contraction. Theorem (see Theorem 1.4 [33]) follows that on $\mathcal{E}$ the equation (1) and (3) has an unique solution. □

4. Multi-valued case for the problem (2) and (3)

Theorem 4.1. *Suppose* $(\mathcal{K}_3)$,$(\mathcal{K}_4)$, *and* $(\mathcal{K}_5)$ *holds. Then, the* (2) *and* (3) *problems contain at least one solution on* $\mathcal{J}$.

Proof. Let us describe an operator $\Psi_{\mathcal{H}} : \mathcal{W} \to \mathcal{S}(\mathcal{W})$ by

$$\Psi_{\mathcal{H}}(z) = \left\{ \begin{array}{l} s \in \mathcal{W} : \\ s(\tau) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(\varrho)} \displaystyle\int_1^\tau \Big(\log \frac{\tau}{\theta}\Big)^{\varrho-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \\ + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^\upsilon \Big(\log \frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \\ \quad + \zeta \displaystyle\sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta}\Big)^{\varrho-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \\ \quad - \frac{1}{\Gamma(\varrho)} \int_1^T \Big(\log \frac{T}{\theta}\Big)^{\varrho-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \Bigg], \end{array} \right. \end{array} \right\}$$

for $\alpha \in \mathcal{M}_{\mathcal{H},z}$. We must prove that $\Psi_{\mathcal{H}}$ fulfills the premises of the Theorem (see Theorem 8.5 [12]). First, for every $z \in \mathcal{W}$, we present that $\Psi_{\mathcal{H}}$ is convex. Next, in $\mathcal{W}$ bounded sets, we prove $\Psi_{\mathcal{H}}$ maps to bound sets. Let $\mathcal{B}_\varepsilon = \{z \in \mathcal{W} : \|z\| \leq \varepsilon\}$ be a bounded ball in $\mathcal{W}$ for the positive number ε. Then, for each $s \in \Psi_{\mathcal{H}}(z)$, $z \in \mathcal{B}_\varepsilon \ \exists \ \alpha \in \mathcal{M}_{\mathcal{H},z}$ such that

$$\begin{aligned} s(\tau) = & \frac{1}{\Gamma(\varrho)} \int_1^\tau \Big(\log \frac{\tau}{\theta}\Big)^{\varrho-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \\ & + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^\upsilon \Big(\log \frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \\ & \quad + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta}\Big)^{\varrho-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \\ & \quad - \frac{1}{\Gamma(\varrho)} \int_1^T \Big(\log \frac{T}{\theta}\Big)^{\varrho-1} h(\theta, z(\theta)) \frac{d\theta}{\theta} \Bigg]. \end{aligned}$$

For $\tau \in \mathcal{E}$, we have

$$\begin{aligned} |s(\tau)| = & \frac{1}{\Gamma(\varrho)} \int_1^\tau \Big(\log \frac{\tau}{\theta}\Big)^{\varrho-1} |\alpha(\theta)| \frac{d\theta}{\theta} + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^\upsilon \Big(\log \frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1} |\alpha(\theta)| \frac{d\theta}{\theta} \\ & + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta}\Big)^{\varrho-1} |\alpha(\theta)| \frac{d\theta}{\theta} + \frac{1}{\Gamma(\varrho)} \int_1^T \Big(\log \frac{T}{\theta}\Big)^{\varrho-1} |\alpha(\theta)| \frac{d\theta}{\theta} \Bigg] \\ \leq & \phi(\|z\|)\|\kappa\| \Bigg\{ \frac{1}{\Gamma(\varrho)} \int_1^\tau \Big(\log \frac{\tau}{\theta}\Big)^{\varrho-1} \frac{d\theta}{\theta} + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^\upsilon \Big(\log \frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1} \frac{d\theta}{\theta} \\ & + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta}\Big)^{\varrho-1} \frac{d\theta}{\theta} + \frac{1}{\Gamma(\varrho)} \int_1^T \Big(\log \frac{T}{\theta}\Big)^{\varrho-1} \frac{d\theta}{\theta} \Bigg] \Bigg\} \\ \leq & \phi(\|z\|)\|\kappa\|\Omega. \end{aligned}$$

Consequently,

$$\|s\| \leq \phi(\|z\|)\|\kappa\|\Omega.$$

We demonstrate that the maps bounded sets into equicontinuous sets of $\mathcal{W}$. Let $\tau_1, \tau_2 \in \mathcal{E}$ for $\tau_1 < \tau_2$ and $z \in \mathcal{B}_\varepsilon$. For each $s \in \Psi_{\mathcal{H}}(z)$, we get

$$|s(\tau_2) - s(\tau_1)| \leq \frac{\phi(\varepsilon)\|\kappa\|}{\Gamma(\varrho)} \Bigg| \int_0^{\tau_1} \Big[\Big(\log \frac{\tau_2}{\theta}\Big)^{\varrho-1} - \Big(\log \frac{\tau_1}{\theta}\Big)^{\varrho-1} \Big] \frac{d\theta}{\theta}$$

$$
\begin{aligned}
& + \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{\theta} \right)^{\varrho-1} \frac{d\theta}{\theta} \Bigg| + \frac{\phi(\varepsilon)\|\kappa\| |(\log \tau_2)^2 - (\log \tau_1)^2|}{\Delta} \\
& \times \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^{\upsilon} \Big(\log \frac{\upsilon}{\theta} \Big)^{\varrho+\varsigma-1} \frac{d\theta}{\theta} \\
& + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta} \Big)^{\varrho-1} \frac{d\theta}{\theta} + \frac{1}{\Gamma(\varrho)} \int_1^{T} \Big(\log \frac{T}{\theta} \Big)^{\varrho-1} \frac{d\theta}{\theta} \Bigg].
\end{aligned}
$$

The RHS of the above inequality obviously tends to be zero, independently of $z \in \mathcal{B}_\varepsilon$ as $\tau_2 - \tau_1 \to 0$. Because $\Psi_{\mathcal{H}}$ meets the $(\mathcal{K}_3)$, $(\mathcal{K}_4)$, and $(\mathcal{K}_5)$ then the Theorem (see Lemma 1.2 [33]) follows that $\Psi_{\mathcal{H}} : \mathcal{W} \to \mathcal{S}(\mathcal{W})$ is completely continuous. When we conclude that it has a closed graph, $\Psi_{\mathcal{H}}$ is shown to be upper semicontinuous (USC), as $\Psi_{\mathcal{H}}$ is already shown to be completely continuous. Thus we'll prove that $\Psi_{\mathcal{H}}$ has a closed graph. Let $z_n \to z_*$, $s_n \in \Psi_{\mathcal{H}}(z_n)$ and $s_n \to s_*$. We will prove then that $\Psi_{\mathcal{H}}(z_*)$. Joined with $s_n \in \Psi_{\mathcal{H}}(z_n)$, $\exists\ \alpha_n \in \Psi_{\mathcal{H}}(z_n)$, such that for each $\tau \in \mathcal{E}$.

$$
\begin{aligned}
s_n(\tau) \;=\; & \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \Big(\log \frac{\tau}{\theta} \Big)^{\varrho-1} \alpha_n(\theta) \frac{d\theta}{\theta} + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^{\upsilon} \Big(\log \frac{\upsilon}{\theta} \Big)^{\varrho+\varsigma-1} \alpha_n(\theta) \frac{d\theta}{\theta} \\
& + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta} \Big)^{\varrho-1} \alpha_n(\theta) \frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)} \int_1^{T} \Big(\log \frac{T}{\theta} \Big)^{\varrho-1} \alpha_n(\theta) \frac{d\theta}{\theta} \Bigg].
\end{aligned}
$$

Thus it is necessary to show that $\exists\ \alpha_* \in \mathcal{M}_{\mathcal{H},z_*} \ni$ for each $\tau \in \mathcal{E}$,

$$
\begin{aligned}
s_*(\tau) \;=\; & \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \Big(\log \frac{\tau}{\theta} \Big)^{\varrho-1} \alpha_*(\theta) \frac{d\theta}{\theta} + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^{\upsilon} \Big(\log \frac{\upsilon}{\theta} \Big)^{\varrho+\varsigma-1} \alpha_*(\theta) \frac{d\theta}{\theta} \\
& + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta} \Big)^{\varrho-1} \alpha_*(\theta) \frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)} \int_1^{T} \Big(\log \frac{T}{\theta} \Big)^{\varrho-1} \alpha_*(\theta) \frac{d\theta}{\theta} \Bigg].
\end{aligned}
$$

Consider a $\mathcal{Y}$ linear operator : $\mathcal{L}^1(\mathcal{E}, \mathbb{R}) \to \mathcal{W}$ as

$$
\begin{aligned}
s \mapsto \mathcal{Y}(\alpha)(\tau) \;=\; & \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \Big(\log \frac{\tau}{\theta} \Big)^{\varrho-1} \alpha(\theta) \frac{d\theta}{\theta} + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^{\upsilon} \Big(\log \frac{\upsilon}{\theta} \Big)^{\varrho+\varsigma-1} \alpha(\theta) \frac{d\theta}{\theta} \\
& + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta} \Big)^{\varrho-1} \alpha(\theta) \frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)} \int_1^{T} \Big(\log \frac{T}{\theta} \Big)^{\varrho-1} \alpha(\theta) \frac{d\theta}{\theta} \Bigg].
\end{aligned}
$$

Remember that

$$
\begin{aligned}
\|s_n(\tau) - s_*(\tau)\| \;=\; & \Bigg\| \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \Big(\log \frac{\tau}{\theta} \Big)^{\varrho-1} (\alpha_n(\theta) - \alpha_*(\theta)) \frac{d\theta}{\theta} \\
& + \frac{(\log \tau)^2}{\Delta} \Bigg[\xi \frac{1}{\Gamma(\varrho+\varsigma)} \int_1^{\upsilon} \Big(\log \frac{\upsilon}{\theta} \Big)^{\varrho+\varsigma-1} (\alpha_n(\theta) - \alpha_*(\theta)) \frac{d\theta}{\theta} \\
& + \zeta \sum_{i=1}^{m-2} \nu_i \frac{1}{\Gamma(\varrho)} \int_1^{\vartheta_i} \Big(\log \frac{\vartheta_i}{\theta} \Big)^{\varrho-1} (\alpha_n(\theta) - \alpha_*(\theta)) \frac{d\theta}{\theta}
\end{aligned}
$$

$$-\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}(\alpha_n(\theta)-\alpha_*(\theta))\frac{d\theta}{\theta}\Bigg]\Bigg\|\to 0,$$

as $n\to\infty$. The $\mathcal{Y}\circ\mathcal{M}_{\mathcal{H},z}$ is a closed graph operator according to Lemma 2.5. We have $s_n(\tau)\in\mathcal{Y}(\mathcal{M}_{\mathcal{H},z_n})$. So, because of $z_n\to z_*$, we have

$$\begin{aligned} s_*(\tau) &= \frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\alpha_*(\theta)\frac{d\theta}{\theta}+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^\upsilon\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\alpha_*(\theta)\frac{d\theta}{\theta}\\ &\quad+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\alpha_*(\theta)\frac{d\theta}{\theta}-\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\alpha_*(\theta)\frac{d\theta}{\theta}\Bigg], \end{aligned}$$

for some $\alpha_*\in\mathcal{M}_{\mathcal{H},z_*}$. There is an open set $\mathcal{P}\subseteq\mathcal{W}$ with $z\notin\Psi_{\mathcal{H}}(z)$ for every $\epsilon\in(0,1)$ and all $z\in\partial\mathcal{P}$. Then there's $\alpha\in\mathcal{L}^1(\mathcal{E},\mathbb{R})$ with $\alpha\in\mathcal{M}_{\mathcal{H},z}$ so we have

$$\begin{aligned} z(\tau) &= \frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\alpha(\theta)\frac{d\theta}{\theta}+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^\upsilon\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\alpha(\theta)\frac{d\theta}{\theta}\\ &\quad+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\alpha(\theta)\frac{d\theta}{\theta}-\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\alpha(\theta)\frac{d\theta}{\theta}\Bigg],\ \tau\in\mathcal{E}. \end{aligned}$$

As in step 2, it is possible to have $\|z\|\le\phi(\|z\|)\|\kappa\|\Omega$, which means $\frac{\|z\|}{\phi(\|z\|)\|\kappa\|\Omega}\le 1$. With regard to $(\mathcal{K}_5)$, $\exists\ \mathcal{Q}$ such that $\|z\|\neq\mathcal{Q}$. Allow us to set $\mathcal{P}=\{z\in\mathcal{W}:\|z\|<\mathcal{Q}\}$. Notice that the operator $\Psi_{\mathcal{H}}:\overline{\mathcal{P}}\to\mathcal{S}(\mathcal{W})$ is USC and completely continuous. There is no $z\in\partial\mathcal{P}$ from $\mathcal{P}$'s choice to $z\in\epsilon\Psi_{\mathcal{H}}(z)$ for some $\epsilon\in(0,1)$. Thus, we conclude from the Theorem (see Theorem 8.5 [12]) that the $\Psi_{\mathcal{H}}$ has a fixed point $z\in\overline{\mathcal{P}}$, which is a solution for this (2) and (3) problem. □

Theorem 4.2. *Suppose $(\mathcal{K}_6)$, $(\mathcal{K}_7)$ holds. Additionally, it assumes $\Phi:=\|\eta\|\Omega<1$, where Ω is specified in (8). Then, there exists at least one solution for the problem (2) and (3) on $\mathcal{E}$.*

Proof. Consider that the set $\mathcal{M}_{\mathcal{H},z}$ is nonempty by presumption of $(\mathcal{K}_6)$ for every $z\in\mathcal{W}$, so $\mathcal{H}$ has a measurable selection (see Theorem III.6 [8]). We demonstrate that the $\Psi_{\mathcal{H}}$, specified at the beginning of the Theorem 4.1 statement, satisfies the hypotheses of Theorem (see [9]). To show that $\Psi_{\mathcal{H}}(z)\in\mathcal{S}_{cld}(\mathcal{W})$ for every $z\in\mathcal{W}$, let $\{p_n\}_{n\ge0}\in\Psi_{\mathcal{H}}(z)$ be such that $p_n\to p\in\mathcal{W}$ as $n\to\infty$. Then $p\in\mathcal{W}$ and $\exists\ \alpha_n\in\mathcal{M}_{\mathcal{H},z_n}$ such that, for every $\tau\in\mathcal{E}$,

$$\begin{aligned} p_n(\tau) =&\frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\alpha_n(\theta)\frac{d\theta}{\theta}+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^\upsilon\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\alpha_n(\theta)\frac{d\theta}{\theta}\\ &+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\alpha_n(\theta)\frac{d\theta}{\theta}-\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\alpha_n(\theta)\frac{d\theta}{\theta}\Bigg]. \end{aligned}$$

We shift a sub-sequence to get α_n converging to α in $\mathcal{L}^1(\mathcal{E},\mathbb{R})$, when $\mathcal{H}$ has compact values, so we have

$$\begin{aligned} \alpha_n(\tau)\to\alpha(\tau) &= \frac{1}{\Gamma(\varrho)}\int_1^\tau\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\alpha(\theta)\frac{d\theta}{\theta}+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^\upsilon\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\alpha(\theta)\frac{d\theta}{\theta}\\ &\quad+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\alpha(\theta)\frac{d\theta}{\theta}-\frac{1}{\Gamma(\varrho)}\int_1^T\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\alpha(\theta)\frac{d\theta}{\theta}\Bigg]. \end{aligned}$$

Therefore, $p \in \Psi_{\mathcal{H}}(z)$. We show that $\exists\ \Phi < 1$ such that

$$\mathcal{F}_g(\Psi_{\mathcal{H}}(z), \Psi_{\mathcal{H}}(\overline{z})) \leq \Phi\|z-\overline{z}\|, \quad \text{for each} \quad z, \overline{z} \in \mathcal{W}.$$

Let $z, \overline{z} \in \mathcal{W}$ and $p_1 \in \Psi_{\mathcal{H}}(z)$. Then $\exists\ \alpha_1(\tau) \in \mathcal{H}(\tau, z(\tau))$ such that, for each $\tau \in \mathcal{E}$,

$$\begin{aligned} p_1(\tau) =& \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\alpha_1(\theta)\frac{d\theta}{\theta} + \frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\alpha_1(\theta)\frac{d\theta}{\theta} \\ &+ \zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\alpha_1(\theta)\frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\alpha_1(\theta)\frac{d\theta}{\theta}\Bigg]. \end{aligned}$$

With the $(\mathcal{K}_7)$ hypothesis, we have $\mathcal{F}_g(\mathcal{H}(\tau, z), \mathcal{H}(\tau, \overline{z})) \leq g(\tau)\|z(\tau)-\overline{z}(\tau)\|$. So, $\exists\ q \in \mathcal{H}(\tau, \overline{z}(\tau))$ such that $|\alpha_1(\tau-q)| \leq g(\tau)\|z(\tau)-\overline{z}(\tau)\|,\ \tau \in \mathcal{E}$. Define $\mathcal{G} : \mathcal{E} \to \mathcal{S}(\mathbb{R})$ by

$$\mathcal{G}(\tau) = \{q \in \mathbb{R} : |\alpha_1(\tau) - q| \leq g(\tau)\|z(\tau)-\overline{z}(\tau)\|\}.$$

Since the $\mathcal{G}(\tau) \cap \mathcal{H}(\tau, \overline{z}(\tau))$ multivalued operator is measurable, a $\alpha_2(\tau)$ function exists, which is a measurable $\mathcal{G}$ selection. So $\alpha_2(\tau) \in \mathcal{H}(\tau, \overline{z}(\tau))$ and $|\alpha_1(\tau) - \alpha_2(\tau)| \leq g(\tau)|z(\tau)-\overline{z}(\tau)|$ for each $\tau \in \mathcal{E}$. Determining

$$\begin{aligned} p_2(\tau) =& \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\alpha_2(\theta)\frac{d\theta}{\theta} + \frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\alpha_2(\theta)\frac{d\theta}{\theta} \\ &+ \zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\alpha_2(\theta)\frac{d\theta}{\theta} - \frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\alpha_2(\theta)\frac{d\theta}{\theta}\Bigg], \text{ for each } \tau \in \mathcal{E}. \end{aligned}$$

Therefore,

$$\begin{aligned} |p_1(\tau) - p_2(\tau)| \ =& \ \frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}|\alpha_1(\theta)-\alpha_2(\theta)|\frac{d\theta}{\theta} \\ &+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}|\alpha_1(\theta)-\alpha_2(\theta)|\frac{d\theta}{\theta} \\ &+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}|\alpha_1(\theta)-\alpha_2(\theta)|\frac{d\theta}{\theta} \\ &-\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}|\alpha_1(\theta)-\alpha_2(\theta)|\frac{d\theta}{\theta}\Bigg] \\ \leq& \ \|\rho\|\Bigg\{\frac{1}{\Gamma(\varrho)}\int_1^{\tau}\Big(\log\frac{\tau}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta} \\ &+\frac{(\log\tau)^2}{\Delta}\Bigg[\xi\frac{1}{\Gamma(\varrho+\varsigma)}\int_1^{\upsilon}\Big(\log\frac{\upsilon}{\theta}\Big)^{\varrho+\varsigma-1}\frac{d\theta}{\theta} \\ &+\zeta\sum_{i=1}^{m-2}\nu_i\frac{1}{\Gamma(\varrho)}\int_1^{\vartheta_i}\Big(\log\frac{\vartheta_i}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta} \\ &+\frac{1}{\Gamma(\varrho)}\int_1^{T}\Big(\log\frac{T}{\theta}\Big)^{\varrho-1}\frac{d\theta}{\theta}\Bigg]\Bigg\}\|z-\overline{z}\| \end{aligned}$$

$$\leq \quad (\|\rho\|\Omega)\|z-\overline{z}\|.$$

Hence,

$$\|p_1 - p_2\| \leq (\|\rho\|\Omega)\|z-\overline{z}\|.$$

Thus, we have

$$\mathcal{F}_g(\Psi_{\mathcal{H}}(z), \Psi_{\mathcal{H}}(\overline{z})) \leq (\|\rho\|\Omega)\|z-\overline{z}\|,$$

when we interchange the z and $\overline{z}$ functions. Since $\Psi_{\mathcal{H}}$ is a contraction, it follows from Theorem (see [9]) that $\Psi_{\mathcal{H}}$ has a fixed point z which is a solution to the problem (2) and (3). □

5. Examples

Example 5.1. *Consider the following BVP*

$$ {}^{CH}\mathfrak{D}^{\frac{62}{25}} z(\tau) \;=\; \frac{1}{(\log\tau)^2+1} + \frac{|z(\tau)|}{1+|z(\tau)|}\cdot\frac{e^{(\log\tau)^2}}{(3+\tau)^2}, \quad \tau\in[1,2], \tag{12}$$

$$z(1) \;=\; 0, \quad z^{'}(1) \;=\; 0, \quad y(T) \;=\; \xi^{H}\mathfrak{I}^{\frac{42}{25}}z(\upsilon) + \zeta\sum_{i=1}^{m-2}\nu_i z(\vartheta_i). \tag{13}$$

Here, $\varrho = \frac{62}{25}$, $\varsigma = \frac{42}{25}$, $\xi = \frac{1}{40}$, $\zeta = \frac{1}{30}$, $\upsilon = \frac{5}{4}$, $T = 2$, $\nu_1 = \frac{19}{50}$, $\nu_2 = \frac{29}{50}$, $\nu_3 = \frac{39}{50}$, $\vartheta_1 = \frac{37}{25}$, $\vartheta_2 = \frac{42}{25}$, $\vartheta_3 = \frac{47}{25}$.
Moreover, we find that

$$\begin{aligned} |h(\tau, z(\tau))| \;&=\; \frac{1}{(\log\tau)^2+1} + \frac{|z|}{1+|z|}\frac{1}{(3+\tau)^2} \qquad as \\ |h(\tau, z_1(\tau)) - h(\tau, z_2(\tau))| \;&\leq\; \frac{1}{16}\|z_1 - z_2\|. \end{aligned}$$

With the above specifics, we find that $\Delta \cong 0.08731557467270329$, $\Omega \cong 0.03246116745330087$. *Thus, the presumptions of Theorem 3.1 are satisfied. Hence, by Theorem 3.1, the problem (12)-(13) has at least one solution on* $\mathcal{E}$.

Example 5.2. *Consider the following BVP*

$$ {}^{CH}\mathfrak{D}^{\frac{72}{25}} z(\tau) \;=\; \frac{\sqrt{\tau}}{1+\tau} + \frac{|z(\tau)|}{1+|z(\tau)|}\cdot\frac{e^{\log\tau}}{(4+\tau^2)}, \quad \tau\in[1,2], \tag{14}$$

with the boundary conditions (13). Additionally, we find that

$$\begin{aligned} |h(\tau, y(\tau))| \;&=\; \frac{\sqrt{\tau}}{1+\tau} + \frac{|z|}{1+|z|}\frac{e^{\log\tau}}{(4+\tau^2)} \qquad as \\ |h(\tau, z_1(\tau)) - h(\tau, z_2(\tau))| \;&\leq\; \frac{1}{5}\|z_1 - z_2\|. \end{aligned}$$

With the above specifics, we find that $\Delta = 0.08731557467270329$ *and* $\Omega = 0.012610111547411674$. *Thus, the presumptions of Theorem 3.2 are satisfied. Hence, by Theorem 3.2, the problem (14) with (13) has a unique solution on* $\mathcal{E}$.

6. Conclusion

Through fixed-point theorems of Krasnoselskii, and Banach for equations, we discussed the existence and uniqueness results for CHFDEs and inclusions supplemented by non-local boundary conditions, and alternative of Leray-Schauder for multivalued maps and fixed-point theorem due to Covitz for inclusions. When we have fixed the parameters involved in the problem (ξ, ζ) (1)-(3), our results correspond to certain specific problems. Suppose that taking $\xi = 0$ in the results provided, we are given the problems (1) and (2) with the form: $y(T) = \zeta \sum_{i=1}^{m-2} \nu_i z(\vartheta_i)$, while the results are $y(T) = \xi^H \mathcal{I}^\varsigma z(\upsilon)$, followed by $\zeta = 0$.

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