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Fractional Integral Inequalities for Exponentially Nonconvex Functions and Their Applications

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Abstract: In this paper, the authors define a new generic class of functions involving a certain modified Fox–Wright function. A useful identity using fractional integrals and this modified Fox–Wright function with two parameters is also found. Applying this as an auxiliary result, we establish some Hermite–Hadamard-type integral inequalities by using the above-mentioned class of functions. Some special cases are derived with relevant details. Moreover, in order to show the efficiency of our main results, an application for error estimation is obtained as well.

Keywords: Hermite–Hadamard inequality; exponentially nonconvex function; modified Fox–Wright function; fractional integrals; error estimation

1. Introduction and Preliminaries

In many problems in mathematics and its applications, fractional calculus has a crucial role (see [1–6]). The analysis of the uniqueness of fractional ordinary differential equations can be accomplished by using fractional integral inequalities (see [7–9]).

Integral inequalities play a major role in the fields of differential equations and applied mathematics (see [10,11]). Moreover, they are linked with such other areas as differential equations, difference equations, mathematical analysis, mathematical physics, convexity theory, and discrete fractional calculus (see [12–18]).

Convexity is a fascinating and natural concept; it is beneficial in optimization theory, the theory of inequalities, numerical analysis, economics, and in other fields of pure and applied mathematics.

The notion of the h -convex function is introduced below.

Definition 1 (see [19]). Let $h : [0, 1] \rightarrow [0, \infty)$ be a function. A function $\psi : I \rightarrow \mathbb{R}$ is said to be h -convex if

$$\psi(\iota\zeta_1 + (1 - \iota)\zeta_2) \leq h(\iota)\psi(\zeta_1) + h(1 - \iota)\psi(\zeta_2)$$

holds true for every $\zeta_1, \zeta_2 \in I$ and $\iota \in [0, 1]$.

The following class of functions was introduced by Awan et al. (see [20]) and was demonstrated to play an important role in optimization theory and mathematical economics.

Definition 2. A function $\psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called exponentially convex if

$$\psi(\iota\zeta_1 + (1-\iota)\zeta_2) \leq \iota \frac{\psi(\zeta_1)}{e^{\omega\zeta_1}} + (1-\iota) \frac{\psi(\zeta_2)}{e^{\omega\zeta_2}}$$

holds true for all $\zeta_1, \zeta_2 \in I$, $\omega \in \mathbb{R}$ and $\iota \in [0, 1]$.

The most significant inequality about a convex function ψ on the closed interval $[\zeta_1, \zeta_2]$ is the Hermite–Hadamard integral inequality (that is, the trapezium inequality). This two-sided inequality is expressed as follows:

$$\psi\left(\frac{\zeta_1 + \zeta_2}{2}\right) \leq \frac{1}{\zeta_2 - \zeta_1} \int_{\zeta_1}^{\zeta_2} \psi(\iota) d\iota \leq \frac{\psi(\zeta_1) + \psi(\zeta_2)}{2}. \quad (1)$$

The two-sided inequality (1) has become a very important foundation within the field of mathematical analysis and optimization. Several applications of inequalities of this type have been derived in a number of different settings (see [21–29]).

In the context of fractional calculus, the standard left and right-sided Riemann–Liouville (RL) fractional integrals of order $\alpha > 0$ are given, respectively, by

$$\begin{aligned} \mathcal{I}_{\zeta_1^+}^\alpha \psi(x) &= \frac{1}{\Gamma(\alpha)} \int_{\zeta_1}^x (x-\iota)^{\alpha-1} \psi(\iota) d\iota \quad (x > \zeta_1) \\ \text{and} \\ \mathcal{I}_{\zeta_2^-}^\alpha \psi(x) &= \frac{1}{\Gamma(\alpha)} \int_x^{\zeta_2} (\iota-x)^{\alpha-1} \psi(\iota) d\iota \quad (x < \zeta_2), \end{aligned} \quad (2)$$

where ψ is a function defined on the closed interval $[\zeta_1, \zeta_2]$ and $\Gamma(\cdot)$ is the classical (Euler’s) gamma function.

Regarding information for some of the fractional integral operators, including those that are known as Erdélyi–Kober, Riemann–Liouville (RL), Weyl and Liouville–Caputo (LC) operators, see [30–34].

There are many directions in which one can introduce a new definition of fractional derivatives and fractional integrals, which are related to or inspired by (for example) the RL definitions (see [35,36]), with reference to some general classes into which such fractional calculus operators can be classified. In applied mathematics, it is important to consider particular types of fractional calculus operators which are suited to the fractional-order modeling of a given real-world problem.

We now recall the familiar Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ (with p numerator and q denominator parameters), which is given by the following series (see [5] (p. 67, Equation (1.12(68))) and [37] (p. 21, Equation (1.2(38)))):

$${}_p\Psi_q \left[\begin{matrix} (\alpha_1, \mathcal{U}_1), \dots, (\alpha_p, \mathcal{U}_p); \\ (\beta_1, \mathcal{V}_1), \dots, (\beta_q, \mathcal{V}_q); \end{matrix} z \right] := \sum_{n=0}^{\infty} \frac{\prod_{\ell=1}^p \Gamma(\alpha_\ell + n\mathcal{U}_\ell)}{\prod_{j=1}^q \Gamma(\beta_j + n\mathcal{V}_j)} \frac{z^n}{n!}, \quad (3)$$

where the parameters

$$\alpha_\ell, \beta_j \in \mathbb{C} \quad (\ell = 1, \dots, p; j = 1, \dots, q)$$

and the coefficients

$$\mathcal{U}_1, \dots, \mathcal{U}_p \in \mathbb{R}^+ \quad \text{and} \quad \mathcal{V}_1, \dots, \mathcal{V}_q \in \mathbb{R}^+$$

satisfy the following condition:

$$1 + \sum_{j=1}^q \mathcal{V}_j - \sum_{\ell=1}^p \mathcal{U}_\ell \geq 0. \tag{4}$$

Here and in what follows, we have made use of the general Pochhammer symbol $(\eta)_\nu$ ($\eta, \nu \in \mathbb{C}$) defined by

$$(\eta)_\nu := \frac{\Gamma(\eta + \nu)}{\Gamma(\eta)} = \begin{cases} 1 & (\nu = 0; \eta \in \mathbb{C} \setminus \{0\}) \\ \eta(\eta + 1) \cdots (\eta + \nu - 1) & (\nu = n \in \mathbb{N}; \eta \in \mathbb{C}), \end{cases} \tag{5}$$

it being assumed *conventionally* that $(0)_0 := 1$ and understand *tacitly* that the Γ -quotient in (5) exists.

The following modified version of the Fox–Wright function ${}_p\Psi_q(z)$ in (3) was introduced, as long ago as 1940, by Wright [38] (p. 424), who partially and formally replaced the Γ -quotient in (3) by a sequence $\{\sigma(n)\}_{n=0}^\infty$ based upon a suitably-restricted function $\sigma(\tau)$ as follows (see also [39], where the same definition is reproduced without giving credit to Wright [38]):

$$\mathcal{F}_{\rho,\varsigma}^\sigma(z) = \mathcal{F}_{\rho,\varsigma}^{\sigma(0),\sigma(1),\dots}(z) = \sum_{\ell=0}^\infty \frac{\sigma(\ell)}{\Gamma(\rho\ell + \varsigma)} z^\ell \quad (\rho > 0; \varsigma > 0). \tag{6}$$

If, in Wright’s definition (6) from 1940 (see [38] (p. 424)), we take $\rho = \varsigma = 1$ and

$$\sigma(\ell) = \frac{\prod_{j=1}^p \Gamma(\alpha_j + \mathcal{U}_j\ell)}{\prod_{k=1}^q \Gamma(\beta_k + \mathcal{V}_k\ell)} \quad (\ell = 0, 1, 2, \dots),$$

then Wright’s definition (6) would immediately yield the familiar Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (3). The one- and two-parameter Mittag–Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$, and indeed also almost all of the parametric generalizations of the Mittag–Leffler type functions, can be deduced as obvious special cases of the Fox–Wright hypergeometric function ${}_p\Psi_q(z)$ defined by (3) (see [40] for details).

We are now in the position to introduce a new generic class of functions involving the modified Fox–Wright function $\mathcal{F}_{\rho,\varsigma}^\sigma(\cdot)$.

Definition 3. Let $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$ be two functions and $\psi : \mathbb{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$. If ψ satisfies the following inequality,

$$\psi\left(\xi_1 + \iota \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)\right) \leq h_1(\iota) \frac{\psi(\xi_1)}{e^{\omega_1 \xi_1}} + h_2(\iota) \frac{\psi(\xi_2)}{e^{\omega_2 \xi_2}}$$

for all $\iota \in [0, 1]$, $\omega_1, \omega_2 \in \mathbb{R}$, and $\xi_1, \xi_2 \in \mathbb{I}$, where $\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) > 0$, then ψ is called an exponentially $(\omega_1, \omega_2, h_1, h_2)$ -nonconvex function.

Remark 1. Upon setting

$$\omega_1 = \omega_2 = \omega, \quad h_1(\iota) = 1 - \iota, \quad h_2(\iota) = \iota$$

and

$$\mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1) = \xi_2 - \xi_1$$

in Definition 3, we then obtain Definition 2.

Remark 2. Some special cases of our Definition 3 are listed below:

- (I) Taking $h_1(t) = h_2(t) = 1$, we have an exponentially (ω_1, ω_2, P) -nonconvex function.
- (II) Choosing $h_1(t) = h(1-t)$ and $h_2(t) = h(t)$, we obtain an exponentially (ω_1, ω_2, h) -nonconvex function.
- (III) Setting $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$ for $s \in (0, 1]$, we obtain an exponentially (s, ω_1, ω_2) -Breckner-nonconvex function.
- (IV) Putting $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ for $s \in (0, 1)$, we obtain an exponentially (s, ω_1, ω_2) -Godunova-Levin-Dragomir-nonconvex function.
- (V) Taking $h_1(t) = h_2(t) = t(1-t)$, we obtain an exponentially $(\omega_1, \omega_2, tgs)$ -nonconvex function.

Our paper has the following structure: in Section 2, we first find a useful identity using fractional integrals with two parameters λ and μ involving the modified Fox–Wright function $\mathcal{F}_{\rho,\varsigma}^{\sigma}(\cdot)$. Applying this as an auxiliary result, we give some Hermite–Hadamard-type integral inequalities pertaining to exponentially $(\omega_1, \omega_2, h_1, h_2)$ -nonconvex functions, and some special cases are derived in details. In Section 3, the efficiency of our main results is demonstrated with an application for error estimation. Section 4 presents the conclusion of this paper.

2. Main Results and Their Consequences

The following notations are used below:

$$\Delta := [\xi_1, \xi_1 + \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)],$$

where

$$\mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1) > 0$$

and Δ° is the interior of the closed interval Δ with $\omega_1, \omega_2 \in \mathbb{R}$. We denote by $\mathcal{L}_1(\Delta)$ the space of integrable functions over Δ . We need to prove the following basic lemma.

Lemma 1. Let the function $\psi : \Delta \rightarrow \mathbb{R}$ be differentiable on Δ° and $\lambda, \mu \in (0, 1]$. If $\psi' \in \mathcal{L}_1(\Delta)$, then, for $\alpha > 0$,

$$\begin{aligned} & \frac{\mu^\alpha \psi(\xi_1) + \lambda^\alpha \psi\left(\xi_1 + \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right)}{(\lambda + \mu)^\alpha} - \frac{\Gamma(\alpha + 1)}{\left[\mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right]^\alpha} \\ & \cdot \left[\mathcal{I}_{\xi_1^+}^\alpha \psi\left(\xi_1 + \frac{\mu}{\lambda + \mu} \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right) + \mathcal{I}_{\left(\xi_1 + \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right)^-}^\alpha \psi\left(\xi_1 + \frac{\mu}{\lambda + \mu} \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right) \right] \\ & = \frac{\mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \\ & \cdot \left[\int_0^\lambda t^\alpha \psi'\left(\xi_1 + \frac{\mu + t}{\lambda + \mu} \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right) dt - \int_0^\mu t^\alpha \psi'\left(\xi_1 + \frac{\mu - t}{\lambda + \mu} \mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)\right) dt \right]. \quad (7) \end{aligned}$$

Proof. We define

$$\mathcal{T}_{\lambda,\mu}^\alpha(\xi_1, \xi_2) := \frac{\mathcal{F}_{\rho,\varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} [\mathcal{I}_2 - \mathcal{I}_1], \quad (8)$$

where

$$\mathcal{I}_1 := \int_0^\mu t^\alpha \psi' \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) dt,$$

and

$$\mathcal{I}_2 := \int_0^\lambda t^\alpha \psi' \left(\xi_1 + \frac{\mu+t}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) dt,$$

which, upon integrating by parts, would yield

$$\begin{aligned} \mathcal{I}_1 &= - \frac{(\lambda+\mu)}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} t^\alpha \psi \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) \Big|_0^\mu \\ &\quad + \frac{\alpha(\lambda+\mu)}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \int_0^\mu t^{\alpha-1} \psi \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) dt \\ &= \Gamma(\alpha+1) \left(\frac{\lambda+\mu}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \right)^{\alpha+1} \\ &\quad \cdot \mathcal{I}_{\xi_1^+}^\alpha \psi \left(\xi_1 + \frac{\mu}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) - \frac{\mu^\alpha(\lambda+\mu)}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \psi(\xi_1). \quad (9) \end{aligned}$$

Similarly, we find that

$$\begin{aligned} \mathcal{I}_2 &= \frac{\lambda^\alpha(\lambda+\mu)}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \psi \left(\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) \\ &\quad - \Gamma(\alpha+1) \left(\frac{\lambda+\mu}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \right)^{\alpha+1} \cdot \mathcal{I}_{(\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1))}^\alpha \psi \left(\xi_1 + \frac{\mu}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right). \quad (10) \end{aligned}$$

Substituting from (9) and (10) into (8), we obtain the desired result (7). \square

From Lemma 1, we can derive the following case:

Remark 3. Taking $\alpha = 1$ in Lemma 1, we have

$$\begin{aligned} &\frac{\mu\psi(\xi_1) + \lambda\psi \left(\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right)}{\lambda+\mu} - \frac{1}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \int_{\xi_1}^{\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \psi(t) dt \\ &= \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda+\mu)^2} \left[\int_0^\lambda t \psi' \left(\xi_1 + \frac{\mu+t}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) dt \right. \\ &\quad \left. - \int_0^\mu t \psi' \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1) \right) dt \right]. \quad (11) \end{aligned}$$

Our first main result is stated as Theorem 1 below.

Theorem 1. Assume that $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$ are two continuous functions and let $\psi : \Delta \rightarrow \mathbb{R}$ be a differentiable function on Δ° with $\lambda, \mu \in (0, 1]$. Furthermore, let $\psi' \in \mathcal{L}_1(\Delta)$. If $|\psi'|^q$ is an exponentially $(\omega_1, \omega_2, h_1, h_2)$ -nonconvex function, then, for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > 0$, it is asserted that

$$|\mathcal{T}_{\lambda, \mu}^{\alpha}(\xi_1, \xi_2)| \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\mathcal{A}_1^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{H}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{H}_{1,2} \right)^{\frac{1}{q}} \right. \\ \left. + \mathcal{A}_2^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{H}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{H}_{2,2} \right)^{\frac{1}{q}} \right], \quad (12)$$

where

$$\mathcal{A}_1 := \frac{\lambda^{p\alpha+1}}{p\alpha+1}, \quad \mathcal{A}_2 := \frac{\mu^{p\alpha+1}}{p\alpha+1}, \\ \mathcal{H}_{1,1} := \int_0^{\lambda} h_1 \left(\frac{\mu+t}{\lambda+\mu} \right) dt, \quad \mathcal{H}_{1,2} := \int_0^{\lambda} h_2 \left(\frac{\mu+t}{\lambda+\mu} \right) dt, \\ \mathcal{H}_{2,1} := \int_0^{\mu} h_1 \left(\frac{\mu-t}{\lambda+\mu} \right) dt \quad \text{and} \quad \mathcal{H}_{2,2} := \int_0^{\mu} h_2 \left(\frac{\mu-t}{\lambda+\mu} \right) dt.$$

Proof. Applying Lemma 1, the property of the modulus, Hölder's inequality, and the exponential $(\omega_1, \omega_2, h_1, h_2)$ -nonconvexity of $|\psi'|^q$, we have

$$|\mathcal{T}_{\lambda, \mu}^{\alpha}(\xi_1, \xi_2)| \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\int_0^{\lambda} t^{\alpha} \left| \psi' \left(\xi_1 + \frac{\mu+t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right| dt \right. \\ \left. + \int_0^{\mu} t^{\alpha} \left| \psi' \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right| dt \right] \\ \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\left(\int_0^{\lambda} t^{p\alpha} dt \right)^{\frac{1}{p}} \left(\int_0^{\lambda} \left| \psi' \left(\xi_1 + \frac{\mu+t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_0^{\mu} t^{p\alpha} dt \right)^{\frac{1}{p}} \left(\int_0^{\mu} \left| \psi' \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\ \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\left(\int_0^{\lambda} t^{p\alpha} dt \right)^{\frac{1}{p}} \left(\int_0^{\lambda} \left\{ h_1 \left(\frac{\mu+t}{\lambda+\mu} \right) \frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + h_2 \left(\frac{\mu+t}{\lambda+\mu} \right) \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right\} dt \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_0^{\mu} t^{p\alpha} dt \right)^{\frac{1}{p}} \left(\int_0^{\mu} \left\{ h_1 \left(\frac{\mu-t}{\lambda+\mu} \right) \frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + h_2 \left(\frac{\mu-t}{\lambda+\mu} \right) \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right\} dt \right)^{\frac{1}{q}} \right] \\ = \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\mathcal{A}_1^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{H}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{H}_{1,2} \right)^{\frac{1}{q}} \right. \\ \left. + \mathcal{A}_2^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{H}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{H}_{2,2} \right)^{\frac{1}{q}} \right],$$

which completes the proof of Theorem 1. \square

Some corollaries and consequences of Theorem 1 are listed below:

Corollary 1. Upon setting $\alpha = 1$, Theorem 1 yields

$$\begin{aligned} & \left| \frac{\mu\psi(\xi_1) + \lambda\psi\left(\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)\right)}{\lambda + \mu} - \frac{1}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \int_{\xi_1}^{\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \psi(t) dt \right| \\ & \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^2} \left[\left(\frac{\lambda^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{H}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{H}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{\mu^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{H}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{H}_{2,2} \right)^{\frac{1}{q}} \right]. \quad (13) \end{aligned}$$

Corollary 2. Choosing $h_1(t) = h_2(t) = 1$ in Theorem 1, it is asserted that

$$\left| \mathcal{T}_{\lambda,\mu}^\alpha(\xi_1, \xi_2) \right| \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left(\mathcal{A}_1^{\frac{1}{p}} \lambda^{\frac{1}{q}} + \mathcal{A}_2^{\frac{1}{p}} \mu^{\frac{1}{q}} \right) \left[\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right]^{\frac{1}{q}}. \quad (14)$$

Corollary 3. Choosing

$$h_1(t) = (1-t)^s \quad \text{and} \quad h_2(t) = t^s \quad (s \in (0, 1])$$

in Theorem 1, it is asserted that

$$\begin{aligned} \left| \mathcal{T}_{\lambda,\mu}^\alpha(\xi_1, \xi_2) \right| & \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\mathcal{A}_1^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{D}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{D}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \mathcal{A}_2^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{D}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{D}_{2,2} \right)^{\frac{1}{q}} \right], \quad (15) \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_{1,1} & := \frac{\lambda^{s+1}}{(s+1)(\lambda + \mu)^s}, & \mathcal{D}_{1,2} & := \frac{(\lambda + \mu)^{s+1} - \mu^{s+1}}{(s+1)(\lambda + \mu)^s}, \\ \mathcal{D}_{2,1} & := \frac{(\lambda + \mu)^{s+1} - \lambda^{s+1}}{(s+1)(\lambda + \mu)^s} & \text{and} & \quad \mathcal{D}_{2,2} := \frac{\mu^{s+1}}{(s+1)(\lambda + \mu)^s}. \end{aligned}$$

Corollary 4. Taking

$$h_1(t) = (1-t)^{-s} \quad \text{and} \quad h_2(t) = t^{-s} \quad (s \in (0, 1))$$

in Theorem 1, the following inequality is deduced:

$$\begin{aligned} \left| \mathcal{T}_{\lambda,\mu}^\alpha(\xi_1, \xi_2) \right| & \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\mathcal{A}_1^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{F}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{F}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \mathcal{A}_2^{\frac{1}{p}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{F}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{F}_{2,2} \right)^{\frac{1}{q}} \right], \quad (16) \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}_{1,1} & := \frac{\lambda^{1-s}(\lambda + \mu)^s}{1-s}, & \mathcal{F}_{1,2} & := \frac{((\lambda + \mu)^{1-s} - \mu^{1-s})(\lambda + \mu)^s}{1-s}, \\ \mathcal{F}_{2,1} & := \frac{((\lambda + \mu)^{1-s} - \lambda^{1-s})(\lambda + \mu)^s}{1-s} & \text{and} & \quad \mathcal{F}_{2,2} := \frac{\mu^{1-s}(\lambda + \mu)^s}{1-s}. \end{aligned}$$

Corollary 5. For

$$h_1(t) = h_2(t) = t(1-t),$$

Theorem 1 yields

$$\left| \mathcal{T}_{\lambda, \mu}^{\alpha}(\xi_1, \xi_2) \right| \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left(\mathcal{A}_1^{\frac{1}{p}} \mathcal{G}_1^{\frac{1}{q}} + \mathcal{A}_2^{\frac{1}{p}} \mathcal{G}_2^{\frac{1}{q}} \right) \left[\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right]^{\frac{1}{q}}, \quad (17)$$

where

$$\mathcal{G}_1 := \frac{\lambda^2(3\mu + \lambda)}{6(\lambda + \mu)^2} \quad \text{and} \quad \mathcal{G}_2 := \frac{\mu^2(3\lambda + \mu)}{6(\lambda + \mu)^2}.$$

Our second main result is stated as Theorem 2 below.

Theorem 2. Assume that $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$ are two continuous functions and $\psi : \Delta \rightarrow \mathbb{R}$ is a differentiable function on Δ° with $\lambda, \mu \in (0, 1]$. Furthermore, let $\psi' \in \mathcal{L}_1(\Delta)$. If $|\psi'|^q$ be an exponentially $(\omega_1, \omega_2, h_1, h_2)$ -nonconvex function; then, for $q \geq 1$ and $\alpha > 0$, it is asserted that

$$\left| \mathcal{T}_{\lambda, \mu}^{\alpha}(\xi_1, \xi_2) \right| \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \left[\mathcal{B}_1^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{S}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{S}_{1,2} \right)^{\frac{1}{q}} \right. \\ \left. + \mathcal{B}_2^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{S}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{S}_{2,2} \right)^{\frac{1}{q}} \right], \quad (18)$$

where

$$\mathcal{B}_1 := \frac{\lambda^{\alpha+1}}{\alpha+1}, \quad \mathcal{B}_2 := \frac{\mu^{\alpha+1}}{\alpha+1}, \\ \mathcal{S}_{1,1} := \int_0^{\lambda} t^{\alpha} h_1 \left(\frac{\mu+t}{\lambda+\mu} \right) dt, \quad \mathcal{S}_{1,2} := \int_0^{\lambda} t^{\alpha} h_2 \left(\frac{\mu+t}{\lambda+\mu} \right) dt, \\ \mathcal{S}_{2,1} := \int_0^{\mu} t^{\alpha} h_1 \left(\frac{\mu-t}{\lambda+\mu} \right) dt \quad \text{and} \quad \mathcal{S}_{2,2} := \int_0^{\mu} t^{\alpha} h_2 \left(\frac{\mu-t}{\lambda+\mu} \right) dt.$$

Proof. Applying Lemma 1, the property of the modulus, power-mean inequality and the exponential $(\omega_1, \omega_2, h_1, h_2)$ -nonconvexity of $|\psi'|^q$, we obtain

$$\left| \mathcal{T}_{\lambda, \mu}^{\alpha}(\xi_1, \xi_2) \right| \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \\ \cdot \left[\int_0^{\lambda} t^{\alpha} \left| \psi' \left(\xi_1 + \frac{\mu+t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right| dt + \int_0^{\mu} t^{\alpha} \left| \psi' \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right| dt \right] \\ \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \\ \cdot \left[\left(\int_0^{\lambda} t^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\lambda} t^{\alpha} \left| \psi' \left(\xi_1 + \frac{\mu+t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right|^q dt \right)^{\frac{1}{q}} \right. \\ \left. + \left(\int_0^{\mu} t^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\mu} t^{\alpha} \left| \psi' \left(\xi_1 + \frac{\mu-t}{\lambda+\mu} \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1) \right) \right|^q dt \right)^{\frac{1}{q}} \right] \\ \leq \frac{\mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \\ \cdot \left[\left(\int_0^{\lambda} t^{\alpha} dt \right)^{1-\frac{1}{q}} \left(\int_0^{\lambda} t^{\alpha} \left\{ h_1 \left(\frac{\mu+t}{\lambda+\mu} \right) \frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + h_2 \left(\frac{\mu+t}{\lambda+\mu} \right) \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right\} dt \right)^{\frac{1}{q}} \right]$$

$$\begin{aligned}
 & + \left(\int_0^\mu t^\alpha dt \right)^{1-\frac{1}{q}} \left(\int_0^\mu t^\alpha \left\{ h_1 \left(\frac{\mu-t}{\lambda-\mu} \right) \frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + h_2 \left(\frac{\mu-t}{\lambda+\mu} \right) \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right\} dt \right)^{\frac{1}{q}} \\
 & = \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} \\
 & \cdot \left[\mathcal{B}_1^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{S}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{S}_{1,2} \right)^{\frac{1}{q}} + \mathcal{B}_2^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{S}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{S}_{2,2} \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

The proof of Theorem 2 is completed. \square

We now state several corollaries and consequences of Theorem 2.

Corollary 6. Upon setting $\alpha = 1$, Theorem 2 yields

$$\begin{aligned}
 & \left| \frac{\mu\psi(\xi_1) + \lambda\psi(\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1))}{\lambda + \mu} - \frac{1}{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \int_{\xi_1}^{\xi_1 + \mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)} \psi(t) dt \right| \\
 & \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^2} \left[\left(\frac{\lambda^2}{2} \right)^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{M}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{M}_{1,2} \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \left(\frac{\mu^2}{2} \right)^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{M}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{M}_{2,2} \right)^{\frac{1}{q}} \right], \quad (19)
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{M}_{1,1} & := \int_0^\lambda ih_1 \left(\frac{\mu+t}{\lambda+\mu} \right) dt, & \mathcal{M}_{1,2} & := \int_0^\lambda ih_2 \left(\frac{\mu+t}{\lambda+\mu} \right) dt, \\
 \mathcal{M}_{2,1} & := \int_0^\mu ih_1 \left(\frac{\mu-t}{\lambda+\mu} \right) dt & \text{and} & \quad \mathcal{M}_{2,2} := \int_0^\mu ih_2 \left(\frac{\mu-t}{\lambda+\mu} \right) dt.
 \end{aligned}$$

Corollary 7. Choosing $h_1(t) = h_2(t) = 1$ in Theorem 2, the following inequality holds true:

$$\left| \mathcal{T}_{\lambda,\mu}^\alpha(\xi_1, \xi_2) \right| \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+1}} (\mathcal{B}_1 + \mathcal{B}_2) \left[\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right]^{\frac{1}{q}}. \quad (20)$$

Corollary 8. Choosing

$$h_1(t) = (1-t)^s \quad \text{and} \quad h_2(t) = t^s \quad (s \in (0, 1]),$$

Theorem 2 is reduced to the following inequality:

$$\begin{aligned}
 \left| \mathcal{T}_{\lambda,\mu}^\alpha(\xi_1, \xi_2) \right| & \leq \frac{\mathcal{F}_{\rho,\varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha+\frac{s}{q}+1}} \left[\mathcal{B}_1^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{P}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{P}_{1,2} \right)^{\frac{1}{q}} \right. \\
 & \quad \left. + \mathcal{B}_2^{1-\frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{P}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{P}_{2,2} \right)^{\frac{1}{q}} \right], \quad (21)
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{P}_{1,1} & := \int_0^\lambda t^\alpha (\lambda-t)^s dt, & \mathcal{P}_{1,2} & := \int_0^\lambda t^\alpha (\mu+t)^s dt, \\
 \mathcal{P}_{2,1} & := \int_0^\mu t^\alpha (\lambda+t)^s dt & \text{and} & \quad \mathcal{P}_{2,2} := \int_0^\mu t^\alpha (\mu-t)^s dt.
 \end{aligned}$$

Corollary 9. *By putting*

$$h_1(t) = (1 - t)^{-s} \quad \text{and} \quad h_2(t) = t^{-s} \quad (s \in (0, 1)),$$

Theorem 2 yields the following inequality:

$$\begin{aligned} |\mathcal{T}_{\lambda, \mu}^\alpha(\xi_1, \xi_2)| \leq & \frac{\mathcal{F}_{\rho, \varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha - \frac{s}{q} + 1}} \left[\mathcal{B}_1^{1 - \frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{R}_{1,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{R}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \left. + \mathcal{B}_2^{1 - \frac{1}{q}} \left(\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} \mathcal{R}_{2,1} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \mathcal{R}_{2,2} \right)^{\frac{1}{q}} \right], \end{aligned} \quad (22)$$

where

$$\begin{aligned} \mathcal{R}_{1,1} &:= \int_0^\lambda \frac{t^\alpha}{(\lambda - t)^s} dt, & \mathcal{R}_{1,2} &:= \int_0^\lambda \frac{t^\alpha}{(\mu + t)^s} dt, \\ \mathcal{R}_{2,1} &:= \int_0^\mu \frac{t^\alpha}{(\lambda + t)^s} dt & \text{and} & \mathcal{R}_{2,2} := \int_0^\mu \frac{t^\alpha}{(\mu - t)^s} dt. \end{aligned}$$

Corollary 10. *Upon letting*

$$h_1(t) = h_2(t) = t(1 - t),$$

Theorem 2 yields the following inequality:

$$|\mathcal{T}_{\lambda, \mu}^\alpha(\xi_1, \xi_2)| \leq \frac{\mathcal{F}_{\rho, \varsigma}^\sigma(\xi_2 - \xi_1)}{(\lambda + \mu)^{\alpha + 1}} \left(\mathcal{B}_1^{1 - \frac{1}{q}} \mathcal{K}_1^{\frac{1}{q}} + \mathcal{B}_2^{1 - \frac{1}{q}} \mathcal{K}_2^{\frac{1}{q}} \right) \left[\frac{|\psi'(\xi_1)|^q}{e^{\omega_1 \xi_1}} + \frac{|\psi'(\xi_2)|^q}{e^{\omega_2 \xi_2}} \right]^{\frac{1}{q}}, \quad (23)$$

where

$$\mathcal{K}_1 := \frac{1}{(\lambda + \mu)^2} \left[\frac{\mu \lambda^{\alpha + 2}}{\alpha + 1} - \frac{\mu \lambda^{\alpha + 2}}{\alpha + 2} + \frac{\lambda^{\alpha + 3}}{\alpha + 2} - \frac{\lambda^{\alpha + 3}}{\alpha + 3} \right],$$

and

$$\mathcal{K}_2 := \frac{1}{(\lambda + \mu)^2} \left[\frac{\lambda \mu^{\alpha + 2}}{\alpha + 1} - \frac{\lambda \mu^{\alpha + 2}}{\alpha + 2} + \frac{\mu^{\alpha + 3}}{\alpha + 2} - \frac{\mu^{\alpha + 3}}{\alpha + 3} \right].$$

Remark 4. *If we take $\lambda = \mu = 1$ or $\mathcal{F}_{\rho, \varsigma}^\sigma(\xi_2 - \xi_1) = \xi_2 - \xi_1$ or $h_1(t) = h(1 - t)$ and $h_2(t) = h(t)$ in Theorem 1 and Theorem 2, then we can obtain some interesting results immediately. We omit their proofs here, and the details are left to the interested reader.*

Remark 5. *If we choose $\omega_1 = \omega_2 = 0$ in our results in this paper, then all of the consequent results will hold true for the (h_1, h_2) -nonconvex functions.*

3. Application

In this section, we present an application involving a new error estimation for the trapezoidal formula by using the inequalities obtained in Section 2. We fix the parameters ρ and ς . We also suppose that the bounded sequence $\{\sigma(\ell)\}_{\ell=0}^\infty$ of positive real numbers is given.

Let

$$U : \xi_1 = \chi_0 < \chi_1 < \dots < \chi_{n-1} < \chi_n = \xi_1 + \mathcal{F}_{\rho, \varsigma}^\sigma(\xi_2 - \xi_1)$$

be a partition of the closed interval Δ .

For $\lambda, \mu \in (0, 1]$, let us define

$$T(U, \psi) := \sum_{i=0}^{n-1} \left(\frac{\mu \psi(\chi_i) + \lambda \psi\left(\chi_i + \mathcal{F}_{\rho, \varsigma}^\sigma(\hbar_i)\right)}{\lambda + \mu} \right) \mathcal{F}_{\rho, \varsigma}^\sigma(\hbar_i),$$

and

$$\int_{\xi_1}^{\xi_1 + \mathcal{F}_{\rho, \varsigma}^{\sigma}(\xi_2 - \xi_1)} \psi(t) dt = T(U, \psi) + R(U, \psi),$$

where $R(U, \psi)$ is the remainder term and

$$\hbar_i = \chi_{i+1} - \chi_i \quad (\forall i = 0, 1, 2, \dots, n-1).$$

From the above notations, we can obtain some new bounds regarding error estimation.

Proposition 1. Assume that $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$ are two continuous functions. Furthermore, let $\psi : \Delta \rightarrow \mathbb{R}$ be a differentiable function on Δ° with $\lambda, \mu \in (0, 1]$. Suppose that $\psi' \in \mathcal{L}_1(\Delta)$ and that $|\psi'|^q$ is an exponentially $(\omega_1, \omega_2, h_1, h_2)$ -nonconvex function. Then, for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, it is asserted that

$$\begin{aligned} |R(U, \psi)| \leq & \frac{1}{(\lambda + \mu)^2} \sum_{i=0}^{n-1} \left[\mathcal{F}_{\rho, \varsigma}^{\sigma}(\hbar_i) \right]^2 \\ & \cdot \left[\left(\frac{\lambda^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\chi_i)|^q}{e^{\omega_1 \chi_i}} \mathcal{H}_{1,1} + \frac{|\psi'(\chi_{i+1})|^q}{e^{\omega_2 \chi_{i+1}}} \mathcal{H}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{\mu^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\chi_i)|^q}{e^{\omega_1 \chi_i}} \mathcal{H}_{2,1} + \frac{|\psi'(\chi_{i+1})|^q}{e^{\omega_2 \chi_{i+1}}} \mathcal{H}_{2,2} \right)^{\frac{1}{q}} \right]. \quad (24) \end{aligned}$$

Proof. Applying Theorem 1 on the subinterval $[\chi_i, \chi_{i+1}]$ of the closed interval Δ ($\forall i = 0, 1, 2, \dots, n-1$), and taking $\alpha = 1$, we obtain

$$\begin{aligned} \left| \left(\frac{\mu \psi(\chi_i) + \lambda \psi(\chi_i + \mathcal{F}_{\rho, \varsigma}^{\sigma}(\hbar_i))}{\lambda + \mu} \right) \mathcal{F}_{\rho, \varsigma}^{\sigma}(\hbar_i) - \int_{\chi_i}^{\chi_i + \mathcal{F}_{\rho, \varsigma}^{\sigma}(\hbar_i)} \psi(t) dt \right| \leq & \frac{[\mathcal{F}_{\rho, \varsigma}^{\sigma}(\hbar_i)]^2}{(\lambda + \mu)^2} \\ & \cdot \left[\left(\frac{\lambda^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\chi_i)|^q}{e^{\omega_1 \chi_i}} \mathcal{H}_{1,1} + \frac{|\psi'(\chi_{i+1})|^q}{e^{\omega_2 \chi_{i+1}}} \mathcal{H}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{\mu^{p+1}}{p+1} \right)^{\frac{1}{p}} \left(\frac{|\psi'(\chi_i)|^q}{e^{\omega_1 \chi_i}} \mathcal{H}_{2,1} + \frac{|\psi'(\chi_{i+1})|^q}{e^{\omega_2 \chi_{i+1}}} \mathcal{H}_{2,2} \right)^{\frac{1}{q}} \right]. \quad (25) \end{aligned}$$

Upon summing the inequality (25) over i from 0 to $n-1$ and using the property of the modulus, we obtain inequality (24). \square

Proposition 2. Assume that $h_1, h_2 : [0, 1] \rightarrow [0, \infty)$ are two continuous functions. Furthermore, let $\psi : \Delta \rightarrow \mathbb{R}$ be a differentiable function on Δ° with $\lambda, \mu \in (0, 1]$. Suppose that $\psi' \in \mathcal{L}_1(\Delta)$ and that $|\psi'|^q$ is an exponentially $(\omega_1, \omega_2, h_1, h_2)$ -nonconvex function. Then, for $q \geq 1$, the following inequality holds true:

$$\begin{aligned} |R(U, \psi)| \leq & \frac{1}{(\lambda + \mu)^2} \sum_{i=0}^{n-1} \left[\mathcal{F}_{\rho, \varsigma}^{\sigma}(\hbar_i) \right]^2 \\ & \cdot \left[\left(\frac{\lambda^2}{2} \right)^{1-\frac{1}{q}} \left(\frac{|\psi'(\chi_i)|^q}{e^{\omega_1 \chi_i}} \mathcal{M}_{1,1} + \frac{|\psi'(\chi_{i+1})|^q}{e^{\omega_2 \chi_{i+1}}} \mathcal{M}_{1,2} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\frac{\mu^2}{2} \right)^{1-\frac{1}{q}} \left(\frac{|\psi'(\chi_i)|^q}{e^{\omega_1 \chi_i}} \mathcal{M}_{2,1} + \frac{|\psi'(\chi_{i+1})|^q}{e^{\omega_2 \chi_{i+1}}} \mathcal{M}_{2,2} \right)^{\frac{1}{q}} \right]. \quad (26) \end{aligned}$$

Proof. Choosing $\alpha = 1$ in Theorem 2 and using the same technique as in our demonstration of Proposition 1, we obtain the desired inequality (26). \square

Remark 6. In view of Remark 2, we can establish new error estimations by using Proposition 1 and Proposition 2.

4. Conclusions

In this paper, the authors have defined a new generic class of functions involving the modified Fox–Wright function $\mathcal{F}_{\rho,\xi}^{\sigma}(\cdot)$ as well as the so-called exponentially $(\omega_1, \omega_2, h_1, h_2)$ –nonconvex function. A useful identity has also been found by using fractional integrals and the function $\mathcal{F}_{\rho,\xi}^{\sigma}(\cdot)$ with two parameters λ and μ . We have established some Hermite–Hadamard-type integral inequalities by using the above class of functions and the aforementioned identity as an auxiliary result. Several special cases have been deduced as corollaries including relevant details. We have also outlined the derivations of several other corollaries and consequences for the interested reader. The efficiency of our main results has been shown by proving an application for error estimation.

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References

1. Adjabi, Y.; Jarad, F.; Baleanu, D.; Abdeljawad, T. On Cauchy problems with Caputo Hadamard fractional derivatives. *Math. Methods Appl. Sci.* **2016**, *40*, 661–681.
2. Tan, W.; Jiang, F.L.; Huang, C.X.; Zhou, L. Synchronization for a class of fractional-order hyperchaotic system and its application. *J. Appl. Math.* **2012**, *2012*, 974639. [[CrossRef](#)]
3. Zhou, X.S.; Huang, C.X.; Hu, H.J.; Liu, L. Inequality estimates for the boundedness of multilinear singular and fractional integral operators. *J. Inequal. Appl.* **2013**, *2013*, 303. [[CrossRef](#)]
4. Liu, F.W.; Feng, L.B.; Anh, V.; Li, J. Unstructured-mesh Galerkin finite element method for the two-dimensional multi-term time-space fractional Bloch-Torrey equations on irregular convex domains. *Comput. Math. Appl.* **2019**, *78*, 1637–1650. [[CrossRef](#)]
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematical Studies; Elsevier: Amsterdam, The Netherlands; Science Publishers: Amsterdam, The Netherlands; London, UK; New York, NY, USA, 2006; Volume 204
6. Srivastava, H.M. Fractional-order derivatives and integrals: Introductory overview and recent developments. *Kyungpook Math. J.* **2020**, *60*, 73–116.
7. Cai, Z.W.; Huang, J.H.; Huang, L.H. Periodic orbit analysis for the delayed Filippov system. *Proc. Am. Math. Soc.* **2018**, *146*, 4667–4682. [[CrossRef](#)]
8. Chen, T.; Huang, L.H.; Yu, P.; Huang, W.T. Bifurcation of limit cycles at infinity in piecewise polynomial systems. *Nonlinear Anal. Real World Appl.* **2018**, *41*, 82–106. [[CrossRef](#)]
9. Wang, J.F.; Chen, X.Y.; Huang, L.H. The number and stability of limit cycles for planar piecewise linear systems of node-saddle type. *J. Math. Anal. Appl.* **2019**, *469*, 405–427. [[CrossRef](#)]
10. Houas, M. Certain weighted integral inequalities involving the fractional hypergeometric operators. *Sci. Ser. A Math. Sci.* **2016**, *27*, 87–97.
11. Houas, M. On some generalized integral inequalities for Hadamard fractional integrals. *Mediterr. J. Model. Simul.* **2018**, *9*, 43–52.

12. Baleanu, D.; Mohammed, P.O.; Vivas-Cortez, M.; Rangel-Oliveros, Y. Some modifications in conformable fractional integral inequalities. *Adv. Differ. Equ.* **2020**, *2020*, 374. [[CrossRef](#)]
13. Abdeljawad, T.; Mohammed, P.O.; Kashuri, A. New modified conformable fractional integral inequalities of Hermite-Hadamard type with applications. *J. Funct. Spaces* **2020**, *2020*, 4352357. [[CrossRef](#)]
14. Mohammed, P.O.; Abdeljawad, T. Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel. *Adv. Differ. Equ.* **2020**, *2020*, 363. [[CrossRef](#)]
15. Mohammed, P.O.; Brevik, I. A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. *Symmetry* **2020**, *12*, 610. [[CrossRef](#)]
16. Cloud, M.J.; Drachman, B.C.; Lebedev, L. *Inequalities*, 2nd ed.; Springer: Cham, Switzerland, 2014.
17. Atici, F.M.; Yaldiz, H. Convex functions on discrete time domains. *Can. Math. Bull.* **2016**, *59*, 225–233. [[CrossRef](#)]
18. Tomar, M.; Agarwal, P.; Jleli, M.; Samet, B. Certain Ostrowski type inequalities for generalized s -convex functions. *J. Nonlinear Sci. Appl.* **2017**, *10*, 5947–5957. [[CrossRef](#)]
19. Varošanec, S. On h -convexity. *J. Math. Anal. Appl.* **2007**, *326*, 303–311. [[CrossRef](#)]
20. Awan, M.U.; Noor, M.A.; Noor, K.I. Hermite-Hadamard inequalities for exponentially convex functions. *Appl. Math. Inf. Sci.* **2018**, *12*, 405–409. [[CrossRef](#)]
21. Baleanu, D.; Kashuri, A.; Mohammed, P.O.; Meftah, B. General Raina fractional integral inequalities on coordinates of convex functions. *Adv. Differ. Equ.* **2021**, *2021*, 82. [[CrossRef](#)]
22. Mohammed, P.O.; Abdeljawad, T.; Zeng, S.; Kashuri, A. Fractional Hermite-Hadamard integral inequalities for a new class of convex functions. *Symmetry* **2020**, *12*, 1485. [[CrossRef](#)]
23. Kashuri, A.; Liko, R. Some new Hermite-Hadamard type inequalities and their applications. *Stud. Sci. Math. Hung.* **2019**, *56*, 103–142. [[CrossRef](#)]
24. Alqudah, M.A.; Kashuri, A.; Mohammed, P.O.; Abdeljawad, T.; Raees, M.; Anwar, M.; Hamed, Y.S. Hermite-Hadamard integral inequalities on coordinated convex functions in quantum calculus. *Adv. Differ. Equ.* **2021**, *2021*, 264. [[CrossRef](#)]
25. Delavar, M.R.; De La Sen, D. Some generalizations of Hermite-Hadamard type inequalities. *SpringerPlus* **2016**, *5*, 1661. [[CrossRef](#)] [[PubMed](#)]
26. Khan, M.B.; Mohammed, P.O.; Noor, B.; Hamed, Y.S. New Hermite-Hadamard inequalities in fuzzy-interval fractional calculus and related inequalities. *Symmetry* **2021**, *13*, 673. [[CrossRef](#)]
27. Mohammed, P.O.; Abdeljawad, T.; Alqudah, M.A.; Jarad, F. New discrete inequalities of Hermite-Hadamard type for convex functions. *Adv. Differ. Equ.* **2021**, *2021*, 122. [[CrossRef](#)]
28. Srivastava, H.M.; Zhang, Z.-H.; Wu, Y.-D. Some further refinements and extensions of the Hermite-Hadamard and Jensen inequalities in several variables. *Math. Comput. Model.* **2011**, *54*, 2709–2717. [[CrossRef](#)]
29. Mohammed, P.O. New generalized Riemann-Liouville fractional integral inequalities for convex functions. *J. Math. Inequal.* **2021**, *15*, 511–519. [[CrossRef](#)]
30. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; Wiley: New York, NY, USA, 1993.
31. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications*; Gordon & Breach Science Publishers: Yverdon, Switzerland, 1993.
32. Herrmann, R. *Fractional Calculus: An Introduction for Physicists*; World Scientific Publishing Company: Singapore; Hackensack, NJ, USA; London, UK; Hong Kong, China, 2011
33. Katugampola, U.N. A new approach to generalized fractional derivatives. *Bull. Math. Anal. Appl.* **2014**, *6*, 1–15.
34. Katugampola, U.N. New fractional integral unifying six existing fractional integrals. *arXiv* **2016**, arXiv:1612.08596.
35. Baleanu, D.; Fernandez, A. On fractional operators and their classifications. *Mathematics* **2019**, *7*, 830. [[CrossRef](#)]
36. Hilfer, R.; Luchko, Y. Desiderata for fractional derivatives and integrals. *Mathematics* **2019**, *7*, 149. [[CrossRef](#)]
37. Srivastava, H.M.; Karlsson, P.W. *Multiple Gaussian Hypergeometric Series*; Halsted Press, Ellis Horwood Limited: Chichester, UK; John Wiley and Sons: New York, NY, USA; Chichester, UK; Brisbane, Australia; Toronto, ON, Canada, 1985
38. Wright, E.M. The asymptotic expansion of integral functions defined by Taylor series. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1940**, *238*, 423–451.
39. Raina, R.K. On generalized Wright's hypergeometric functions and fractional calculus operators. *East Asian Math. J.* **2005**, *21*, 191–203.
40. Srivastava, H.M.; Bansal, M.K.; Harjule, P. A study of fractional integral operators involving a certain generalized multi-index Mittag-Leffler function. *Math. Meth. Appl. Sci.* **2018**, *41*, 6108–6121. [[CrossRef](#)]