FICITIOUS TIME INTEGRATION METHOD FOR SOLVING THE TIME FRACTIONAL GAS DYNAMICS EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI190421365P

In this work a poweful approach is presented to solve the time-fractional gas dynamics equation. In fact, we use a fictitious time variable y to convert the dependent variable w(x, t) into a new one with one more dimension. Then by taking a initial guess and implementing the group preserving scheme we solve the problem. Finally four examples are solved to illustrate the power of the offered method.

Key words: *fictitious time integration method, group preserving scheme, time fractional gas dynamics equation, Caputo derivative*

Introduction

Calculus of fractional order is increasingly being worked to model various physical systems. Since many physical phenomena growing in engineering as well as in allied sciences can be depicted by developing models with the help of the fractional calculus. The fractional partial equations response ultimately converges to the non-fractional equations, fulfilling a notable care in the present times. The fractional derivatives are important due to broad scope of applications for mathematical modelling of problems such as traffic flow models, control, and relaxation processes [1-11]. There are some some analytical and numerical methods which are implemented to solve the fractional equations such as Group preserving scheme [12, 13], differential transform methods [14-16], homotopy pertubation methods [17-20]. This presented work is dedicated to study the following time fractional gas dynamics equation (TFGD):

$$\begin{cases} {}^{C}\mathcal{D}^{\alpha}_{0^+,t} w(x,t) + w(x,t)w_x(x,t) - w(x,t)(1-w(x,t)) = \mathcal{K}(x,t) \\ w(x,0) = g_1(x),\ x \in \Omega_{\mathbf{x}} \\ w(x,T) = g_2(x),\ x \in \Omega_{\mathbf{x}} \\ w(0,t) = h_1(t),\ \ t \in \Omega_t \\ w(b,t) = h_2(t),\ t \in \Omega_t \\ \Omega := \{(x,t) : a \le x \le b,\ 0 \le t \le T\} \end{cases} \tag{1}$$

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The gas dynamics equations are mathematical terminology that are adjunct on the physical laws of conservation such as the conservation of momentum, conservation of mass, and conservation of energy. Many authors solved the fractional gas dynamics equations using different numerical and analytical methods [21-29]. The differential transform method is implemented for solving TFGD [30, 31] and Fractional homotopy analysis transform method [32].

In this presented work, we create a powerful and reliable numerical approach to obtain the numerical solution of TFGD equation. This approach is firstly presented by Liu [33].

The fictitious time integration method (FTIM)

The Caputo fractional derivative of for fractional order $\alpha > 0$ is described by [34, 35]:

$$
{}^{C}\mathcal{D}^{\alpha}_{0^+,t} w(x,t) = \frac{\partial^\alpha w(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\sigma)^{m-\alpha-1} \dfrac{\partial^m w(x,\sigma)}{\partial \sigma^m} \mathrm{d}\sigma, m-1 < \alpha < m \\ \dfrac{\partial w(x,t)}{\partial t^m}, \qquad \alpha = m \end{cases} \tag{2}
$$

By using Caputo fractional derivative definition and $0 \leq \alpha < 1$ for eq. (1):

$$
\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w_\sigma(x,\sigma)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + w w_x - w(1-w) - \mathcal{K}(x,t) = 0 \tag{3}
$$

Now, we multiply the eq. (3) into the parameter η as a fictitious damping coefficient which can help our to raise the stability of numerical integration:

$$
\frac{\eta}{\Gamma(1-\alpha)} \int_0^t \frac{w_\sigma(x,\sigma)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + \eta w w_x - \eta w(1-w) - \eta \mathcal{K}(x,t) = 0 \tag{4}
$$

Now, we impose the following transformation:

$$
z(x,t,y) = (1+y)^d w(x,t), 0 < d \leq 1 \tag{5}
$$

By using previous transformation, eq. (4) converts to a new form:

$$
\frac{\eta}{(1+y)^d} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + \eta z z_x - \eta z(1-z) - \eta \mathcal{K}(x,t) = 0 \right. \tag{6}
$$

From eq. (5) we can get:

$$
\frac{\partial z}{\partial y} = d(1+y)^{d-1} w(x,t) \tag{7}
$$

A combination of eqs. (7) and (6), concludes:

$$
\frac{\partial z}{\partial y} = \frac{\eta}{(1+y)^d} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + z z_x - z(1-z) \right] - \eta \mathcal{K}(x,t) + d(1+y)^{d-1} w \tag{8}
$$

Then, eq. (8) can be transformed into a new form of PDE for z, by using $w = z/(1+y)^d$:

$$
\frac{\partial z}{\partial y} = \frac{\eta}{(1+y)^d} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha} \mathrm{d}\sigma + z z_x - z(1-z) \right] - \eta \mathcal{K}(x,t) + \frac{kz}{1+y} \tag{9}
$$

By using:

$$\frac{\partial}{\partial y}\left[\frac{z}{(1+y)^d}\right]=\frac{z_y}{(1+y)^d}-\frac{dz}{(1+y)^{1+d}}$$

Next, by multiplying $1/(1+y)^d$ on both sides of eq. (9), we obtain:

$$\frac{\partial}{\partial y}\left(\frac{z}{(1+y)^d}\right)=\frac{\eta}{(1+y)^{2d}}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{z_\sigma(x,\sigma,y)}{(t-\sigma)^\alpha}\mathrm{d}\sigma+zz_x-z(1-z)\right]-\frac{\eta\mathcal{K}(x,t)}{(1+y)^d} \tag{11}$$

Using again the transformation $w = z/(1+y)^d$, we get:

$$\begin{aligned}w_y=&\frac{\eta}{(1+y)^d}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{z_s(x,\sigma,y)}{(t-\sigma)^\alpha}\mathrm{d}\sigma+w(x,t,y)w_x(x,t,y)-\right.\\&\left.-w(x,t,y)[1-w(x,t,y)]\right]-\eta\mathcal{K}(x,t)\end{aligned} \tag{12}$$

We have to emphasis that y plays the fictitious co-ordinate role which able us to embed eq. (3) into a new PDE form in a space called 3-space, denoted R^3. As well as, by a initially guess $w(x, t, 0)$, for all $y \geq 0$, $w = w(x, t, y)$ is an undetermined function with regard to the conditions in eq. (1).

Supposing $w_i^j(\xi) := w(x_i, t_j, y)$ and $\mathcal{K}_i^j := \mathcal{K}(x_i, t_j)$ as the discrete values of w and $\mathcal{K}$ at a point (x_i, t_j). Implementing a semi-discretization to the eq. (12) concludes:

$$\begin{aligned}&\frac{d}{dy}w_i^j(y)=\frac{\eta}{(1+y)^d}\cdot\\&\cdot\left[\frac{1}{\Gamma(1-\alpha)}\int_0^{t_j}\frac{y_\sigma(x_i,\sigma,y)}{(t_j-\sigma)^\alpha}\mathrm{d}\sigma+w_i^j(y)\frac{w_i^{j+1}(y)-w_i^j(y)}{\Delta x}-w_i^j(y)(1-w_i^j(y))-\mathcal{K}_i^j\right]\end{aligned} \tag{13}$$

To calculate the aforementioned integral terms we cam write the following approximation:

$$\int_0^{t_j}\frac{w_\sigma(x_i,\sigma,y)}{(t_j-\sigma)^\alpha}\mathrm{d}\sigma\approx\sum_{l=1}^{j-1}\frac{w(x_i,t_{l+1},y)-w(x_i,t_l,y)}{\Delta t(t_j-t_l)^\alpha} \tag{14}$$

Which stepsize Δx is $(b-a)/M_1$, $\Delta t = T/M_2$, $x_i = a + i\Delta x$ and $t_j = j\Delta t$.

The GPS for extracted system of ODE

In this stage, with $\mathbf{w} = (w_1^1, w_1^2, ..., w_m^n)^T$ we can write the eq. (13) in the following form:

$$\mathbf{w}'=\mathbf{E}(\mathbf{w},y),\ \mathbf{w}\in\mathbb{R}^N,\ y\in\mathbb{R} \tag{15}$$

where $\mathbf{E}$ indicates a vector with ij-elements being the right-hand side of eq. (13) and $\mathbf{w}'$ denotes the differential of $\mathbf{w}$ with regard to y, and $N = M_1 \times M_2$ is the number of total grid point.

In this step we can use of group-preserving scheme (GPS) that introduced by Liu [33]. Let:

$$\mathbf{X}_{l+1}=\mathbf{B}(l)\mathbf{X}_l \tag{16}$$

where $\mathbf{X}_l$ indicates the value of $\mathbf{X}$ at the y_l and $\mathbf{B}(l)$ is a component of $SO_0(N, 1)$ which represents the group value of $\mathbf{B}$ at y_l.

The Lie group can be created from $\mathbf{C}$ which is a element of $so(N, 1)$:

$$\mathbf{B}_l = \exp[\Delta y \mathbf{C}(l)] = \begin{bmatrix} I_N + \dfrac{(\Psi_l - 1)}{\|\mathbf{E}_l\|^2}\mathbf{E}_l\mathbf{E}_l^T & \dfrac{\Phi_l \mathbf{E}_l}{\|\mathbf{E}_l\|} \\ \dfrac{\Phi_l \mathbf{E}_l^T}{\|\mathbf{E}_l\|} & \Psi_l \end{bmatrix}$$

where

$$\Psi_l = \cosh\left(\frac{\Delta y \|\mathbf{E}_l\|}{\|\mathbf{w}_l\|}\right)$$
$$\Phi_l = \sinh\left(\frac{\Delta y \|\mathbf{E}_l\|}{\|\mathbf{w}_l\|}\right) \tag{17}$$

The $\mathbf{X} := (\mathbf{w}^{T,} \|\mathbf{w}\|)^T$ is a vector in Minkowskian space which converts eq. (15) into $\partial X/\partial y = \mathbf{CX}$.

Where

$$\mathbf{C} := \begin{pmatrix} \mathbf{0}_{N\times N} & \dfrac{\mathbf{E}(\mathbf{w},\theta)}{\|\mathbf{w}\|} \\ \dfrac{\mathbf{E}^T(\mathbf{w},y)}{\|\mathbf{w}\|} & 0 \end{pmatrix} \in so(N,1) \tag{18}$$

is a Liu algebra of the proper orthochronous Lorentz group $SO_0(N, 1)$. By replacing eq. (17) for B_l into eq. (16), we have:

$$\mathbf{w}_{l+1} = \mathbf{w}_l + \frac{(\Psi_l - 1)\mathbf{E}_l\mathbf{w}_l + \Phi_l \|\mathbf{w}_l\| \ \|\mathbf{E}_l\|}{\|\mathbf{E}_l\|^2}\mathbf{E}_l = \mathbf{w}_l + \Pi_l \mathbf{E}_l \tag{19}$$

in this stage, by selecting an initial value u_i^j (0) we can apply GPS to solve numerical solution of eq. (15) from the initial fictitious y_0 to a chosen final fictitious time y_f. Moreover, we can control the convergence of w_i^j at the l and $l + 1$ steps by the following criterion:

$$\sqrt{\sum_{i,j=1}^{M_1,M_2}\left[\mathbf{w}_i^j(l+1) - \mathbf{w}_i^j(l)\right]^2} \le \varepsilon \tag{20}$$

where ε is the convergence criterion.

Numerical examples

To show the power of our method four examples are solved.

Example 1: In order to show the ability of presented method we consider the following fractional TFGD equation with fractional order $\alpha = 0.9$.

$${}^C\mathcal{D}_{0^+,t}^{\alpha} w(x,t) + w(x,t)w_x(x,t) - w(x,t)(1 - w(x,t)) = 0$$

We implement the presented method to solve this problem under parameters $\eta = 35$ and $d = 0.1$. The initial guess and stepsize for y are supposed as $w_i^j(0) = 1\mathrm{e}^{-5}$ and $\Delta y = 1\mathrm{e}^{-3}$. We use the number of knots $M_1 = 25$ and $M_2 = 25$ in each co-ordinates of space and time, respectively. Also, considered domain in this example is $\Omega = [0,1] \times [0,1]$. Figure 1 is dedicated to show the exact solution $w(x,t) = \mathrm{e}^{-x+t}$ and approximate solutions obtained by presented scheme. Power of the method with maximum absolute error $1.4 \cdot 10^{-17}$ is shown in fig. 2.

Example 2: Suppose following problem of TFGD with $\alpha = 1.5$ and $a = 2$.

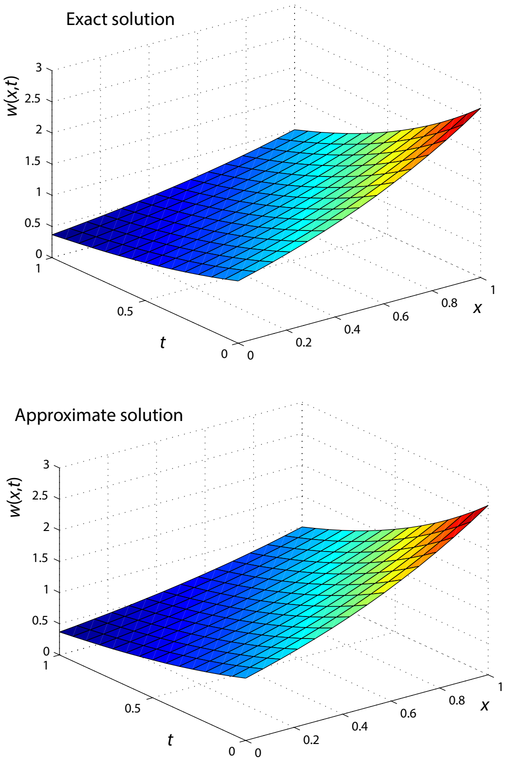

Figure 1. Plots of the exact and approximate solutions for *Example 1*

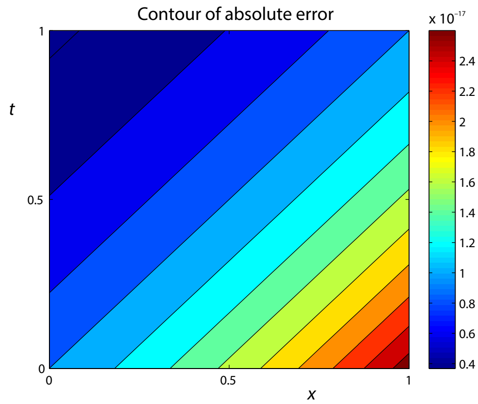

Figure 2. Plot of error for *Example 1*

In order to manage the stability and convergency of the approach we choose $\eta = 5$, $d = 0.001$, respectively. Initial guess is $w^{j}_{i}(0) = 0.001$ and stepsize of method is same with *Example 1*. For $M_1 = M_2 = 39$ and $\Delta y = 10^{-10}$. Exact $w(x,\ t) = a^{t-x}$ and numerical solutions are plotted in fig. 3. Absolute numerical errors for this example $1 \cdot 10^{-17}$ which are depicted in fig. 4.

Example 3: We take the TFGD equation with:

$$ {}^{C}\mathcal{D}^{\alpha}_{0^{+},t} w(x,t) + w(x,t) w_x(x,t) - (1+t)^2 w^2(x,t) - x^2 = 0, \quad a > 0 $$

Under parameters $\alpha = 0.3$, $\eta = 2$, $d = 0.001$, $\Delta y = 10^{-5}$, $M_1 = M_2 = 19$ and initial guess $w_i^j(0) = 0.0001$. The solutions and maximum absolute errors are demonstrated in figs. 5 and 6, respectively. Moreover, the exact solution of this example is $w(x,\ t) = x/(1 + t)$ and $\Omega = [0,\ 1] \times [0,\ 1]$.

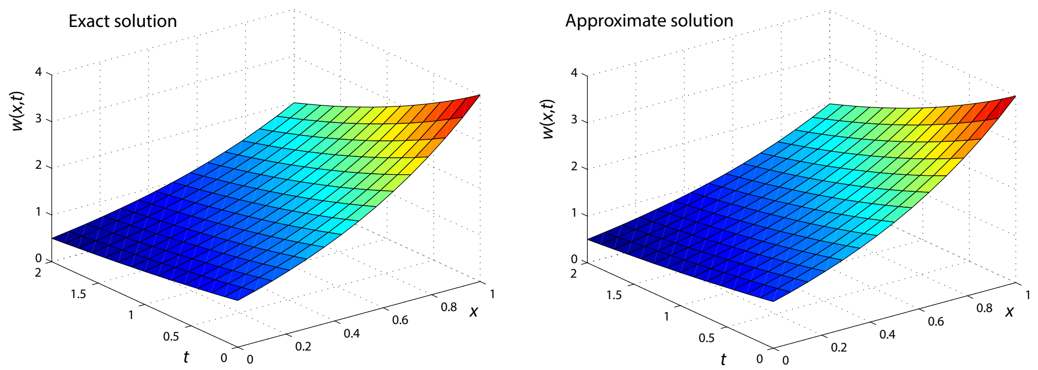

Figure 3. Plots of the exact and approximate solutions for *Example 2*

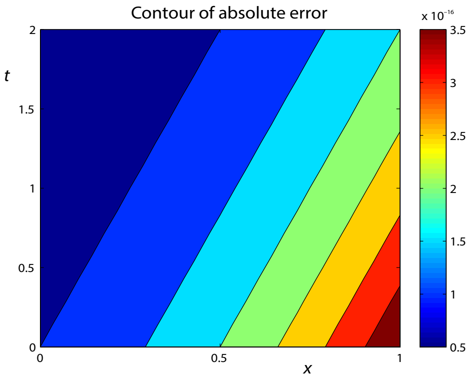

Figure 4. Plot of error for *Example 2*

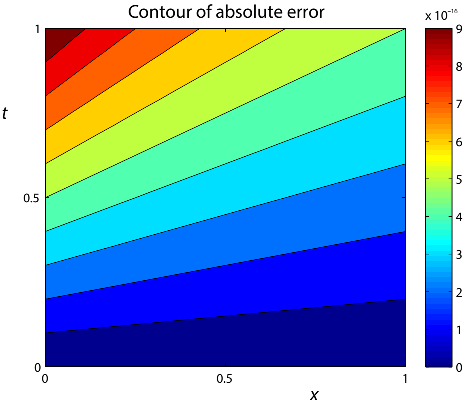

Figure 6. Plot of error for *Example 3*

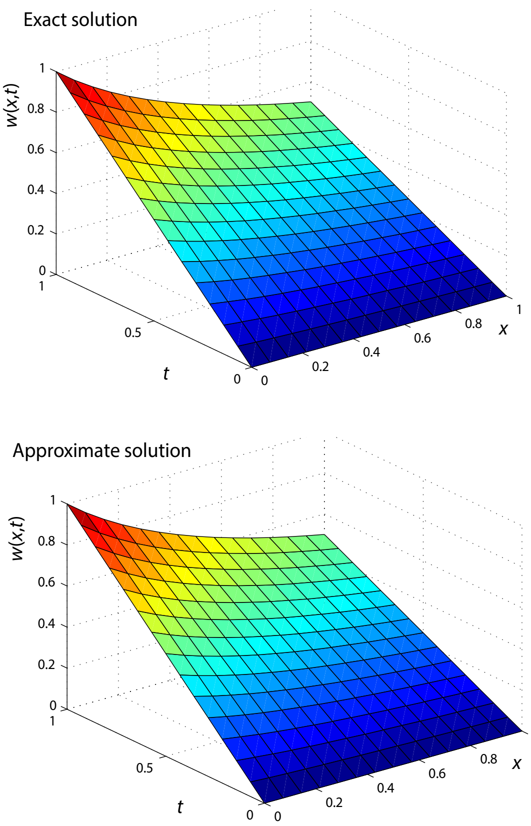

Figure 5. Plots of the exact and approximate solutions for *Example 3*

Conclusion

In this work we have converted TFGD equation into a new type of functional PDE in one more dimension by implementing a fictitious co-ordinate. Then by using a semi-discretization for original equation, the GPS as a geometric approach is imposed to solve the obtained system of first order ODE. Four numerical examples are solved, which demonstrate that our presented scheme is powerful and applicable to gain the numerical solutions of TFGD equation.

Nomenclature

B – an element of Lorentz group
C – augmented matrix
d – convergence rate parameter
E – M-dimensional vector field in eq. (15)
g_1 – initial solute concentration
I_M – M-dimensional unit matrix
h – boundary solute concentration
$\mathcal{K}$ – source term
N – M number of discretized points
$SO_0(M,1)$ – M-dimensional Lorentz group
T – time
Δt – time stepsize
w – solute concentration
x – space dimension
Δx – space stepsize

Greek symbols

α – fractional derivative order
η – fictitious damping coefficient

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Paper submitted: April 21, 2019
Paper revised: August 2, 2019
Paper accepted: August 9, 2019