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Existence, uniqueness and stability analysis of a coupled fractional-order differential systems involving Hadamard derivatives and associated with multi-point boundary conditions

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Abstract

In this paper, we examine the consequences of existence, uniqueness and stability of a multi-point boundary value problem defined by a system of coupled fractional differential equations involving Hadamard derivatives. To prove the existence and uniqueness, we use the techniques of fixed point theory. Stability of Hyers-Ulam type is also discussed. Furthermore, we investigate variations of the problem in the context of different boundary conditions. The current results are verified by illustrative examples.

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1 Introduction

Due to its broad applications in mathematical modeling of many complex and non-local nonlinear systems, fractional calculus has emerged as an important field of investigation. A significant feature of fractional-order operators is their non-local nature, which accounts for the underlying phenomena's inherited properties. In 1892, in contrast to his Riemann–Liouville counterpart of the form $(d/d\tau)^q$, Hadamard [1] proposed a concept of fractional integro-differentiation in terms of the type $(\tau d/d\tau)^q$ fractional power. The Hadamard fractional derivative contains a logarithmic function of an arbitrary exponent in the integral kernel, which appears in its description. We refer the reader to [2–9] for details of Hadamard's fractional calculus. In recent years, for example, fractional differential equations involving Hadamard derivatives have attracted considerable attention; see [10–17], and the references cited therein.

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There was significant attention given to coupled systems of differential equations of fractional order with different boundary conditions. Such structures were used in various real-world experiments such as [18] infection modeling [19] and chaotic system control. Recent research in the coupled fractional-order BVPs is in [20–29] and the references cited therein. Recently, some authors have started studying a coupled fractional BVPs. Agarwal *et al.* [30] dealt with the consequences of coupled fractional-order systems with discrete and integral boundary conditions. In a fractional BVP involving the Caputo derivatives, Subramanian *et al.* [31] studied coupled non-local slit–strip conditions. Similarly, under periodic/antiperiodic boundary conditions, Ahmad *et al.* [32] studied the coupled system of sequential fractional BVP. Recently, Muthaiah *et al.* [10] have studied the existence of the following Hadamard-type FDE solutions with integral boundary conditions:

$$\begin{cases} {}^H\mathcal{D}^\varrho y(\tau) = g(\tau, y(\tau)), & 1 < \tau < T, 2 < \varrho \leq 3, \\ y(1) = 0, \quad y'(1) = 0, \quad {}^H\mathcal{D}^\zeta y(T) = \omega {}^H\mathcal{I}^\gamma y(\varphi), \end{cases}$$

where ${}^H\mathcal{D}^\varrho, {}^H\mathcal{D}^\zeta$ denote the Hadamard fractional derivatives (HFDs) of order $\varrho, \zeta, g : [1, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function, and ω is a positive real constant. The results are provided by the effects of appropriate fixed point theorems. The nonlinear coupled system of Riemann–Liouville FDEs

$$\begin{cases} {}_{RL}\mathcal{D}^q x(t) = f(t, x(t), y(t)), \\ {}_{RL}\mathcal{D}^p y(t) = g(t, x(t), y(t)), \\ x(0) = 0, \quad x(T) = \sum_{i=1}^n \alpha_i {}_{iH}\mathcal{I}^{\rho_i} y(\eta_i), \quad \eta_i \in (0, T), \\ y(0) = 0, \quad y(T) = \sum_{i=1}^n \beta_i {}_{iH}\mathcal{I}^{\gamma_i} x(\theta_i), \quad \theta_i \in (0, T), \end{cases}$$

for $0 < t < T$ and $1 < q, p \leq 2$, have been studied in [33], where ${}_{RL}\mathcal{D}^q, {}_{RL}\mathcal{D}^p$ denote the Riemann–Liouville fractional derivatives (RLFDs) of order $q, p, f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given continuous functions, and $\alpha_i, \beta_i \in \mathbb{R}, i = 1, 2, \dots, n$ are positive real constants. The results of the study are made possible via the consequences of appropriate fixed point theorems. The nonlinear coupled system of Hadamard FDEs

$$\begin{cases} \mathcal{D}^\alpha u(t) = f(t, u(t), v(t)), \\ \mathcal{D}^\beta v(t) = g(t, u(t), v(t)), \\ u(1) = 0, \quad u(e) = \mathcal{I}^\gamma u(\sigma_1), \quad \sigma_1 \in (1, e), \\ v(1) = 0, \quad v(e) = \mathcal{I}^\gamma v(\sigma_2), \quad \sigma_2 \in (1, e), \end{cases}$$

for $1 < t < e, 1 < \alpha, \beta \leq 2$ and $\gamma > 0$, has been discussed in [34], where $\mathcal{D}^\alpha, \mathcal{D}^\beta$ denote the HFDs of order $\alpha, \beta, f, g : [1, e] \times \mathbb{R}^2 \rightarrow \mathbb{R}$. The existence and uniqueness of solutions are proved by the Leray–Schauder alternative and contraction mapping principle. The Caputo-type FDE nonlinear coupled system

$$\begin{cases} {}^c\mathcal{D}^{\alpha_1} u(t) + \lambda_1 f_1(t, u(t), v(t)) = 0, \\ {}^c\mathcal{D}^{\alpha_2} v(t) + \lambda_2 f_2(t, u(t), v(t)) = 0, \\ u'(0) = u''(0) = \dots = u^{n-1}(0) = 0, \quad u(1) = \mu_1 \int_0^1 a(s)v(s) \, dA_1(s), \\ v'(0) = v''(0) = \dots = v^{n-1}(0) = 0, \quad v(1) = \mu_2 \int_0^1 b(s)u(s) \, dA_2(s), \end{cases}$$

for $0 < t < 1, n - 1 < \alpha_1 \leq n, m - 1 < \alpha_2 \leq m$, and $n, m \geq 2$, has been examined in [35], where $\lambda_i > 0$ is a parameter, $D_{0+}^{\alpha_i}$ is the standard Caputo derivative; $\mu_i > 0$ is a constant,

$$\int_0^1 a(s)v(s) \, dA_1(s), \quad \int_0^1 b(s)u(s) \, dA_2(s),$$

denote the Riemann–Stieltjes integrals. Leray–Schauder’s alternative and the contraction mapping principle prove the existence and uniqueness of solutions.

More recently, in [36], the authors discussed Hadamard-type fractional differential equations with Hadamard integral and discrete boundary conditions on a half-line of the form

$$\begin{cases} {}^H\mathcal{D}^q x(t) + \sigma(t)f(t, x(t)) = 0, & 2 < q \leq 3, t \in [1, +\infty], \\ x(1) = x'(1) = 0, & {}^H\mathcal{D}^{q-1}x(\infty) = a {}^H\mathcal{I}^\beta x(\xi) + b \sum_{i=1}^{m-2} \alpha_i x(\eta_i). \end{cases}$$

Additionally, [37] contains several intriguing results on Hadamard-type fractional differential equations, inclusions, and inequalities. We refer the reader to [38, 39] for applications of Hadamard fractional differential equations. Another area of study that has gained considerable attention from researchers in the field of fractional differential equations is the concept of Ulam stability. Ulam [40] proposed the Ulam-type stability in 1940, and it has since been studied and generalized using a variety of methods [17, 28, 29, 41, 42].

In this article, we extend the boundary value problem of Wang *et al.* [36] to nonlinear coupled system of Hadamard fractional differential equations having the Hadamard derivative value of the unknown function at T is propositional to the sum of Hadamard integral values of the unknown function on the strips $(1, \nu)$, $(1, \theta)$ and multi-points values of the unknown functions with different strip lengths $(1, \nu)$, $(1, \theta)$ and with different multi-point $\delta_j, \gamma_j, j = 1, 2, \dots, k - 2$. In [36], a monotone iterative method was applied to study the existence of positive solutions for Hadamard fractional differential equations complemented with non-local multi-point discrete and Hadamard integral boundary conditions.

In the present paper, inspired by [36], we introduce and investigate the existence and stability of solutions for a coupled system of nonlinear Hadamard-type fractional differential equations:

$$\begin{cases} {}^H\mathcal{D}^{\varrho} y(\tau) = h_1(\tau, y(\tau), z(\tau)), \\ {}^H\mathcal{D}^{\varsigma} z(\tau) = h_2(\tau, y(\tau), z(\tau)), \end{cases} \tag{1}$$

for $\tau \in [1, T] := \mathcal{H}$, enhanced with boundary conditions defined by

$$\begin{cases} y(1) = y'(1) = 0, & {}^H\mathcal{D}^{\varrho_1} y(T) = \nu_1 {}^H\mathcal{I}^{\varsigma_2} z(\nu) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, & {}^H\mathcal{D}^{\varsigma_1} z(T) = \nu_2 {}^H\mathcal{I}^{\varrho_2} y(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j y(\gamma_j), \end{cases} \tag{2}$$

and

$$1 < \theta < \nu < \gamma_1 < \delta_1 < \gamma_2 < \delta_2 < \dots < \gamma_{k-2} < \delta_{k-2} < T,$$

where ${}^H\mathcal{D}^{(\cdot)}$ denotes the Hadamard fractional derivatives (HFDs) of order (\cdot) , $2 < \varrho, \varsigma \leq 3$ and $0 < \varrho_1, \varsigma_1 \leq 1$. ${}^H\mathcal{I}^{(\cdot)}$ denotes the Hadamard fractional integrals (HFIs) of order (\cdot) ,

$0 < \varrho_2, \varsigma_2 < 1, f, g : \mathcal{H} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions, $\alpha_1, \alpha_2, \beta_1$ and β_2 are real constants and $\xi_j, \nu_j, j = 1, 2, \dots, k - 2$ are positive real constants. Notice that the multi-point strip boundary conditions in (2) are new and can be regarded as the HFDs value of unknown functions with the right end-point T is proportional with the sum of HFIs of unknown functions with different strip lengths $(1, \nu), (1, \theta)$ and with different multi-point values of unknown functions with $\delta_j, \gamma_j, j = 1, 2, \dots, k - 2$. By using the fixed point theory, we obtain the existence and uniqueness results. Furthermore, we investigate the Hyers–Ulam-type stability. We also discuss some variants of the given problem. Examples are given to support the theoretical outcomes.

Section 2 focuses on the basic principles of fractional calculus with the accompanying fundamental definitions and lemmas. The consequences of existence and uniqueness can be explored in Sect. 3 using fixed point theorems of Leray–Schauder, Krasnoselskii, and Banach. Section 4 addresses the stability of Hyers–Ulam solutions and establishes sufficient conditions for stability. In Sect. 5, we consider two new problems analog to (1)–(2). Section 6 gives examples of verifying the results.

2 Preliminaries

In this section, we recall some preliminary concepts related to our work concerning Hadamard fractional calculus.

Definition 1 ([2]) The Hadamard fractional integral of order $\varrho \in \mathbb{R}^+$ of a function $h \in \mathcal{L}^p[b, c], 0 \leq b \leq \tau \leq c \leq \infty$ is defined as

$$({}^H\mathcal{I}^\varrho h)(\tau) = \frac{1}{\Gamma(\varrho)} \int_b^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} h(\sigma) \frac{d\sigma}{\sigma}.$$

Definition 2 ([2]) Let $0 < b < c < \infty, \delta = \tau \frac{d}{d\tau}$ and

$$\mathcal{AC}_\delta^n[b, c] = \{h : [b, c] \rightarrow \mathbb{R} : \delta^{n-1}[h(\tau)] \in \mathcal{AC}[b, c]\}.$$

The Hadamard derivative of fractional order ϱ for a function $h \in \mathcal{AC}_\delta^n[b, c]$ is defined as

$$({}^H\mathcal{D}^\varrho h)(\tau) = \delta^n (\mathcal{I}^{n-\varrho})(\tau) = \frac{1}{\Gamma(n-\varrho)} \left(\tau \frac{d}{d\tau}\right)^n \int_b^\tau \left(\log \frac{\tau}{\sigma}\right)^{n-\varrho-1} h(\sigma) \frac{d\sigma}{\sigma},$$

where $n - 1 < \varrho < n, n = [\varrho] + 1, [\varrho]$ denotes the integer part of the real number ϱ and $\log(\cdot) = \log_e(\cdot)$. Recall that the Hadamard fractional derivative is the left-inverse operator to Hadamard fractional integral in the space $\mathcal{L}^p[b, c], 1 \leq p \leq \infty$, (i.e.), ([2])

$${}^H\mathcal{D}^\varrho ({}^H\mathcal{I}^\varrho h(\tau)) = h(\tau).$$

Lemma 3 ([2, p. 113]) Let $\varrho > 0$ and $\varsigma > 0$. Then the formulas

$${}^H\mathcal{I}^\varrho \tau^\varsigma = \varsigma^{-\varrho} \tau^\varsigma \quad \text{and} \quad {}^H\mathcal{D}^\varrho \tau^\varsigma = \varsigma^\varrho \tau^\varsigma$$

hold. For Hadamard fractional integrals, the semigroup property holds:

$${}^H\mathcal{I}^\varrho {}^H\mathcal{I}^\varsigma h(\tau) = {}^H\mathcal{I}^{\varrho+\varsigma} h(\tau), \quad \varrho \geq 0, \varsigma \geq 0,$$

which leads to the commutative property:

$${}^H\mathcal{I}^\varrho {}^H\mathcal{I}^\varsigma h(\tau) = {}^H\mathcal{I}^\varsigma {}^H\mathcal{I}^\varrho h(\tau).$$

Lemma 4 ([2, Property 2.24]) *If $b, \varrho, \varsigma > 0$, then*

$$\begin{aligned} \left({}^H\mathcal{D}^\varrho \left(\log \frac{\tau}{b}\right)^{\varsigma-1}\right)(y) &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\varrho)} \left(\log \frac{y}{b}\right)^{\varsigma-\varrho-1}, \\ \left({}^H\mathcal{I}^\varrho \left(\log \frac{\tau}{b}\right)^{\varsigma-1}\right)(y) &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma+\varrho)} \left(\log \frac{y}{b}\right)^{\varsigma+\varrho-1}. \end{aligned}$$

Lemma 5 ([2]) *Let $\varrho > 0$ and $y \in \mathcal{C}[1, \infty) \cap \mathcal{L}^1[1, \infty)$. Then the solution of the Hadamard fractional differential equation ${}^H\mathcal{D}^\varrho y(\tau) = 0$ is given by*

$$y(\tau) = \sum_{j=1}^m a_j (\log \tau)^{\varrho-j},$$

and the following formula holds:

$${}^H\mathcal{I}^\varrho {}^H\mathcal{D}^\varrho y(\tau) = y(\tau) + \sum_{j=1}^m a_j (\log \tau)^{\varrho-j},$$

where $a_i \in \mathbb{R}, i = 1, 2, \dots, m$ and $n-1 < \varrho < n$.

Lemma 6 *Let $\widehat{h}_1, \widehat{h}_2 \in C[1, T]$. Then the solution of the linear system of FDEs:*

$$\begin{cases} {}^H\mathcal{D}^\varrho y(\tau) = \widehat{h}_1(\tau), \\ {}^H\mathcal{D}^\varsigma z(\tau) = \widehat{h}_2(\tau), \end{cases} \tag{3}$$

augmented with the boundary conditions:

$$\begin{cases} y(1) = y'(1) = 0, & {}^H\mathcal{D}^{\varrho_1} y(T) = \nu_1 {}^H\mathcal{I}^{\varsigma_2} z(u) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, & {}^H\mathcal{D}^{\varsigma_1} z(T) = \nu_2 {}^H\mathcal{I}^{\varrho_2} y(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j y(\gamma_j), \end{cases} \tag{4}$$

and

$$1 < \theta < u < \gamma_1 < \delta_1 < \gamma_2 < \delta_2 < \dots < \gamma_{k-2} < \delta_{k-2} < T,$$

is given by

$$\begin{aligned} y(\tau) &= {}^H\mathcal{I}^\varrho \widehat{h}_1(\tau) + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \\ &\times \left[\vartheta_4 \left\{ \nu_1 {}^H\mathcal{I}^{\varsigma+\varsigma_2} \widehat{h}_2(u) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j {}^H\mathcal{I}^\varsigma \widehat{h}_2(\delta_j) - {}^H\mathcal{I}^{\varrho-\varrho_1} \widehat{h}_1(T) \right\} \right. \\ &\left. + \vartheta_3 \left\{ \nu_2 {}^H\mathcal{I}^{\varrho+\varrho_2} \widehat{h}_1(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j {}^H\mathcal{I}^\varrho \widehat{h}_1(\gamma_j) - {}^H\mathcal{I}^{\varsigma-\varsigma_1} \widehat{h}_2(T) \right\} \right] \end{aligned} \tag{5}$$

and

$$\begin{aligned}
 z(\tau) = & {}^H\mathcal{I}^\zeta \hat{h}_2(\tau) + \frac{(\log \tau)^{\zeta-1}}{\vartheta} \\
 & \times \left[\vartheta_1 \left\{ v_2 {}^H\mathcal{I}^{\varrho+\varrho_2} \hat{h}_1(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j {}^H\mathcal{I}^\varrho \hat{h}_1(\gamma_j) - {}^H\mathcal{I}^{\zeta-\varsigma_1} \hat{h}_2(T) \right\} \right. \\
 & \left. + \vartheta_2 \left\{ v_1 {}^H\mathcal{I}^{\zeta+\zeta_2} \hat{h}_2(\nu) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j {}^H\mathcal{I}^\zeta \hat{h}_2(\delta_j) - {}^H\mathcal{I}^{\varrho-\varrho_1} \hat{h}_1(T) \right\} \right], \tag{6}
 \end{aligned}$$

where

$$\begin{aligned}
 \vartheta_1 = & \frac{\Gamma(\varrho)}{\Gamma(\varrho - \varrho_1)} (\log T)^{\varrho-\varrho_1-1}, \\
 \vartheta_2 = & \frac{v_2 \Gamma(\varrho)}{\Gamma(\varrho + \varrho_2)} (\log \theta)^{\varrho+\varrho_2-1} + \omega_2 \sum_{j=1}^{k-2} \varpi_j (\log \gamma_j)^{\varrho-1}, \\
 \vartheta_3 = & \frac{v_1 \Gamma(\zeta)}{\Gamma(\zeta + \zeta_2)} (\log \nu)^{\zeta+\zeta_2-1} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j (\log \delta_j)^{\zeta-1}, \\
 \vartheta_4 = & \frac{\Gamma(\zeta)}{\Gamma(\zeta - \varsigma_1)} (\log T)^{\zeta-\varsigma_1-1},
 \end{aligned} \tag{7}$$

and

$$\vartheta = \vartheta_1 \vartheta_4 - \vartheta_2 \vartheta_3. \tag{8}$$

Proof As argued in [3], the general solution of the system (3) can be written as

$$y(\tau) = {}^H\mathcal{I}^\varrho \hat{h}_1(\tau) + a_1 (\log \tau)^{\varrho-1} + a_2 (\log \tau)^{\varrho-2} + a_3 (\log \tau)^{\varrho-3}, \tag{9}$$

$$z(\tau) = {}^H\mathcal{I}^\zeta \hat{h}_2(\tau) + b_1 (\log \tau)^{\zeta-1} + b_2 (\log \tau)^{\zeta-2} + b_3 (\log \tau)^{\zeta-3}, \tag{10}$$

where $a_i, b_i \in \mathbb{R}$ ($i = 1, 2, 3$) are arbitrary constants. Making use of the conditions $y(1) = y'(1) = 0$, $z(1) = z'(1) = 0$, we find that $a_i = b_i = 0$, ($i = 1, 2$). Now, using the coupled non-separated boundary conditions given by (4) in (9) and (10), we obtain

$$a_1 \vartheta_1 - b_1 \vartheta_3 = \mathcal{A}_1, \quad b_1 \vartheta_4 - a_1 \vartheta_2 = \mathcal{A}_2, \tag{11}$$

where ϑ_i ($i = 1, 2, 3, 4$) are, respectively, given by (7), and

$$\mathcal{A}_1 = v_1 {}^H\mathcal{I}^{\zeta+\zeta_2} \hat{h}_2(\nu) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j {}^H\mathcal{I}^\zeta \hat{h}_2(\delta_j) - {}^H\mathcal{I}^{\varrho-\varrho_1} \hat{h}_1(T), \tag{12}$$

$$\mathcal{A}_2 = v_2 {}^H\mathcal{I}^{\varrho+\varrho_2} \hat{h}_1(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j {}^H\mathcal{I}^\varrho \hat{h}_1(\gamma_j) - {}^H\mathcal{I}^{\zeta-\varsigma_1} \hat{h}_2(T). \tag{13}$$

Solving the system (11) for a_1 and b_1 and using the notation (8), we find that

$$a_1 = \frac{1}{\vartheta} \left[\vartheta_4 \left\{ v_1 {}^H\mathcal{I}^{\zeta+\varsigma_2} \hat{h}_2(v) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j {}^H\mathcal{I}^{\zeta} \hat{h}_2(\delta_j) - {}^H\mathcal{I}^{\varrho-\varrho_1} \hat{h}_1(T) \right\} + \vartheta_3 \left\{ v_2 {}^H\mathcal{I}^{\varrho+\varrho_2} \hat{h}_1(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j {}^H\mathcal{I}^{\varrho} \hat{h}_1(\gamma_j) - {}^H\mathcal{I}^{\zeta-\varsigma_1} \hat{h}_2(T) \right\} \right], \tag{14}$$

$$b_1 = \frac{1}{\vartheta} \left[\vartheta_1 \left\{ v_2 {}^H\mathcal{I}^{\varrho+\varrho_2} \hat{h}_1(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j {}^H\mathcal{I}^{\varrho} \hat{h}_1(\gamma_j) - {}^H\mathcal{I}^{\zeta-\varsigma_1} \hat{h}_2(T) \right\} + \vartheta_2 \left\{ v_1 {}^H\mathcal{I}^{\zeta+\varsigma_2} \hat{h}_2(v) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j {}^H\mathcal{I}^{\zeta} \hat{h}_2(\delta_j) - {}^H\mathcal{I}^{\varrho-\varrho_1} \hat{h}_1(T) \right\} \right]. \tag{15}$$

Substituting the values of a_1 and b_1 in (11), we get the desired solution (5)–(6). □

3 Existence results for the problem (1)–(2)

We define the spaces

$$\mathcal{Y} = \{y(\tau) : y(\tau) \in C(\mathcal{H}, \mathbb{R})\},$$

endowed with the norm $\|y\| = \sup\{|y(\tau)|, \tau \in \mathcal{H}\}$. Obviously $(\mathcal{Y}, \|\cdot\|)$ is a Banach space. Also

$$\mathcal{Z} = \{z(\tau) : z(\tau) \in C(\mathcal{H}, \mathbb{R})\},$$

endowed with the norm $\|z\| = \sup\{|z(\tau)|, \tau \in \mathcal{H}\}$ is a Banach space. Then the product space $(\mathcal{Y} \times \mathcal{Z}, \|(y, z)\|)$ is also a Banach space equipped with norm $\|(y, z)\| = \|y\| + \|z\|$.

In relation to the problem, we introduce an operator $\Pi : \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{Y} \times \mathcal{Z}$ by Lemma 6 as follows:

$$\Pi(y, z)(\tau) := (\Pi_1(y, z)(\tau), \Pi_2(y, z)(\tau)), \tag{16}$$

where

$$\begin{aligned} \Pi_1(y, z)(\tau) = & \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\ & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ v_1 \frac{1}{\Gamma(\zeta+\varsigma_2)} \int_1^v \left(\log \frac{v}{\sigma}\right)^{\zeta+\varsigma_2-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \right. \\ & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\ & \left. \left. - \frac{1}{\Gamma(\varrho-\varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho-\varrho_1-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\ & \left. + \vartheta_3 \left\{ v_2 \frac{1}{\Gamma(\varrho+\varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho+\varrho_2-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & - \frac{1}{\Gamma(\zeta - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \varsigma_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \Bigg\}, \tag{17}
 \end{aligned}$$

$$\begin{aligned}
 \Pi_2(y, z)(\tau) = & \frac{1}{\Gamma(\zeta)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \frac{(\log \tau)^{\zeta-1}}{\vartheta} \left[\vartheta_1 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\zeta - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \varsigma_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\
 & + \vartheta_2 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \varsigma_2)} \int_1^v \left(\log \frac{v}{\sigma}\right)^{\zeta + \varsigma_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right]. \tag{18}
 \end{aligned}$$

For convenience of computation, we set

$$\begin{aligned}
 \mathcal{S}_1 = & \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} + \frac{(\log T)^{\varrho-1}}{\vartheta} \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\
 & \left. + \vartheta_3 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right], \tag{19}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{T}_1 = & \frac{(\log T)^{\zeta-1}}{\vartheta} \left[\frac{\vartheta_3 (\log T)^{\zeta - \varsigma_1}}{\Gamma(\zeta - \varsigma_1 + 1)} \right. \\
 & \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log v)^{\zeta + \varsigma_2}}{\Gamma(\zeta + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right\} \right], \tag{20}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{S}_2 = & \frac{(\log T)^{\zeta-1}}{\vartheta} \left[\frac{\vartheta_2 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\
 & \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right], \tag{21}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{T}_2 = & \frac{(\log T)^\zeta}{\Gamma(\zeta + 1)} + \frac{(\log T)^{\zeta-1}}{\vartheta} \left[\frac{\vartheta_1 (\log T)^{\zeta - \varsigma_1}}{\Gamma(\zeta - \varsigma_1 + 1)} \right. \\
 & \left. + \vartheta_2 \left\{ \nu_1 \frac{(\log v)^{\zeta + \varsigma_2}}{\Gamma(\zeta + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right\} \right], \tag{22}
 \end{aligned}$$

$$\Psi = \min\{1 - [\varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2)], 1 - [\varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2)]\}. \tag{23}$$

Next, we present the hypotheses that we need in the sequel.

Let the functions $h_1, h_2 : \mathcal{H} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous.

(\mathcal{F}_1) There exist real constants $\varphi_i, \widehat{\varphi}_i \geq 0, i = 1, 2$ and $\varphi_0 > 0, \widehat{\varphi}_0 > 0$ such that

$$\begin{aligned} |h_1(\tau, w_1, w_2)| &\leq \varphi_0 + \varphi_1|w_1| + \varphi_2|w_2|, \\ |h_2(\tau, w_1, w_2)| &\leq \widehat{\varphi}_0 + \widehat{\varphi}_1|w_1| + \widehat{\varphi}_2|w_2|, \end{aligned} \tag{24}$$

for all $w_i \in \mathbb{R}$ and $i = 1, 2$.

(\mathcal{F}_2) There exist positive constants $\widetilde{\vartheta}_i, \widehat{\vartheta}_i$, with $i = 1, 2$ such that

$$\begin{aligned} |h_1(\tau, w_1, w_2) - h_1(\tau, \widehat{w}_1, \widehat{w}_2)| &\leq \widetilde{\vartheta}_1|w_1 - \widehat{w}_1| + \widetilde{\vartheta}_2|w_2 - \widehat{w}_2|, \\ |h_2(\tau, w_1, w_2) - h_2(\tau, \widehat{w}_1, \widehat{w}_2)| &\leq \widehat{\vartheta}_1|\widehat{w}_1 - \widehat{w}_1| + \widehat{\vartheta}_2|w_2 - \widehat{w}_2|, \end{aligned} \tag{25}$$

for all $\tau \in \mathcal{H}, w_i, \widehat{w}_i \in \mathbb{R}$ and $i = 1, 2$.

Lemma 7 (Leray–Schauder alternative, [43, p.4]) *Let $\mathcal{F} : \mathcal{E} \rightarrow \mathcal{E}$ be a completely continuous operator. Let*

$$\psi(\mathcal{F}) = \{y \in \mathcal{E} : y = \eta \mathcal{F}y \text{ for some } 0 < \eta < 1\}.$$

Then either the set $\psi(\mathcal{F})$ is unbounded or \mathcal{F} has at least one fixed point.

Theorem 8 *Suppose that (\mathcal{F}_1) holds. If*

$$\begin{aligned} \varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2) &< 1, \\ \varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2) &< 1, \end{aligned} \tag{26}$$

then the system (1)–(2) on \mathcal{H} has at least one solution, where (19)–(22) refer to $\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2,$ and \mathcal{T}_2 , respectively.

Proof We define that in the first step the operator $\Pi : \mathcal{Y} \times \mathcal{Z} \rightarrow \mathcal{Y} \times \mathcal{Z}$ is completely continuous, indicating the continuity of the h_1 and h_2 functions of the Π_1 and Π_2 operators. The operator Π is also continuous. Let $\Omega \subset \mathcal{Y} \times \mathcal{Z}$ be a bounded set to show the uniformly bounded operator Π . Then there exist positive constants \mathcal{U}_1 and \mathcal{U}_2 such that

$$|h_1(\tau, y(\tau), z(\tau))| \leq \mathcal{U}_1, \quad |h_2(\tau, y(\tau), z(\tau))| \leq \mathcal{U}_2,$$

for all $(y, z) \in \Omega$. Then we have, for any $(y, z) \in \Omega$,

$$\begin{aligned} &|\Pi_1(y, z)(\tau)| \\ &\leq \frac{\mathcal{U}_1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} d\sigma + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \\ &\quad \times \left[\vartheta_4 \left\{ \nu_1 \frac{\mathcal{U}_2}{\Gamma(\zeta + \varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\zeta + \varsigma_2 - 1} d\sigma \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{\mathcal{U}_2}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} d\sigma + \frac{\mathcal{U}_1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} d\sigma \Bigg\} \\
 & + \vartheta_3 \left\{ \nu_2 \frac{\mathcal{U}_1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} d\sigma \right. \\
 & \left. + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{\mathcal{U}_1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma}\right)^{\varrho-1} d\sigma + \frac{\mathcal{U}_2}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \zeta_1 - 1} d\sigma \Bigg\} \\
 & \leq \frac{(\log T)^{\varrho-1}}{\vartheta} \left\{ \mathcal{U}_2 \left[\frac{\vartheta_3 (\log T)^{\zeta - \zeta_1}}{\Gamma(\zeta - \zeta_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_4 \left(\nu_1 \frac{(\log \nu)^{\zeta + \zeta_2}}{\Gamma(\zeta + \zeta_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right) \right] \right. \\
 & \left. + \frac{\vartheta \mathcal{U}_1 (\log T)^\varrho}{(\log T)^{\varrho-1} \Gamma(\varrho + 1)} + \mathcal{U}_1 \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_3 \left(\nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right) \right] \right\},
 \end{aligned}$$

which yields when taking the $\tau \in \mathcal{H}$ norm and using (19) and (20),

$$\|\Pi_1(y, z)\| \leq \mathcal{S}_1 \mathcal{U}_1 + \mathcal{T}_1 \mathcal{U}_2. \tag{27}$$

Likewise we obtain

$$\begin{aligned}
 |\Pi_2(y, z)(\tau)| & \leq \frac{(\log T)^{\zeta-1}}{\vartheta} \left\{ \mathcal{U}_1 \left[\frac{\vartheta_2 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_1 \left(\nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right) \right] \right. \\
 & \left. + \frac{\vartheta \mathcal{U}_2 (\log T)^\zeta}{(\log T)^{\zeta-1} \Gamma(\zeta + 1)} + \mathcal{U}_2 \left[\frac{\vartheta_1 (\log T)^{\zeta - \zeta_1}}{\Gamma(\zeta - \zeta_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_2 \left(\nu_1 \frac{(\log \nu)^{\zeta + \zeta_2}}{\Gamma(\zeta + \zeta_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right) \right] \right\} \\
 & \leq \mathcal{S}_2 \mathcal{U}_1 + \mathcal{T}_2 \mathcal{U}_2, \tag{28}
 \end{aligned}$$

by using (21) and (22). We deduce that Π_1 and Π_2 are uniformly bound from inequalities (27) and (28), which means that the operator Π is uniformly bounded. Next we show Π 's equicontinuity. Let $\tau_1, \tau_2 \in \mathcal{H}$ with $\tau_1 < \tau_2$. Then we have

$$\begin{aligned}
 & |\Pi_1(y, z)(\tau_2) - \Pi_1(y, z)(\tau_1)| \\
 & \leq \frac{1}{\Gamma(\varrho)} \left| \int_1^{\tau_1} \left[\left(\log \frac{\tau_2}{\sigma}\right)^{\varrho-1} - \left(\log \frac{\tau_1}{\sigma}\right)^{\varrho-1} \right] |h_1(\sigma, y(\sigma), z(\sigma))| d\sigma \right. \\
 & \left. + \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{\sigma}\right)^{\varrho-1} |h_1(\sigma, y(\sigma), z(\sigma))| d\sigma \right|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \right. \right. \\
 & \times \int_1^{\nu} \left(\log \frac{\nu}{\sigma} \right)^{\zeta + \zeta_2 - 1} |h_2(\sigma, y(\sigma), z(\sigma))| \, d\sigma \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\zeta-1} |h_2(\sigma, y(\sigma), z(\sigma))| \, d\sigma \\
 & + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} |h_1(\sigma, y(\sigma), z(\sigma))| \, d\sigma \left. \right\} \\
 & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^{\theta} \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} |h_1(\sigma, y(\sigma), z(\sigma))| \, d\sigma \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho-1} |h_1(\sigma, y(\sigma), z(\sigma))| \, d\sigma \\
 & \left. \left. + \frac{1}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\zeta - \zeta_1 - 1} |h_2(\sigma, y(\sigma), z(\sigma))| \, d\sigma \right\} \right] \\
 & \rightarrow 0,
 \end{aligned}$$

as $\tau_2 \rightarrow \tau_1$, independent of (y, z) with respect to $|h_1(\tau, y(\tau), z(\tau))| \leq \mathcal{U}_1$ and $|h_2(\tau, y(\tau), z(\tau))| \leq \mathcal{U}_2$. Similarly,

$$|\Pi_2(y, z)(\tau_2) - \Pi_2(y, z)(\tau_1)| \rightarrow 0,$$

as $\tau_2 \rightarrow \tau_1$ independent of (y, z) with respect to the boundedness of h_1 and h_2 . And because of the equicontinuity of Π_1 and Π_2 , the operator Π is equicontinuous. Thus, by the Arzelà–Ascoli theorem the operator Π is compact. The set

$$\Lambda(\Pi) = \{(y, z) \in \mathcal{Y} \times \mathcal{Z} : (y, z) = \lambda \Pi(y, z); 0 < \lambda < 1\}$$

is finally shown to be bounded. Then $(y, z) = \varepsilon \Pi(y, z)$. For any $\tau \in \mathcal{H}$, we have $y(\tau) = \lambda \Pi_1(y, z)(\tau)$, $z(\tau) = \lambda \Pi_2(y, z)(\tau)$. Using (\mathcal{F}_1) in (17), we get

$$\begin{aligned}
 |y(\tau)| \leq & \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \left(\log \frac{\tau}{\sigma} \right)^{\varrho-1} (\varphi_0 + \varphi_1 |y(\sigma)| + \varphi_2 |z(\sigma)|) \, d\sigma \\
 & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \frac{\nu_1}{\Gamma(\zeta + \zeta_2)} \right. \right. \\
 & \times \int_1^{\nu} \left(\log \frac{\nu}{\sigma} \right)^{\zeta + \zeta_2 - 1} (\widehat{\varphi}_0 + \widehat{\varphi}_1 |y(\sigma)| + \widehat{\varphi}_2 |z(\sigma)|) \, d\sigma \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\zeta-1} (\widehat{\varphi}_0 + \widehat{\varphi}_1 |y(\sigma)| + \widehat{\varphi}_2 |z(\sigma)|) \, d\sigma \\
 & \left. \left. + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} (\varphi_0 + \varphi_1 |y(\sigma)| + \varphi_2 |z(\sigma)|) \, d\sigma \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
 &+ \vartheta_3 \left\{ \frac{\nu_2}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} (\varphi_0 + \varphi_1 |y(\sigma)| + \varphi_2 |z(\sigma)|) \, d\sigma \right. \\
 &+ \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho - 1} (\varphi_0 + \varphi_1 |y(\sigma)| + \varphi_2 |z(\sigma)|) \, d\sigma \\
 &\left. + \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma - \varsigma_1 - 1} (\widehat{\varphi}_0 + \widehat{\varphi}_1 |y(\sigma)| + \widehat{\varphi}_2 |z(\sigma)|) \, d\sigma \right\},
 \end{aligned}$$

which yields, when taking the norm for $\tau \in \mathcal{H}$,

$$\|y\| \leq (\varphi_0 + \varphi_1 |y(\sigma)| + \varphi_2 |z(\sigma)|) \mathcal{S}_1 + (\widehat{\varphi}_0 + \widehat{\varphi}_1 |y(\sigma)| + \widehat{\varphi}_2 |z(\sigma)|) \mathcal{T}_1. \tag{29}$$

We also obtain

$$\|z\| \leq (\varphi_0 + \varphi_1 |y(\sigma)| + \varphi_2 |z(\sigma)|) \mathcal{S}_2 + (\widehat{\varphi}_0 + \widehat{\varphi}_1 |y(\sigma)| + \widehat{\varphi}_2 |z(\sigma)|) \mathcal{T}_2. \tag{30}$$

From (29) and (30), we get

$$\begin{aligned}
 \|y\| + \|z\| &= \varphi_0(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_0(\mathcal{T}_1 + \mathcal{T}_2) \\
 &+ \|y\| [\varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2)] \\
 &+ \|z\| [\varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2)],
 \end{aligned}$$

which yields, with $\|y, z\| = \|y\| + \|z\|$,

$$\|y, z\| \leq \frac{\varphi_0(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_0(\mathcal{T}_1 + \mathcal{T}_2)}{\Psi}.$$

This means that $\Lambda(\Pi)$ is bounded. Thus, by Lemma 7, the operator Π has at least one fixed point. This indicates that the BVP (1)–(2) has at least one solution on \mathcal{H} . \square

Theorem 9 *Suppose that (\mathcal{F}_2) hold. Then the BVP (1)–(2) has a unique solution on \mathcal{H} , provided that*

$$(\mathcal{S}_1 + \mathcal{S}_2)(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) < 1, \tag{31}$$

where (19)–(22) refer to $\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2,$ and $\mathcal{T}_2,$ respectively.

Proof Let us fix

$$\varepsilon \geq \frac{\mathcal{E}_1(\mathcal{S}_1 + \mathcal{S}_2) + \mathcal{E}_2(\mathcal{T}_1 + \mathcal{T}_2)}{1 - (\mathcal{S}_1 + \mathcal{S}_2)(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) - (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2)},$$

and show that $\Pi \mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$, when the operator Π is given by (16) and

$$\mathcal{B}_\varepsilon = \{(y, z) \in \mathcal{Y} \times \mathcal{Z} : \|(y, z)\| \leq \varepsilon\}.$$

For $(y, z) \in \mathcal{B}_\varepsilon$, $\tau \in \mathcal{H}$, we have

$$|h_1(\tau, y(\tau), z(\tau))| \leq \tilde{\vartheta}_1 |y(\tau)| + \tilde{\vartheta}_2 |z(\tau)| + \mathcal{E}_1 \leq \tilde{\vartheta}_1 \|y\| + \tilde{\vartheta}_2 \|z\| + \mathcal{E}_1$$

and $|h_2(\tau, y(\tau), z(\tau))| \leq \hat{\vartheta}_1 \|y\| + \hat{\vartheta}_2 \|z\| + \mathcal{E}_2$. This leads to

$$\begin{aligned} |\Pi_1(y, z)(\tau)| &\leq \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} \\ &\quad \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \\ &\quad + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ v_1 \frac{1}{\Gamma(\varsigma + \varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\varsigma + \varsigma_2 - 1} \right. \right. \\ &\quad \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \\ &\quad + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varsigma-1} \\ &\quad \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \\ &\quad \left. \left. + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} \right. \right. \\ &\quad \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \left. \right\} \\ &\quad + \vartheta_3 \left\{ v_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} \right. \\ &\quad \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \\ &\quad + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma}\right)^{\varrho-1} \\ &\quad \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \\ &\quad \left. \left. + \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varsigma - \varsigma_1 - 1} \right. \right. \\ &\quad \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \left. \right\} \Bigg] \\ &\leq (\tilde{\vartheta}_1 \|y\| + \tilde{\vartheta}_2 \|z\| + \mathcal{E}_1) \left\{ \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} \right. \\ &\quad + \frac{(\log T)^{\varrho-1}}{\vartheta} \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\ &\quad \left. \left. + \vartheta_3 \left\{ v_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & + (\widehat{\vartheta}_1 \|y\| + \widehat{\vartheta}_2 \|z\| + \mathcal{E}_2) \left\{ \frac{(\log T)^{\varrho-1}}{\vartheta} \left[\frac{\vartheta_3 (\log T)^{\varsigma-\varsigma_1}}{\Gamma(\varsigma-\varsigma_1+1)} \right. \right. \\
 & \left. \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma+\varsigma_2}}{\Gamma(\varsigma+\varsigma_2+1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma+1)} \right\} \right] \right\} \\
 & \leq (\widetilde{\vartheta}_1 \|y\| + \widetilde{\vartheta}_2 \|z\| + \mathcal{E}_1) \mathcal{S}_1 + (\widehat{\vartheta}_1 \|y\| + \widehat{\vartheta}_2 \|z\| + \mathcal{E}_2) \mathcal{T}_1. \tag{32}
 \end{aligned}$$

Similarly, we get

$$\begin{aligned}
 \Pi_2(y, z)(\tau) & \leq \frac{1}{\Gamma(\varsigma)} \int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\varsigma-1} \\
 & \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \\
 & + \frac{(\log \tau)^{\varsigma-1}}{\vartheta} \left[\vartheta_1 \left\{ \nu_2 \frac{1}{\Gamma(\varrho+\varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho+\varrho_2-1} \right. \right. \\
 & \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho-1} \\
 & \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \\
 & \left. \left. + \frac{1}{\Gamma(\varsigma-\varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma-\varsigma_1-1} \right. \right. \\
 & \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \left. \left. \right\} \right. \\
 & + \vartheta_2 \left\{ \nu_1 \frac{1}{\Gamma(\varsigma+\varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma} \right)^{\varsigma+\varsigma_2-1} \right. \\
 & \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\varsigma-1} \\
 & \times |h_2(\sigma, y(\sigma), z(\sigma)) - h_2(\sigma, 0, 0)| + |h_2(\sigma, 0, 0)| \, d\sigma \\
 & \left. \left. + \frac{1}{\Gamma(\varrho-\varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho-\varrho_1-1} \right. \right. \\
 & \times |h_1(\sigma, y(\sigma), z(\sigma)) - h_1(\sigma, 0, 0)| + |h_1(\sigma, 0, 0)| \, d\sigma \left. \left. \right\} \right] \\
 & \leq (\widehat{\vartheta}_1 \|y\| + \widehat{\vartheta}_2 \|z\| + \mathcal{E}_2) \mathcal{T}_2 + (\widetilde{\vartheta}_1 \|y\| + \widetilde{\vartheta}_2 \|z\| + \mathcal{E}_1) \mathcal{S}_2. \tag{33}
 \end{aligned}$$

Thus, (32) and (33) imply $\|\Pi(y, z)\| \leq \varepsilon$, and hence $\Pi\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$. Now, for $(y_1, z_1), (y_2, z_2) \in \mathcal{Y} \times \mathcal{Z}$ and any $\tau \in \mathcal{H}$, we get

$$\begin{aligned}
 & \left| \Pi_1(y_1, z_1)(\tau) - \Pi_1(y_2, z_2)(\tau) \right| \\
 & \leq \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\varrho-1} \left| h_1(\sigma, y_1(\sigma), z_1(\sigma)) - h_1(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \\
 & \quad + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ v_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma} \right)^{\zeta + \zeta_2 - 1} \right. \right. \\
 & \quad \times \left| h_2(\sigma, y_1(\sigma), z_1(\sigma)) - h_2(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \\
 & \quad + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\zeta-1} \\
 & \quad \times \left| h_2(\sigma, y_1(\sigma), z_1(\sigma)) - h_2(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \\
 & \quad + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} \\
 & \quad \times \left. \left| h_1(\sigma, y_1(\sigma), z_1(\sigma)) - h_1(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \right\} \\
 & \quad + \vartheta_3 \left\{ v_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} \right. \\
 & \quad \times \left| h_1(\sigma, y_1(\sigma), z_1(\sigma)) - h_1(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \\
 & \quad + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho-1} \\
 & \quad \times \left| h_1(\sigma, y_1(\sigma), z_1(\sigma)) - h_1(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \\
 & \quad + \frac{1}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\zeta - \zeta_1 - 1} \\
 & \quad \times \left. \left| h_2(\sigma, y_1(\sigma), z_1(\sigma)) - h_2(\sigma, y_2(\sigma), z_2(\sigma)) \right| d\sigma \right\} \\
 & \leq (\tilde{\vartheta}_1 \|y_1 - y_2\| + \tilde{\vartheta}_2 \|z_1 - z_2\|) \left\{ \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} \right. \\
 & \quad + \frac{(\log T)^{\varrho-1}}{\vartheta} \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} + \vartheta_3 \left\{ v_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right] \\
 & \quad + (\hat{\vartheta}_1 \|y_1 - y_2\| + \hat{\vartheta}_2 \|z_1 - z_2\|) \\
 & \quad \times \left[\frac{(\log T)^{\varrho-1}}{\vartheta} \left[\frac{\vartheta_3 (\log T)^{\zeta - \zeta_1}}{\Gamma(\zeta - \zeta_1 + 1)} + \vartheta_4 \left\{ v_1 \frac{(\log \nu)^{\zeta + \zeta_2}}{\Gamma(\zeta + \zeta_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right\} \right] \right] \\
 & \leq (\mathcal{S}_1(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + \mathcal{T}_1(\hat{\vartheta}_1 + \hat{\vartheta}_2)) (\|y_1 - y_2\| + \|z_1 - z_2\|).
 \end{aligned}$$

Likewise, we obtain

$$\begin{aligned}
 |\Pi_2(y_1, z_1)(\tau) - \Pi_2(y_2, z_2)(\tau)| &\leq (\widehat{\vartheta}_1 \|y_1 - y_2\| + \widehat{\vartheta}_2 \|z_1 - z_2\|) \\
 &\quad \times \left\{ \frac{(\log T)^\zeta}{\Gamma(\zeta + 1)} + \frac{(\log T)^{\zeta-1}}{\vartheta} \left[\frac{\vartheta_1 (\log T)^{\zeta-\varsigma_1}}{\Gamma(\zeta - \varsigma_1 + 1)} \right. \right. \\
 &\quad \left. \left. + \vartheta_2 \left\{ \nu_1 \frac{(\log \nu)^{\zeta+\varsigma_2}}{\Gamma(\zeta + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right\} \right] \right\} \\
 &\quad + (\widetilde{\vartheta}_1 \|y_1 - y_2\| + \widetilde{\vartheta}_2 \|z_1 - z_2\|) \\
 &\quad \times \left\{ \frac{(\log T)^{\varrho-1}}{\vartheta} \left[\frac{\vartheta_2 (\log T)^{\varrho-\varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \right. \\
 &\quad \left. \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \theta)^{\varrho+\varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right] \right\} \\
 &\leq (\mathcal{S}_2(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + \mathcal{T}_2(\widehat{\vartheta}_1 + \widehat{\vartheta}_2))(\|y_1 - y_2\| + \|z_1 - z_2\|),
 \end{aligned}$$

so we obtain

$$\begin{aligned}
 \|\Pi_1(y_1, z_1) - \Pi_1(y_2, z_2)\| &\leq (\mathcal{S}_1(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + \mathcal{T}_1(\widehat{\vartheta}_1 + \widehat{\vartheta}_2)) \\
 &\quad \times (\|y_1 - y_2\| + \|z_1 - z_2\|). \tag{34}
 \end{aligned}$$

In the same way,

$$\begin{aligned}
 \|\Pi_2(y_1, z_1) - \Pi_2(y_2, z_2)\| &\leq (\mathcal{S}_2(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + \mathcal{T}_2(\widehat{\vartheta}_1 + \widehat{\vartheta}_2)) \\
 &\quad \times (\|y_1 - y_2\| + \|z_1 - z_2\|). \tag{35}
 \end{aligned}$$

So, we infer from (34) and (35)

$$\begin{aligned}
 \|\Pi(y_1, z_1) - \Pi(y_2, z_2)\| &\leq (\mathcal{S}_1 + \mathcal{S}_2)(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) \\
 &\quad \times [\|y_1 - y_2\| + \|z_1 - z_2\|].
 \end{aligned}$$

It follows, thus, from the condition

$$(\mathcal{S}_1 + \mathcal{S}_2)(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) < 1,$$

that Π is a contraction operator. This shows that Π is a contraction. Hence, by Banach’s fixed point theorem, the operator Π has a unique fixed point which corresponds to a unique solution of problem (1)–(2). This completes the proof. \square

Theorem 10 (Krasnoselskii, [44, p. 31]) *Let \mathcal{M} be a closed convex non-empty subset of a Banach space \mathcal{E} . Suppose that \mathcal{A} and \mathcal{B} map \mathcal{M} into \mathcal{E} and that*

- (i) $\mathcal{A}y + \mathcal{B}z \in \mathcal{M} \forall x, y \in \mathcal{M}$,
- (ii) \mathcal{A} is compact and continuous.

- (iii) \mathcal{B} is a contraction mapping.

Then there exists $x \in \mathcal{M}$ such that $\mathcal{A}x + \mathcal{B}x = x$.

Theorem 11 *Suppose that (\mathcal{F}_2) hold. In addition, there exist positive constants $\mathcal{V}_1, \mathcal{V}_2$ such that*

$$|h_1(\tau, y, z)| \leq \mathcal{V}_1, \quad |h_2(\tau, y, z)| \leq \mathcal{V}_2, \tag{36}$$

for all $\tau \in \mathcal{H}$ and $y, z \in \mathbb{R}$. Then the BVP (1)–(2) has at least one solution on \mathcal{H} , if

$$\frac{(\log T)^\varrho (\tilde{\vartheta}_1 + \tilde{\vartheta}_2)}{\Gamma(\varrho + 1)} + \frac{(\log T)^\varsigma (\widehat{\vartheta}_1 + \widehat{\vartheta}_2)}{\Gamma(\varsigma + 1)} < 1. \tag{37}$$

Proof Let us define a ball $\mathcal{B}_\varepsilon = \{(y, z) \in \mathcal{Y} \times \mathcal{Z} : \|(y, z)\| \leq \varepsilon\}$ to be closed as follows:

$$\begin{aligned} & \Pi_{1,1}(y, z)(\tau) \\ &= \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \frac{\nu_1}{\Gamma(\varsigma + \varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma} \right)^{\varsigma + \varsigma_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \right. \\ & \quad + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\varsigma-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\ & \quad \left. \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\ & \quad + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\ & \quad + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\ & \quad \left. \left. - \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma - \varsigma_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right], \\ & \Pi_{1,2}(y, z)(\tau) = \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma} \right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma, \end{aligned}$$

and

$$\begin{aligned} & \Pi_{2,1}(y, z)(\tau) \\ &= \frac{(\log \tau)^{\varsigma-1}}{\vartheta} \left[\vartheta_1 \left\{ \frac{\nu_2}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \right. \\ & \quad + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\ & \quad \left. \left. - \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma - \varsigma_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & + \vartheta_2 \left\{ \nu_1 \frac{1}{\Gamma(\varsigma + \varsigma_2)} \int_1^{\nu} \left(\log \frac{\nu}{\sigma} \right)^{\varsigma + \varsigma_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\varsigma - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\}, \\
 \Pi_{2,2}(y, z)(\tau) & = \frac{1}{\Gamma(\varsigma)} \int_1^{\tau} \left(\log \frac{\tau}{\sigma} \right)^{\varsigma - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma.
 \end{aligned}$$

Note that

$$\Pi_1(y, z)(\tau) = \Pi_{1,1}(y, z)(\tau) + \Pi_{1,2}(y, z)(\tau),$$

$$\Pi_2(y, z)(\tau) = \Pi_{2,1}(y, z)(\tau) + \Pi_{2,2}(y, z)(\tau),$$

on \mathcal{B}_ε is a closed, bounded, and convex subset of Banach space $\mathcal{Y} \times \mathcal{Z}$ and that we have a ball \mathcal{B}_ε . Now, let us choose

$$\varepsilon \geq \max\{\mathcal{S}_1 \mathcal{V}_1 + \mathcal{T}_1 \mathcal{V}_2, \mathcal{S}_2 \mathcal{V}_1 + \mathcal{T}_2 \mathcal{V}_2\},$$

and demonstrate that $\Pi \mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$; to test Krasnoselskii theorem’s condition (i), if we set $y = (y_1, y_2), z = (z_1, z_2) \in \mathcal{B}_\varepsilon$, and using condition (36), we get

$$\begin{aligned}
 & |\Pi_{1,1}(y, z)(\tau) + \Pi_{1,2}(y, z)(\tau)| \\
 & \leq \frac{1}{\Gamma(\varrho)} \int_1^{\tau} \left(\log \frac{\tau}{\sigma} \right)^{\varrho - 1} \mathcal{V}_1 \, d\sigma \\
 & + \frac{(\log \tau)^{\varrho - 1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\varsigma + \varsigma_2)} \int_1^{\nu} \left(\log \frac{\nu}{\sigma} \right)^{\varsigma + \varsigma_2 - 1} \mathcal{V}_2 \, d\sigma \right. \right. \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\varsigma - 1} \mathcal{V}_2 \, d\sigma \\
 & \left. + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} \mathcal{V}_1 \, d\sigma \right\} \\
 & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^{\theta} \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} \mathcal{V}_1 \, d\sigma \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho - 1} \mathcal{V}_1 \, d\sigma \\
 & \left. + \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma - \varsigma_1 - 1} \mathcal{V}_2 \, d\sigma \right\} \\
 & \leq \mathcal{S}_1 \mathcal{V}_1 + \mathcal{T}_1 \mathcal{V}_2 \leq \varepsilon.
 \end{aligned}$$

Similarly, we find that

$$|\Pi_{2,1}(y, z)(\tau) + \Pi_{2,2}(y, z)(\tau)| \leq \mathcal{S}_2 \mathcal{V}_1 + \mathcal{T}_2 \mathcal{V}_2 \leq \varepsilon.$$

The two above inequalities contribute to the assumption that

$$\Pi_1(y, z) + \Pi_2(\hat{y}, \hat{z}) \in \mathcal{B}_\varepsilon.$$

So we define that operator $(\Pi_{1,2}, \Pi_{2,2})$ obeys condition (iii) of the Krasnoselskii theorem that satisfies contraction. For $(y_1, z_1), (y_2, z_2) \in \mathcal{B}_\varepsilon$, we have

$$\begin{aligned} & |\Pi_{1,2}(y_1, z_1)(\tau) - \Pi_{1,2}(y_2, z_2)(\tau)| \\ & \leq \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} |h_1(\sigma, y_1(\sigma), z_1(\sigma)) - h_1(\sigma, y_2(\sigma), z_2(\sigma))| \, d\sigma \\ & \leq \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} (\tilde{\vartheta}_1 \|y_1 - y_2\| + \tilde{\vartheta}_2 \|z_1 - z_2\|) \end{aligned} \tag{38}$$

and

$$\begin{aligned} & |\Pi_{2,2}(y_1, z_1)(\tau) - \Pi_{2,2}(y_2, z_2)(\tau)| \\ & \leq \frac{1}{\Gamma(\zeta)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\zeta-1} |h_2(\sigma, y_1(\sigma), z_1(\sigma)) - h_2(\sigma, y_2(\sigma), z_2(\sigma))| \, d\sigma \\ & \leq \frac{(\log T)^\zeta}{\Gamma(\zeta + 1)} (\hat{\vartheta}_1 \|y_1 - y_2\| + \hat{\vartheta}_2 \|z_1 - z_2\|). \end{aligned} \tag{39}$$

From (38) and (39) it follows that

$$\begin{aligned} & |(\Pi_{1,2}, \Pi_{2,2})(y_1, z_1)(\tau) - (\Pi_{1,2}, \Pi_{2,2})(y_2, z_2)(\tau)| \\ & \leq \left(\frac{(\log T)^\varrho (\tilde{\vartheta}_1 + \tilde{\vartheta}_2)}{\Gamma(\varrho + 1)} + \frac{(\log T)^\zeta (\hat{\vartheta}_1 + \hat{\vartheta}_2)}{\Gamma(\zeta + 1)} \right) \\ & \quad \times (\|y_1 - y_2\| + \|z_1 - z_2\|), \end{aligned}$$

which is a contraction by (37). Hence Krasnoselskii theorem’s condition (iii) is satisfied. Next we can demonstrate that the operator $(\Pi_{1,1}, \Pi_{2,1})$ fulfills the Krasnoselskii theorem’s condition (ii). By applying the continuity of the $h_1, h_2 : \mathcal{H} \times \mathbb{R} \times \mathbb{R}$ functions, we can infer that the $(\Pi_{1,1}, \Pi_{2,1})$ operator is continuous. For each $(y, z) \in \mathcal{B}_\varepsilon$ we have

$$\begin{aligned} |\Pi_{1,1}(y, z)(\tau)| & \leq \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\zeta + \zeta_2 - 1} \nu_2 \, d\sigma \right. \right. \\ & \quad + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} \nu_2 \, d\sigma \\ & \quad \left. \left. + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} \nu_1 \, d\sigma \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} \nu_1 \, d\sigma \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho - 1} \nu_1 \, d\sigma \\
 & \left. + \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma - \varsigma_1 - 1} \nu_2 \, d\sigma \right\} \\
 & = \widehat{\Delta}_1
 \end{aligned}$$

and

$$\begin{aligned}
 |\Pi_{2,1}(y, z)(\tau)| & \leq \frac{(\log \tau)^{\varsigma - 1}}{\vartheta} \left[\vartheta_1 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma} \right)^{\varrho + \varrho_2 - 1} \nu_1 \, d\sigma \right. \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma} \right)^{\varrho - 1} \nu_1 \, d\sigma \\
 & + \left. \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varsigma - \varsigma_1 - 1} \nu_2 \, d\sigma \right\} \\
 & + \vartheta_2 \left\{ \nu_1 \frac{1}{\Gamma(\varsigma + \varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma} \right)^{\varsigma + \varsigma_2 - 1} \nu_2 \, d\sigma \right. \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma} \right)^{\varsigma - 1} \nu_2 \, d\sigma \\
 & \left. + \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma} \right)^{\varrho - \varrho_1 - 1} \nu_1 \, d\sigma \right\} \\
 & = \widehat{\Delta}_2,
 \end{aligned}$$

which leads to

$$\left\| (\Pi_{1,1}, \Pi_{2,1})(y, z) \right\| \leq \widehat{\Delta}_1 + \widehat{\Delta}_2.$$

Therefore the set $(\Pi_{1,1}, \Pi_{2,1})\mathcal{B}_\varepsilon$ is bounded uniformly. We shall demonstrate in the next phase that the $(\Pi_{1,1}, \Pi_{2,1})\mathcal{B}_\varepsilon$ set is equicontinuous. For $\tau_1, \tau_2 \in \mathcal{H}$ with $\tau_1 < \tau_2$ and for any $(y, z) \in \mathcal{B}_\varepsilon$ we obtain

$$\begin{aligned}
 \left| \Pi_{1,1}(y, z)(\tau_2) - \Pi_{1,1}(y, z)(\tau_1) \right| & \leq \frac{(\log \tau_2)^{\varrho - 1} - (\log \tau_1)^{\varrho - 1}}{\vartheta} \\
 & \times \left\{ \nu_1 \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} + \vartheta_3 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} \right. \right. \right. \\
 & + \left. \left. \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right] + \nu_2 \left[\frac{\vartheta_3 (\log T)^{\varsigma - \varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \\
 & \left. \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right] \right\}.
 \end{aligned}$$

In a similar manner, we can get

$$\begin{aligned} |\Pi_{2,1}(y, z)(\tau_2) - \Pi_{2,1}(y, z)(\tau_1)| &\leq \frac{(\log \tau_2)^{\zeta-1} - (\log \tau_1)^{\zeta-1}}{\vartheta} \\ &\times \left\{ \mathcal{V}_2 \left[\frac{\vartheta_1 (\log T)^{\zeta-\zeta_1}}{\Gamma(\zeta - \zeta_1 + 1)} + \vartheta_2 \left\{ \nu_1 \frac{(\log \nu)^{\zeta+\zeta_2}}{\Gamma(\zeta + \zeta_2 + 1)} \right. \right. \right. \\ &+ \left. \left. \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta + 1)} \right\} \right] + \mathcal{V}_1 \left[\frac{\vartheta_2 (\log T)^{\varrho-\varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\ &\left. \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \theta)^{\varrho+\varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right] \right\}. \end{aligned}$$

Thus

$$|(\Pi_{2,1,1}, \Pi_{2,1})(y, z)(\tau_2) - (\Pi_{1,1}, \Pi_{2,1})(y, z)(\tau_1)|$$

tends to zero as $\tau_1 \rightarrow \tau_1$ independent of $(y, z) \in \mathcal{B}_\epsilon$. Therefore the set $(\Pi_{1,1}, \Pi_{2,1})\mathcal{B}_\epsilon$ is equicontinuous. Therefore it implies from the Arzelà–Ascoli theorem that the operator $(\Pi_{1,1}, \Pi_{2,1})$ is compact on \mathcal{B}_ϵ . We conclude from Krasnoselskii fixed point theorem’s statement that the problem (1)–(2) has at least one solution on \mathcal{H} . \square

4 Stability results for the problem (1)–(2)

Let us define nonlinear operators $\mathcal{G}_1, \mathcal{G}_2 \in C(\mathcal{H}, \mathbb{R}) \times C(\mathcal{H}, \mathbb{R}) \rightarrow C(\mathcal{H}, \mathbb{R})$; where Π_1 and Π_2 are defined by (17) and (18):

$$\begin{cases} {}^H\mathcal{D}^\varrho y(\tau) - h_1(\tau, y(\tau), z(\tau)) = \mathcal{G}_1(y, z)(\tau), \\ {}^H\mathcal{D}^\zeta z(\tau) - h_2(\tau, y(\tau), z(\tau)) = \mathcal{G}_2(y, z)(\tau), \end{cases}$$

for $\tau \in \mathcal{H}$. For some $\pi_1, \pi_2 > 0$, we consider the following inequalities:

$$\|\mathcal{G}_1(y, z)\| \leq \pi_1, \quad \|\mathcal{G}_2(y, z)\| \leq \pi_2. \tag{40}$$

Definition 12 The coupled system (1)–(2) is said to be stable in the Hyers–Ulam sense, if $\mathcal{K}_1, \mathcal{K}_2 > 0$ exist such that there is a unique solution $(y, z) \in C(\mathcal{H}, \mathbb{R}) \times C(\mathcal{H}, \mathbb{R})$ of problems (1)–(2) with

$$\|(y, z) - (\widehat{y}, \widehat{z})\| \leq \mathcal{K}_1\pi_1 + \mathcal{K}_2\pi_2,$$

for every solution $(\widehat{y}, \widehat{z})$ belongs to $C(\mathcal{H}, \mathbb{R}) \times C(\mathcal{H}, \mathbb{R})$ of inequality (40).

Theorem 13 Suppose that (\mathcal{F}_2) hold. Then the BVP (1)–(2) is Hyers–Ulam stable.

Proof Let $(y, z) \in C(\mathcal{H}, \mathbb{R}) \times C(\mathcal{H}, \mathbb{R})$ be the (1)–(2) the solution of the problems that satisfy (17) and (18). Let $(\widehat{y}, \widehat{z})$ be any satisfying solution (40):

$${}^H\mathcal{D}^\varrho y(\tau) = h_1(\tau, y(\tau), z(\tau)) + \mathcal{G}_1(y, z)(\tau),$$

$${}^H\mathcal{D}^\zeta z(\tau) = h_2(\tau, y(\tau), z(\tau)) + \mathcal{G}_2(y, z)(\tau),$$

for $\tau \in \mathcal{H}$. Therefore,

$$\begin{aligned} \widehat{y}(\tau) = & \Pi_1(\widehat{y}, \widehat{z})(\tau) + \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} \mathcal{G}_1(y, z)(\sigma) \, d\sigma \\ & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\varsigma + \varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\varsigma + \varsigma_2 - 1} \mathcal{G}_2(y, z)(\sigma) \, d\sigma \right. \right. \\ & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varsigma-1} \mathcal{G}_2(y, z)(\sigma) \, d\sigma \\ & - \left. \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} \mathcal{G}_1(y, z)(\sigma) \, d\sigma \right\} \\ & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} \mathcal{G}_1(y, z)(\sigma) \, d\sigma \right. \\ & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma}\right)^{\varrho-1} \mathcal{G}_1(y, z)(\sigma) \, d\sigma \\ & \left. \left. - \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varsigma - \varsigma_1 - 1} \mathcal{G}_2(y, z)(\sigma) \, d\sigma \right\} \right]. \end{aligned}$$

It follows that

$$\begin{aligned} |\Pi_1(\widehat{y}, \widehat{z})(\tau) - \widehat{y}(\tau)| \leq & \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} \pi_1 \, d\sigma \\ & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\varsigma + \varsigma_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\varsigma + \varsigma_2 - 1} \pi_2 \, d\sigma \right. \right. \\ & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\varsigma)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varsigma-1} \pi_2 \, d\sigma \\ & + \left. \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} \pi_1 \, d\sigma \right\} \\ & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} \pi_1 \, d\sigma \right. \\ & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\gamma_j} \left(\log \frac{\gamma_j}{\sigma}\right)^{\varrho-1} \pi_1 \, d\sigma \\ & \left. \left. + \frac{1}{\Gamma(\varsigma - \varsigma_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varsigma - \varsigma_1 - 1} \pi_2 \, d\sigma \right\} \right] \\ \leq & \left[\frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} + \frac{(\log T)^{\varrho-1}}{\vartheta} \left(\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \right. \end{aligned}$$

$$\begin{aligned}
 & + \vartheta_3 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \pi_1 \\
 & + \left[\frac{(\log T)^{\varrho-1}}{\vartheta} \left(\frac{\vartheta_3 (\log T)^{\varrho-\varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right) \right] \mu_2 \\
 & \leq \mathcal{S}_1 \pi_1 + \mathcal{T}_1 \pi_2.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 |\Pi_2(\widehat{y}, \widehat{z})(\tau) - \widehat{z}(\tau)| & \leq \left[\frac{(\log T)^\varsigma}{\Gamma(\varsigma + 1)} + \frac{(\log T)^{\varsigma-1}}{\vartheta} \left(\frac{\vartheta_1 (\log T)^{\varsigma-\varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_2 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right) \right] \pi_2 \\
 & + \frac{(\log T)^2}{\nu} \left[\frac{(\log T)^{\varrho-1}}{\vartheta} \left(\frac{\vartheta_2 (\log T)^{\varrho-\varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \right. \\
 & \left. \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right) \right] \pi_1 \\
 & \leq \mathcal{T}_2 \pi_2 + \mathcal{S}_2 \pi_1,
 \end{aligned}$$

where in (19)–(22) we describe $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1$ and \mathcal{T}_2 . The Π operator, given by (17) and (18), can therefore be excluded as follows from the fixed point property. We have

$$\begin{aligned}
 |y(\tau) - \widehat{y}(\tau)| & = |y(\tau) - \Pi_1(\widehat{y}, \widehat{z})(\tau) + \Pi_1(\widehat{y}, \widehat{z})(\tau) - \widehat{y}(\tau)| \\
 & \leq |\Pi_1(y, z)(\tau) - \Pi_1(\widehat{y}, \widehat{z})(\tau)| + |\Pi_1(\widehat{y}, \widehat{z})(\tau) - \widehat{y}(\tau)| \\
 & \leq (\mathcal{S}_1 \widehat{\vartheta}_1 + \mathcal{T}_1 \widehat{\vartheta}_1) + (\mathcal{S}_1 \widetilde{\vartheta}_2 + \mathcal{T}_1 \widehat{\vartheta}_2) \| (y, z) - (\widehat{y} - \widehat{z}) \| \\
 & \quad + \mathcal{S}_1 \pi_1 + \mathcal{T}_1 \pi_2, \tag{41}
 \end{aligned}$$

$$\begin{aligned}
 |z(\tau) - \widehat{z}(\tau)| & = |z(\tau) - \Pi_2(\widehat{y}, \widehat{z})(\tau) + \Pi_2(\widehat{y}, \widehat{z})(\tau) - \widehat{z}(\tau)| \\
 & \leq |\Pi_2(y, z)(\tau) - \Pi_2(\widehat{y}, \widehat{z})(\tau)| + |\Pi_2(\widehat{y}, \widehat{z})(\tau) - \widehat{z}(\tau)| \\
 & \leq (\mathcal{T}_2 \widehat{\vartheta}_1 + \mathcal{S}_2 \widetilde{\vartheta}_1) + (\mathcal{T}_2 \widehat{\vartheta}_2 + \mathcal{S}_2 \widetilde{\vartheta}_2) \| (y, z) - (\widehat{y} - \widehat{z}) \| \\
 & \quad + \mathcal{T}_2 \pi_2 + \mathcal{S}_2 \pi_1. \tag{42}
 \end{aligned}$$

From (41) and (42) it follows that

$$\begin{aligned}
 \| (y, z) - (\widehat{y} - \widehat{z}) \| & \leq (\mathcal{S}_1 + \mathcal{S}_2) \pi_1 + (\mathcal{T}_1 + \mathcal{T}_2) \pi_2 + (\mathcal{S}_1 + \mathcal{S}_2) (\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) \\
 & \quad + (\mathcal{T}_1 + \mathcal{T}_2) (\widehat{\vartheta}_1 + \widehat{\vartheta}_2) \| (y, z) - (\widehat{y} - \widehat{z}) \|,
 \end{aligned}$$

$$\begin{aligned} \|(y, z) - (\widehat{y} - \widehat{z})\| &\leq \frac{(\mathcal{S}_1 + \mathcal{S}_2)\pi_1 + (\mathcal{T}_1 + \mathcal{T}_2)\pi_2}{1 - ((\mathcal{S}_1 + \mathcal{S}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2))} \\ &\leq \mathcal{K}_1\pi_1 + \mathcal{K}_2\pi_2, \end{aligned}$$

with

$$\begin{aligned} \mathcal{K}_1 &= \frac{(\mathcal{S}_1 + \mathcal{S}_2)}{1 - ((\mathcal{S}_1 + \mathcal{S}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2))}, \\ \mathcal{K}_2 &= \frac{(\mathcal{T}_1 + \mathcal{T}_2)}{1 - ((\mathcal{S}_1 + \mathcal{S}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2))}. \end{aligned}$$

Therefore, the BVP (1)–(2) is Hyers–Ulam stable. □

5 Variants of the problem

Further, we can solve some problems similar to problem (1)–(2) by using the methodology employed in the previous section. For example, we consider two new problems by replacing the condition. Note that the boundary conditions (2) include the strips of the different lengths when modifying the strips in boundary conditions like the same lengths (2), then the problem reduces to the form

$$\begin{cases} y(1) = y'(1) = 0, \\ {}^H\mathcal{D}^{q_1}y(T) = v_1 {}^H\mathcal{I}^{s_2}z(v) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, \\ {}^H\mathcal{D}^{s_1}z(T) = v_2 {}^H\mathcal{I}^{q_2}y(v) + \omega_2 \sum_{j=1}^{k-2} \varpi_j y(\delta_j), \end{cases} \tag{43}$$

with

$$1 < v < \delta_1 < \delta_2 < \dots < \delta_{k-2} < T.$$

On the other hand, the multi-point boundary conditions in (2) contain different multi-points in contrast to the multi-point boundary conditions in (43). We have

$$\begin{cases} y(1) = y'(1) = 0, \\ {}^H\mathcal{D}^{q_1}y(T) = v_1 {}^H\mathcal{I}^{s_2}z(v) + \omega_1 \sum_{j=1}^{k-2} \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, \\ {}^H\mathcal{D}^{s_1}z(T) = v_2 {}^H\mathcal{I}^{q_2}y(\theta) + \omega_2 \sum_{j=1}^{k-2} \varpi_j y(\delta_j), \end{cases} \tag{44}$$

and

$$1 < \theta < v < \delta_1 < \delta_2 < \dots < \delta_{k-2} < T.$$

Concerning problem (1) with (43) instead of (2), we obtained the operator defined by

$$\Pi(y, z)(\tau) = (\Pi_1(y, z)(\tau), \Pi_2(y, z)(\tau)), \tag{45}$$

where

$$\begin{aligned}
 \Pi_1(y, z)(\tau) = & \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \right. \right. \\
 & \times \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\zeta + \zeta_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\
 & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \zeta_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right], \tag{46}
 \end{aligned}$$

$$\begin{aligned}
 \Pi_2(y, z)(\tau) = & \frac{1}{\Gamma(\zeta)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \frac{(\log \tau)^{\zeta-1}}{\vartheta} \left[\vartheta_1 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \right. \right. \\
 & \times \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \zeta_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\
 & + \vartheta_2 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\zeta + \zeta_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right], \tag{47}
 \end{aligned}$$

where

$$\begin{aligned}
 \vartheta_1 &= \frac{\Gamma(\varrho)}{\Gamma(\varrho - \varrho_1)} (\log T)^{\varrho - \varrho_1 - 1}, \\
 \vartheta_2 &= \frac{\Gamma(\varrho)}{\Gamma(\varrho + \varrho_2)} (\log \nu)^{\varrho + \varrho_2 - 1} + \omega_2 \sum_{j=1}^{k-2} \varpi_j (\log \delta_j)^{\varrho - 1}, \\
 \vartheta_3 &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma + \varsigma_2)} (\log \nu)^{\varsigma + \varsigma_2 - 1} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j (\log \delta_j)^{\varsigma - 1}, \\
 \vartheta_4 &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma - \varsigma_1)} (\log T)^{\varsigma - \varsigma_1 - 1},
 \end{aligned} \tag{48}$$

$\vartheta = \vartheta_1 \vartheta_4 - \vartheta_2 \vartheta_3$, and

$$\begin{aligned}
 \mathcal{S}_1 &= \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} + \frac{(\log T)^{\varrho - 1}}{\vartheta} \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\
 &\quad \left. + \vartheta_3 \left\{ \nu_2 \frac{(\log \nu)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \delta_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right],
 \end{aligned} \tag{49}$$

$$\begin{aligned}
 \mathcal{T}_1 &= \frac{(\log T)^{\varrho - 1}}{\vartheta} \left[\frac{\vartheta_3 (\log T)^{\varsigma - \varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \\
 &\quad \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right],
 \end{aligned} \tag{50}$$

$$\begin{aligned}
 \mathcal{S}_2 &= \frac{(\log T)^{\varsigma - 1}}{\vartheta} \left[\frac{\vartheta_2 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\
 &\quad \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \nu)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \delta_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right],
 \end{aligned} \tag{51}$$

$$\begin{aligned}
 \mathcal{T}_2 &= \frac{(\log T)^\varsigma}{\Gamma(\varsigma + 1)} + \frac{(\log T)^{\varsigma - 1}}{\vartheta} \left[\frac{\vartheta_1 (\log T)^{\varsigma - \varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \\
 &\quad \left. + \vartheta_2 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right].
 \end{aligned} \tag{52}$$

Concerning the problem (1) with (44) instead of (2), we obtain the operator defined by

$$\Pi(y, z)(\tau) = (\Pi_1(y, z)(\tau), \Pi_2(y, z)(\tau)), \tag{53}$$

where

$$\begin{aligned}
 \Pi_1(y, z)(\tau) = & \frac{1}{\Gamma(\varrho)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \frac{(\log \tau)^{\varrho-1}}{\vartheta} \left[\vartheta_4 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \right. \right. \\
 & \times \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\zeta + \zeta_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\
 & + \vartheta_3 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \zeta_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right], \tag{54}
 \end{aligned}$$

$$\begin{aligned}
 \Pi_2(y, z)(\tau) = & \frac{1}{\Gamma(\zeta)} \int_1^\tau \left(\log \frac{\tau}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \frac{(\log \tau)^{\zeta-1}}{\vartheta} \left[\vartheta_1 \left\{ \nu_2 \frac{1}{\Gamma(\varrho + \varrho_2)} \right. \right. \\
 & \times \int_1^\theta \left(\log \frac{\theta}{\sigma}\right)^{\varrho + \varrho_2 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{1}{\Gamma(\varrho)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\varrho-1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\zeta - \zeta_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\zeta - \zeta_1 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right. \\
 & + \vartheta_2 \left\{ \nu_1 \frac{1}{\Gamma(\zeta + \zeta_2)} \int_1^\nu \left(\log \frac{\nu}{\sigma}\right)^{\zeta + \zeta_2 - 1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right. \\
 & + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{1}{\Gamma(\zeta)} \int_1^{\delta_j} \left(\log \frac{\delta_j}{\sigma}\right)^{\zeta-1} h_2(\sigma, y(\sigma), z(\sigma)) \, d\sigma \\
 & \left. \left. - \frac{1}{\Gamma(\varrho - \varrho_1)} \int_1^T \left(\log \frac{T}{\sigma}\right)^{\varrho - \varrho_1 - 1} h_1(\sigma, y(\sigma), z(\sigma)) \, d\sigma \right\} \right], \tag{55}
 \end{aligned}$$

where

$$\vartheta_1 = \frac{\Gamma(\varrho)}{\Gamma(\varrho - \varrho_1)} (\log T)^{\varrho - \varrho_1 - 1},$$

$$\begin{aligned} \vartheta_2 &= \frac{\Gamma(\varrho)}{\Gamma(\varrho + \varrho_2)} (\log \theta)^{\varrho + \varrho_2 - 1} + \omega_2 \sum_{j=1}^{k-2} \varpi_j (\log \delta_j)^{\varrho - 1}, \\ \vartheta_3 &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma + \varsigma_2)} (\log \nu)^{\varsigma + \varsigma_2 - 1} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j (\log \delta_j)^{\varsigma - 1}, \\ \vartheta_4 &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma - \varsigma_1)} (\log T)^{\varsigma - \varsigma_1 - 1}, \end{aligned} \tag{56}$$

$\vartheta = \vartheta_1 \vartheta_4 - \vartheta_2 \vartheta_3$, and

$$\begin{aligned} \mathcal{S}_1 &= \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} + \frac{(\log T)^{\varrho - 1}}{\vartheta} \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\ &\quad \left. + \vartheta_3 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \delta_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right], \end{aligned} \tag{57}$$

$$\begin{aligned} \mathcal{T}_1 &= \frac{(\log T)^{\varrho - 1}}{\vartheta} \left[\frac{\vartheta_3 (\log T)^{\varsigma - \varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \\ &\quad \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right], \end{aligned} \tag{58}$$

$$\begin{aligned} \mathcal{S}_2 &= \frac{(\log T)^{\varsigma - 1}}{\vartheta} \left[\frac{\vartheta_2 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\ &\quad \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \delta_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right], \end{aligned} \tag{59}$$

$$\begin{aligned} \mathcal{T}_2 &= \frac{(\log T)^\varsigma}{\Gamma(\varsigma + 1)} + \frac{(\log T)^{\varsigma - 1}}{\vartheta} \left[\frac{\vartheta_1 (\log T)^{\varsigma - \varsigma_1}}{\Gamma(\varsigma - \varsigma_1 + 1)} \right. \\ &\quad \left. + \vartheta_2 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma + \varsigma_2}}{\Gamma(\varsigma + \varsigma_2 + 1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma + 1)} \right\} \right]. \end{aligned} \tag{60}$$

Now for the problems (1)–(43), (1)–(44), we state the results of existence, uniqueness, and stability. We are not providing the evidence as it is similar to Sects. 3 and 4.

Theorem 14 *Suppose that (\mathcal{F}_1) hold. If*

$$\begin{aligned} \varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2) &< 1, \\ \varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2) &< 1. \end{aligned} \tag{61}$$

Then there exists at least one solution for problem (1)–(43) on \mathcal{H} , where $\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2$, and \mathcal{T}_2 are given by (49) and (52), respectively.

Theorem 15 *Suppose that (\mathcal{F}_2) hold. Then the BVP (1)–(43) has a unique solution on \mathcal{H} , provided that*

$$(\mathcal{S}_1 + \mathcal{S}_2)(\widetilde{\vartheta}_1 + \widetilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) < 1, \tag{62}$$

where $\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2,$ and \mathcal{T}_2 are given by (49) and (52).

Theorem 16 *Suppose that (\mathcal{F}_2) hold. In addition, there exist positive constants $\mathcal{V}_1, \mathcal{V}_2$ such that, for all $\tau \in \mathcal{H}$ and $y, z \in \mathbb{R}$,*

$$|h_1(\tau, y, z)| \leq \mathcal{V}_1, \quad |h_2(\tau, y, z)| \leq \mathcal{V}_2. \tag{63}$$

Then the BVP (1)–(43) has at least one solution on \mathcal{H} , if

$$\frac{(\log T)^\varrho (\tilde{\vartheta}_1 + \tilde{\vartheta}_2)}{\Gamma(\varrho + 1)} + \frac{(\log T)^\varsigma (\widehat{\vartheta}_1 + \widehat{\vartheta}_2)}{\Gamma(\varsigma + 1)} < 1. \tag{64}$$

Theorem 17 *Suppose that (\mathcal{F}_2) hold. Then the BVP (1)–(43) is Hyers–Ulam stable.*

Theorem 18 *Suppose that (\mathcal{F}_1) hold. If*

$$\begin{aligned} \varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2) &< 1, \\ \varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2) &< 1. \end{aligned} \tag{65}$$

Then there exists at least one solution for problem (1)–(44) on \mathcal{H} , where $\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2,$ and \mathcal{T}_2 are given by (57) and (60), respectively.

Theorem 19 *Suppose that (\mathcal{F}_2) hold. Then the BVP (1)–(44) has a unique solution on \mathcal{H} , provided that*

$$(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\widehat{\vartheta}_1 + \widehat{\vartheta}_2) < 1, \tag{66}$$

where $\mathcal{S}_1, \mathcal{T}_1, \mathcal{S}_2,$ and \mathcal{T}_2 are given by (57) and (60).

Theorem 20 *Suppose that (\mathcal{F}_2) hold. In addition, there exist positive constants $\mathcal{V}_1, \mathcal{V}_2$ such that, for all $\tau \in \mathcal{H}$ and $y, z \in \mathbb{R}$,*

$$|h_1(\tau, y, z)| \leq \mathcal{V}_1, \quad |h_2(\tau, y, z)| \leq \mathcal{V}_2. \tag{67}$$

Then the BVP (1)–(44) has at least one solution on \mathcal{H} , if

$$\frac{(\log T)^\varrho (\tilde{\vartheta}_1 + \tilde{\vartheta}_2)}{\Gamma(\varrho + 1)} + \frac{(\log T)^\varsigma (\widehat{\vartheta}_1 + \widehat{\vartheta}_2)}{\Gamma(\varsigma + 1)} < 1. \tag{68}$$

Theorem 21 *Suppose that (\mathcal{F}_2) hold. Then the BVP (1)–(44) is Hyers–Ulam stable.*

6 Applications with illustrative examples

Example 1 Consider the following coupled system of Hadamard-type FDEs:

$$\begin{cases} {}^H\mathcal{D}^{\frac{121}{50}} y(\tau) = h_1(\tau, y(\tau), z(\tau)), \\ {}^H\mathcal{D}^{\frac{71}{25}} z(\tau) = h_2(\tau, y(\tau), z(\tau)), \end{cases} \tag{69}$$

for $\tau \in [1, 2]$, equipped with coupled boundary conditions:

$$\begin{cases} y(1) = y'(1) = 0, \\ {}^H\mathcal{D}^{\frac{11}{25}}y(T) = \frac{3}{125} {}^H\mathcal{I}^{\frac{11}{20}}z\left(\frac{77}{50}\right) + \frac{159}{250} \sum_{j=1}^4 \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, \\ {}^H\mathcal{D}^{\frac{17}{20}}z(T) = \frac{17}{500} {}^H\mathcal{I}^{\frac{19}{20}}y\left(\frac{97}{50}\right) + \frac{7}{200} \sum_{j=1}^4 \varpi_j y(\gamma_j). \end{cases} \tag{70}$$

Here, $\varrho = \frac{121}{50} \in (2, 3]$, $\varsigma = \frac{71}{25} \in (2, 3]$, $\varrho_1 = \frac{11}{25} \in (0, 1]$, $\varsigma_1 = \frac{17}{20} \in (0, 1]$, $\varrho_2 = \frac{19}{20} \in (0, 1)$, $\varsigma_2 = \frac{11}{20} \in (0, 1)$, $\epsilon_1 = \frac{23}{200}$, $\epsilon_2 = \frac{32}{125}$, $\epsilon_3 = \frac{179}{500}$, $\epsilon_4 = \frac{23}{50}$, $\varpi_1 = \frac{13}{100}$, $\varpi_2 = \frac{6}{25}$, $\varpi_3 = \frac{19}{50}$, $\varpi_4 = \frac{53}{125}$, $\delta_1 = \frac{49}{40}$, $\delta_2 = \frac{723}{500}$, $\delta_3 = \frac{417}{250}$, $\delta_4 = \frac{47}{25}$, $\gamma_1 = \frac{267}{200}$, $\gamma_2 = \frac{389}{250}$, $\gamma_3 = \frac{889}{500}$, $\gamma_4 = \frac{379}{200}$, $\omega_1 = \frac{159}{200}$, $\omega_2 = \frac{7}{200}$, $\nu_1 = \frac{3}{125}$, $\nu_2 = \frac{17}{500}$, $T = 2$, $\nu = \frac{77}{50}$, $\theta = \frac{369}{200}$ and we have the continuous functions

$$h_1(\tau, y(\tau), z(\tau)) = \frac{1}{6(\tau^2 + 9)} \left(3\tau + \frac{|y(\tau)|}{1 + |y(\tau)|} + \frac{1}{5} \cos(z(\tau)) \right),$$

$$h_2(\tau, y(\tau), z(\tau)) = \frac{1}{16\tau} \left(\frac{\sqrt{\tau}}{3} + \cos(y(\tau)) + \frac{1}{5} \frac{|z(\tau)|}{1 + |z(\tau)|} \right).$$

Then by using (24), for $\tau \in \mathcal{H}$, we have

$$\begin{aligned} |h_1(\tau, w_1, w_2)| &= \left| \frac{1}{6(\tau^2 + 9)} \left(3\tau + \frac{|w_1(\tau)|}{1 + |w_1(\tau)|} + \frac{1}{5} \cos(w_2(\tau)) \right) \right| \\ &\leq \frac{1\tau}{2(\tau^2 + 9)} + \frac{|w_1(\tau)|}{6(\tau^2 + 9)(1 + |w_1(\tau)|)} + \frac{|\cos(w_2(\tau))|}{30(\tau^2 + 9)} \\ &\leq \frac{1}{20} + \frac{|w_1(\tau)|}{60(1 + |w_1(\tau)|)} + \frac{|w_2(\tau)|}{300} \end{aligned}$$

and

$$\begin{aligned} |h_2(\tau, w_1, w_2)| &= \left| \frac{1}{16\tau} \left(\frac{\sqrt{\tau}}{3} + \cos(w_1(\tau)) + \frac{1}{5} \frac{|w_2(\tau)|}{1 + |w_2(\tau)|} \right) \right| \\ &\leq \frac{1}{48\sqrt{\tau}} + \frac{|\cos(w_1(\tau))|}{16\tau} + \frac{|w_2(\tau)|}{80\tau(1 + |w_2(\tau)|)} \\ &\leq \frac{1}{48} + \frac{|w_1(\tau)|}{16} + \frac{|w_2(\tau)|}{80(1 + |w_2(\tau)|)}. \end{aligned}$$

Therefore, the functions h_1 and h_2 obviously satisfy the (\mathcal{F}_1) condition with

$$\varphi_0 = \frac{1}{20}, \quad \varphi_1 = \frac{1}{60}, \quad \varphi_2 = \frac{1}{300}, \quad \widehat{\varphi}_0 = \frac{1}{48}, \quad \widehat{\varphi}_1 = \frac{1}{16}, \quad \widehat{\varphi}_2 = \frac{1}{80}.$$

From Eqs. (7) and (8), with the data given, we find that

$$\begin{aligned} \vartheta_1 &= \frac{\Gamma(\varrho)}{\Gamma(\varrho - \varrho_1)} (\log T)^{\varrho - \varrho_1 - 1} \\ &= \frac{\Gamma\left(\frac{121}{50}\right)}{\Gamma\left(\frac{121}{50} - \frac{11}{25}\right)} (\log 2)^{\frac{121}{50} - \frac{11}{25} - 1} = \frac{\Gamma\left(\frac{121}{50}\right)}{\Gamma\left(\frac{99}{50}\right)} (\log 2)^{\frac{49}{50}}, \end{aligned}$$

$$\begin{aligned}
 \vartheta_2 &= \frac{\nu_2 \Gamma(\varrho)}{\Gamma(\varrho + \varrho_2)} (\log \theta)^{\varrho + \varrho_2 - 1} + \omega_2 \sum_{j=1}^{k-2} \varpi_j (\log \gamma_j)^{\varrho - 1} \\
 &= \frac{17}{500} \frac{\Gamma(\frac{121}{50})}{\Gamma(\frac{121}{50} + \frac{19}{20})} \left(\log \frac{369}{200} \right)^{\frac{121}{50} + \frac{19}{20} - 1} + \frac{7}{200} \sum_{j=1}^{k-2} \varpi_j (\log \gamma_j)^{\frac{121}{50} - 1} \\
 &= \frac{17}{500} \frac{\Gamma(\frac{121}{50})}{\Gamma(\frac{337}{100})} \left(\log \frac{369}{200} \right)^{\frac{237}{100}} + \frac{7}{200} \left[\frac{13}{100} \left(\log \frac{267}{200} \right)^{\frac{71}{50}} \right. \\
 &\quad \left. + \frac{6}{25} \left(\log \frac{289}{250} \right)^{\frac{71}{50}} + \frac{19}{50} \left(\log \frac{889}{500} \right)^{\frac{71}{50}} + \frac{53}{125} \left(\log \frac{379}{200} \right)^{\frac{71}{50}} \right], \\
 \vartheta_3 &= \frac{\nu_1 \Gamma(\varsigma)}{\Gamma(\varsigma + \varsigma_2)} (\log \nu)^{\varsigma + \varsigma_2 - 1} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j (\log \delta_j)^{\varsigma - 1} \\
 &= \frac{3}{125} \frac{\Gamma(\frac{71}{25})}{\Gamma(\frac{71}{25} + \frac{11}{20})} \left(\log \frac{77}{50} \right)^{\frac{71}{25} + \frac{11}{20} - 1} + \frac{159}{250} \sum_{j=1}^{k-2} \epsilon_j (\log \delta_j)^{\frac{71}{25} - 1} \\
 &= \frac{3}{125} \frac{\Gamma(\frac{71}{25})}{\Gamma(\frac{339}{100})} \left(\log \frac{77}{50} \right)^{\frac{239}{100}} + \frac{159}{250} \left[\frac{23}{200} \left(\log \frac{49}{40} \right)^{\frac{46}{25}} \right. \\
 &\quad \left. + \frac{32}{125} \left(\log \frac{723}{500} \right)^{\frac{46}{25}} + \frac{179}{500} \left(\log \frac{417}{250} \right)^{\frac{46}{25}} + \frac{23}{50} \left(\log \frac{47}{25} \right)^{\frac{46}{25}} \right], \\
 \vartheta_4 &= \frac{\Gamma(\varsigma)}{\Gamma(\varsigma - \varsigma_1)} (\log T)^{\varsigma - \varsigma_1 - 1} \\
 &= \frac{\Gamma(\frac{71}{25})}{\Gamma(\frac{71}{25} - \frac{17}{20})} (\log 2)^{\frac{71}{25} - \frac{17}{20} - 1} = \frac{\Gamma(\frac{71}{25})}{\Gamma(\frac{199}{100})} (\log 2)^{\frac{99}{100}}.
 \end{aligned}$$

Table 1 shows the numerical results of ϑ_j for $j = 1, 2, \dots, k - 2$ and ϑ . These results are shown in Fig. 1. Now by employing Algorithm 1, we obtain $\vartheta_1 \approx 0.88617886$, $\vartheta_2 \approx 0.02198308$, $\vartheta_3 \approx 0.22359817$, $\vartheta_4 \approx 1.21168646$ and

$$\vartheta = \vartheta_1 \vartheta_4 - \vartheta_2 \vartheta_3 \approx 1.06885555.$$

On this level, by applying Eqs. (19), (20), (21) and (22), we got

$$\begin{aligned}
 \mathcal{S}_1 &= \frac{(\log T)^\varrho}{\Gamma(\varrho + 1)} + \frac{(\log T)^{\varrho - 1}}{\vartheta} \left[\frac{\vartheta_4 (\log T)^{\varrho - \varrho_1}}{\Gamma(\varrho - \varrho_1 + 1)} \right. \\
 &\quad \left. + \vartheta_3 \left\{ \frac{(\log \theta)^{\varrho + \varrho_2}}{\Gamma(\varrho + \varrho_2 + 1)} + \omega_2 \sum_{j=1}^{k-2} \frac{\varpi_j (\log \gamma_j)^\varrho}{\Gamma(\varrho + 1)} \right\} \right] \\
 &= \frac{(\log 2)^{\frac{121}{50}}}{\Gamma(\frac{121}{50} + 1)} + \frac{(\log 2)^{\frac{121}{50} - 1}}{1.06885555} \left[\frac{1.21168646 (\log 2)^{\frac{121}{50} - \frac{11}{25}}}{\Gamma(\frac{121}{50} - \frac{11}{25} + 1)} \right. \\
 &\quad \left. + 0.22359817 \left\{ \frac{17}{500} \frac{(\log \frac{369}{200})^{\varrho + \varrho_2}}{\Gamma(\frac{121}{50} + \frac{19}{20} + 1)} + \frac{7}{200} \sum_{j=1}^{k-2} \frac{\varpi_j (\log \gamma_j)^{\frac{121}{50}}}{\Gamma(\frac{121}{50} + 1)} \right\} \right] \\
 &\approx 0.30176768,
 \end{aligned}$$

Table 1 Numerical results of ϑ_j for $j = 1, 2, 3, 4$ and ϑ , in Example 1

n	τ	ϑ_j				ϑ
		ϑ_1	ϑ_2	ϑ_3	ϑ_4	
1	1.00	0.0000	0.0220	0.2236	0.0000	-0.0049
2	1.05	0.0658	0.0220	0.2236	0.0876	0.0008
3	1.10	0.1268	0.0220	0.2236	0.1699	0.0166
4	1.15	0.1845	0.0220	0.2236	0.2483	0.0409
5	1.20	0.2394	0.0220	0.2236	0.3230	0.0724
6	1.25	0.2918	0.0220	0.2236	0.3945	0.1102
7	1.30	0.3420	0.0220	0.2236	0.4631	0.1535
8	1.35	0.3902	0.0220	0.2236	0.5290	0.2015
9	1.40	0.4364	0.0220	0.2236	0.5925	0.2537
10	1.45	0.4810	0.0220	0.2236	0.6536	0.3095
11	1.50	0.5240	0.0220	0.2236	0.7126	0.3685
12	1.55	0.5655	0.0220	0.2236	0.7696	0.4303
13	1.60	0.6056	0.0220	0.2236	0.8248	0.4946
14	1.65	0.6444	0.0220	0.2236	0.8783	0.5610
15	1.70	0.6820	0.0220	0.2236	0.9301	0.6294
16	1.75	0.7185	0.0220	0.2236	0.9804	0.6995
17	1.80	0.7540	0.0220	0.2236	1.0292	0.7711
18	1.85	0.7884	0.0220	0.2236	1.0767	0.8439
19	1.90	0.8219	0.0220	0.2236	1.1229	0.9179
20	1.95	0.8544	0.0220	0.2236	1.1679	0.9930
21	2.00	0.8862	0.0220	0.2236	1.2117	1.0689

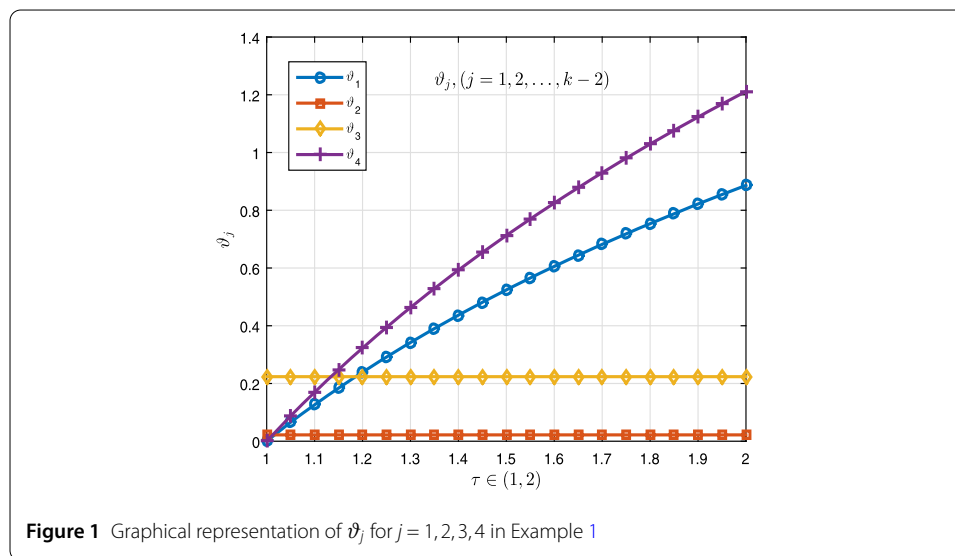


Figure 1 Graphical representation of ϑ_j for $j = 1, 2, 3, 4$ in Example 1

$$\begin{aligned}
 \mathcal{T}_1 &= \frac{(\log T)^{\epsilon-1}}{\vartheta} \left[\frac{\vartheta_3 (\log T)^{\zeta-\varsigma_1}}{\Gamma(\zeta-\varsigma_1+1)} \right. \\
 &\quad \left. + \vartheta_4 \left\{ \nu_1 \frac{(\log \nu)^{\zeta+\varsigma_2}}{\Gamma(\zeta+\varsigma_2+1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\zeta}{\Gamma(\zeta+1)} \right\} \right] \\
 &= \frac{(\log 2)^{\frac{121}{50}-1}}{1.06885555} \left[\frac{0.22359817 (\log 2)^{\frac{71}{25}-\frac{17}{20}}}{\Gamma(\frac{71}{25}-\frac{17}{20}+1)} \right. \\
 &\quad \left. + 1.21168646 \left\{ \frac{3}{125} \frac{(\log \frac{77}{50})^{\frac{71}{25}+\frac{11}{20}}}{\Gamma(\frac{77}{50}+\frac{11}{20}+1)} + \frac{159}{250} \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^{\frac{71}{25}}}{\Gamma(\frac{71}{25}+1)} \right\} \right] \approx 0.04723924,
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{S}_2 &= \frac{(\log T)^{\varsigma-1}}{\vartheta} \left[\frac{\vartheta_2(\log T)^{\varrho-\varrho_1}}{\Gamma(\varrho-\varrho_1+1)} \right. \\
 &\quad \left. + \vartheta_1 \left\{ \nu_2 \frac{(\log \theta)^{\varrho+\varrho_2}}{\Gamma(\varrho+\varrho_2+1)} + \omega_2 \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^\varrho}{\Gamma(\varrho+1)} \right\} \right] \\
 &= \frac{(\log 2)^{\frac{71}{25}-1}}{1.06885555} \left[\frac{0.02198308(\log 2)^{\frac{121}{50}-\frac{11}{25}}}{\Gamma(\frac{121}{50}-\frac{11}{25}+1)} \right. \\
 &\quad \left. + 0.88617886 \left\{ \frac{17}{500} \frac{(\log \frac{369}{200})^{\frac{121}{50}+\frac{19}{20}}}{\Gamma(\frac{121}{50}+\frac{19}{20}+1)} + \frac{7}{200} \sum_{j=1}^{k-2} \varpi_j \frac{(\log \gamma_j)^{\frac{121}{50}}}{\Gamma(\frac{121}{50}+1)} \right\} \right] \\
 &\approx 0.00423958, \\
 \mathcal{T}_2 &= \frac{(\log T)^\varsigma}{\Gamma(\varsigma+1)} + \frac{(\log T)^{\varsigma-1}}{\vartheta} \left[\frac{\vartheta_1(\log T)^{\varsigma-\varsigma_1}}{\Gamma(\varsigma-\varsigma_1+1)} \right. \\
 &\quad \left. + \vartheta_2 \left\{ \nu_1 \frac{(\log \nu)^{\varsigma+\varsigma_2}}{\Gamma(\varsigma+\varsigma_2+1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^\varsigma}{\Gamma(\varsigma+1)} \right\} \right] \\
 &= \frac{(\log 2)^{\frac{71}{25}}}{\Gamma(\frac{71}{25}+1)} + \frac{(\log 2)^{\frac{71}{25}-1}}{1.06885555} \left[\frac{0.88617886(\log 2)^{\frac{71}{25}-\frac{17}{20}}}{\Gamma(\frac{71}{25}-\frac{17}{20}+1)} \right. \\
 &\quad \left. + 0.02198308 \left\{ \frac{3}{125} \frac{(\log \frac{77}{50})^{\frac{71}{25}+\frac{11}{20}}}{\Gamma(\frac{71}{25}+\frac{11}{20}+1)} + \omega_1 \sum_{j=1}^{k-2} \epsilon_j \frac{(\log \delta_j)^{\frac{71}{25}}}{\Gamma(\frac{71}{25}+1)} \right\} \right] \\
 &\approx 0.17474270,
 \end{aligned}$$

with

$$\begin{aligned}
 \eta_1 &= \varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2) \\
 &\approx \frac{1}{60}(0.30176768 + 0.00423958) + \frac{1}{16}(0.04723924 + 0.17474270) \\
 &\approx 0.01897399 < 1, \\
 \eta_2 &= \varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2) \\
 &\approx \frac{1}{300}(0.30176768 + 0.00423958) + \frac{1}{80}(0.04723924 + 0.17474270) \\
 &\approx 0.00379479 < 1.
 \end{aligned}$$

Table 2 shows the numerical results of \mathcal{S}_i , \mathcal{T}_i and η_i for $i = 1, 2$ and Ψ . These results are shown in Figs. 2 and 3. Hence, again by using the Algorithm 1, we obtain

$$\begin{aligned}
 \Psi &= \min\{1 - [\varphi_1(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_1(\mathcal{T}_1 + \mathcal{T}_2)], 1 - [\varphi_2(\mathcal{S}_1 + \mathcal{S}_2) + \widehat{\varphi}_2(\mathcal{T}_1 + \mathcal{T}_2)]\} \\
 &= \min\{1 - 0.01897399, 1 - 0.00379479\} \\
 &= 0.98102600.
 \end{aligned}$$

All of Theorem 8's requirements are fulfilled. Problem (69)–(70) therefore has a solution on $\mathcal{H} = [1, 2]$.

Table 2 Numerical results of $\mathcal{S}_i, \mathcal{T}_i, \eta_i$ for $i = 1, 2$ and Ψ in Example 1

τ	\mathcal{S}_i		\mathcal{T}_i		η_i		Ψ
	\mathcal{S}_1	\mathcal{S}_2	\mathcal{T}_1	\mathcal{T}_2	η_1	η_2	
1.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000
1.05	0.0163	0.0013	0.0403	0.0029	0.0030	0.0006	0.9970
1.10	0.0047	0.0005	0.0114	0.0012	0.0009	0.0002	0.9991
1.15	0.0080	0.0006	0.0127	0.0023	0.0011	0.0002	0.9989
1.20	0.0134	0.0008	0.0147	0.0044	0.0014	0.0003	0.9986
1.25	0.0208	0.0010	0.0169	0.0075	0.0019	0.0004	0.9981
1.30	0.0300	0.0012	0.0191	0.0115	0.0024	0.0005	0.9976
1.35	0.0410	0.0014	0.0212	0.0167	0.0031	0.0006	0.9969
1.40	0.0536	0.0016	0.0234	0.0229	0.0038	0.0008	0.9962
1.45	0.0677	0.0018	0.0255	0.0302	0.0046	0.0009	0.9954
1.50	0.0834	0.0020	0.0276	0.0385	0.0056	0.0011	0.9944
1.55	0.1003	0.0022	0.0297	0.0479	0.0066	0.0013	0.9934
1.60	0.1186	0.0025	0.0318	0.0583	0.0077	0.0015	0.9923
1.65	0.1381	0.0027	0.0338	0.0698	0.0088	0.0018	0.9912
1.70	0.1587	0.0029	0.0358	0.0821	0.0101	0.0020	0.9899
1.75	0.1803	0.0031	0.0378	0.0954	0.0114	0.0023	0.9886
1.80	0.2029	0.0033	0.0397	0.1096	0.0128	0.0026	0.9872
1.85	0.2264	0.0036	0.0416	0.1247	0.0142	0.0028	0.9858
1.90	0.2507	0.0038	0.0435	0.1406	0.0157	0.0031	0.9843
1.95	0.2759	0.0040	0.0454	0.1573	0.0173	0.0035	0.9827
2.00	0.3018	0.0042	0.0472	0.1747	0.0190	0.0038	0.9810

Example 2 Consider the following coupled system of Hadamard-type FDEs:

$$\begin{cases} {}^H\mathcal{D}^{\frac{78}{33}}y(\tau) = \frac{\tau}{3} + \frac{1}{90} \cos(y(\tau)) + \frac{7}{200} \frac{|z(\tau)|}{1+|z(\tau)|}, \\ {}^H\mathcal{D}^{\frac{55}{26}}z(\tau) = \frac{\sqrt{\tau}+1}{4} + \frac{3}{5(\tau+36)} \frac{|y(\tau)|}{1+|y(\tau)|} + \frac{13}{250} \cos(z(\tau)), \end{cases} \tag{71}$$

for $\tau \in \mathcal{H}$, equipped with coupled boundary conditions:

$$\begin{cases} y(1) = y'(1) = 0, \\ {}^H\mathcal{D}^{\frac{11}{25}}y(T) = \frac{3}{125} {}^H\mathcal{I}^{\frac{11}{20}}z\left(\frac{77}{50}\right) + \frac{159}{250} \sum_{j=1}^4 \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, \\ {}^H\mathcal{D}^{\frac{17}{30}}z(T) = \frac{17}{500} {}^H\mathcal{I}^{\frac{19}{20}}y\left(\frac{97}{50}\right) + \frac{7}{200} \sum_{j=1}^4 \varpi_j y(\gamma_j). \end{cases} \tag{72}$$

Here, $\varrho = \frac{78}{33} \in (2, 3], \varsigma = \frac{55}{26} \in (2, 3], \varrho_1 = \frac{11}{25} \in (0, 1], \varsigma_1 = \frac{17}{20} \in (0, 1], \varrho_2 = \frac{19}{20} \in (0, 1], \varsigma_2 = \frac{11}{20} \in (0, 1], \epsilon_1 = \frac{23}{200}, \epsilon_2 = \frac{32}{125}, \epsilon_3 = \frac{179}{500}, \epsilon_4 = \frac{23}{50}, \varpi_1 = \frac{13}{100}, \varpi_2 = \frac{6}{25}, \varpi_3 = \frac{19}{50}, \varpi_4 = \frac{53}{125}, \delta_1 = \frac{49}{40}, \delta_2 = \frac{723}{500}, \delta_3 = \frac{417}{250}, \delta_4 = \frac{47}{25}, \gamma_1 = \frac{267}{200}, \gamma_2 = \frac{389}{250}, \gamma_3 = \frac{889}{500}, \gamma_4 = \frac{379}{200}, \omega_1 = \frac{159}{250}, \omega_2 = \frac{7}{200}, \nu_1 = \frac{3}{125}, \nu_2 = \frac{17}{500}, T = 2, \nu = \frac{77}{50}, \theta = \frac{369}{200}$ and

$$h_1(\tau, y(\tau), z(\tau)) = \frac{\tau}{3} + \frac{1}{90} \cos(y(\tau)) + \frac{7}{200} \frac{|z(\tau)|}{1+|z(\tau)|},$$

$$h_2(\tau, y(\tau), z(\tau)) = \frac{\sqrt{\tau}+1}{4} + \frac{3}{5(\tau+36)} \frac{|y(\tau)|}{1+|y(\tau)|} + \frac{13}{250} \cos(z(\tau)).$$

First we show that condition (\mathcal{F}_2) holds. In this case, we have

$$\begin{aligned} & \left| h_1(\tau, w_1, w_2) - h_1(\tau, \widehat{w}_1, \widehat{w}_2) \right| \\ &= \left| \frac{\tau}{3} + \frac{1}{90} \cos(w_1(\tau)) + \frac{7}{200} \frac{|w_2(\tau)|}{1+|w_2(\tau)|} - \left(\frac{\tau}{3} + \frac{1}{90} \cos(\widehat{w}_1(\tau)) + \frac{7}{200} \frac{|\widehat{w}_2(\tau)|}{1+|\widehat{w}_2(\tau)|} \right) \right| \end{aligned}$$

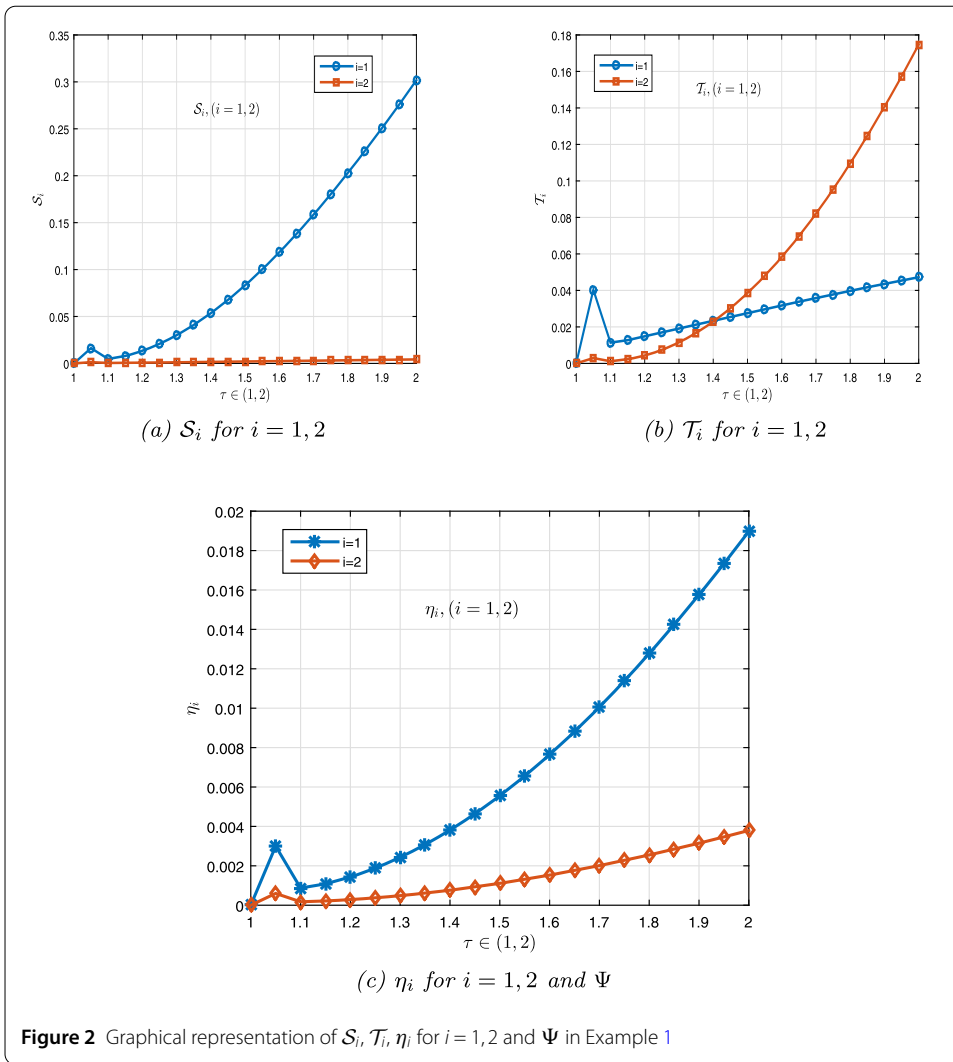


Figure 2 Graphical representation of S_i, T_i, η_i for $i = 1, 2$ and Ψ in Example 1

$$\begin{aligned} &\leq \frac{1}{90} |\cos(w_1(\tau)) - \cos(\widehat{w}_1(\tau))| + \frac{7}{200} \left| \frac{|w_2(\tau)|}{1 + |w_2(\tau)|} - \frac{|\widehat{w}_2(\tau)|}{1 + |\widehat{w}_2(\tau)|} \right| \\ &\leq \frac{1}{90} |w_1(\tau) - \widehat{w}_1(\tau)| + \frac{7}{200} |w_2(\tau) - \widehat{w}_2(\tau)| \end{aligned}$$

and

$$\begin{aligned} &|h_2(\tau, w_1, w_2) - h_2(\tau, \widehat{w}_1, \widehat{w}_2)| \\ &= \left| \frac{\sqrt{\tau} + 1}{4} + \frac{3}{5(\tau + 36)} \frac{|w_1(\tau)|}{1 + |w_1(\tau)|} + \frac{13}{250} \cos(w_2(\tau)) \right. \\ &\quad \left. - \left(\frac{\sqrt{\tau} + 1}{4} + \frac{3}{5(\tau + 36)} \frac{|\widehat{w}_1(\tau)|}{1 + |\widehat{w}_1(\tau)|} + \frac{13}{250} \cos(\widehat{w}_2(\tau)) \right) \right| \\ &\leq \frac{3}{5(\tau + 36)} \left| \frac{|w_1(\tau)|}{1 + |w_1(\tau)|} - \frac{|\widehat{w}_1(\tau)|}{1 + |\widehat{w}_1(\tau)|} \right| + \frac{13}{250} |\cos(w_2(\tau)) - \cos(\widehat{w}_2(\tau))| \\ &\leq \frac{3}{5(\tau + 36)} |w_1(\tau) - \widehat{w}_1(\tau)| + \frac{13}{250} |w_2(\tau) - \widehat{w}_2(\tau)|. \end{aligned}$$

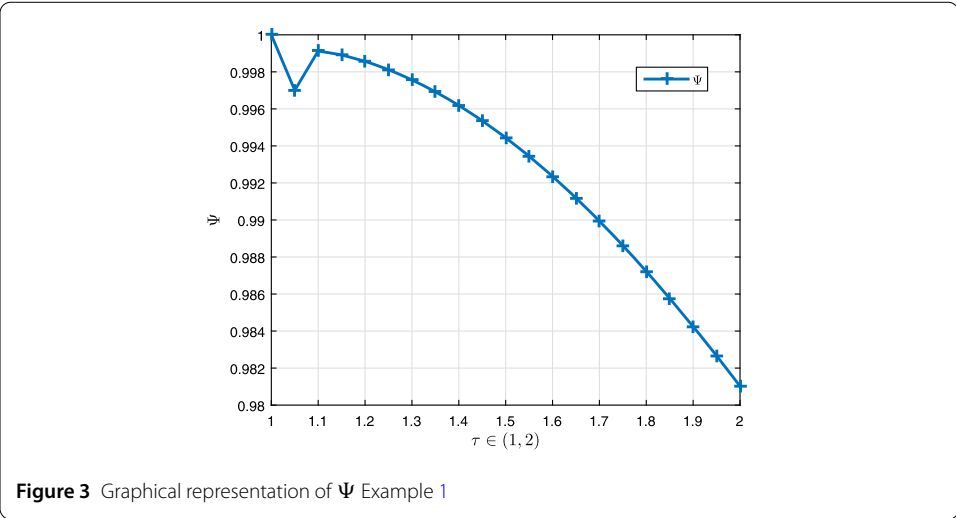


Figure 3 Graphical representation of Ψ Example 1

Table 3 Numerical results of ϑ_j for $j = 1, 2, 3, 4$ and ϑ , in Example 2

n	τ	ϑ_j				ϑ
		ϑ_1	ϑ_2	ϑ_3	ϑ_4	
1	1.00	0.0000	0.0228	0.3530	0.0000	-0.0080
2	1.05	0.0769	0.0228	0.3530	0.5237	0.0322
3	1.10	0.1427	0.0228	0.3530	0.6255	0.0812
4	1.15	0.2032	0.0228	0.3530	0.6924	0.1326
5	1.20	0.2597	0.0228	0.3530	0.7430	0.1849
6	1.25	0.3130	0.0228	0.3530	0.7839	0.2373
7	1.30	0.3635	0.0228	0.3530	0.8183	0.2894
8	1.35	0.4115	0.0228	0.3530	0.8480	0.3409
9	1.40	0.4574	0.0228	0.3530	0.8742	0.3918
10	1.45	0.5013	0.0228	0.3530	0.8975	0.4419
11	1.50	0.5434	0.0228	0.3530	0.9185	0.4911
12	1.55	0.5838	0.0228	0.3530	0.9377	0.5394
13	1.60	0.6228	0.0228	0.3530	0.9553	0.5869
14	1.65	0.6604	0.0228	0.3530	0.9715	0.6335
15	1.70	0.6967	0.0228	0.3530	0.9865	0.6792
16	1.75	0.7317	0.0228	0.3530	1.0005	0.7241
17	1.80	0.7657	0.0228	0.3530	1.0137	0.7681
18	1.85	0.7986	0.0228	0.3530	1.0260	0.8113
19	1.90	0.8305	0.0228	0.3530	1.0376	0.8537
20	1.95	0.8615	0.0228	0.3530	1.0486	0.8953
21	2.00	0.8916	0.0228	0.3530	1.0590	0.9362

Thus, we obtain

$$\tilde{\vartheta}_1 = \frac{1}{90}, \quad \tilde{\vartheta}_2 = \frac{7}{200}, \quad \hat{\vartheta}_1 = \frac{3}{185}, \quad \hat{\vartheta}_2 = \frac{13}{250}.$$

From Eqs. (7) and (8) and employing Algorithm 2 with the data given, we find that $\vartheta_1 \approx 0.89164224$, $\vartheta_2 \approx 0.02279537$, $\vartheta_3 \approx 0.35298423$, $\vartheta_4 \approx 1.05901515$, and

$$\vartheta = \vartheta_1\vartheta_4 - \vartheta_2\vartheta_3 \approx 0.93621624.$$

Table 3 shows the numerical results of ϑ_j for $j = 1, 2, \dots, k - 2$ and ϑ . These results are shown in Fig. 4. On this level, by applying Eqs. (19), (20), (21) and (22), we got $\mathcal{S}_1 \approx$

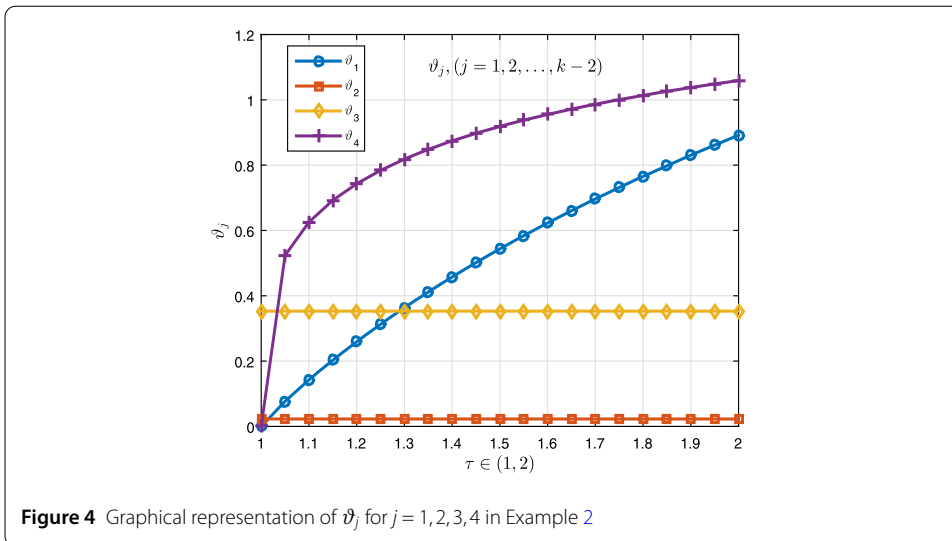


Figure 4 Graphical representation of ϑ_j for $j = 1, 2, 3, 4$ in Example 2

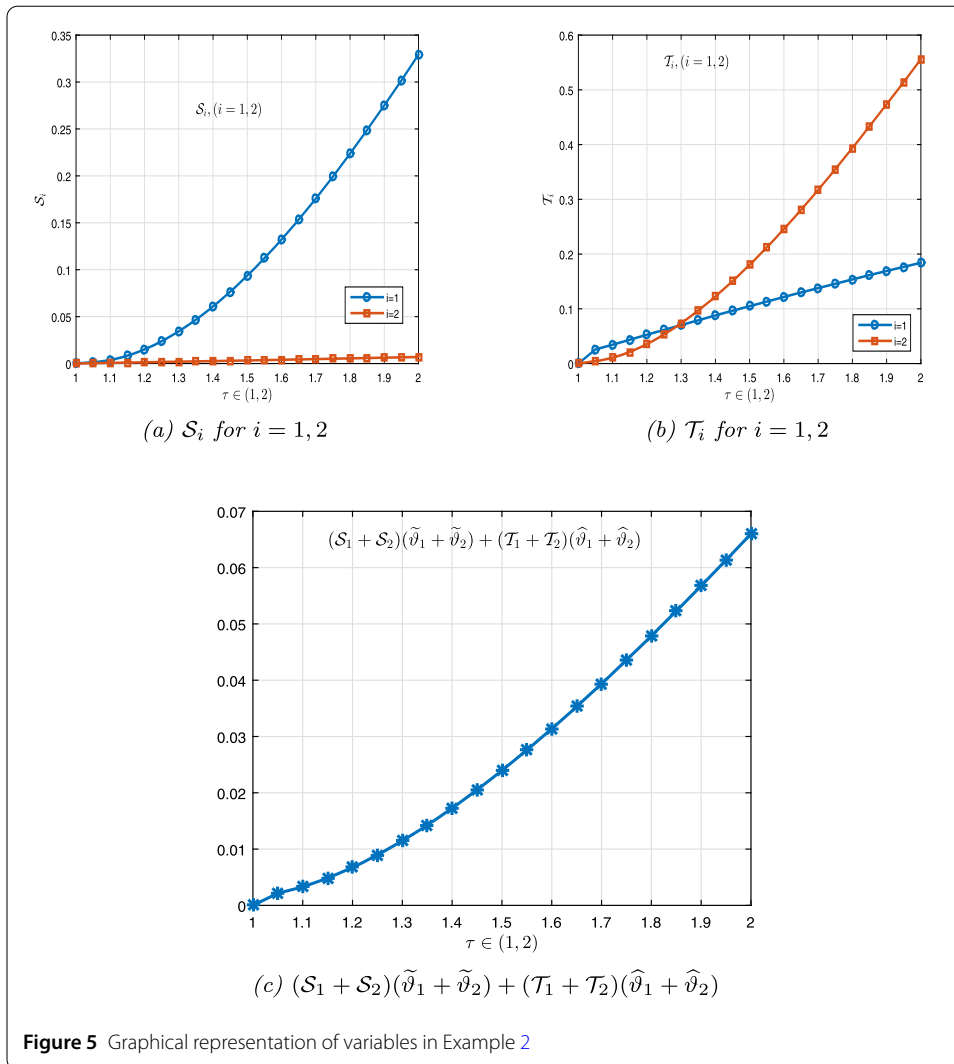
Table 4 Numerical results of $\mathcal{S}_i, \mathcal{T}_i$ and $(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2)$ in Example 2

τ	\mathcal{S}_i		\mathcal{T}_i		$(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2)$
	\mathcal{S}_1	\mathcal{S}_2	\mathcal{T}_1	\mathcal{T}_2	
1.00	0.0000	0.0000	0.0000	0.0000	0.0000
1.05	0.0015	0.0004	0.0259	0.0044	0.0022
1.10	0.0039	0.0007	0.0344	0.0105	0.0033
1.15	0.0085	0.0010	0.0435	0.0210	0.0048
1.20	0.0151	0.0013	0.0525	0.0352	0.0067
1.25	0.0237	0.0016	0.0615	0.0528	0.0090
1.30	0.0342	0.0019	0.0704	0.0734	0.0115
1.35	0.0466	0.0023	0.0792	0.0968	0.0143
1.40	0.0608	0.0026	0.0879	0.1226	0.0173
1.45	0.0765	0.0030	0.0965	0.1507	0.0205
1.50	0.0937	0.0033	0.1050	0.1807	0.0240
1.55	0.1124	0.0037	0.1134	0.2126	0.0276
1.60	0.1324	0.0041	0.1216	0.2460	0.0314
1.65	0.1535	0.0044	0.1298	0.2810	0.0353
1.70	0.1758	0.0048	0.1379	0.3172	0.0394
1.75	0.1992	0.0052	0.1459	0.3547	0.0436
1.80	0.2235	0.0055	0.1537	0.3932	0.0479
1.85	0.2488	0.0059	0.1615	0.4327	0.0523
1.90	0.2749	0.0063	0.1691	0.4730	0.0568
1.95	0.3017	0.0066	0.1767	0.5142	0.0613
2.00	0.3293	0.0070	0.1841	0.5560	0.0660

$0.32929244, \mathcal{T}_1 \approx 0.18410940, \mathcal{S}_2 \approx 0.00700804, \mathcal{T}_2 \approx 0.55601261$. Table 4 shows the numerical results of $\mathcal{S}_i, \mathcal{T}_i$ for $i = 1, 2$. These results are shown in Figs. 5. Hence, again by using the Algorithm 2, we obtain

$$(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2) \approx 0.06599551 < 1.$$

Table 4 shows the results. Also, one can see the graphical representation of results in Figs. 5. Therefore all of the Theorem 9 assumptions are fulfilled. Consequently, on $\mathcal{H} = [1, 2]$ a unique solution exists for the problem (71)–(72) by Theorem 9.



Example 3 Consider the following coupled system of Hadamard-type FDEs:

$$\begin{cases} {}^H\mathcal{D}^{\frac{88}{35}} y(\tau) = e^{-\log \tau} + \frac{9}{250} \sin(y(\tau)) + \frac{4}{9(23+\tau^2)} \frac{|z(\tau)|}{1+|z(\tau)|}, \\ {}^H\mathcal{D}^{\frac{32}{13}} z(\tau) = \frac{1}{\sqrt{(4+\tau)^2}} + \frac{3}{7(\tau+20)} \frac{|y(\tau)|}{1+|y(\tau)|} + \frac{12}{225} \cos(z(\tau)), \end{cases} \tag{73}$$

for $\tau \in [1, 2]$, equipped with coupled boundary conditions:

$$\begin{cases} y(1) = y'(1) = 0, \\ {}^H\mathcal{D}^{\frac{11}{25}} y(T) = \frac{3}{125} {}^H\mathcal{I}^{\frac{11}{20}} z\left(\frac{77}{50}\right) + \frac{159}{250} \sum_{j=1}^4 \epsilon_j z(\delta_j), \\ z(1) = z'(1) = 0, \\ {}^H\mathcal{D}^{\frac{17}{20}} z(T) = \frac{17}{500} {}^H\mathcal{I}^{\frac{19}{20}} y\left(\frac{97}{50}\right) + \frac{7}{200} \sum_{j=1}^4 \varpi_j y(\gamma_j). \end{cases} \tag{74}$$

Here, $\varrho = \frac{88}{35} \in (2, 3]$, $\varsigma = \frac{32}{13} \in (2, 3]$, $\varrho_1 = \frac{11}{25} \in (0, 1]$, $\varsigma_1 = \frac{17}{20} \in (0, 1]$, $\varrho_2 = \frac{19}{20} \in (0, 1]$, $\varsigma_2 = \frac{11}{20} \in (0, 1]$, $\epsilon_1 = \frac{23}{200}$, $\epsilon_2 = \frac{32}{125}$, $\epsilon_3 = \frac{179}{500}$, $\epsilon_4 = \frac{23}{50}$, $\varpi_1 = \frac{13}{100}$, $\varpi_2 = \frac{6}{25}$, $\varpi_3 = \frac{19}{50}$, $\varpi_4 = \frac{53}{125}$, $\delta_1 = \frac{49}{40}$, $\delta_2 = \frac{723}{500}$, $\delta_3 = \frac{417}{250}$, $\delta_4 = \frac{47}{25}$, $\gamma_1 = \frac{267}{200}$, $\gamma_2 = \frac{389}{250}$, $\gamma_3 = \frac{889}{500}$, $\gamma_4 = \frac{379}{200}$, $\omega_1 = \frac{159}{250}$, $\omega_2 = \frac{7}{200}$, $\nu_1 = \frac{3}{125}$,

$v_2 = \frac{17}{500}$, $T = 2$, $v = \frac{77}{50}$, $\theta = \frac{369}{200}$ and

$$h_1(\tau, y(\tau), z(\tau)) = e^{-\log \tau} + \frac{9}{250} \sin(y(\tau)) + \frac{4}{9(23 + \tau^2)} \frac{|z(\tau)|}{1 + |z(\tau)|},$$

$$h_2(\tau, y(\tau), z(\tau)) = \frac{1}{\sqrt{(4 + \tau)^2}} + \frac{3}{7(\tau + 20)} \frac{|y(\tau)|}{1 + |y(\tau)|} + \frac{12}{225} \cos(z(\tau)).$$

Now, we show that condition (\mathcal{F}_2) holds. In this case, we have

$$\begin{aligned} & |h_1(\tau, w_1, w_2) - h_1(\tau, \widehat{w}_1, \widehat{w}_2)| \\ &= \left| e^{-\log \tau} + \frac{9}{250} \sin(w_1(\tau)) + \frac{4}{9(23 + \tau^2)} \frac{|w_2(\tau)|}{1 + |w_2(\tau)|} \right. \\ &\quad \left. - \left(e^{-\log \tau} + \frac{9}{250} \sin(\widehat{w}_1(\tau)) + \frac{4}{9(23 + \tau^2)} \frac{|\widehat{w}_2(\tau)|}{1 + |\widehat{w}_2(\tau)|} \right) \right| \\ &\leq \frac{9}{250} |\sin(w_1(\tau)) - \sin(\widehat{w}_1(\tau))| \\ &\quad + \frac{4}{9(23 + \tau^2)} \left| \frac{|w_2(\tau)|}{1 + |w_2(\tau)|} - \frac{|\widehat{w}_2(\tau)|}{1 + |\widehat{w}_2(\tau)|} \right| \\ &\leq \frac{9}{250} |w_1(\tau) - \widehat{w}_1(\tau)| + \frac{1}{54} |w_2(\tau) - \widehat{w}_2(\tau)| \end{aligned}$$

and

$$\begin{aligned} & |h_2(\tau, w_1, w_2) - h_2(\tau, \widehat{w}_1, \widehat{w}_2)| \\ &= \left| \frac{1}{\sqrt{(4 + \tau)^2}} + \frac{3}{7(\tau + 20)} \frac{|w_1(\tau)|}{1 + |w_1(\tau)|} + \frac{12}{225} \cos(w_2(\tau)) \right. \\ &\quad \left. - \left(\frac{1}{\sqrt{(4 + \tau)^2}} + \frac{3}{7(\tau + 20)} \frac{|\widehat{w}_1(\tau)|}{1 + |\widehat{w}_1(\tau)|} + \frac{12}{225} \cos(\widehat{w}_2(\tau)) \right) \right| \\ &\leq \frac{3}{7(\tau + 20)} \left| \frac{|w_1(\tau)|}{1 + |w_1(\tau)|} - \frac{|\widehat{w}_1(\tau)|}{1 + |\widehat{w}_1(\tau)|} \right| \\ &\quad + \frac{12}{225} |\cos(w_2(\tau)) - \cos(\widehat{w}_2(\tau))| \\ &\leq \frac{1}{49} |w_1(\tau) - \widehat{w}_1(\tau)| + \frac{12}{225} |w_2(\tau) - \widehat{w}_2(\tau)|. \end{aligned}$$

So, we get

$$\begin{aligned} \widetilde{\vartheta}_1 &= \frac{9}{250}, & \widetilde{\vartheta}_2 &= \frac{1}{54}, \\ \widehat{\vartheta}_1 &= \frac{1}{49}, & \widehat{\vartheta}_2 &= \frac{12}{225}. \end{aligned}$$

From Eqs. (7) and (8) and employing Algorithm 2 with the data given, we find that $\vartheta_1 \approx 0.87623493$, $\vartheta_2 \approx 0.02070006$, $\vartheta_3 \approx 0.28269454$, $\vartheta_4 \approx 1.15597829$ and

$$\vartheta = \vartheta_1 \vartheta_4 - \vartheta_2 \vartheta_3 \approx 1.00705677.$$

Table 5 Numerical results of ϑ_j for $j = 1, 2, 3, 4$ and ϑ , in Example 3

n	τ	ϑ_j				ϑ
		ϑ_1	ϑ_2	ϑ_3	ϑ_4	
1	1.00	0.0000	0.0207	0.2827	0.0000	-0.0059
2	1.05	0.0506	0.0207	0.2827	0.2281	0.0057
3	1.10	0.1040	0.0207	0.2827	0.3436	0.0299
4	1.15	0.1569	0.0207	0.2827	0.4342	0.0623
5	1.20	0.2087	0.0207	0.2827	0.5108	0.1008
6	1.25	0.2593	0.0207	0.2827	0.5780	0.1440
7	1.30	0.3086	0.0207	0.2827	0.6382	0.1911
8	1.35	0.3565	0.0207	0.2827	0.6928	0.2411
9	1.40	0.4031	0.0207	0.2827	0.7430	0.2937
10	1.45	0.4484	0.0207	0.2827	0.7895	0.3482
11	1.50	0.4925	0.0207	0.2827	0.8328	0.4043
12	1.55	0.5355	0.0207	0.2827	0.8734	0.4618
13	1.60	0.5772	0.0207	0.2827	0.9115	0.5203
14	1.65	0.6179	0.0207	0.2827	0.9476	0.5797
15	1.70	0.6576	0.0207	0.2827	0.9817	0.6397
16	1.75	0.6963	0.0207	0.2827	1.0142	0.7003
17	1.80	0.7340	0.0207	0.2827	1.0451	0.7613
18	1.85	0.7708	0.0207	0.2827	1.0746	0.8225
19	1.90	0.8068	0.0207	0.2827	1.1029	0.8839
20	1.95	0.8419	0.0207	0.2827	1.1300	0.9455
21	2.00	0.8762	0.0207	0.2827	1.1560	1.0071

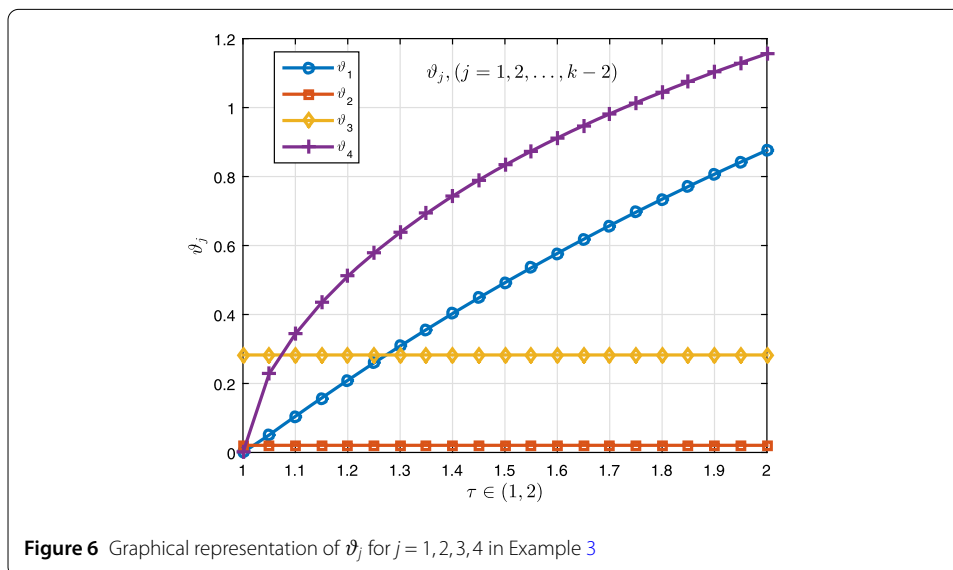


Table 5 shows the numerical results of ϑ_j for $j = 1, 2, \dots, k - 2$ and ϑ . These results are shown in Fig. 6. On this level, by applying Eqs. (19), (20), (21) and (22), we got $\mathcal{S}_1 \approx 0.26208346$, $\mathcal{T}_1 \approx 0.09384668$, $\mathcal{S}_2 \approx 0.00432903$, $\mathcal{T}_2 \approx 0.32353228$. Table 6 shows the numerical results of $\mathcal{S}_i, \mathcal{T}_i$ for $i = 1, 2$. These results are shown in Figs. 7a and 7b. Again by using the Algorithm 2, we find that the problem (73)–(74) has a unique solution for $[1, 2]$, which is stable for Hyers–Ulam, with

$$(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2) \approx 0.04075659 < 1.$$

Table 6 Numerical results of $\mathcal{S}_i, \mathcal{T}_i$ and $(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2)$ in Example 3

τ	\mathcal{S}_i		\mathcal{T}_i		$(\mathcal{S}_1 + \mathcal{S}_2)(\tilde{\vartheta}_1 + \tilde{\vartheta}_2) + (\mathcal{T}_1 + \mathcal{T}_2)(\hat{\vartheta}_1 + \hat{\vartheta}_2)$
	\mathcal{S}_1	\mathcal{S}_2	\mathcal{T}_1	\mathcal{T}_2	
1.00	0.0000	0.0000	0.0000	0.0000	0.0000
1.05	0.0022	0.0004	0.0228	0.0029	0.0020
1.10	0.0029	0.0005	0.0201	0.0038	0.0019
1.15	0.0057	0.0006	0.0239	0.0075	0.0027
1.20	0.0101	0.0008	0.0282	0.0133	0.0037
1.25	0.0161	0.0010	0.0326	0.0210	0.0049
1.30	0.0237	0.0012	0.0370	0.0307	0.0064
1.35	0.0328	0.0014	0.0414	0.0423	0.0080
1.40	0.0434	0.0016	0.0457	0.0557	0.0099
1.45	0.0555	0.0019	0.0500	0.0707	0.0120
1.50	0.0688	0.0021	0.0543	0.0874	0.0143
1.55	0.0835	0.0023	0.0585	0.1056	0.0168
1.60	0.0993	0.0025	0.0626	0.1252	0.0194
1.65	0.1163	0.0027	0.0667	0.1461	0.0222
1.70	0.1344	0.0030	0.0708	0.1683	0.0251
1.75	0.1535	0.0032	0.0747	0.1916	0.0282
1.80	0.1735	0.0034	0.0787	0.2161	0.0314
1.85	0.1944	0.0036	0.0825	0.2415	0.0347
1.90	0.2162	0.0039	0.0864	0.2680	0.0381
1.95	0.2388	0.0041	0.0901	0.2953	0.0417
2.00	0.2621	0.0043	0.0938	0.3235	0.0453

Table 6 shows the results. Also, one can see the graphical representation of results in Figs. 7. So all requirements of Theorem 13 are fulfilled.

7 Discussion

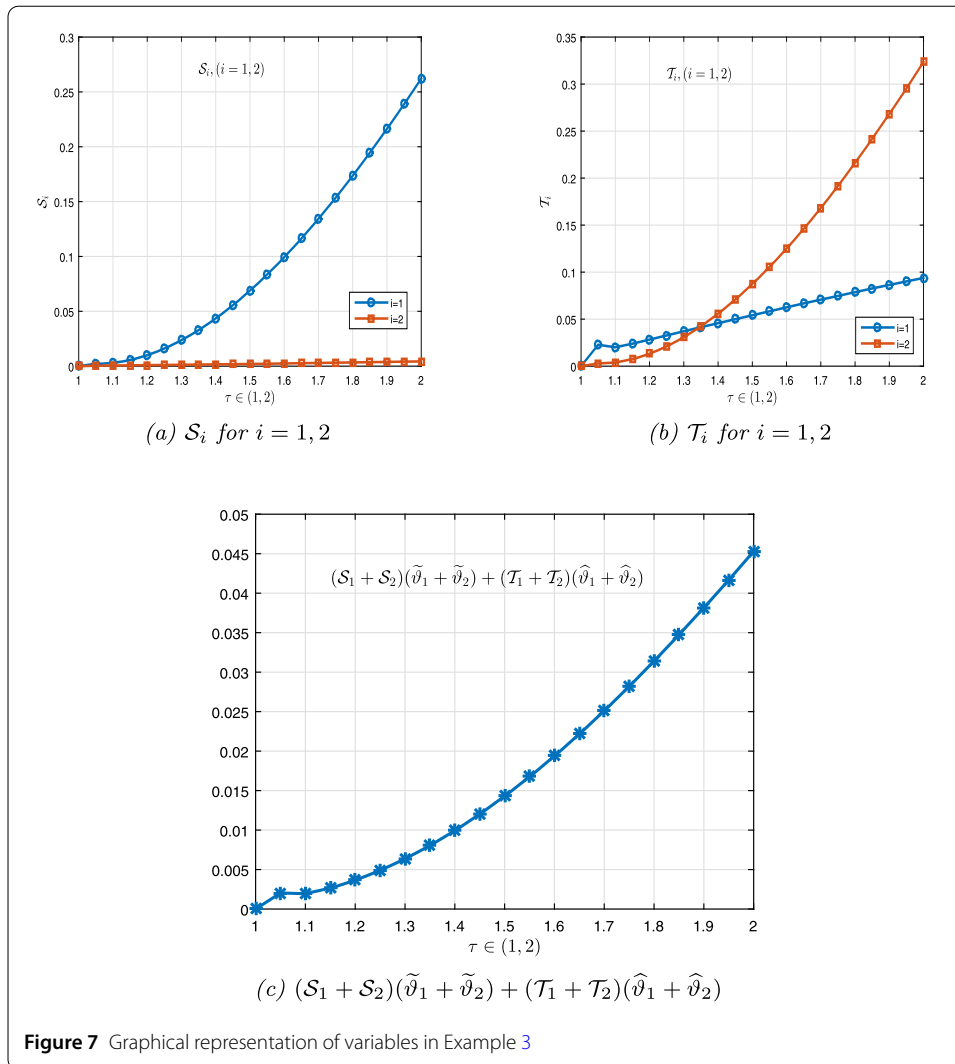
We addressed the existence, uniqueness, and stability of solutions for a coupled system of Hadamard-type FDEs augmented by Hadamard fractional integral and multi-point conditions via the alternatives of Leray–Schauder, Banach, fixed point theorems of Krasnosel’skii, and Hyer–Ulam stability. One can observe that, for particular values of $(\nu_1, \nu_2, \omega_1, \omega_2)$, our problem reduces to some special problems. Suppose we present the problems (1)–(2) in the form: to take $\omega_1 = \omega_2 = 0$ in the results provided;

$$\begin{aligned}
 y(1) = y'(1) = 0, & \quad {}^H\mathcal{D}^{\varrho_1} y(T) = \nu_1 {}^H\mathcal{I}^{\varsigma_2} z(\nu), \\
 z(1) = z'(1) = 0, & \quad {}^H\mathcal{D}^{\varsigma_1} z(T) = \nu_2 {}^H\mathcal{I}^{\varrho_2} y(\theta),
 \end{aligned}$$

with $1 < \theta < \nu < T$, while the results are

$$\begin{aligned}
 y(1) = y'(1) = 0, & \quad {}^H\mathcal{D}^{\varrho_1} y(T) = \omega_1 \sum_{j=1}^{k-2} \epsilon_j z(\delta_j), \\
 z(1) = z'(1) = 0, & \quad {}^H\mathcal{D}^{\varsigma_1} z(T) = \omega_2 \sum_{j=1}^{k-2} \varpi_j y(\gamma_j),
 \end{aligned}$$

with $1 < \gamma_1 < \delta_1 < \gamma_2 < \delta_2 < \dots < \gamma_{k-2} < \delta_{k-2} < T$, followed by $\nu_1 = \nu_2 = 0$. We can solve the above problems similar to problem (1)–(2) by using the methodology employed in the previous section. We emphasize that the problems discussed above are new.



Supporting informations

Algorithm 1 MATLAB lines for calculating the numerical results of variable in Example 1

```

1 function Matrixout = Funcexample1(varrho, varsigma, varrhoi,...
2 varsigmai, epsilonj, varpij, deltaj, gammaj, omegai, nui,...
3 T, upsilon, theta, varphim, widehatvarphim)
4     k=6;
5     %section 1
6     vartheta1 = (gamma(varrho))/(gamma(varrho - varrhoi(1)))...
7     * (log(T))^(varrho - varrhoi(1) -1);
8     vartheta2 = (nui(2)*gamma(varrho))/gamma(varrho + varrhoi(2))...
9     * (log(theta))^(varrho + varrhoi(2)-1);
10    s=0;
11    for j=1:k-2
12        s = s + varpij(j) * (log(gammaj(j)))^(varrho -1);
13    end;
14    vartheta2= vartheta2 + omegai(2) * s;
15    vartheta3 = (nui(1) * gamma(varsigma))/gamma(varsigma + varsigmai(2))...
16    *(log(upsilon))^(varsigma + varsigmai(2) -1);
17    s=0;
18    for j=1:k-2
19        s=s + epsilonj(j)*(log(deltaj(j)))^(varsigma-1);
20    end;

```

```

21 vartheta3= vartheta3+ omegai(1) * s;
22 vartheta4 = gamma(varsigma)/gamma(varsigma - varsigmai(1))...
23 * (log(T))^(varsigma - varsigmai(1)-1);
24 varthetaj(1,1)=vartheta1;
25 varthetaj(1,2)=vartheta2;
26 varthetaj(1,3)=vartheta3;
27 varthetaj(1,4)=vartheta4;
28 vartheta=vartheta1* vartheta4- vartheta2* vartheta3;
29 Matrixout=varthetaj;
30 Matrixout(1,5)=vartheta;
31 %section 2
32 S1 = (log(T))^varrho / gamma(varrho+1);
33 s=0;
34 for j=1:k-2
35     s= s + varpij(j) * (log(gammaj(j)))^varrho;
36 end;
37 s= s * omegai(2)/gamma(varrho+1);
38 S1= S1 + (log(T))^(varrho-1)/ vartheta...
39 * ( varthetaj(4)* (log(T))^(varrho - varrhoi(1)))/gamma(varrho - varrhoi
(1)+1)...
40 + varthetaj(3)*( nui(2)* (log(theta))^(varrho + varrhoi(2)))/gamma(varrho
+varrhoi(2)+1)...
41 + s));
42 Matrixout(1,6)=S1;
43 T1 = (log(T))^(varrho-1)/vartheta;
44 s=0;
45 for j=1:k-2
46     s= s + epsilonj(j)*(log(deltaj(j)))^(varsigma);
47 end;
48 s= s * omegai(1)/ gamma(varsigma+1);
49 T1= T1 * ( varthetaj(3)* (log(T))^(varsigma- varsigmai(1)))/gamma(varsigma
- varsigmai(1)+1)...
50 + varthetaj(4) * ( nui(1)* (log(upsilon))^(varsigma + varsigmai(2))...
51 / gamma(varsigma+ varsigmai(2)+1) + s ));
52 Matrixout(1,7)=T1;
53 S2 = (log(T))^(varsigma-1)/vartheta;
54 s=0;
55 for j=1:k-2
56     s= s + varpij(j) * (log(gammaj(j)))^varrho;
57 end;
58 s= s * omegai(2)/gamma(varrho+1);
59 S2= S2 * ( varthetaj(2) * (log(T))^(varrho - varrhoi(1))...
60 / gamma(varrho - varrhoi(1)+1) + varthetaj(1)...
61 * (nui(2) * (log(theta))^(varrho + varrhoi(2)) ...
62 / gamma(varrho + varrhoi(2) + 1) + s ));
63 Matrixout(1,8)=S2;
64 T2 = (log(T))^(varsigma) / gamma(varsigma+1);
65 s=0;
66 for j=1:k-2
67     s= s + epsilonj(j)*(log(deltaj(j)))^(varsigma);
68 end;
69 s= s * omegai(1)/ gamma(varsigma+1);
70 T2= T2 + (log(T))^(varsigma-1)/vartheta...
71 * (varthetaj(1) * (log(T))^(varsigma- varsigmai(1))...
72 / gamma(varsigma- varsigmai(1)+1) + varthetaj(2)*( nui(1)...
73 *(log(upsilon))^(varsigma + varsigmai(2))...
74 / gamma(varsigma + varsigmai(2)+1) + s));
75 Matrixout(1,9)=T2;
76 Matrixout(1,10)=varphim(2)*(S1+S2)+ widehatvarphim(2)* (T1+T2);
77 Matrixout(1,11)=varphim(3)*(S1+S2)+ widehatvarphim(3)* (T1+T2);
78 Psi=min(1-Matrixout(1,10),1-Matrixout(1,11) );
79 Matrixout(1,12)=Psi;
80 end

```

Algorithm 2 MATLAB lines for calculating the numerical results of variable in Examples 2 and 3

```

1 function Matrixout = Funcexample21(varrho, varsigma, varrhoi,...
2 varsigmai, epsilonj, varpij, deltaj, gammaj, omegai, nui,...
3 T, upsilon, theta, widetildevarthetai, widehatvarthetai)
4 k=6;
5 %section 1
6 vartheta1 = (gamma(varrho))/(gamma(varrho - varrhoi(1)))...
7 * (log(T))^(varrho - varrhoi(1) -1);
8 vartheta2 = (nui(2)*gamma(varrho))/gamma(varrho + varrhoi(2))...
9 * (log(theta))^(varrho + varrhoi(2)-1);
10 s=0;
11 for j=1:k-2
12     s= s + varpij(j) * (log(gammaj(j)))^(varrho -1);
13 end;

```

```

14 vartheta2= vartheta2 + omegai(2) * s;
15 vartheta3 = (nui(1) * gamma(varsigma))/gamma(varsigma + varsigmai(2))...
16 * (log(upsilon))^(varsigma + varsigmai(2) -1);
17 s=0;
18 for j=1:k-2
19 s=s + epsilonj(j)*(log(deltaj(j)))^(varsigma-1);
20 end;
21 vartheta3= vartheta3+ omegai(1) * s;
22 vartheta4 = gamma(varsigma)/gamma(varsigma - varsigmai(1))...
23 * (log(T))^(varsigma - varsigmai(1)-1);
24 varthetaj(1,1)=vartheta1;
25 varthetaj(1,2)=vartheta2;
26 varthetaj(1,3)=vartheta3;
27 varthetaj(1,4)=vartheta4;
28 vartheta=vartheta1* vartheta4- vartheta2* vartheta3;
29 Matrixout=varthetaj;
30 Matrixout(1,5)=vartheta;
31 %section 2
32 S1 = (log(T))^varrho / gamma(varrho+1);
33 s=0;
34 for j=1:k-2
35 s= s + varpij(j) * (log(gammaj(j)))^varrho;
36 end;
37 s= s * omegai(2)/gamma(varrho+1);
38 S1= S1 + (log(T))^(varrho-1)/ vartheta...
39 * ( (varthetaj(4)* (log(T))^(varrho - varrhoi(1)))/gamma(varrho - varrhoi(1)
+1)...
40 + varthetaj(3)*( nui(2)* (log(theta))^(varrho + varrhoi(2)))/gamma(varrho+
varrhoi(2)+1)...
41 + s) );
42 Matrixout(1,6)=S1;
43 T1 = (log(T))^(varrho-1)/vartheta;
44 s=0;
45 for j=1:k-2
46 s=s + epsilonj(j)*(log(deltaj(j)))^(varsigma);
47 end;
48 s= s* omegai(1)/ gamma(varsigma+1);
49 T1= T1 * ( varthetaj(3)* (log(T))^(varsigma- varsigmai(1))/gamma(varsigma
- varsigmai(1)+1)...
50 + varthetaj(4) * ( nui(1)* (log(upsilon))^(varsigma + varsigmai(2))...
51 / gamma(varsigma+ varsigmai(2)+1) + s) );
52 Matrixout(1,7)=T1;
53 S2 = (log(T))^(varsigma-1)/vartheta;
54 s=0;
55 for j=1:k-2
56 s= s + varpij(j) * (log(gammaj(j)))^varrho;
57 end;
58 s= s * omegai(2)/gamma(varrho+1);
59 S2= S2 * ( varthetaj(2) * (log(T))^(varrho - varrhoi(1))...
60 / gamma(varrho - varrhoi(1)+1) + varthetaj(1)...
61 * (nui(2) * (log(theta))^(varrho + varrhoi(2)) ...
62 / gamma(varrho + varrhoi(2) + 1) + s) );
63 Matrixout(1,8)=S2;
64 T2 = (log(T))^(varsigma) / gamma(varsigma+1);
65 s=0;
66 for j=1:k-2
67 s=s + epsilonj(j)*(log(deltaj(j)))^(varsigma);
68 end;
69 s= s* omegai(1)/ gamma(varsigma+1);
70 T2= T2 + (log(T))^(varsigma-1)/vartheta...
71 * (varthetaj(1) * (log(T))^(varsigma- varsigmai(1))...
72 / gamma(varsigma- varsigmai(1)+1) + varthetaj(2)*( nui(1)...
73 * (log(upsilon))^(varsigma + varsigmai(2))...
74 / gamma(varsigma + varsigmai(2)+1) + s));
75 Matrixout(1,9)=T2;
76 Matrixout(1,10) = (widetildevartheta(1) + widetildevartheta(2))...
77 * (S1+S2) + (T1+T2)...
78 * (widehatvartheta(1)+ widehatvartheta(2));
79 end

```

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