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Research article

Existence results for coupled differential equations of non-integer order with Riemann-Liouville, Erdélyi-Kober integral conditions

Dumitru Baleanu[1,2,3], **S. Hemalatha**[4], **P. Duraisamy**[5], **P. Pandiyan**[6] **and Subramanian Muthaiah**[7,*]

[1] Department of Mathematics, Cankaya University, Ankara, Turkey
[2] Institute of Space Science, Magurele-Bucharest, Romania
[3] Department of Medical Research, China Medical University, Taichung, Taiwan
[4] Department of Mathematics, Sasurie College of Arts and Science, Vijayamangalam, India
[5] Department of Mathematics, Gobi Arts and Science College, Gobichettipalayam, India
[6] Department of Electrical and electronics engineering, KPR Institute of Engineering and Technology, Coimbatore, India
[7] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore, India

* **Correspondence:** Email: subramanianmcbe@gmail.com; Tel: +919003832474.

Abstract: This paper proposes the existence and uniqueness of a solution for a coupled system that has fractional differential equations through Erdélyi-Kober and Riemann-Liouville fractional integral boundary conditions. The existence of the solution for the coupled system by adopting the Leray-Schauder alternative. The uniqueness of solution for the problem can be found using Banach fixed point theorem. In order to verify the proposed criterion, some numerical examples are also discussed.

Keywords: Caputo derivatives; Erdélyi-Kober integrals; Riemann-Liouville integrals; coupled system; existence; fixed point
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1. Introduction

Fractional calculus is one of the most widely used mathematical analysis which deals with different ways to represent the real and complex number powers of the differentiation or integration operator and creating a calculus for the same operators in the generalized form. This calculus has numerous applications in the fields of science and engineering viz viscoelasticity, engineering mechanics, control systems, biological population models, etc. In specific, this branch of mathematics involves the methods and notion to solve the differential equations concern with a fractional derivative of unknown function which is also called fractional differential equations (FDEs). Moreover, this fractional calculus has been widely employed for modeling the engineering and physical processes which are possibly represented in terms of FDEs. This type of fractional derivative model is utilized in order to provide accurate modeling of those systems which needs to be accurate modeling of damping and also has the capability of modeling the complex engineering problems [10–12,14–17,23]. In recent years, a variety of numerical and analytical modeling approaches with their applications to new problems have been addressed in the research field of mechanics, electrodynamics of complex medium, and aerodynamics, etc. The application of Erdélyi-Kober fractional integrals is discussed in detail with the examples in [7, 11, 12, 25, 27, 31]. Unlike integer derivatives, fractional derivatives access the system's global evolution rather than just its local characteristics; as a result, when dealing with certain phenomena, they provide more accurate models of real-world behaviour than standard derivatives. In real life, differential equations of fractional order are used to calculate the movement or flow of electricity, the motion of an object back and forth like a pendulum, and to explain thermodynamic concepts, etc. Additionally, in medical terms, they are used to visualize the progression of diseases. They represent real-world behaviour more accurately than standard derivatives. The coupled system consists of a couple of differential equations with pair of dependent variables and a single independent variable. The coupled system of FDEs becomes a more popular research field due to its vast applications in real-time problems namely anomalous diffusion, ecological models, chaotic systems, and disease models [1, 2, 8, 20, 24, 26]. Boundary value problems (BVPs) applied to a coupled system with non-linear differential equations attracting researchers because of its applications in plasma physics and heat conduction; see [3–6, 18, 19, 21, 22, 28, 29, 32], and the references cited therein. The nonlinear coupled system of Riemann-Liouville FDEs

$$\begin{cases} {}_{RL}\mathcal{D}^{q}x(t) = f(t, x(t), y(t)), \\ {}_{RL}\mathcal{D}^{p}y(t) = g(t, x(t), y(t)), \\ x(0) = 0, \quad x(T) = \displaystyle\sum_{i=1}^{n} \alpha_{iH}\mathcal{I}^{\rho_i} y(\eta_i), \quad \eta_i \in (0, T), \\ y(0) = 0, \quad y(T) = \displaystyle\sum_{i=1}^{n} \beta_{iH}\mathcal{I}^{\gamma_i} x(\theta_i), \quad \theta_i \in (0, T), \end{cases} \tag{1.1}$$

for $0 < t < T$ and $1 < q, p \leq 2$, was investigated in [30], where ${}_{RL}\mathcal{D}^{q}$, ${}_{RL}\mathcal{D}^{p}$ denote the Riemann-Liouville fractional derivatives (RLFDs) of order q, p, f, $g : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2, \cdots, n$ are positive real constants. Fixed-point theorems were also employed to prove the main results. The Caputo type FDEs nonlinear coupled system

$$\begin{cases} {}^{c}\mathcal{D}^{\alpha_1}u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \\ {}^{c}\mathcal{D}^{\alpha_2}v(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \\ u'(0) = u''(0) = \cdots = u^{n-1}(0) = 0, \quad u(1) = \mu_1 \displaystyle\int_0^1 a(s)v(s)dA_1(s), \\ v'(0) = v''(0) = \cdots = v^{n-1}(0) = 0, \quad v(1) = \mu_2 \displaystyle\int_0^1 b(s)u(s)dA_2(s), \end{cases} \tag{1.2}$$

for $0 < t < 1$, $n - 1 < \alpha_1 \leq n$, $m - 1 < \alpha_2 \leq m$, and $n, m \geq 2$, were examined in [34], where $\lambda_i > 0$ is a parameter, $\mathcal{D}_{0+}^{\alpha_i}$ is the standard Caputo derivative; $\mu_i > 0$ is a constant, $\int_0^1 a(s)v(s)dA_1(s)$,

$\int_0^1 b(s)u(s)dA_2(s)$ denote the Riemann-Stieltjes integrals. Leray-Schauder's alternative and the contraction mapping principle proved the existence and uniqueness of solutions.

In this study, a coupled system with non-linear FDEs is considered and which is represented as in (1.3).

$$\begin{cases} {}^c\mathcal{D}^{\varsigma}u(\tau) = f(\tau, v(\tau), {}^c\mathcal{D}^{\varrho_1}v(\tau)), \tau \in [0,1] : \mathcal{H}, 1 < \varsigma \leq 2, 0 < \varrho_1 < 1, \\ {}^c\mathcal{D}^{\varrho}v(\tau) = g(\tau, u(\tau), {}^c\mathcal{D}^{\varsigma_1}u(\tau)), \tau \in [0,1] : \mathcal{H}, 1 < \varrho \leq 2, 0 < \varsigma_1 < 1, \end{cases} \tag{1.3}$$

Equation (1.3) is subjected to the Erdélyi-Kober, Riemann-Liouville integral boundary conditions are given in Eq (1.4).

$$\begin{cases} u(0) = \mu_1 \mathcal{J}^p u(\omega), u(1) = \tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1} u(\xi), 0 < \omega, \xi < 1, \\ v(0) = \mu_2 \mathcal{J}^q v(\zeta), v(1) = \tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} v(\eta), 0 < \zeta, \eta < 1, \end{cases} \tag{1.4}$$

where ${}^c\mathcal{D}^j$ represents the Caputo derivatives of order $j, \{j = \varsigma, \varrho, \varrho_1, \varsigma_1\}$, $\mathcal{J}^p$ and $\mathcal{J}^q$ are the Riemann-Liouville integrals of order $p, q > 0$ and $\mathcal{I}_{\sigma_i}^{\epsilon_i,\theta_i}(i = 1,2)$ is the Erdélyi-Kober integrals of order $\sigma_i > 0, \theta_i > 0, \epsilon_i \in \mathbb{R}(i = 1,2), f, g : \mathcal{H} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\mu_i, \tau_i (i = 1,2)$ are real constants. The structure of this proposed work is as follows: Section 2 deals with some facts and definitions related to this study. Section 3 gives a solution for the system described in Eq (2) and (3). The examples of the proposed problem are drawn to validate the applications in Section 4. Finally, the discussion is presented.

2. Preliminaries

This section recollects the definitions and some basics facts related to the proposed study are presented [9, 12, 23, 33].

Definition 2.1. *The Riemann-Liouville integral of order $\varrho > 0$ for a function $f(\tau)$ is defined as*

$$\mathcal{J}^{\varrho} f(\tau) = \frac{1}{\Gamma(\varrho)} \int_0^{\tau} (\tau - \theta)^{\varrho - 1} f(\theta)\, d\theta, \tau > 0,$$

provided that the right hand side is point wise defined on $[0, \infty)$.

Definition 2.2. *The Caputo derivative of order $\varrho > 0$ of a function $f : [0, \infty) \to \mathbb{R}$ is defined as*

$${}^c\mathcal{D}^{\varrho} f(\tau) = \frac{1}{\Gamma(n - \varrho)} \int_0^{\tau} (\tau - \theta)^{n - \varrho - 1} f^{(n)}(\theta)\, d\theta, n - 1 < \varrho < n,$$

where $n = [\varrho] + 1$ and $[\varrho]$ denotes the integral part of the real number.

Definition 2.3. *The Erdélyi-Kober fractional integral of order $\varsigma_1 > 0$ with $\eta > 0$ and $\varrho_1 \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$\mathcal{I}_{\eta}^{\varrho_1,\varsigma_1} f(\tau) = \frac{\eta \tau^{-\eta(\varsigma_1 + \varrho_1)}}{\Gamma(\varsigma_1)} \int_0^{\tau} \frac{\theta^{\eta\varrho_1 + \eta - 1}}{(\tau^{\eta} - \theta^{\eta})^{1 - \varsigma_1}} f(\theta)\, d\theta,$$

provided the right hand side is point wise defined on $\mathbb{R}_+$.

Remark 2.4. *For $\eta = 1$, the above operator is reduced to the Kober operator*

$$\mathcal{I}_1^{\varrho_1,\varsigma_1} f(\tau) = \frac{\tau^{-(\varsigma_1+\varrho_1)}}{\Gamma(\varsigma_1)} \int_0^\tau \frac{\theta^{\varrho_1}}{(\tau-\theta)^{1-\varsigma_1}} f(\theta)\, d\theta, \eta, \varsigma_1 > 0,$$

that was introduced for the first time by Kober in [13]. For $\varrho_1 = 0$, the kober operator is reduced to the Riemann-Liouville integral with a power weight:

$$\mathcal{I}_1^{0,\varsigma_1} f(\tau) = \frac{\tau^{-\varsigma_1}}{\Gamma(\varsigma_1)} \int_0^\tau \frac{1}{(\tau-\theta)^{1-\varsigma_1}} f(\theta)\, d\theta, \varsigma_1 > 0.$$

Lemma 2.5. *Given the functions $\nu, \rho \in C(\mathcal{H}, \mathbb{R})$, the solution of the problem*

$$\begin{aligned} {}^c\mathcal{D}^\varsigma u(\tau) &= \nu(\tau), \tau \in \mathcal{H}, 1 < \varsigma \leq 2, \\ {}^c\mathcal{D}^\varrho \nu(\tau) &= \rho(\tau), \tau \in \mathcal{H}, 1 < \varrho \leq 2, \\ u(0) &= \mu_1 \mathcal{J}^p u(\omega), u(1) = \tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1} u(\xi), \\ \nu(0) &= \mu_2 \mathcal{J}^q \nu(\zeta), \nu(1) = \tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} \nu(\eta), \end{aligned} \tag{2.1}$$

is equivalent to the fractional integral equations

$$u(\tau) = \mathcal{J}^\varsigma \nu(\tau) + \frac{\mu_1}{\Lambda_1}(a_4 - a_3\tau)\mathcal{J}^{\varsigma+p}\nu(\omega) + \frac{1}{\Lambda_1}(a_1\tau + a_2)\Big[\tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^\varsigma \nu(\xi) - \mathcal{J}^\varsigma \nu(1)\Big], \tag{2.2}$$

and

$$\nu(\tau) = \mathcal{J}^\varrho \rho(\tau) + \frac{\mu_2}{\Lambda_2}(b_4 - b_3\tau)\mathcal{J}^{\varrho+q}\rho(\zeta) + \frac{1}{\Lambda_2}(b_1\tau + b_2)\Big[\tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2}\mathcal{J}^\varrho \rho(\eta) - \mathcal{J}^\varrho \rho(1)\Big]. \tag{2.3}$$

Here the non zero constants Λ_1 and Λ_2 are

$$\Lambda_1 = a_1a_4 + a_3a_2 \neq 0, \Lambda_2 = b_1b_4 + b_3b_2 \neq 0, \tag{2.4}$$

where

$$a_1 = 1 - \mu_1 \frac{\omega^p}{\Gamma(p+1)}, \ a_2 = \mu_1 \frac{\omega^{p+1}}{\Gamma(p+2)}, \tag{2.5}$$

$$a_3 = 1 - \tau_1 \frac{\Gamma(\epsilon_1+1)}{\Gamma(\epsilon_1+\theta_1+1)}, a_4 = 1 - \tau_1 \frac{\xi\Gamma(\epsilon_1 + \left(\frac{1}{\sigma_1}\right) + 1)}{\Gamma\left(\epsilon_1 + \left(\frac{1}{\sigma_1}\right) + \theta_1 + 1\right)}, \tag{2.6}$$

and

$$b_1 = 1 - \mu_2 \frac{\zeta^q}{\Gamma(q+1)}, \ b_2 = \mu_2 \frac{\zeta^{q+1}}{\Gamma(q+2)}, \tag{2.7}$$

$$b_3 = 1 - \tau_2 \frac{\Gamma(\epsilon_2+1)}{\Gamma(\epsilon_2+\theta_2+1)}, \ b_4 = 1 - \tau_2 \frac{\eta\Gamma(\epsilon_2 + \left(\frac{1}{\sigma_2}\right) + 1)}{\Gamma\left(\epsilon_2 + \left(\frac{1}{\sigma_2}\right) + \theta_2 + 1\right)}. \tag{2.8}$$

Proof. The general solution for the Eq (2.1) can be expressed as

$$u(\tau) = \mathcal{J}^{\varsigma} v(\tau) + c_0 + c_1\tau, \tag{2.9}$$

$$v(\tau) = \mathcal{J}^{\varrho} \rho(\tau) + d_0 + d_1\tau, \tag{2.10}$$

where c_0, c_1, d_0, d_1 are arbitrary constants.
Substituting the (1.4) in Eqs (2.9) and (2.10), the following equations will be obtained.

$$c_0 = \frac{1}{\Lambda_1}\Big(\mu_1 a_4 \mathcal{J}^{\varsigma+p} v(\omega) + a_2\Big[\tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1} \mathcal{J}^{\varsigma} v(\xi) - \mathcal{J}^{\varsigma} v(1)\Big]\Big),$$

$$c_1 = \frac{1}{\Lambda_1}\Big(a_1\Big[\tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1} \mathcal{J}^{\varsigma} v(\xi) - \mathcal{J}^{\varsigma} v(1)\Big] - \mu_1 a_3 \mathcal{J}^{\varsigma+p} v(\omega)\Big),$$

$$d_0 = \frac{1}{\Lambda_2}\Big(\mu_2 b_4 \mathcal{J}^{\varrho+q} \rho(\zeta) + b_2\Big[\tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} \mathcal{J}^{\varrho} \rho(\eta) - \mathcal{J}^{\varrho} \rho(1)\Big]\Big),$$

$$d_1 = \frac{1}{\Lambda_2}\Big(b_1\Big[\tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} \mathcal{J}^{\varrho} \rho(\eta) - \mathcal{J}^{\varrho} \rho(1)\Big] - \mu_2 b_3 \mathcal{J}^{\varrho+q} \rho(\zeta)\Big),$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are given by (2.5)–(2.8). Substituting the values of c_0, c_1, d_0, d_1 in (2.9) and (2.10) respectively, we get the solution for (2.1). □

3. Main results

Let us introduce the space $\mathcal{U} = \{u : u \in C(\mathcal{H}, \mathbb{R}) \text{ and } {}^{c}\mathcal{D}^{\varsigma_1} u \in C(\mathcal{H}, \mathbb{R})\}$ with the norm defined by

$$\|u\|_{\mathcal{U}} = \|u\| + \|{}^{c}\mathcal{D}^{\varsigma_1} u\| = \sup_{\tau\in\mathcal{H}} |u(\tau)| + \sup_{\tau\in\mathcal{H}} |{}^{c}\mathcal{D}^{\varsigma_1} u(\tau)| .$$

Then $(\mathcal{U}, \|.\|_{\mathcal{U}})$ is a Banach space and also let us introduce the space $\mathcal{V} = \{v : v \in C(\mathcal{H}, \mathbb{R}) \text{ and } {}^{c}\mathcal{D}^{\varrho_1} v \in C(\mathcal{H}, \mathbb{R})\}$ with the norm defined by

$$\|v\|_{\mathcal{V}} = \|v\| + \|{}^{c}\mathcal{D}^{\varrho_1} v\| = \sup_{\tau\in\mathcal{H}} |v(\tau)| + \sup_{\tau\in\mathcal{H}} |{}^{c}\mathcal{D}^{\varrho_1} v(\tau)| .$$

Then $(\mathcal{V}, \|.\|_{\mathcal{V}})$ is a Banach space.

Clearly, the product space $(\mathcal{U}\times\mathcal{V}, \|.\|_{\mathcal{U}\times\mathcal{V}})$ is a Banach space with the norm defined by

$$\|(u, v)\|_{\mathcal{U}\times\mathcal{V}} = \|u\|_{\mathcal{U}} + \|v\|_{\mathcal{V}}\, for\, (u, v) \in \mathcal{U}\times\mathcal{V}.$$

In view of Lemma 2.5, we define an operator $\mathcal{F} : \mathcal{U}\times\mathcal{V} \to \mathcal{U}\times\mathcal{V}$ by

$$\mathcal{F}(u, v)(\tau) = (\mathcal{F}_1(u, v)(\tau), \mathcal{F}_2(u, v)(\tau)),$$

where

$$\mathcal{F}_1(u,v)(\tau) = \mathcal{J}^{\varsigma} f(\theta, v(\theta), {}^{c}\mathcal{D}^{\varrho_1} v(\theta))(\tau) + \frac{\mu_1}{\Lambda_1}(a_4 - a_3\tau)\mathcal{J}^{\varsigma+p} f(\theta, v(\theta), {}^{c}\mathcal{D}^{\varrho_1} v(\theta))(\omega)$$
$$+\frac{1}{\Lambda_1}(a_1\tau + a_2)\left[\tau_1 \mathcal{I}^{\epsilon_1,\theta_1}_{\sigma_1}\mathcal{J}^{\varsigma} f(\theta, v(\theta), {}^{c}\mathcal{D}^{\varrho_1} v(\theta))(\xi) - \mathcal{J}^{\varsigma} f(\theta, v(\theta), {}^{c}\mathcal{D}^{\varsigma} v(\theta))(1)\right],$$

and

$$\mathcal{F}_2(u,v)(\tau) = \mathcal{J}^{\varrho} g(\theta, u(\theta), {}^{c}\mathcal{D}^{\varsigma_1} u(\theta))(\tau) + \frac{\mu_2}{\Lambda_2}(b_4 - b_3\tau)\mathcal{J}^{\varrho+q} g(\theta, u(\theta), {}^{c}\mathcal{D}^{\varsigma_1} u(\theta))(\zeta)$$
$$+\frac{1}{\Lambda_2}(b_1\tau + b_2)\left[\tau_2 \mathcal{I}^{\epsilon_2,\theta_2}_{\sigma_2}\mathcal{J}^{\varrho} g(\theta, u(\theta), {}^{c}\mathcal{D}^{\varsigma_1} u(\theta))(\eta) - \mathcal{J}^{\varrho} g(\theta, u(\theta), {}^{c}\mathcal{D}^{\varrho} u(\theta))(1)\right].$$

Let us present the following assumptions that are used afterward here:
(H_1) Assume that $f, g\colon \mathcal{H}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions and there exists constants $K_1, K_2 > 0$, such that

$$(i)\,|f(\tau,u_1,v_1) - f(\tau,u_2,v_2)| \le K_1(|u_1-u_2| + |v_1-v_2|),$$
$$(ii)\,|g(\tau,u_1,v_1) - g(\tau,u_2,v_2)| \le K_2(|u_1-u_2| + |v_1-v_2|),$$

for each $\tau \in \mathcal{H}$ and all $u_i, v_i \in \mathbb{R}, i = 1,2$.
(H_2) $\chi_1 = \Delta_1 + \frac{\Delta_2}{\Gamma(2-\varsigma_1)}, \chi_2 = \Delta_1^{'} + \frac{\Delta_2^{'}}{\Gamma(2-\varrho_1)}$, where $\chi = max\{\chi_1, \chi_2\}$.
(H_3) Assume that $f, g\colon \mathcal{H}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions and there exists real constants $l_i, \lambda_i \ge 0\,(i=1,2)$ and $l_0, \lambda_0 > 0$ such that for all $u_i \in \mathbb{R}\,(i=1,2)$. We have

$$(i)\,|f(\tau,u_1,u_2)| \le l_0 + l_1|u_1| + l_2|u_2|,$$
$$(ii)\,|g(\tau,u_1,u_2)| \le \lambda_0 + \lambda_1|u_1| + \lambda_2|u_2|.$$

For making a simplified expression, the following terms are introduced throughout this study:

$$\mathcal{G}_1 = \frac{1}{\Gamma(\varsigma+1)} + \frac{|\mu_1|(|a_4|+|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)} + \frac{(|a_2|+|a_1|)}{|\Lambda_1|}$$
$$\left[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)} + \frac{1}{\Gamma(\varsigma+1)}\right], \quad (3.1)$$

$$\mathcal{G}_2 = \frac{1}{\Gamma(\varsigma)} + \frac{|\mu_1|(|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)} + \frac{|a_1|}{|\Lambda_1|}$$
$$\left[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)} + \frac{1}{\Gamma(\varsigma+1)}\right], \quad (3.2)$$

$$\mathcal{G}_1^{'} = \frac{1}{\Gamma(\varrho+1)} + \frac{|\mu_2|(|b_4|+|b_3|)}{|\Lambda_2|}\frac{\zeta^{\varrho+q}}{\Gamma(\varrho+q+1)} + \frac{(|b_2|+|b_1|)}{|\Lambda_2|}$$

$$\left[|\tau_2|\frac{\eta^{\varrho}\Gamma\left(\epsilon_2+\left(\frac{\varrho}{\sigma_2}\right)+1\right)}{\Gamma(\varrho+1)\Gamma\left(\epsilon_2+\left(\frac{\varrho}{\sigma_2}\right)+\theta_2+1\right)}+\frac{1}{\Gamma(\varrho+1)}\right], \tag{3.3}$$

$$\begin{aligned}\mathcal{G}_2^{'} &= \frac{1}{\Gamma(\varrho)}+\frac{|\mu_2|(|b_3|)}{|\Lambda_2|}\frac{\zeta^{\varrho+q}}{\Gamma(\varrho+q+1)}+\frac{|b_1|}{|\Lambda_2|}\\ &\quad \left[|\tau_2|\frac{\eta^{\varrho}\Gamma\left(\epsilon_2+\left(\frac{\varrho}{\sigma_2}\right)+1\right)}{\Gamma(\varrho+1)\Gamma\left(\epsilon_2+\left(\frac{\varrho}{\sigma_2}\right)+\theta_2+1\right)}+\frac{1}{\Gamma(\varrho+1)}\right],\end{aligned} \tag{3.4}$$

$\Delta_1 = K_1\mathcal{G}_1$, $P_1 = \mathcal{G}_1 M_1$, $\Delta_2 = K_1\mathcal{G}_2$, $P_2 = \mathcal{G}_2 M_1$,$\Delta_1^{'} = K_2\mathcal{G}_1^{'}$, $P_1^{'} = \mathcal{G}_1^{'} M_2$, $\Delta_2^{'} = K_2\mathcal{G}_2^{'}$, $P_2^{'} = \mathcal{G}_2^{'} M_2$,

$$\mathcal{A} = \left(\mathcal{G}_1+\frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\right)l_0+\left(\mathcal{G}_1^{'}+\frac{\mathcal{G}_2^{'}}{\Gamma(2-\varrho_1)}\right)\lambda_0, \tag{3.5}$$

$$\mathcal{B} = \left(\mathcal{G}_1^{'}+\frac{\mathcal{G}_2^{'}}{\Gamma(2-\varrho_1)}\right)max\{\lambda_1,\lambda_2\}, \tag{3.6}$$

$$C = \left(\mathcal{G}_1+\frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\right)max\{l_1,l_2\}. \tag{3.7}$$

Theorem 3.1. *Suppose that (H_3) condition holds. Furthermore, it is assumed that $max\{\mathcal{B},C\}<1$. Then, on $\mathcal{H}$, the BVP (1.3) and (1.4) have at least one solution.*

Proof. The $\mathcal{F}:\mathcal{U}\times\mathcal{V}\to\mathcal{U}\times\mathcal{V}$ operator is shown to be completely continuous. It follows that the $\mathcal{F}$ operator is continuous by the continuity of the f and g functions.

Let $\Theta\subset\mathcal{U}\times\mathcal{V}$ be bounded. Then there exists positive constants N_1 and N_2 such that $f(\tau,v(\tau),{}^cD^{\varrho_1}v(\tau))\le N_1$ and $g(\tau,u(\tau),{}^cD^{\varsigma_1}u(\tau))\le N_2$ for all $(u,v)\in\Theta$.

Step 1: To show that $\mathcal{F}$ is uniformly bounded.

For each $\tau\in\mathcal{H}$, we have

$$\begin{aligned}&|\mathcal{F}_1(u,v)(\tau)|\\ &\le \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^c\mathcal{D}^{\varrho_1}v(\theta))|(\tau)+\frac{|\mu_1|}{|\Lambda_1|}(|a_4-a_3\tau|)\mathcal{J}^{\varsigma+p}|f(\theta,v(\theta),{}^c\mathcal{D}^{\varrho_1}v(\theta))|(\omega)\\ &\quad+\frac{1}{|\Lambda_1|}(|a_1\tau+a_2|)\Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^c\mathcal{D}^{\varrho_1}v(\theta))|(\xi)+\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^c\mathcal{D}^{\varrho_1}v(\theta))|(1)\Big]\Big\}\\ &\le N_1\Big\{\mathcal{J}^{\varsigma}(1)+\frac{|\mu_1|}{|\Lambda_1|}(|a_4-a_3\tau|)\mathcal{J}^{\varsigma+p}(\omega)+\frac{1}{|\Lambda_1|}(|a_1\tau+a_2|)\Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}(\xi)+\mathcal{J}^{\varsigma}(1)\Big]\Big\}\\ &\le N_1\Big\{\frac{1}{\Gamma(\varsigma+1)}+\frac{|\mu_1|(|a_4|+|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)}+\frac{(|a_2|+|a_1|)}{|\Lambda_1|}\\ &\quad\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)}+\frac{1}{\Gamma(\varsigma+1)}\Big]\Big\}\end{aligned}$$

$$\leq \quad N_1\mathcal{G}_1,$$

and

$$\begin{aligned}
&\left|\mathcal{F}_1(u,v)'(\tau)\right| \\
&\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma-1}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\tau)+\frac{|\mu_1|}{|\Lambda_1|}(|a_3|)\mathcal{J}^{\varsigma+p}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\omega) \\
&\quad+\frac{1}{|\Lambda_1|}(|a_1|)\Big[|\tau_1|\,\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\xi)+\mathcal{J}^{\varsigma}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(1)\Big]\Big\} \\
&\leq N_1\Big\{\frac{1}{\Gamma(\varsigma)}+\frac{|\mu_1|(|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)}+\frac{(|a_1|)}{|\Lambda_1|} \\
&\quad\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)}+\frac{1}{\Gamma(\varsigma+1)}\Big]\Big\} \\
&\leq N_1\mathcal{G}_2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\left|{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)(\tau)\right| &\leq \frac{1}{\Gamma(1-\varsigma_1)}\int_0^{\tau}(\tau-\theta)^{-\varsigma_1}\left|\mathcal{F}_1(u,v)'(\theta)\right|d\theta \\
&\leq \frac{N_1\mathcal{G}_2}{\Gamma(1-\varsigma_1)}\int_0^{\tau}(\tau-\theta)^{-\varsigma_1}\,d\theta \\
&\leq \frac{1}{\Gamma(2-\varsigma_1)}(N_1\mathcal{G}_2).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\mathcal{F}_1(u,v)\|_{\mathcal{U}} &= \|\mathcal{F}_1(u,v)\|+\|{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)\| \\
&\leq N_1\Big(\mathcal{G}_1+\frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\Big).
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&|\mathcal{F}_2(u,v)(\tau)| \\
&\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varrho}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(\tau)+\frac{|\mu_2|}{|\Lambda_2|}(|b_4-b_3\tau|)\mathcal{J}^{\varrho+q}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(\zeta) \\
&\quad+\frac{1}{|\Lambda_2|}(|b_1\tau+b_2|)\Big[|\tau_2|\,\mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2}\mathcal{J}^{\varrho}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(\eta)+\mathcal{J}^{\varrho}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(1)\Big]\Big\} \\
&\leq N_2\mathcal{G}_1',
\end{aligned}$$

and

$$\left|\mathcal{F}_2(u,v)'(\tau)\right|$$

$$\begin{aligned}
&\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varrho-1}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(\tau)+\frac{|\mu_2|}{|\Lambda_2|}(|b_3|)\mathcal{J}^{\varrho+q}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varrho}u(\theta))\right|(\zeta)\\
&\quad+\frac{1}{|\Lambda_2|}(|b_1|)\Big[\,|\tau_2|\,\mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2}\mathcal{J}^{\varrho}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(\eta)+\mathcal{J}^{\varrho}\left|g(\theta,u(\theta),{}^{c}\mathcal{D}^{\varsigma_1}u(\theta))\right|(1)\Big]\Big\}\\
&\leq N_2\mathcal{G}_2^{'},
\end{aligned}$$

which implies that

$$\left|{}^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u,v)(\tau)\right| \leq \frac{1}{\Gamma(2-\varrho_1)}\left(N_2\mathcal{G}_2^{'}\right).$$

As a result, the following expression is obtained,

$$\begin{aligned}
\|\mathcal{F}_2(u,v)\|_{\mathcal{V}} &= \|\mathcal{F}_2(u,v)\|+\|{}^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u,v)\|\\
&\leq N_2\Big(\mathcal{G}_1^{'}+\frac{\mathcal{G}_2^{'}}{\Gamma(2-\varrho_1)}\Big).
\end{aligned}$$

Therefore, the above equation follows the inequalities in which operator $\mathcal{F}$ is uniformly bounded.
Step 2: To show that $\mathcal{F}$ is equicontinuous. Let $\tau_1,\tau_2\in\mathcal{H}$ with $\tau_1<\tau_2$. Then we have

$$\begin{aligned}
&|\mathcal{F}_1(u,v)(\tau_2)-\mathcal{F}_1(u,v)(\tau_1)|\\
&\leq \frac{1}{\Gamma(\varsigma)}\Bigg[\Bigg|\int_0^{\tau_1}\Big[(\tau_2-\theta)^{\varsigma-1}-(\tau_1-\theta)^{\varsigma-1}\Big]f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\,d\theta\\
&\quad+\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{\varsigma-1}f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\,d\theta\Bigg|\Bigg]+\frac{|\mu_1|}{|\Lambda_1|}(|a_3|\,|\tau_2-\tau_1|)\\
&\quad\mathcal{J}^{\varsigma+p}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\omega)+\frac{1}{|\Lambda_1|}(|a_1|\,|\tau_2-\tau_1|)\\
&\quad\Big[\,|\tau_1|\,\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\xi)+\mathcal{J}^{\varsigma}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(1)\Big]\\
&\leq N_1\Bigg\{\frac{1}{\Gamma(\varsigma+1)}\Big[2\,|\tau_2-\tau_1|^{\varsigma}+\left|\tau_2^{\varsigma}-\tau_1^{\varsigma}\right|\Big]+\frac{|\mu_1|}{|\Lambda_1|}(|a_3|\,|\tau_2-\tau_1|)\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)}\\
&\quad+\frac{1}{|\Lambda_1|}(|a_1|\,|\tau_2-\tau_1|)\Bigg[\,|\tau_1|\frac{\xi^{\varsigma}\Gamma(\epsilon_1+\frac{\varsigma}{\sigma_1}+1)}{\Gamma(\varsigma+1)\Gamma(\epsilon_1+\frac{\varsigma}{\sigma_1}+\theta_1+1)}+\frac{1}{\Gamma(\varsigma+1)}\Bigg]\Bigg\},
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
\left|{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)(\tau_2)-{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)(\tau_1)\right| &\leq \frac{1}{\Gamma(1-\varsigma_1)}\Bigg[\int_0^{\tau_1}\left[(\tau_1-\theta)^{-\varsigma_1}-(\tau_2-\theta)^{-\varsigma_1}\right]\left|\mathcal{F}_1(u,v)^{'}(\theta)\right|d\theta\\
&\quad-\int_{\tau_1}^{\tau_2}(\tau_2-\theta)^{-\varsigma_1}\left|\mathcal{F}_1(u,v)^{'}(\theta)\right|d\theta\Bigg]\\
&\leq \frac{N_1\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\Big[2\,|\tau_2-\tau_1|^{1-\varsigma_1}+\left|\tau_2^{1-\varsigma_1}-\tau_1^{1-\varsigma_1}\right|\Big].
\end{aligned}\tag{3.9}$$

Also, we obtain

$$\begin{aligned}
&|\mathcal{F}_2(u,v)(\tau_2)-\mathcal{F}_2(u,v)(\tau_1)| \\
&\le N_2\Big\{\frac{1}{\Gamma(\varrho+1)}[2|\tau_2-\tau_1|^{\varrho}+\left|\tau_2^{\varrho}-\tau_1^{\varrho}\right|]+\frac{|\mu_2|}{|\Lambda_2|}(|b_3||\tau_2-\tau_1|)\frac{\zeta^{\varrho+q}}{\Gamma(\varrho+q+1)} \\
&\quad+\frac{1}{|\Lambda_2|}(|b_1||\tau_2-\tau_1|)\Big[|\tau_2|\frac{\eta^{\varrho}\Gamma(\epsilon_2+\frac{\varrho}{\sigma_2}+1)}{\Gamma(\varrho+1)\Gamma(\epsilon_2+\frac{\varrho}{\sigma_2}+\theta_2+1)}+\frac{1}{\Gamma(\varrho+1)}\Big]\Big\},
\end{aligned} \tag{3.10}$$

and

$$|{}^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u,v)(\tau_2)-{}^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u,v)(\tau_1)| \le \frac{N_2\mathcal{G}'_2}{\Gamma(2-\varrho_1)}\Big[2|\tau_2-\tau_1|^{1-\varrho_1}+\left|\tau_2^{1-\varrho_1}-\tau_1^{1-\varrho_1}\right|\Big]. \tag{3.11}$$

This operator with Eqs (3.8)–(3.11) tends to zero when $\tau_2 \to \tau_1$. Subsequently, the $\mathcal{F}$ operator is equicontinuous and completely continuous according to Arzel*á*-Ascoli Theorem.
Step 3: To prove that the set $\varpi = \{(u,v)\in\mathcal{U}\times\mathcal{V} : (u,v)=\mu\mathcal{F}(u,v), 0<\mu\le 1\}$ is bounded.
Let $(u,v)\in\varpi$. Then $(u,v)=\mu\mathcal{F}(u,v)$. For any $\tau\in[0,1]$, we have

$$u(\tau)=\mu\mathcal{F}_1(u,v)(\tau), v(\tau)=\mu\mathcal{F}_2(u,v)(\tau).$$

Then

$$\begin{aligned}
|u(\tau)| &\le \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(\tau)+\frac{|\mu_1|}{|\Lambda_1|}(|a_4-a_3\tau|)\mathcal{J}^{\varsigma+p}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(\omega) \\
&\quad+\frac{1}{|\Lambda_1|}(|a_1\tau+a_2|)\Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(\xi)+\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(1)\Big]\Big\} \\
&\le (l_0+l_1\|v\|+l_2\|{}^{c}\mathcal{D}^{\varrho_1}v\|)\Big\{\frac{1}{\Gamma(\varsigma+1)}+\frac{|\mu_1|(|a_4|+|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)} \\
&\quad+\frac{(|a_2|+|a_1|)}{|\Lambda_1|}\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)}+\frac{1}{\Gamma(\varsigma+1)}\Big]\Big\} \\
&\le (l_0+l_1\|v\|+l_2\|{}^{c}\mathcal{D}^{\varrho_1}v\|)\mathcal{G}_1 \\
&\le \mathcal{G}_1(l_0+max\{l_1,l_2\}\|v\|_{\mathcal{V}}),
\end{aligned}$$

and

$$\begin{aligned}
\left|u'(\tau)\right| &\le \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma-1}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(\tau)+\frac{|\mu_1|}{|\Lambda_1|}(|a_3|)\mathcal{J}^{\varsigma+p}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(\omega) \\
&\quad+\frac{1}{|\Lambda_1|}(|a_1|)\Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(\xi)+\mathcal{J}^{\varsigma}|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))|(1)\Big]\Big\} \\
&\le (l_0+l_1\|v\|+l_2\|{}^{c}\mathcal{D}^{\varrho_1}v\|)\Big\{\frac{1}{\Gamma(\varsigma)}+\frac{|\mu_1|(|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)} \\
&\quad+\frac{(|a_1|)}{|\Lambda_1|}\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)}+\frac{1}{\Gamma(\varsigma+1)}\Big]\Big\}
\end{aligned}$$

$$\leq \mathcal{G}_2\left(l_0 + max\{l_1, l_2\}\|v\|_{\mathcal{V}}\right),$$

which implies that

$$\begin{aligned}|{}^c\mathcal{D}^{\varsigma_1}u(\tau)| &\leq \frac{\mathcal{G}_2\left(l_0 + max\{l_1, l_2\}\|v\|_{\mathcal{V}}\right)}{\Gamma(1-\varsigma_1)}\int_0^\tau (\tau-\theta)^{-\varsigma_1}\,d\theta \\ &\leq \frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\left(l_0 + max\{l_1, l_2\}\|v\|_{\mathcal{V}}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}\|u\|_{\mathcal{U}} &\leq \|u\| + \|{}^c\mathcal{D}^{\varsigma_1}u\| \\ &\leq \left(\mathcal{G}_1 + \frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\right)l_0 + \left(\mathcal{G}_1 + \frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\right)max\{l_1, l_2\}\|v\|_{\mathcal{V}}.\end{aligned}$$

We can have in a similar way,

$$\begin{aligned}|v(\tau)| &\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varrho}|g(\theta, u(\theta), {}^c\mathcal{D}^{\varsigma_1}u(\theta))|(\tau) + \frac{|\mu_2|}{|\Lambda_2|}(|b_4 - b_3\tau|)\mathcal{J}^{\varrho+q}|g(\theta, u(\theta), {}^c\mathcal{D}^{\varsigma_1}u(\theta))|(\zeta) \\ &\quad + \frac{1}{|\Lambda_2|}(|b_1\tau + b_2|)\Big[|\tau_2|\,\mathcal{I}^{\epsilon_2,\theta_2}_{\sigma_2}\mathcal{J}^{\varrho}|g(\theta, u(\theta), {}^c\mathcal{D}^{\varsigma_1}u(\theta))|(\eta) + \mathcal{J}^{\varrho}|g(\theta, u(\theta), {}^c\mathcal{D}^{\varsigma_1}u(\theta))|(1)\Big]\Big\} \\ &\leq \mathcal{G}_1'\left(\lambda_0 + max\{\lambda_1, \lambda_2\}\|u\|_{\mathcal{U}}\right),\end{aligned}$$

and

$$\left|v'(\tau)\right| \leq \mathcal{G}_2'\left(\lambda_0 + max\{\lambda_1, \lambda_2\}\|u\|_{\mathcal{U}}\right),$$

which implies that

$$|{}^c\mathcal{D}^{\varrho_1}v(\tau)| \leq \frac{\mathcal{G}_2'}{\Gamma(2-\varrho_1)}\left(\lambda_0 + max\{\lambda_1, \lambda_2\}\|u\|_{\mathcal{U}}\right).$$

Hence we have

$$\begin{aligned}\|v\|_{\mathcal{V}} &\leq \|v\| + \|{}^c\mathcal{D}^{\varrho_1}v\| \\ &\leq \left(\mathcal{G}_1' + \frac{\mathcal{G}_2'}{\Gamma(2-\varrho_1)}\right)\lambda_0 + \left(\mathcal{G}_1' + \frac{\mathcal{G}_2'}{\Gamma(2-\varrho_1)}\right)max\{\lambda_1, \lambda_2\}\|u\|_{\mathcal{U}}.\end{aligned}$$

Thus, we find that

$$\begin{aligned}\|u\|_{\mathcal{U}} + \|v\|_{\mathcal{V}} &\leq \left(\mathcal{G}_1' + \frac{\mathcal{G}_2'}{\Gamma(2-\varrho_1)}\right)\lambda_0 + \left(\mathcal{G}_1 + \frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\right)l_0 + \left(\mathcal{G}_1 + \frac{\mathcal{G}_2}{\Gamma(2-\varsigma_1)}\right)max\{l_1, l_2\}\|v\|_{\mathcal{V}} \\ &\quad + \left(\mathcal{G}_1' + \frac{\mathcal{G}_2'}{\Gamma(2-\varrho_1)}\right)max\{\lambda_1, \lambda_2\}\|u\|_{\mathcal{U}} \\ &\leq \mathcal{A} + \mathcal{B}\|u\|_{\mathcal{U}} + C\|v\|_{\mathcal{V}}\end{aligned}$$

$$\leq \mathcal{A} + max\{\mathcal{B}, C\} \|(u, v)\|_{\mathcal{U}\times\mathcal{V}},$$

which implies that

$$\|(u, v)\|_{\mathcal{U}\times\mathcal{V}} \leq \frac{\mathcal{A}}{1 - max\{\mathcal{B}, C\}}.$$

The above equation proves that the set ϖ is bounded. Therefore, the $\mathcal{F}$ operator consists of at least a single fixed point according to the (see [33] Theorem 1.9). As a result, the boundary value problem is represented in Eqs (1.3) and (1.4) also, consist of at least a single solution on $\mathcal{H}$. □

Theorem 3.2. *Suppose that* (H_1)*,* (H_2) *and* $\chi < \frac{1}{2}$ *hold, then the BVP (1.3) and (1.4) has a unique solution on* $\mathcal{H}$*.*

Proof. Let us fix $M_1 = \sup_{\tau\in[0,1]} |f(\tau, 0, 0)| < \infty$ and $M_2 = \sup_{\tau\in[0,1]} |g(\tau, 0, 0)| < \infty$ and we define

$$\hat{\rho} \geq max\left\{\frac{P_1 + \frac{P_2}{\Gamma(2-\varsigma_1)}}{\frac{1}{2} - \left(\Delta_1 + \frac{\Delta_2}{\Gamma(2-\varsigma_1)}\right)}, \frac{P_1' + \frac{P_2'}{\Gamma(2-\varrho_1)}}{\frac{1}{2} - \left(\Delta_1' + \frac{\Delta_2'}{\Gamma(2-\varrho_1)}\right)}\right\}.$$

Consider the set $B_{\hat{\rho}} = \{(u, v) \in \mathcal{U} \times \mathcal{V} : \|(u, v)\|_{\mathcal{U}\times\mathcal{V}} \leq \hat{\rho}\}$.
Now, to prove that $FB_{\hat{\rho}} \subset B_{\hat{\rho}}$. For $(u, v) \in B_{\hat{\rho}}$, we have

$$\begin{aligned}
&|\mathcal{F}_1(u, v)(\tau)| \\
&\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma}|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta))|(\tau) + \frac{|\mu_1|}{|\Lambda_1|}(|a_4 - a_3\tau|)\mathcal{J}^{\varsigma+p}|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta))|(\omega) \\
&\quad + \frac{1}{|\Lambda_1|}(|a_1\tau + a_2|)\Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta))|(\xi) + \mathcal{J}^{\varsigma}|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta))|(1)\Big]\Big\} \\
&\leq \mathcal{J}^{\varsigma}(|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta)) - f(\theta, 0, 0)| + |f(\theta, 0, 0)|)(1) \\
&\quad + \frac{|\mu_1|}{|\Lambda_1|}(|a_4 - a_3\tau|)\mathcal{J}^{\varsigma+p}(|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta)) - f(\theta, 0, 0)| + |f(\theta, 0, 0)|)(\omega) \\
&\quad + \frac{1}{|\Lambda_1|}(|a_1\tau + a_2|)\Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}(|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta)) - f(\theta, 0, 0)| + |f(\theta, 0, 0)|)(\xi) \\
&\quad + \mathcal{J}^{\varsigma}(|f(\theta, v(\theta),^c \mathcal{D}^{\varrho_1} v(\theta)) - f(\theta, 0, 0)| + |f(\theta, 0, 0)|)(1)\Big] \\
&\leq \Big[K_1\|v\|_{\mathcal{V}} + M_1\Big]\Big\{\frac{1}{\Gamma(\varsigma + 1)} + \frac{|\mu_1|(|a_4| + |a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma + p + 1)} \\
&\quad + \frac{(|a_2| + |a_1|)}{|\Lambda_1|}\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1 + \left(\frac{\varsigma}{\sigma_1}\right) + 1\right)}{\Gamma(\varsigma + 1)\Gamma\left(\epsilon_1 + \left(\frac{\varsigma}{\sigma_1}\right) + \theta_1 + 1\right)} + \frac{1}{\Gamma(\varsigma + 1)}\Big]\Big\} \\
&\leq \Delta_1\hat{\rho} + P_1,
\end{aligned}$$

and

$$\left|\mathcal{F}_1(u, v)'(\tau)\right|$$

$$
\begin{aligned}
&\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma-1}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\tau)+\frac{|\mu_1|}{|\Lambda_1|}(|a_3|)\mathcal{J}^{\varsigma+p}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\omega)\\
&\quad+\frac{1}{|\Lambda_1|}(|a_1|)\Big[|\tau_1|\,\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(\xi)+\mathcal{J}^{\varsigma}\left|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))\right|(1)\Big]\Big\}\\
&\leq \mathcal{J}^{\varsigma-1}\left(|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\right)(1)\\
&\quad+\frac{|\mu_1|}{|\Lambda_1|}(|a_3|)\mathcal{J}^{\varsigma+p}\left(|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\right)(\omega)\\
&\quad+\frac{1}{|\Lambda_1|}(|a_1|)\Big[|\tau_1|\,\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}\left(|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\right)(\xi)\\
&\quad+\mathcal{J}^{\varsigma}\left(|f(\theta,v(\theta),{}^{c}\mathcal{D}^{\varrho_1}v(\theta))-f(\theta,0,0)|+|f(\theta,0,0)|\right)(1)\Big]\\
&\leq \Big[K_1\|v\|_{\mathcal{V}}+M_1\Big]\Big\{\frac{1}{\Gamma(\varsigma)}+\frac{|\mu_1|(|a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)}\\
&\quad+\frac{(|a_1|)}{|\Lambda_1|}\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1+\left(\frac{\varsigma}{\sigma_1}\right)+\theta_1+1\right)}+\frac{1}{\Gamma(\varsigma+1)}\Big]\Big\}\\
&\leq \Delta_2\hat{\rho}+P_2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\left|{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)(\tau)\right| &\leq \frac{1}{\Gamma(1-\varsigma_1)}\int_0^{\tau}(\tau-\theta)^{-\varsigma_1}\left|\mathcal{F}_1(u,v)'(\theta)\right|d\theta\\
&\leq \frac{\Delta_2\hat{\rho}+P_2}{\Gamma(2-\varsigma_1)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\mathcal{F}_1(u,v)\|_{\mathcal{U}} &= \|\mathcal{F}_1(u,v)\|+\|{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)\|\\
&\leq \left(\Delta_1+\frac{\Delta_2}{\Gamma(2-\varsigma_1)}\right)\hat{\rho}+\left(P_1+\frac{P_2}{\Gamma(2-\varsigma_1)}\right)\\
&\leq \frac{\hat{\rho}}{2}.
\end{aligned}
$$

In this same way, we have

$$
|\mathcal{F}_2(u,v)(\tau)| \leq \Delta_1'\hat{\rho}+P_1',
$$

and

$$
|\mathcal{F}_2(u,v)'(\tau)| \leq \Delta_2'\hat{\rho}+P_2',
$$

which implies that

$$
\left|{}^{c}\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u,v)(\tau)\right| \leq \frac{\Delta_2'\hat{\rho}+P_2'}{\Gamma(2-\varrho_1)}.
$$

In consequence, we get

$$\begin{aligned}\|\mathcal{F}_2(u,v)\|_{\mathcal{V}} &= \|\mathcal{F}_2(u,v)\| + \|{}^c\mathcal{D}^{\varrho_1}\mathcal{F}_2(u,v)\| \\ &\leq \left(\Delta_1^{'} + \frac{\Delta_2^{'}}{\Gamma(2-\varrho_1)}\right)\hat{\rho} + \left(P_1^{'} + \frac{P_2^{'}}{\Gamma(2-\varrho_1)}\right) \\ &\leq \frac{\hat{\rho}}{2}.\end{aligned}$$

Hence, we get

$$\|\mathcal{F}(u,v)\|_{\mathcal{U}\times\mathcal{V}} = \|\mathcal{F}_1(u,v)\|_{\mathcal{U}} + \|\mathcal{F}_2(u,v)\|_{\mathcal{V}} \leq \hat{\rho}.$$

Hence, $\mathcal{F}B_{\hat{\rho}} \subset B_{\hat{\rho}}$.

Next to prove that $\mathcal{F}$ is a contraction mapping on $B_{\hat{\rho}}$.

For $u_i, v_i \in B_{\hat{\rho}}, i = 1,2$ and for each $\tau \in \mathcal{H}$, we have

$$\begin{aligned}&|\mathcal{F}_1(u_1,v_1)(\tau) - \mathcal{F}_1(u_2,v_2)(\tau)| \\ &\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(\tau) \\ &\quad + \frac{|\mu_1|}{|\Lambda_1|}(|a_4 - a_3\tau|)\mathcal{J}^{\varsigma+p}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(\omega) \\ &\quad + \frac{1}{|\Lambda_1|}(|a_1\tau + a_2|) \\ &\quad \Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(\xi) \\ &\quad + \mathcal{J}^{\varsigma}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(1)\Big]\Big\} \\ &\leq (K_1\|v_1 - v_2\|_{\mathcal{V}})\Big\{\frac{1}{\Gamma(\varsigma+1)} + \frac{|\mu_1|(|a_4| + |a_3|)}{|\Lambda_1|}\frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)} \\ &\quad + \frac{(|a_2| + |a_1|)}{|\Lambda_1|}\Big[|\tau_1|\frac{\xi^{\varsigma}\Gamma\left(\epsilon_1 + \left(\frac{\varsigma}{\sigma_1}\right) + 1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1 + \left(\frac{\varsigma}{\sigma_1}\right) + \theta_1 + 1\right)} + \frac{1}{\Gamma(\varsigma+1)}\Big]\Big\} \\ &\leq \Delta_1\|v_1 - v_2\|_{\mathcal{V}},\end{aligned}$$

and

$$\begin{aligned}&\left|\mathcal{F}_1(u_1,v_1)^{'}(\tau) - \mathcal{F}_1(u_1,v_1)^{'}(\tau)\right| \\ &\leq \sup_{\tau\in\mathcal{H}}\Big\{\mathcal{J}^{\varsigma-1}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(\tau) \\ &\quad + \frac{|\mu_1|}{|\Lambda_1|}(|a_3|)\mathcal{J}^{\varsigma+p}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(\omega) \\ &\quad + \frac{1}{|\Lambda_1|}(|a_1|) \\ &\quad \Big[|\tau_1|\mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1}\mathcal{J}^{\varsigma}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(\xi) \\ &\quad + \mathcal{J}^{\varsigma}|f(\theta,v_1(\theta),{}^c\mathcal{D}^{\varrho_1}v_1(\theta)) - f(\theta,v_2(\theta),{}^c\mathcal{D}^{\varrho_1}v_2(\theta))|(1)\Big]\Big\}\end{aligned}$$

$$\begin{aligned}
&\leq (K_1 \|v_1 - v_2\|_{\mathcal{V}}) \Bigg\{ \frac{1}{\Gamma(\varsigma)} + \frac{|\mu_1|(|a_3|)}{|\Lambda_1|} \frac{\omega^{\varsigma+p}}{\Gamma(\varsigma+p+1)} \\
&\quad + \frac{(|a_1|)}{|\Lambda_1|} \Bigg[|\tau_1| \frac{\xi^{\varsigma} \Gamma\left(\epsilon_1 + \left(\frac{\varsigma}{\sigma_1}\right) + 1\right)}{\Gamma(\varsigma+1)\Gamma\left(\epsilon_1 + \left(\frac{\varsigma}{\sigma_1}\right) + \theta_1 + 1\right)} + \frac{1}{\Gamma(\varsigma+1)} \Bigg] \Bigg\} \\
&\leq \Delta_2 \|v_1 - v_2\|_{\mathcal{V}}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&|{}^c\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u_1, v_1)(\tau) -{}^c \mathcal{D}^{\varsigma_1}\mathcal{F}_1(u_2, v_2)(\tau)| \\
&\quad \leq \frac{1}{\Gamma(1-\varsigma_1)} \int_0^{\tau} (\tau - \theta)^{-\varsigma_1} \left| \mathcal{F}_1(u_1, v_1)'(\theta) - \mathcal{F}_1(u_2, v_2)'(\theta) \right| d\theta \\
&\quad \leq \frac{1}{\Gamma(2-\varsigma_1)} (\Delta_2 \|v_1 - v_2\|_{\mathcal{V}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\mathcal{F}_1(u_1, v_1) - \mathcal{F}_1(u_2, v_2)\|_{\mathcal{U}} \\
&\quad = \|\mathcal{F}_1(u_1, v_1) - \mathcal{F}_1(u_2, v_2)\| + \|{}^c\mathcal{D}^{\varsigma_1}\mathcal{F}_1(u_1, v_1) -{}^c \mathcal{D}^{\varsigma_1}\mathcal{F}_1(u_2, v_2)\| \\
&\quad \leq \left(\Delta_1 + \frac{\Delta_2}{\Gamma(2-\varsigma_1)} \right) \|v_1 - v_2\|_{\mathcal{V}}. \\
&\quad \leq \chi_1 \|v_1 - v_2\|_{\mathcal{V}}.
\end{aligned}$$

In a similar way, we can find

$$\begin{aligned}
&|\mathcal{F}_2(u_1, v_1)(\tau) - \mathcal{F}_2(u_2, v_2)(\tau)| \\
&\quad \leq \sup_{\tau \in \mathcal{H}} \Bigg\{ \mathcal{J}^{\varrho} |g(\theta, u_1(\theta), {}^c \mathcal{D}^{\varsigma_1} u_1(\theta)) - g(\theta, u_2(\theta), {}^c \mathcal{D}^{\varsigma_1} u_2(\theta))| (\tau) \\
&\quad + \frac{|\mu_2|}{|\Lambda_2|} (|b_4 - b_3 \tau|) \mathcal{J}^{\varrho+q} |g(\theta, u_1(\theta), {}^c \mathcal{D}^{\varsigma_1} u_1(\theta)) - g(\theta, u_2(\theta), {}^c \mathcal{D}^{\varsigma_1} u_2(\theta))| (\zeta) \\
&\quad + \frac{1}{|\Lambda_2|} (|b_1 \tau + b_2|) \\
&\quad \Big[|\tau_2| \mathcal{I}_{\sigma_2}^{\epsilon_2, \theta_2} \mathcal{J}^{\varrho} |g(\theta, u_1(\theta), {}^c \mathcal{D}^{\varsigma_1} u_1(\theta)) - g(\theta, u_2(\theta), {}^c \mathcal{D}^{\varsigma_1} u_2(\theta))| (\eta) \\
&\quad + \mathcal{J}^{\varrho} |g(\theta, u_1(\theta), {}^c \mathcal{D}^{\varsigma_1} u_1(\theta)) - g(\theta, u_2(\theta), {}^c \mathcal{D}^{\varsigma_1} u_2(\theta))| (1) \Big] \Bigg\} \\
&\quad \leq \Delta_1' \|u_1 - u_2\|_{\mathcal{U}},
\end{aligned}$$

and

$$\begin{aligned}
&\left| \mathcal{F}_2(u_1, v_1)'(\tau) - \mathcal{F}_2(u_2, v_2)'(\tau) \right| \\
&\quad \leq \sup_{\tau \in \mathcal{H}} \Bigg\{ \mathcal{J}^{\varrho-1} |g(\theta, u_1(\theta), {}^c \mathcal{D}^{\varsigma_1} u_1(\theta)) - g(\theta, u_2(\theta), {}^c \mathcal{D}^{\varsigma_1} u_2(\theta))| (\tau) \\
&\quad + \frac{|\mu_2|}{|\Lambda_2|} (|b_3|) \mathcal{J}^{\varrho+q} |g(\theta, u_1(\theta), {}^c \mathcal{D}^{\varsigma_1} u_1(\theta)) - g(\theta, u_2(\theta), {}^c \mathcal{D}^{\varsigma_1} u_2(\theta))| (\zeta)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\Lambda_2|}(|b_1|) \\
&\Big[|\tau_2|\, \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} \mathcal{J}^{\varrho}\left|g(\theta,u_1(\theta),^{c}\mathcal{D}^{\varsigma_1}u_1(\theta)) - g(\theta,u_2(\theta),^{c}\mathcal{D}^{\varsigma_1}u_2(\theta))\right|(\eta) \\
&+\mathcal{J}^{\varrho}\left|g(\theta,u_1(\theta),^{c}\mathcal{D}^{\varsigma_1}u_1(\theta)) - g(\theta,u_2(\theta),^{c}\mathcal{D}^{\varsigma_1}u_2(\theta))\right|(1)\Big]\Big\} \\
\leq\ & \Delta_2^{'}\, \|u_1-u_2\|_{\mathcal{U}},
\end{aligned}
$$

which implies that

$$
\left|^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u_1,v_1)(\tau) -^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u_2,v_2)(\tau)\right| \leq \frac{1}{\Gamma(2-\varrho_1)}\left(\Delta_2^{'}\,\|u_1-u_2\|_{\mathcal{U}}\right).
$$

In consequence, we get

$$
\begin{aligned}
&\|\mathcal{F}_2(u_1,v_1) - \mathcal{F}_2(u_2,v_2)\|_{\mathcal{V}} \\
&= \|\mathcal{F}_2(u_1,v_1) - \mathcal{F}_2(u_2,v_2)\| + \|^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u_1,v_1) -^{c}\mathcal{D}^{\varrho_1}\mathcal{F}_2(u_2,v_2)\| \\
&\leq \left(\Delta_1^{'} + \frac{\Delta_2^{'}}{\Gamma(2-\varrho_1)}\right)\|u_1-u_2\|_{\mathcal{U}} \\
&\leq \chi_2\,\|u_1-u_2\|_{\mathcal{U}}.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\|\mathcal{F}(u,v)\|_{\mathcal{U}\times\mathcal{V}} &\leq \chi_1\,\|v_1-v_2\|_{\mathcal{V}} + \chi_2\,\|u_1-u_2\|_{\mathcal{U}} \\
&\leq max\{\chi_1,\chi_2\}\left(\|u_1-u_2\|_{\mathcal{U}} + \|v_1-v_2\|_{\mathcal{V}}\right) \\
&\leq \chi\,\|(u_1-u_2)+(v_1-v_2)\|_{\mathcal{U}\times\mathcal{V}}.
\end{aligned}
$$

Thus, the $\mathcal{F}$ operator is referred to as a contraction operator (see [33] Theorem 1.4) and produced a unique fixed point that generates a unique solution for the BVP of (1.3) and (1.4) on $\mathcal{H}$. □

4. Examples

Example 4.1. *Consider the following coupled system of non-integer order differential equations subject to the Riemann-Liouville, Erdélyi-Kober integral conditions:*

$$
\begin{cases}
^{c}\mathcal{D}^{\frac{7}{6}}u(\tau) = \frac{1}{\sqrt{36+\tau^2}}\cos\tau + \frac{49}{300}\cos v(\tau) + \frac{35}{3(60+\tau)}\,^{c}\mathcal{D}^{\frac{1}{3}}v(\tau), \\
^{c}\mathcal{D}^{\frac{5}{4}}v(\tau) = \frac{e^{-2\tau}}{\sqrt{16+\tau^2}} + \frac{39}{240}\sin u(\tau) + \frac{25}{2(190+\tau)}\,^{c}\mathcal{D}^{\frac{1}{5}}u(\tau), \\
u(0) = \mathcal{J}^{\frac{4}{3}}u\left(\frac{1}{3}\right), u(1) = 4\mathcal{I}_{\frac{3}{2}}^{\frac{1}{6},\frac{5}{4}}u\left(\frac{1}{5}\right), \\
v(0) = 4\mathcal{J}^{\frac{6}{5}}v\left(\frac{1}{2}\right), v(1) = \mathcal{I}_{\frac{7}{6}}^{\frac{6}{4},\frac{1}{8}}v\left(\frac{1}{7}\right).
\end{cases}
\tag{4.1}
$$

Here, $\varsigma = 7/6, \varrho = 5/4, \varrho_1 = 1/3, \varsigma_1 = 1/5, \mu_1 = 1, \mu_2 = 4, \tau_1 = 4, \tau_2 = 1, p = 4/3, q = 6/5, \sigma_1 = 3/2, \epsilon_1 = 1/6, \theta_1 = 5/4, \sigma_2 = 7/6, \epsilon_2 = 6/4, \theta_2 = 1/8, \omega = 1/3, \zeta = 1/2, \xi = 1/5, \eta = 1/7$*, and also*

$$
f\left(\tau, v(\tau),^{c}\mathcal{D}^{\varrho_1}v(\tau)\right) = \frac{1}{\sqrt{36+\tau^2}}\cos\tau + \frac{49}{300}\cos v(\tau) + \frac{35}{3(60+\tau)}\,^{c}\mathcal{D}^{\frac{1}{3}}v(\tau),
$$

$$g\left(\tau,u\left(\tau\right),{}^{c}\mathcal{D}^{\varsigma_1}u\left(\tau\right)\right) = \frac{e^{-2\tau}}{\sqrt{16+\tau^2}}+\frac{39}{240}e^{-\tau}\sin u(\tau)+\frac{25}{2\left(190+\tau\right)}{}^{c}\mathcal{D}^{\frac{1}{5}}u(\tau).$$

Clearly,

$$\begin{aligned}\left|f\left(\tau,v\left(\tau\right),{}^{c}\mathcal{D}^{\varrho_1}v\left(\tau\right)\right)\right| &\leq \frac{1}{6}+\frac{49}{300}\|v\|+\frac{35}{180}\left\|{}^{c}\mathcal{D}^{\frac{1}{3}}v\right\|\\ \left|g\left(\tau,u\left(\tau\right),{}^{c}\mathcal{D}^{\varsigma_1}u\left(\tau\right)\right)\right| &\leq \frac{1}{4}+\frac{39}{240}\|u\|+\frac{25}{380}\left\|{}^{c}\mathcal{D}^{\frac{1}{5}}u\right\|.\end{aligned}$$

Thus, $l_0 = 1/6, l_1 = 49/300, l_2 = 35/180, \lambda_0 = 1/4, \lambda_1 = 39/240, \lambda_2 = 25/380.$ *Using the given data, we find that* $a_1 = 0.8059, a_2 = 0.0277, a_3 = -1.9549, a_4 = 0.6521, \Lambda_1 = 0.4713$ *and* $b_1 = -0.5802, b_2 = 0.3591, b_3 = 0.0876, b_4 = 0.8751, \Lambda_2 = -0.4763$ *and also* $\mathcal{G}_1 = 2.3068, \mathcal{G}_2 = 2.4740, \mathcal{G}_1^{'} = 2.2666, \mathcal{G}_2^{'} = 2.1105.$ *Furthermore, we can find*

$$\begin{aligned}B &= \left(\mathcal{G}_1^{'}+\frac{\mathcal{G}_2^{'}}{\Gamma\left(2-\varrho_1\right)}\right)max\left\{\lambda_1,\lambda_2\right\} = 0.7482,\\ C &= \left(\mathcal{G}_1+\frac{\mathcal{G}_2}{\Gamma\left(2-\varsigma_1\right)}\right)max\left\{l_1,l_2\right\} = 0.9650.\end{aligned}$$

Thus, $max\left\{B,C\right\} = 0.9650 < 1.$
All of the hypotheses of the theorem 3.1 are satisfied. Therefore, there is a solution for the problem (4.1) on $\mathcal{H}$.

Example 4.2. *Consider the following coupled system of non-integer order differential equations subject to the Riemann- Liouville, Erdélyi-Kober integral conditions:*

$$\begin{cases}{}^{c}\mathcal{D}^{\frac{5}{3}}u\left(\tau\right) = \frac{1}{25}\tau v\left(\tau\right)+\frac{1}{25}\tau^2\,{}^{c}\mathcal{D}^{\frac{4}{5}}v\left(\tau\right)+\tau,\\ {}^{c}\mathcal{D}^{\frac{6}{5}}v\left(\tau\right) = \frac{1}{35}\tau u\left(\tau\right)+\frac{1}{35}\tau^3\,{}^{c}\mathcal{D}^{\frac{3}{5}}u\left(\tau\right)+\tau,\\ u\left(0\right) = 2\mathcal{J}^{\frac{3}{2}}u\left(\frac{1}{4}\right), u\left(1\right) = \mathcal{I}_{\frac{5}{2}}^{\frac{1}{3},\frac{4}{3}}u\left(\frac{1}{6}\right),\\ v\left(0\right) = \mathcal{J}^{\frac{1}{2}}v\left(\frac{1}{5}\right), v\left(1\right) = 2\mathcal{I}_{\frac{5}{4}}^{\frac{7}{6},\frac{6}{4}}v\left(\frac{1}{8}\right).\end{cases} \tag{4.2}$$

Here, $\varsigma = 5/3, \varrho = 6/5, \varrho_1 = 4/5, \varsigma_1 = 3/5, \mu_1 = 2, \mu_2 = 1, \tau_1 = 1, \tau_2 = 2, p = 3/2, q = 1/2, \sigma_1 = 5/2, \epsilon_1 = 1/3, \theta_1 = 4/3, \sigma_2 = 5/4, \epsilon_2 = 7/6, \theta_2 = 6/4, \omega = 1/4, \zeta = 1/5, \xi = 1/6, \eta = 1/8$ *and also* $K_1 = 1/10, K_2 = 1/5.$ *Clearly,*

$$\begin{aligned}\left|f\left(\tau,u_1,v_1\right)-f\left(\tau,u_2,v_2\right)\right| &\leq \frac{1}{25}\left(|u_1-u_2|+|v_1-v_2|\right),\\ \left|g\left(\tau,u_1,v_1\right)-g\left(\tau,u_2,v_2\right)\right| &\leq \frac{1}{35}\left(|u_1-u_2|+|v_1-v_2|\right).\end{aligned}$$

Using the given data, we find that $a_1 = 0.8119, a_2 = 0.0188, a_3 = 0.4065, a_4 = 0.9283, \Lambda_1 = 0.7608, b_1 = 0.4954, b_2 = 0.0673, b_3 = 0.4605, b_4 = 0.9563, \lambda_2 = 0.5047$ *and also* $\mathcal{G}_1 = 1.3483\mathcal{G}_2 = 1.7870, \mathcal{G}_1^{'} = 1.9601, \mathcal{G}_2^{'} = 2.0589, \Delta_1 = 0.0539, \Delta_2 = 0.0715, \Delta_1^{'} = 0.0560, \Delta_2^{'} = 0.0588.$ *We can find*

$$\chi_1 = \left(\Delta_1+\frac{\Delta_2}{\Gamma\left(2-\varsigma_1\right)}\right) = 0.1345,$$

$$\chi_2 = \left(\Delta_1^{'} + \frac{\Delta_2^{'}}{\Gamma(2-\varrho_1)}\right) = 0.1201.$$

Thus, $\chi = max\{\chi_1, \chi_2\} = 0.1345 < 1$.

All of the hypotheses of the theorem 3.2 are satisfied. Therefore, there is a unique solution for the problem (4.2) on $\mathcal{H}$.

5. Discussion

This paper implemented the Riemann-Liouville, Erdélyi-Kober integral conditions with Leray-Schauder and Banach fixed point theorems based solution for a Caputo type coupled differential equations of non-integer order. The results are obtained through fixing the parameters of interest for the proposed problem (1.3) and (1.4), such as $(p, q, \mu_1, \mu_2, \tau_1, \tau_2)$ which makes the distinctive classes of the problem. For example, by applying the value for $p, q = 1$ with a boundary condition in the proposed solution, the following equation will be obtained for the problem (1.3) and (1.4):

$$\begin{cases} u(0) = \mu_1 \int_0^{\omega} u(\theta)d\theta, u(1) = \tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1} u(\xi), 0 < \omega, \xi < 1, \\ v(0) = \mu_2 \int_0^{\zeta} v(\theta)d\theta, v(1) = \tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} v(\eta), 0 < \zeta, \eta < 1, \end{cases}$$

the result will be in the form of (1.3)–(1.4):

$$\begin{cases} u(0) = 0, u(1) = \tau_1 \mathcal{I}_{\sigma_1}^{\epsilon_1,\theta_1} u(\xi), 0 < \xi < 1, \\ v(0) = 0, v(1) = \tau_2 \mathcal{I}_{\sigma_2}^{\epsilon_2,\theta_2} v(\eta), 0 < \eta < 1, \end{cases}$$

when applying the $\mu_1 = \mu_2 = 0$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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