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Research Article

Fractional Grassi–Miller Map Based on the Caputo h-Difference Operator: Linear Methods for Chaos Control and Synchronization

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Investigating dynamic properties of discrete chaotic systems with fractional order has been receiving much attention recently. This paper provides a contribution to the topic by presenting a novel version of the fractional Grassi–Miller map, along with improved schemes for controlling and synchronizing its dynamics. By exploiting the Caputo h-difference operator, at first, the chaotic dynamics of the map are analyzed via bifurcation diagrams and phase plots. Then, a novel theorem is proved in order to stabilize the dynamics of the map at the origin by linear control laws. Additionally, two chaotic fractional Grassi–Miller maps are synchronized via linear controllers by utilizing a novel theorem based on a suitable Lyapunov function. Finally, simulation results are reported to show the effectiveness of the approach developed herein.

1. Introduction

Nonlinear dynamics, chaos control, and chaos synchronization represent important research topics [1–27]. In particular, referring to synchronization and control, new advances have been recently reported, for both integer-order systems and fractional-order systems [28, 29]. In particular, referring to continuous-time systems described by fractional derivative, some interesting techniques involving adaptive synchronization have been recently illustrated in [28, 29]. However, there is a remarkable difference in fractional calculus regarding continuous-time and discrete-time systems. Namely, while fractional derivatives made their first appearance in a letter that Gottfried Wilhelm Leibniz wrote to Guillaume de l'Hopital in 1695, discrete fractional calculus has been introduced by Diaz and Olser only in 1974 [6]. Indeed, the authors of [6] presented the first definition of a discrete fractional operator, obtained by discretizing a continuous-time fractional operator. Over the years, several types of difference operators have been introduced in the field of discrete fractional calculus [3, 7, 8]. In particular, a number of fractional h-difference operators, which represent generalizations of the fractional difference operators, have been investigated in [7].

Based on fractional difference equations, in recent years some chaotic discrete-time systems have been studied [10, 25–27]. These systems are fractional-order maps, which show complex unpredictable behaviors due to the nonlinearities included in their difference equations [7]. With the introduction of fractional chaotic maps, attention has been also focused on the issues related to the synchronization and control of these systems [12]. For example, in [26] the fractional logistic map and its chaotic behaviors have been illustrated, whereas in [27] the presence of chaos in fractional sine and standard maps has been discussed. In [10], discrete chaos in the fractional Hénon map is reported, whereas in [25] the chaotic dynamics of the fractional delayed logistic map are analyzed in detail. In [12], three different discrete-time systems, namely, the fractional Lozi map, the fractional Lorenz map, and the fractional flow map, have been studied, along with the control laws for stabilizing and synchronizing these three maps. In [23], the fractional generalized hyperchaotic Hénon map has been introduced, whereas in [20], the dynamics of the Ikeda map have been investigated via phase plots and bifurcation diagrams. In [13], three fractional chaotic maps, namely, the Stefanski map, the Rossler map, and the Wang map have been studied, along with the synchronization properties of these systems. In [16], dynamics and control of the fractional version of the discrete double-scroll hyperchaotic map are investigated in detail. In [18], bifurcations, entropy, and control of a quadratic fractional map without equilibrium points are analyzed, whereas in [9] the dynamics of fractional maps with fixed points located on closed curves are studied.

A challenging topic in discrete fractional calculus is to study dynamics, synchronization, and control of very complex systems, such as the chaotic three-dimensional (3D) maps [8]. Namely, by computing the approximate entropy, it can be shown that 3D maps highlight a higher degree of complexity with respect to one-dimensional (1D) or two-dimensional (2D) fractional maps [5,21]. Since the increased complexity can enhance the applicability of 3D maps in pseudo-random number generators and image encryption techniques [22], it is important to analyze their dynamics as well as conceive improved synchronization and control schemes for these maps. In this regard, some interesting results have been recently published [11, 17, 19]. In [11], synchronization and control schemes for a new 3D generalized Hénon map have been proposed, whereas in [19] control and synchronization properties of a 3D fractional map without equilibria have been analyzed in detail. In [17], the fractional form of the Grassi–Miller map has been introduced using the ν-Caputo delta difference. In particular, phase portraits and bifurcation diagrams have been illustrated in [17], with the aim of deriving the fractional-order range for which the system is chaotic. In addition, two nonlinear control laws have been proposed in [17], one for stabilizing the system dynamics and the other for synchronizing a master-slave pair of maps. Although the methods developed in [11, 17, 19] are interesting, a drawback is represented by the fact that very complex control laws have been exploited for controlling and synchronizing the corresponding 3D fractional maps. For example, in [11], synchronization and control in the 3D generalized Hénon maps have been achieved using nonlinear control laws. Moreover, in [19], the 3D fractional maps with hidden attractors have been synchronized and controlled via nonlinear control laws that include several nonlinear terms. We would observe that it might be difficult to implement very complex control laws in practical applications of fractional maps. This drawback also regards the Grassi–Miller map in [17], since its introduction via the Caputo delta difference has led to complex nonlinear control laws to achieve synchronization and control of its chaotic dynamics.

Inspired by the mentioned above considerations, this paper provides a further contribution to the topic of dynamics, control, and synchronization of fractional 3D maps by presenting a novel version of the Grassi–Miller map, along with improved schemes for controlling and synchronizing its dynamics. The structure of the article is as follows. In Section 2, definition of the fractional Caputo h-difference operator [7] and a novel fractional Grassi–Miller map is proposed, along with its chaotic dynamic behavior. In Section 3 linear control laws are proposed to stabilize the dynamics of the map at the origin. In particular, a novel theorem is proved, which assures the stability condition via a suitable Lyapunov function. In Section 4, a master-slave system based on two chaotic Grassi–Miller maps is synchronized using linear controllers. The objective is achieved by exploiting a novel theorem involving a Lyapunov-based approach. Note that this paper makes an attempt to overcome the weakness and the difficulties encountered in [11, 17, 19]. Namely, on one hand, this paper focuses on a novel 3D map, with the aim of exploiting the potentials deriving from the higher degree of complexity of 3D maps with respect to simpler 1D and 2D maps. On the other hand, the paper proposes simple linear control laws (with respected to the complex control laws developed [11, 17, 19]), with the aim of making feasible their implementation for potential applications of 3D maps in pseudo-random number generators and image encryption techniques. In addition to these improvements, note that, by virtue of the linearity of the control laws developed herein, the proposed control and synchronization schemes require less control effort with respect to the nonlinear approaches illustrated in [17]. Finally, simulation results are reported to show the effectiveness of the control and synchronization methods developed herein. All the results developed thorough the manuscript clearly highlight the novelty of the conceived approach, consisting in the following: (i) the introduction of a new 3D fractional map characterized by complex dynamics; (ii) the proof of a novel theorem for stabilizing the map via a linear control law; (iii) the proof of a novel theorem for synchronizing the map via linear control law; (iv) comparisons for illustrating the better performances of our method if compared to recent published articles where complex nonlinear control laws have been used.

2. Fractional Grassi–Miller Map Based on the Caputo h-Difference Operator

In this section, a novel version of the fractional Grassi–Miller map is presented. To this purpose, some concepts related to the Caputo h-difference operator are briefly summarized.

Throughout the rest of the paper, we assume that $(h\mathbb{N})_a = \{a, a+h, a+2h, \ldots\}$, where h is a positive real and

$a \in \mathbb{R}$. The forward h-difference operator of a function X defined on $(h\mathbb{N})_a$ is defined as

$$\Delta_h X(t) = \frac{X(t+h) - X(t)}{h}. \quad (1)$$

Definition 1 (see [2]). Let $X\colon (h\mathbb{N})_a \longrightarrow \mathbb{R}$. The fractional h-sum of positive fractional order ν is defined by

$$_h\Delta_a^{-\nu} X(t) = \frac{h}{\Gamma(\nu)} \sum_{s=(a/h)}^{(t/h)-\nu} (t - \sigma(sh))_h^{(\nu-1)} X(sh), \quad (2)$$

where $\sigma(sh) = (s+1)h$, $a \in \mathbb{R}$, and $t \in (h\mathbb{N})_{a+\nu h}$. $t_h^{(\nu)}$ is the h-falling fractional function with two real numbers t, h that can be written in the form

$$t_h^{(\nu)} = \frac{h^\nu \Gamma((t/h)+1)}{\Gamma((t/h)+1-\nu)}. \quad (3)$$

Definition 2 (see [1]). For $X(t)$ defined on $(h\mathbb{N})_a$ and a real order $0 < \nu \leq 1$, the Caputo fractional h-difference operator is given by

$$^C_h\Delta_a^\nu X(t) = \Delta_a^{-(n-\nu)} \Delta^n X(t), \quad t \in (h\mathbb{N})_{a+(n-\nu)h}, \quad (4)$$

in which $n = \lceil \nu \rceil + 1$.

Now, a theorem reported in [4] is briefly illustrated, with the aim to identify the stability conditions of the zero equilibrium point for the fractional nonlinear difference system written in the form

$$^C_h\Delta_a^\nu = F(t + \nu h, X(t + \nu h)). \quad (5)$$

Theorem 1. *The fractional nonlinear discrete system (5) is asymptotically stable if there exists a positive definite and decreasing scalar function $V(t, X(t))$ for the equilibrium point $x = 0$, such that $V(t, X(t)) \leq 0$.*

Lemma 1. *For every $t \in (h\mathbb{N})_{a+(n-\nu)h}$, the following inequality holds:*

$$^C_h\Delta_a^\nu X^2(t) \leq 2X(t+\nu h)\, ^C_h\Delta_a^\nu X(t), \quad 0 < \nu \leq 1. \quad (6)$$

All the details regarding the proof of Lemma 1 can be found in [4].

Referring to the fractional Grassi–Miller map, it was introduced in [17] using the ν-Caputo delta difference operator. The fractional map, which proved to be chaotic for proper values of the system parameters (α, β) and of the fractional order $\nu \in (0, 1]$, possesses only a nonlinear term [17].

Herein, the fractional Caputo h-difference operator is considered, in order to derive a different mathematical model of the 3D Grassi–Miller map. Namely, the following equations are proposed:

$$\left\{ {}^C_h\Delta_a^\nu x(t) = \alpha - y^2(t+\nu h) - \beta z(t+\nu h) - x(t+\nu h), {}^C_h\Delta_a^\nu y(t) = x(t+\nu h) - y(t+\nu h), {}^C_h\Delta_a^\nu z(t) = y(t+\nu h) - z(t+\nu h), \right. \quad (7)$$

where $^C_h\Delta_a^\nu$ denotes the fractional h-difference operator, $t \in (h\mathbb{N})_{a+(n-\nu)h}$, a is the starting point, and (α, β) are system parameters. The fractional map (7) can be considered a generalized model of the map introduced in [17].

The solution of the fractional Grassi–Miller map (7) is obtained by introducing the fractional h-sum operator. According to [15], the equivalent implicit discrete formula can be written in the form

$$\begin{cases} x(n+1) = x(0) + \dfrac{h^\nu}{\Gamma(\nu)} \displaystyle\sum_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} \left(\alpha - y^2(j+1) - \beta z(j+1) - x(j+1)\right), \\ y(n+1) = y(0) + \dfrac{h^\nu}{\Gamma(\nu)} \displaystyle\sum_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (x(j+1) - y(j+1)), \\ z(n+1) = z(0) + \dfrac{h^\nu}{\Gamma(\nu)} \displaystyle\sum_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (y(j+1) - z(j+1)), \end{cases} \quad (8)$$

where $x(0)$, $y(0)$, and $z(0)$ are the initial state values. Based on predictor-corrector method [14], the implicit equation (8) is transformed into its explicit form, which can be used for investigating the dynamic behavior of the Grassi–Miller map (7). By taking the initial state values $x(0) = 1$, $y(0) = 0.1$, and $z(0) = 0$, with the fractional order value $\nu = 0.999$ and the system parameters $\alpha = 1$, $\beta = 0.5$, it can be shown that map (7) displays the attractor reported in Figure 1. The computation of the bifurcation diagram and of the largest Lyapunov exponent, both reported in Figure 2 as a

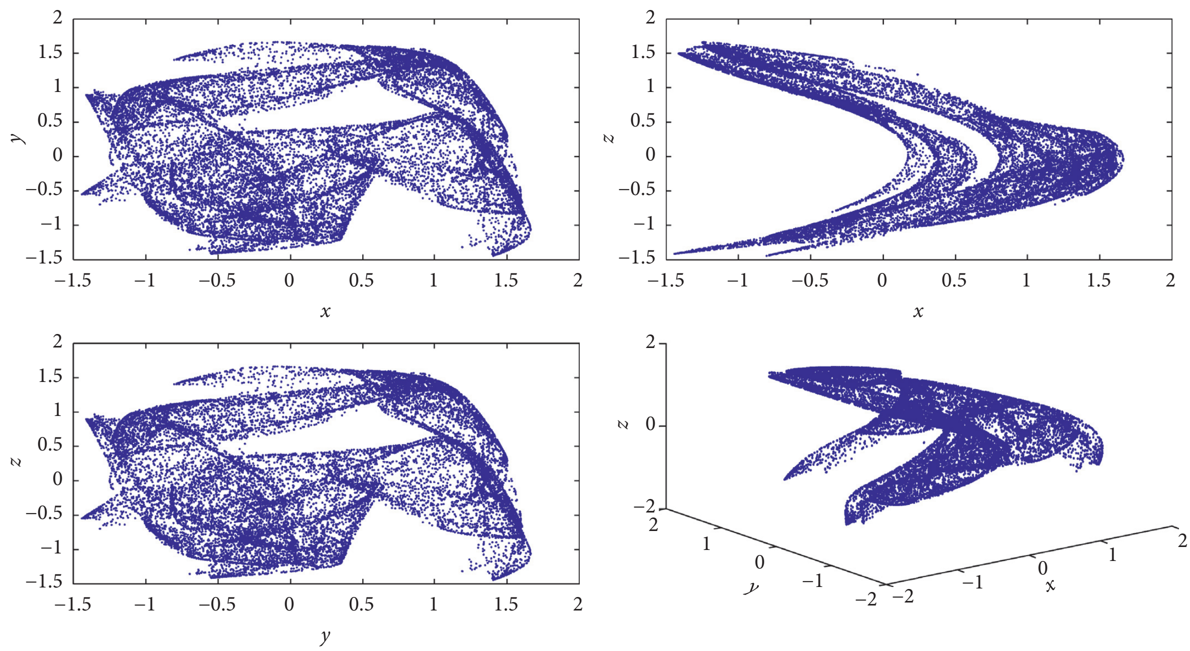

Figure 1: Chaotic attractor of the fractional order Grassi–Miller map for $\alpha = 1$, $\beta = 0.5$, and order $\nu = 0.999$.

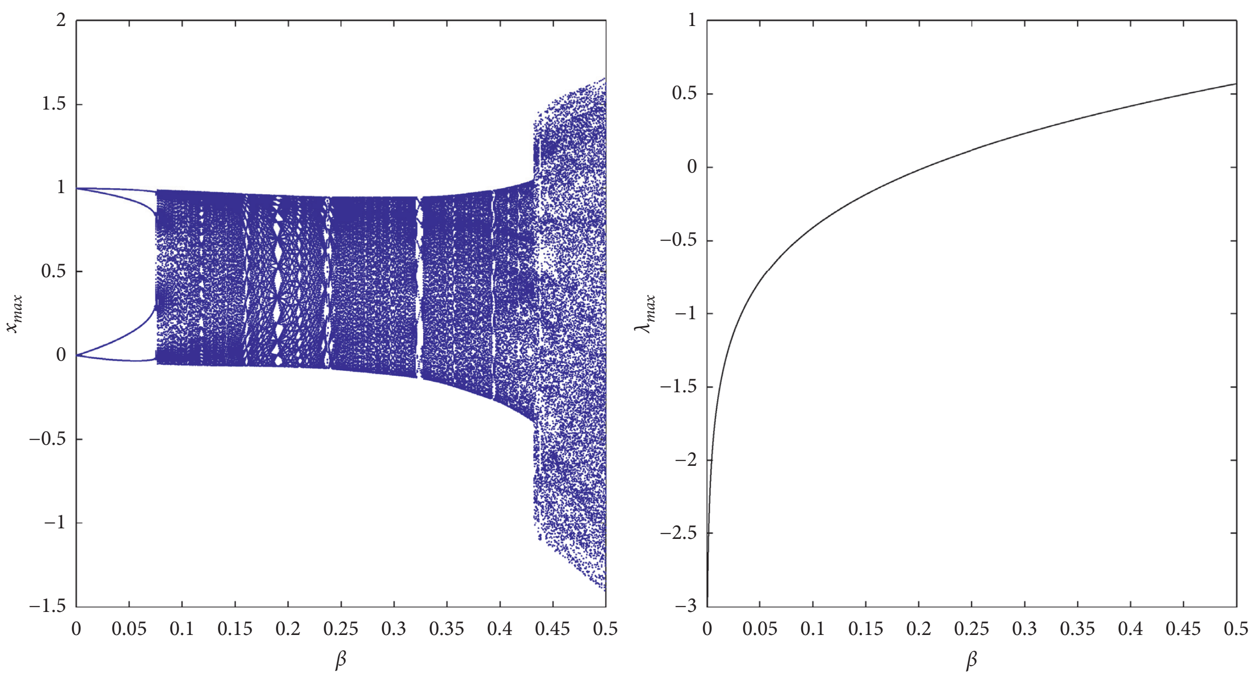

Figure 2: Bifurcation and largest Lyapunov exponent plots versus system parameter β for fractional order $\nu = 0.999$.

function of the system parameter β, clearly highlights the chaotic behavior of the fractional Grassi–Miller map (7) for $\alpha = 1$, $\beta = 0.5$, and $\nu = 0.999$. Regarding the bifurcation diagram reported in Figure 2, it can be noted that the map oscillates when β assumes values around 0.05. When β approaches the value of 0.1, more complex dynamic regimes appear, until β approaches the value of 0.45, when chaotic behaviours are reached. Note that the presence of chaos for $0.45 < \beta < 0.5$ is also confirmed by the positive values assumed by the maximum Lyapunov exponents (see Figure 2). Note that the evolution of states of the fractional map (7), which involves the adoption of the Caputo h-difference operator, is different from those of the map reported in [17], the latter being based on the ν-Caputo delta difference operator. This can be clearly seen by comparing the shapes of the chaotic attractors reported in Figure 1 with those of the attractors reported in [17]. Namely, the adoption of two different fractional operators has led to different shapes in the chaotic attractors as well as different parameter values for generating chaos (see [17]), indicating that the proposed Grassi–Miller map (7) provides a contribution to the topic of 3D discrete-time fractional systems.

Referring to potential applications of the proposed model (7), it should first be noted that 3D maps highlight a higher degree of complexity with respect to 1D and 2D maps [5, 21]. Thus, the applicability of the conceived 3D map (7) would mainly be in pseudo-random number generators and image encryption techniques. This makes perceive the importance of developing simple and feasible control methods, given that master-slave synchronization schemes based on model (7), in combination with encryption algorithms, might be used for experimentally generating and recovering the secret keys.

3. Chaos Control of the New Version of the Grassi–Miller Map

Here, a controller is presented in order to stabilize at zero the chaotic trajectories of the state-variables in the Grassi–Miller map (7) with fractional order. The objective is achieved by adding two linear terms into both first and second equations of the map. Namely, the controlled fractional Grassi–Miller chaotic map is described by

$$\left\{ {}_{h}^{C}\Delta_{a}^{\nu}x(t) = \alpha - y^{2}(t+\nu h) - \beta z(t+\nu h) - x(t+\nu h) + \mathbf{C}_{1}(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu}y(t) = x(t+\nu h) - y(t+\nu h) + \mathbf{C}_{2}(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu}z(t) = y(t+\nu h) - z(t+\nu h), \right. \tag{9}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are suitable controllers to be determined. To this purpose, a theorem is now given for rigorously assuring that the dynamics of (9) can be stabilized at zero.

Theorem 2. *The three-dimensional fractional Grassi–Miller map (9) is controlled at the origin under the following control laws:*

$$\begin{cases} \mathbf{C}_{1}(t) = -\alpha + \beta z(t) - y(t), \\ \mathbf{C}_{2}(t) = -b_{1}y(t) - z(t), \end{cases} \tag{10}$$

where $|x(t)| \le b_1$, $\forall t \in (h\mathbb{N})_{a+(n-\nu)h}$.

Proof of Theorem 2. By subtracting (10) into system (9), we get the following fractional difference equations:

$$\left\{ {}_{h}^{C}\Delta_{a}^{\nu}x(t) = -y^{2}(t+\nu h) - x(t+\nu h) - y(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu}y(t) = x(t+\nu h) - (1+b_{1})y(t+\nu h) - z(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu}z(t) = y(t+\nu h) - z(t+\nu h). \right. \tag{11}$$

By taking a Lyapunov function in the form $V = (1/2)(x^2(t) + y^2(t) + z^2(t))$, the adoption of the Caputo h-difference operator implies that

$${}_{h}^{C}\Delta_{a}^{\nu}xV = \frac{1}{2}{}_{h}^{C}\Delta_{a}^{\nu}xx^{2}(t) + \frac{1}{2}{}_{h}^{C}\Delta_{a}^{\nu}xy^{2}(t) + \frac{1}{2}{}_{h}^{C}\Delta_{a}^{\nu}xz^{2}(t). \tag{12}$$

By using Lemma 1, it follows that

$$\begin{aligned} {}_{h}^{C}\Delta_{a}^{\nu}V &\le x(t+\nu h){}_{h}^{C}\Delta_{a}^{\nu}x(t) + y(t+\nu h){}_{h}^{C}\Delta_{a}^{\nu}y(t) + z(t+\nu h){}_{h}^{C}\Delta_{a}^{\nu}z(t) \\ &= -x(t+\nu h)y^{2}(t+\nu h) - x^{2}(t+\nu h) - x(t+\nu h)y(t+\nu h) + y(t+\nu h)x(t+\nu h) \\ &\quad - (1+b_{1})y^{2}(t+\nu h) - y(t+\nu h)z(t+\nu h) + z(t+\nu h)y(t+\nu h) - z^{2}(t+\nu h) \\ &\le |x(t+\nu h)|y^{2}(t+\nu h) - x^{2}(t+\nu h) - (1+b_{1})y^{2}(t+\nu h) - z^{2}(t+\nu h) \\ &\le b_{1}y^{2}(t+\nu h) - x^{2}(t+\nu h) - (1+b_{1})y^{2}(t+\nu h) - z^{2}(t+\nu h) \\ &= -x^{2}(t+\nu h) - y^{2}(t+\nu h) - z^{2}(t+\nu h) < 0. \end{aligned} \tag{13}$$

From Theorem 1, it can be concluded that the zero equilibrium of (9) is asymptotically stable. As a consequence, it is proved that the dynamics of the proposed 3D Grassi–Miller map (7) are stabilized at the origin by the linear control laws (10). □

Remark 1. Since all the chaotic states of map (9) are bounded, it can be deduced that it is easy to find a parameter b_1 larger than the absolute value of the state variable $x(t)$, as requested by the proof of Theorem 2. Namely, the existence of b_1 is intrinsically justified by the property of boundedness of the state $x(t)$. Thus, the value of b_1 can be easily found by looking at the plots reported in Figure 1, from which it is clear that $-1.6 < x(t) < 1.6$ for any t. Through the paper, the value of b_1 has been selected as $b_1 = 1.7$. Note that the value of b_1 does not significantly affect the time for stabilizing the map dynamics.

Now, we give the numerical simulation to prove the above theory. We select $\alpha = 1$ and $\beta = 0.5$, and we give the evolution of the states and the phase-space plots as shown in Figure 3 for $\nu = 0.999$. These plots clearly show that the new fractional map (7) is driven to the origin by linear control laws in the form (10).

Now comparisons are carried out with recent results regarding 3D fractional maps, with the aim to confirm the effectiveness of the conceived approach when comparing control strategies for maps of similar degree of complexity. For example, the results in [11] show that the 3D fractional map proposed therein is stabilized after more than 20 steps, whereas the map illustrated herein is stabilized in at most 3 steps. On the other hand, the results in [19] highlight that the 3D fractional map proposed therein is stabilized in the same number of steps taken by our method. However, the control law adopted in [19] is complex, since it involves some nonlinear terms, whereas the proposed control strategies is simple and involves only linear terms. Finally, the results in [4] show that the 3D fractional Grassi–Miller map proposed therein, based on the-Caputo delta difference, is stabilized after more than 20 steps, whereas the map illustrated herein, based on the Caputo h-difference operator, is stabilized in at most 3 steps. These comparisons make us perceive the effectiveness of the proposed control strategy with respect to 3D fractional maps of similar complexity published in recent literature.

3.1. Synchronization of the Fractional Grassi–Miller Map. In this paragraph, a master-slave system, based on two identical chaotic fractional Grassi–Miller maps, is synchronized using linear controllers. The dynamics of the master system can be written as follows:

$$\left\{ {}_{h}^{C}\Delta_{a}^{\nu} x_m(t) = \alpha - y_m^2(t+\nu h) - \beta z_m(t+\nu h) - x_m(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} y_m(t) = x_m(t+\nu h) - y_m(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} z_m(t) = y_m(t+\nu h) - z_m(t+\nu h), \right. \tag{14}$$

where $x_m(t)$, $y_m(t)$, and $z_m(t)$ are the system states. The equations of the slave system are given by

$$\left\{ {}_{h}^{C}\Delta_{a}^{\nu} x_s(t) = \alpha - y_s^2(t+\nu h) - \beta z_s(t+\nu h) - x_s(t+\nu h) + \mathbf{L}_1(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} y_s(t) = x_s(t+\nu h) - y_s(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} z_s(t) = y_s(t+\nu h) - z_s(t+\nu h) + \mathbf{L}_2(t+\nu h), \right. \tag{15}$$

where $x_s(t)$, $y_s(t)$, and $z_s(t)$ are the system states, whereas $\mathbf{L}_1$ and $\mathbf{L}_2$ are suitable linear controllers to be determined. We subtract master system (14) from the slave system (15) to get the error system as

$$(e_1(t), e_2(t), e_3(t))^T = (x_s(t), y_s(t), z_s(t))^T - (x_m(t), y_m(t), z_m(t))^T. \tag{16}$$

Now a theorem involving a Lyapunov-based approach is proved, with the aim of synchronizing the master-slave (14) and (15) via linear controllers $\mathbf{L}_1$ and $\mathbf{L}_2$.

Theorem 3. *The master system (14) and the slave system (15) achieve synchronized dynamics, provided that the linear control laws $\mathbf{L}_1$ and $\mathbf{L}_2$ are selected as*

$$\begin{cases} \mathbf{L}_1(t) = \left(1 - \left(b_2 + \dfrac{1}{2}\right)^2\right) e_1(t), \\ \mathbf{L}_2(t) = \beta e_1(t) - e_2(t), \end{cases} \tag{17}$$

where $|y_m(t)| = |y_s(t)| \leq b_2$, $t \in (h\mathbb{N})_{a+(n-\nu)h}$.

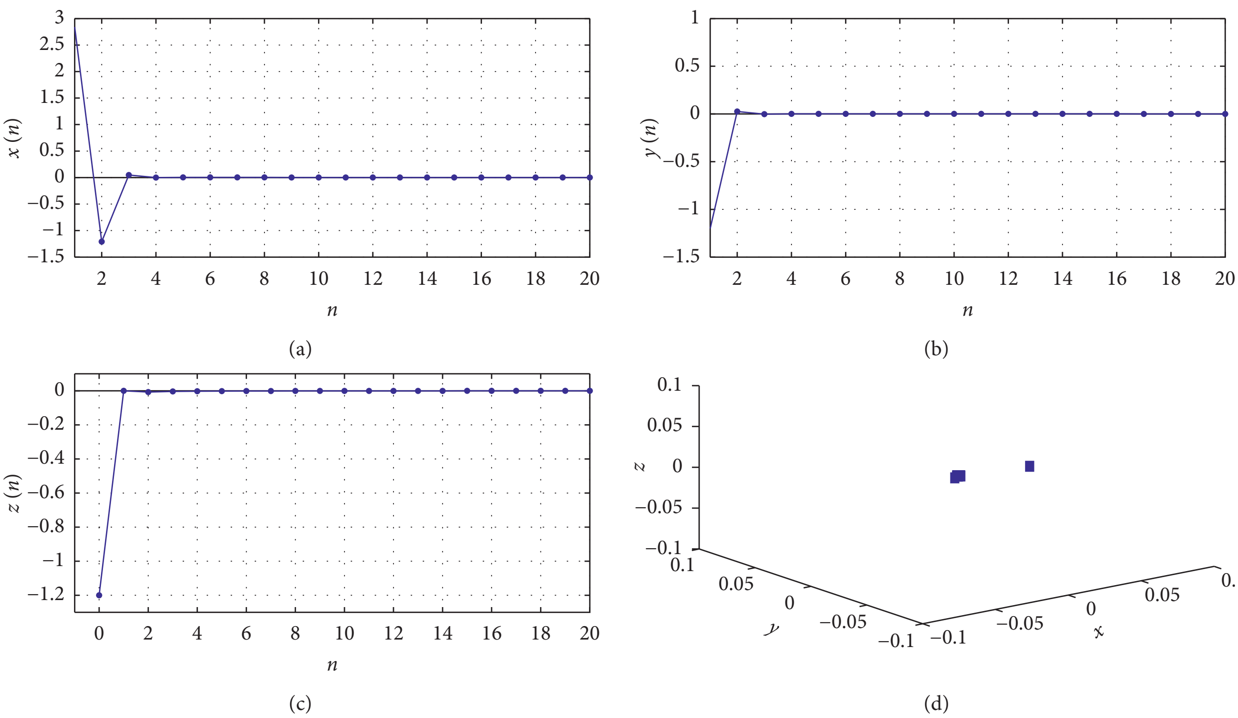

FIGURE 3: Stabilized states of the controlled fractional Grassi–Miller map (9) via linear control laws (10) with $\alpha = 1$, $\beta = 0.5$, and fractional order $\nu = 0.999$.

Proof of Theorem 3. By taking into account (16), the dynamics of the error system can be written as

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu} e_1(t) = y_m^2(t+\nu h) - y_s^2(t+\nu h) - \beta e_3(t+\nu h) - e_1(t+\nu h) + \mathbf{L}_1(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} e_2(t) \\ \qquad = e_1(t+\nu h) - e_2(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} e_3(t) = e_2(t+\nu h) - e_3(t+\nu h) + \mathbf{L}_2(t+\nu h). \end{cases} \tag{18}$$

By substituting the control law (17) into error system (18), we get

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu} e_1(t) = -(y_m(t+\nu h) + y_s(t+\nu h))e_2(t+\nu h) - \beta e_3(t+\nu h) - \left(b_2 + \frac{1}{2}\right)^2 e_1(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} e_2(t) \\ \qquad = e_1(t+\nu h) - e_2(t+\nu h), {}_{h}^{C}\Delta_{a}^{\nu} e_3(t) = \beta e_1(t+\nu h) - e_3(t+\nu h). \end{cases} \tag{19}$$

Now, by taking a Lyapunov function in the form $V = (1/2)(e_1^2(t) + e_2^2(t) + e_3^2(t))$ and by exploiting Lemma 1, it follows that

$$
\begin{aligned}
{}_h^C\Delta_a^{\nu} V &\leq e_1(t+\nu h)\,{}_h^C\Delta_a^{\nu} e_1(t) + e_2(t+\nu h)\,{}_h^C\Delta_a^{\nu} e_2(t) + e_3(t+\nu h)\,{}_h^C\Delta_a^{\nu} e_3(t) \\
&= -\left(b_2 + \frac{1}{2}\right)^2 e_1^2(t+\nu h) - \left(y_m(t+\nu h) + y_s(t+\nu h)\right) e_1(t+\nu h) e_2(t+\nu h) \\
&\quad - \beta e_1(t+\nu h) e_3(t+\nu h) + e_2(t+\nu h) e_1(t+\nu h) - e_2^2(t+\nu h) + \beta e_1(t+\nu h) e_3(t+\nu h) - e_3^2(t+\nu h) \\
&\leq -\left(b_2 + \frac{1}{2}\right)^2 e_1^2(t+\nu h) + \left(1 + \left|y_m(t+\nu h) + y_s(t+\nu h)\right|\right) \\
&\quad \left|e_1(t+\nu h)\right|\left|e_2(t+\nu h)\right| - e_2^2(t+\nu h) - e_3^2(t+\nu h) \\
&\leq -\left(b_2 + \frac{1}{2}\right)^2 e_1^2(t+\nu h) + \left(1 + 2b_2\right)\left|e_1(t+\nu h)\right|\left|e_2(t+\nu h)\right| - e_2^2(t+\nu h) - e_3^2(t+\nu h) \\
&= -\left(\left(b_2 + \frac{1}{2}\right)^2 e_1(t+\nu h) - e_2(t+\nu h)\right)^2 - e_3(t+\nu h) \leq 0.
\end{aligned}
\tag{20}
$$

From Theorem 1, it can be concluded that the dynamics of the error system (18) are stabilized at the origin. As a consequence, it is proved that the master system (14) and the slave system (15) achieve synchronized dynamics via linear control laws in the form (17). □

Remark 2. It is easy to find a parameter b_2 larger than the absolute value of the variables $y_m(t) = y_s(t)$, as requested by the proof of Theorem 3. Namely, the existence of b_2 is intrinsically justified by the property of boundedness of the chaotic states of map (9). Thus, the value of b_2 can be easily found by looking at the plots reported in Figure 1, from which it is clear that $-1.6 < y(t) < 1.6$ for any t. Herein, in order to achieve synchronization, the value of b_2 has been selected as $b_2 = 2$. Note that the value of b_2 does not significantly affect the time for synchronizing the master-slave pair.

In order to show the effectiveness of the proposed approach, Figure 4 displays the chaotic dynamics of the master system states (blue color) and of the slave system states (red color) when $\alpha = 1, \beta = 0.5$, and $\nu = 0.999$. These plots clearly show that two identical Grassi–Miller maps achieve chaos synchronization via linear controllers. Note that, through the manuscript, all the simulation results and the related figures have been obtained using the software MATLAB.

Now, we would discuss the issue regarding the complexity of the proposed method. We would observe that the approach proposed herein is simpler than similar methods reported in literature. For example, the techniques developed in [11, 17, 19] present the drawback that very complex control laws have been exploited for controlling and synchronizing the corresponding 3D fractional maps. For example, in [11, 19] synchronization and control have been achieved using nonlinear control laws that include several nonlinear terms. This drawback also regards the Grassi–Miller map in [17], since complex nonlinear control laws have been used to achieve synchronization and control of its chaotic dynamics. Since it might be difficult to implement very complex control laws in practical applications of fractional maps, this paper has provided a contribution to the topic by developing simple linear control laws for stabilizing and synchronizing 3D fractional maps.

Referring to synchronization issues, now comparisons are carried out with recent results regarding 3D fractional maps. The objective is to highlight the effectiveness of the conceived approach when synchronization involves 3D maps with similar degree of complexity. For example, the results in [11] show that synchronization for the 3D fractional map proposed therein is achieved after more than 10 steps, whereas the map illustrated herein can be synchronized in at most 3 steps. On the other hand, the results in [19] show that synchronization for the 3D fractional map proposed therein is achieved in the same number of steps taken by our method. However, [19] exploits a complex control law that involves some nonlinear terms, whereas the proposed synchronization technique is simple and involves only linear terms. Finally, the results in [17] show that synchronization for 3D fractional Grassi–Miller map proposed therein, based on the ν-Caputo delta difference, is achieved after more than 20 steps, whereas the map illustrated herein, based on the Caputo h-difference operator, achieves synchronized dynamics in at most 3 steps. These comparisons make us perceive the effectiveness of the proposed synchronization strategy with respect to 3D fractional maps of similar complexity published in recent literature.

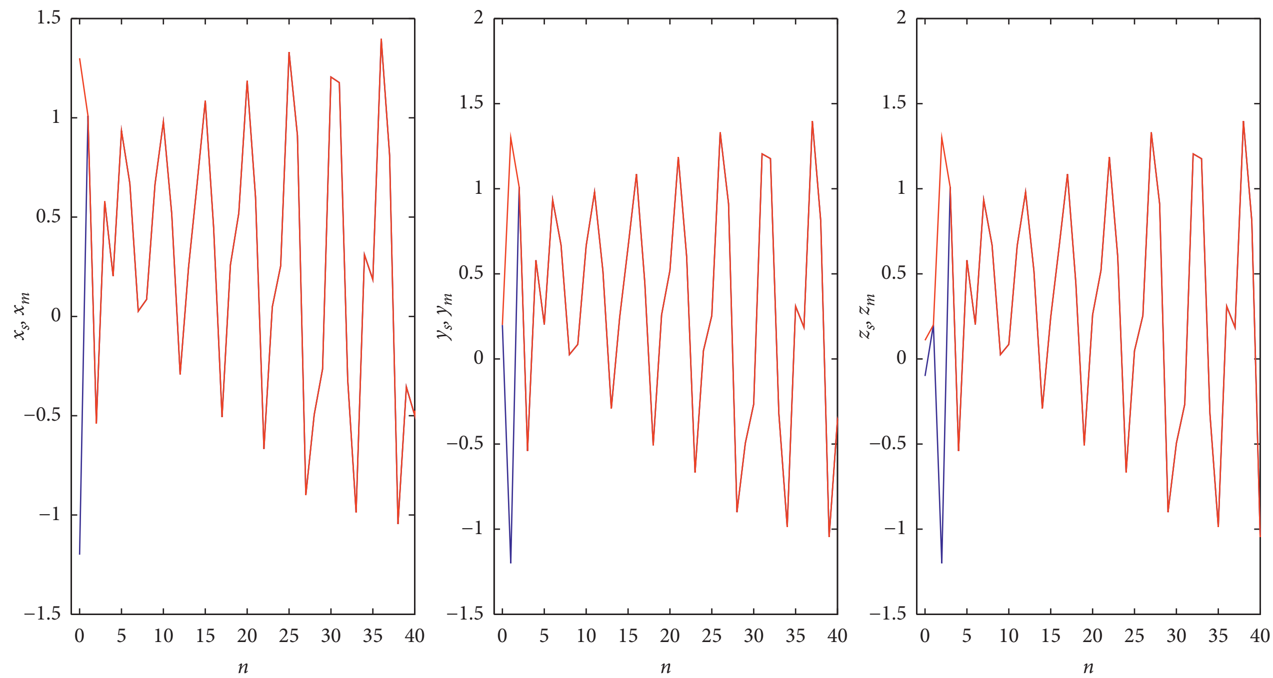

FIGURE 4: Synchronization of two identical Grassi–Miller map with $\alpha = 1$, $\beta = 0.5$, and $\nu = 0.999$: evolutions of the master system states (blue color) and of the slave system states (red color).

Finally, we would briefly discuss the potential applications of the conceived approach in real world. As any 3D map, the Grassi–Miller map highlights a higher degree of complexity with respect to 1D or 2D fractional maps. This increased complexity can be very useful for pseudo-random number generators in chaos-based communications systems. Moreover, since the production of images is increasing day by day in real life, confidentiality and privacy are becoming key issues when transmitting digital images using portable devices. Thus, referring to secure image transmission, the proposed discrete-time synchronization scheme could be utilized for retrieving the secrets keys at the receiver side in chaos-based image encryption systems.

4. Conclusions and Future Work

By including Grassi as a coauthor, this paper has presented a novel version of the chaotic fractional Grassi–Miller map, based on the Caputo h-difference operator. Two novel theorems have been proved, with the aim of deriving improved schemes (with respect to those presented in [17]) for controlling and synchronizing the dynamics of the map. Namely, while synchronization and control in [17] are achieved via more complex nonlinear control laws, herein, simple linear controllers have been conceived. Finally, simulation results have been carried out to highlight the effectiveness of the proposed method. Referring to future improvements of the conceived approach, our plan is to make an attempt to further simplify the control laws developed herein. Specifically, the objective is to reduce the control law (10) to just one term $C(t)$, instead of having two terms $C_1(t)$ and $C_2(t)$. Furthermore, regarding synchronization, we will try to reduce the control law (17) to just one term $L(t)$, instead of having two terms $L_1(t)$ and $L_2(t)$.

By exploiting the results achieved herein, our future work will focus on two main steps. At first, we will implement the proposed Grassi–Miller map using an Arduino board, with the aim of experimentally showing the high degree of complexity generated by fractional 3D maps. Then, the second step will consist in applying the conceived linear controllers to image encryption. Namely, our plan is to implement in hardware the proposed master-slave synchronization scheme, which will be used in combination with an encryption algorithm to experimentally generate and recover the secret keys.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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