

REVIEW

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Fractional calculus in the sky

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Abstract

Fractional calculus was born in 1695 on September 30 due to a very deep question raised in a letter of L'Hospital to Leibniz. The prophetic answer of Leibniz to that deep question encapsulated a huge inspiration for all generations of scientists and is continuing to stimulate the minds of contemporary researchers. During 325 years of existence, fractional calculus has kept the attention of top level mathematicians, and during the last period of time it has become a very useful tool for tackling the dynamics of complex systems from various branches of science and engineering. In this short manuscript, we briefly review the tremendous effect that the main ideas of fractional calculus had in science and engineering and briefly present just a point of view for some of the crucial problems of this interdisciplinary field.

Keywords: Fractional calculus; Fractional differential equations; Fractional modelling

1 The basic question and its consequences

When we deal with classical calculus and its applications, one of the first advantages we have is that we have physical and geometrical meanings of classical derivative and integral. However, we recall that the basic question of L'Hospital was if we can extend the meaning of the classical derivative to noninteger case. The magnificent answer of Leibniz [1] motivated a series of interesting results reported during the last 325 years (see for example Refs. [2–16] and the references therein). Besides, we strongly believe that many valuable answers of the primary question will be reported in the future. During the eighteenth and nineteenth centuries, there were a lot of top level scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus (see for example [2, 11, 17, 18] and the references therein). In the twentieth century, but before 1985, we can mention some of high level researchers in fractional calculus such as Weyl, Hardy, Littlewood, Levy, Zygmud, Riesz, Doetsch, Erdelyi, Widder, Rabotnov, Feller, Maraval, Sneddon, Gorenflo, Caputo, Dzherbashyan, Samko, Srivastava, Oldham, Osler, Mainardi, Love, Spanier, Mathai, Saxena, Ross, McBride, Blair, Nigmatullin, Oustaloup, Bagley, Torvik among others (see for example [11] and the references therein).

Nowadays, fractional calculus deals with the study of so-called fractional order integral and derivative operators over real or complex domains and their applications. Fractional calculus is actually a misnomer; the so-called designation of integration and differentiation of arbitrary order is more appropriate. More details regarding the history of fractional

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calculus (see for more details [2–16] and the references therein) can be seen for example in [2, 17, 18] and the references therein. In our opinion, the physical and geometrical meanings of fractional operators are still not clearly established despite many interesting attempts (see for example [19] and the references therein) to tackle these still contemporary open problems. In our opinion, even for the notion of the fractional order of a fractional derivative [8], we do not have yet a clear meaning. Of course, for given real data corresponding to complex phenomena, let us identify the order of a fractional operator utilised within a related fractional model, but what is the case when the fractional order becomes an irrational number?

During the last decades, fractional calculus started to be used as a powerful tool by many researchers working in several branches of science and engineering, e.g., the fractional control theory [6, 9]. Especially after 2010, in our opinion, we can identify some new viewpoints within fractional calculus. Besides, a lot of real-world applications were treated within fractional calculus, e.g., in biology and related areas (among many new results, see for example [8, 11–16, 21–30] and the references therein).

Very recently, fractional calculus has been passing through a process of filtration of several suggested operators reported during the last decade. In fact, in our opinion, we assist in the process of natural classification in the classes of fractional operators (see for example [17, 20, 25] and the references therein for more details about this topic).

2 The theory

Starting from the earlier stage of fractional calculus, many researchers aimed to extend the classical calculus. Thus, several fractional operators have been introduced during the last 325 years [4, 9]. For example, the Caputo derivative appeared naturally in the work of Abel, Liouville [31], and it was introduced by Caputo in [3] to solve a practical problem.

Definition 2.1 The Riemann–Liouville fractional integral of a function $g \in L^1[a, b]$, with the constant of integration a , is defined as follows for $t \in [a, b]$ and the order of integration $\alpha \in \mathbb{C}$ with $Real(\alpha) > 0$ [4, 7, 9]:

$${}_aI^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - y)^{\alpha-1} g(y) dy. \tag{1}$$

Definition 2.2 The Riemann–Liouville fractional derivative of a function $g \in C^n[a, b]$ is defined as follows for $t \in [a, b]$, the order of differentiation $\alpha \in \mathbb{C}$ with $Real(\alpha) \geq 0$ and $n - 1 \leq Real(\alpha) < n$ [4, 7, 9]:

$${}_aD^\alpha g(t) = \frac{d^n}{dx^n} ({}_aI^{n-\alpha} g(t)). \tag{2}$$

Definition 2.3 The Caputo fractional derivative of a function $g \in C^n[a, b]$ is defined as follows for $t \in [a, b]$, the order of differentiation $\alpha \in \mathbb{C}$ with $Real(\alpha) \geq 0$ and $n - 1 \leq Real(\alpha) < n$ [4, 7, 9]:

$${}^C_aD^\alpha g(t) = {}_aI^{n-\alpha} \left(\frac{d^n}{dx^n} g(t) \right). \tag{3}$$

Also, the integral of Riemann–Liouville [4, 7, 9] type appears in several natural processes, and it played a key factor in fractional calculus. Several numerical methods used

for fractional differential equations have been investigated (see for example [9, 11, 13, 14] and the references therein), but still they are in the process of developing despite of the important progress done within this field. For example, the key role of the initialisation of fractional differential equations has received a huge attention of researchers during the last few years. The most general type of fractional differential equations involved the so-called Caputo like operators containing various types of kernels. In this case, in our opinion, we have to find the answer to the following question: why for a fractional operator not having yet a very clear physical meaning do we have to use the initial conditions similar to the classical case?

In our opinion, the ideas formulated by Liouville in 1832 [31] and related to the fractional calculus have remained contemporary; namely, we need new types of fractional operators for solving real-world problems which were not able to be solved with other types of existing mathematical tools. So far, in our opinion, the answer to the question “which is the most general fractional operator solving all types of complicated dynamical systems with various memory effects?” could not be found, and it remains a big and interesting open question. In our opinion, the development of fractional calculus depends dramatically on the development of numerical methods applied to fractional operators. A new perspective should be added in this area, and it should bypass the difficulties due to memory effect. We recall that almost all numerical techniques for fractional differential equations rely on the approximation of fractional differential or integral operators by specific formulas (see for examples [9, 11, 13, 14] and the references therein); thus, much interest for the future will be concentrated at this point. Besides, an interesting direction within fractional calculus, especially for modelling and in the perspective of artificial intelligence, is its discrete version. Basically, the discrete fractional calculus [32–35] generalises the classical results of difference equations [36].

Definition 2.4 Let $y: N_a \rightarrow R$ and $0 < \nu$. Then the ν order fractional sum becomes [32, 37]

$$\Delta_a^{-\nu} y(t) := \sum_{s=a}^{t-\nu} K_{-\nu}(t, s)y(s), \quad a \in R, t \in N_{a+\nu}, \tag{4}$$

where $K_{-\nu}(t, s) = (t - s - 1)^{(\nu-1)}/\Gamma(\nu)$, and $t^{(\nu)}$ is the discrete factorial functional defined by

$$t^{(\nu)} = \frac{\Gamma(t + 1)}{\Gamma(t + 1 - \nu)}.$$

Definition 2.5 For $y(t)$ defined on N_a and $0 < \nu, \nu \notin N$, the Caputo difference reads as [38, 39]

$${}^C \Delta_a^\nu y(t) := \Delta_a^{-(m-\nu)} \Delta^m y(t), \quad t \in N_{a+m-\nu}, m = [\nu] + 1. \tag{5}$$

When $\nu = m$, we have ${}^C \Delta_a^\nu y(t) := \Delta^m y(t)$.

Here, $\Delta^m y(t)$ denotes the m th order forward difference operator (the expression can be seen for example in [38]).

Definition 2.6 For $x(t)$ defined on N_a and $0 < \nu, \nu \notin N$, the Riemann–Liouville difference has the following expression [32]:

$$\Delta_a^\nu x(t) := \Delta^m \Delta_a^{-(m-\nu)} x(t), \quad t \in N_{a+m-\nu}. \quad (6)$$

3 The experimental data versus the models based on fractional calculus

Fractional calculus has been intensively applied to various types of complex dynamics having a different origin of the memory effect. In some cases, the real data exists; thus, the fractional models have been appropriately tested.

One of the most interesting applications of discrete fractional calculus (see for example Refs [32–35, 37–39] and the references therein) is related to chaotic maps which play an important role in the information encryption. For example, the patent [40] proposed a new fractional chaotic map with more parameters and complicated chaotic behaviour such that the secret space increases in image encryption. Much more important, this new map fast generates chaotic series and can be applied to encryption of big data.

Particularly during the last few years, the fractional models for biological systems have been developed by taking into account the dimensionality issue which always appears in such formulations. Namely among various possibilities to construct a fractional biological model [26, 41], we have to take care of keeping the biological meaning of modified parameters in the model. The new models of fractional calculus take into account the causality condition and the thermodynamic principles for prediction of some valuable consequences. Dealing with real data shows us that there is no single operator describing the dynamics of complex systems from various disciplines. On the contrary, given real data selects naturally the best fractional kernel. Generally, from the experimental viewpoint, a simple fractional operator is required. Using the fractional calculus tool to explore the new and complicated dynamical systems is one target of the new generation of researchers, and the examples from space sciences are one huge possible way. For example, the fading memory concept relating the flux to its gradient for simple materials is modelled by integro-differential equation as a manifestation of the old Boltzmann linear superposition function through a memory kernel [42]. Besides, we have to apply correctly a fractional operator to an appropriate phenomenon; otherwise, the results are incorrect. A typical example is to apply the Caputo–Fabrizio operator [21, 22, 43, 44] to a linear viscoelasticity instead of the nonlinear viscoelastic behaviour of materials. On this line of thought, in our opinion, the replacement of the classical operator with a given operator (fractional or not) should be tackled with a great care in the future.

Besides, in the theory of modelling it is well-accepted to follow a so-called five-step method [45], namely:

(a) Ask the question. (b) Identify the modelling approach. (c) Formulate the model. (d) Solve the model. (e) Answer the initial question [45].

Looking at the dynamics of published papers so far in the area of fractional modelling, we conclude that not all the above-mentioned five steps are taken into account; therefore, obeying all five steps should be a priority for the future studies. The question related to the most suitable kernel [9, 17, 20–23, 25, 27–30, 44, 46–48] for given real data corresponding to a given complex process is still a difficult task, and it is deeply related to the second and third steps presented above. In our opinion, various kernels used today by researchers (some of them related to fractional calculus but some of them not) can be satisfactorily unified under the umbrella of the large spectra of memory.

3.1 An example: tumour–immune surveillance mathematical model

In what follows we give a simple example of using several fractional kernels [29]. The classical model of tumour–immune surveillance system introduced in [49] investigated the interaction between the population of various tumour cells and immune system interceded by activated CD8⁺ CTLs and NK cells. Taking into account that fractional derivatives basically comprise memory, it is argumentative to utilise a fractional extension to this model. Thus, in the classical model all integer derivatives are replaced by their α th order Mittag-Leffler kernel (ABC) fractional derivative [23] where $\alpha \in (0, 1)$. The dimensionality condition on both sides of the fractional model, namely the left- and right-hand sides possess the same dimension (time)^{− α} , implies that the biological parameters having the dimension (time)^{−1} are modified by the power of α , while the other parameters as well as dimensionless ones remain unchanged [41]. According to these attributes, the generalised ABC fractional model of tumour–immune surveillance is proposed as follows [29]:

$$\begin{cases} {}^{ABC}D_t^\alpha L(t) = (\beta_L^+)^{\alpha} L(t) T_0(t) \left(\frac{r_1}{\beta_L^+(T_0(t)+g)} + q_L - 1 \right) \\ \quad + (\beta_N^+)^{\alpha} L(t) T_N(t) \left(\frac{r_2}{\beta_L^+(T_N(t)+g)} + \zeta_L - 1 \right) - \mu_L^\alpha L(t), \\ {}^{ABC}D_t^\alpha N(t) = s^\alpha - (\mu_N^\alpha + (1 - p_N)(\alpha_N^+)^{\alpha} T_0(t) + (1 - \pi_N)(\alpha_L^+)^{\alpha} T_N(t)) N(t), \\ {}^{ABC}D_t^\alpha T_0(t) = -a^\alpha b T_0^2(t) - ((\alpha_N^+)^{\alpha} N(t) + (\beta_L^+)^{\alpha} L(t) - a^\alpha) T_0(t), \\ {}^{ABC}D_t^\alpha T_L(t) = -a^\alpha b T_L^2(t) - ((\alpha_L^+)^{\alpha} N(t) - a^\alpha) T_L(t) + q_T (\beta_L^+)^{\alpha} L(t) T_0(t), \\ {}^{ABC}D_t^\alpha T_N(t) = -a^\alpha b T_N^2(t) - ((\beta_N^+)^{\alpha} L(t) - a^\alpha) T_N(t) + p_T (\alpha_N^+)^{\alpha} N(t) T_0(t), \\ {}^{ABC}D_t^\alpha T_{NL}(t) = -a^\alpha b T_{NL}^2(t) + a^\alpha T_{NL}(t) + \pi_T (\alpha_L^+)^{\alpha} N(t) T_L(t) \\ \quad + \zeta_T (\beta_N^+)^{\alpha} L(t) T_N(t), \\ L(0) = L_0, \quad N(0) = N_0, \quad T_0(0) = T_{00}, \quad T_L(0) = T_N(0) = T_{NL}(0) = 0, \end{cases} \tag{7}$$

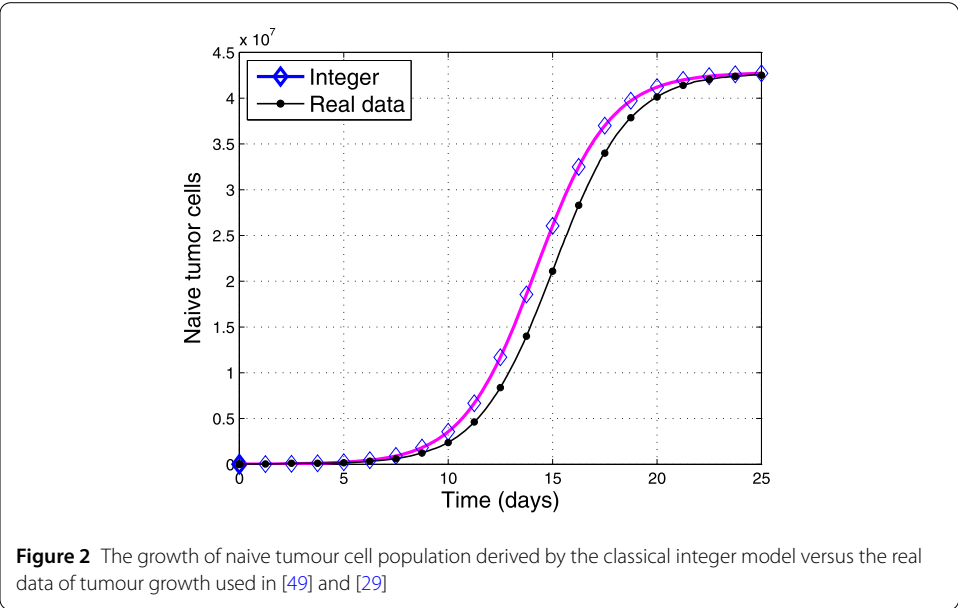
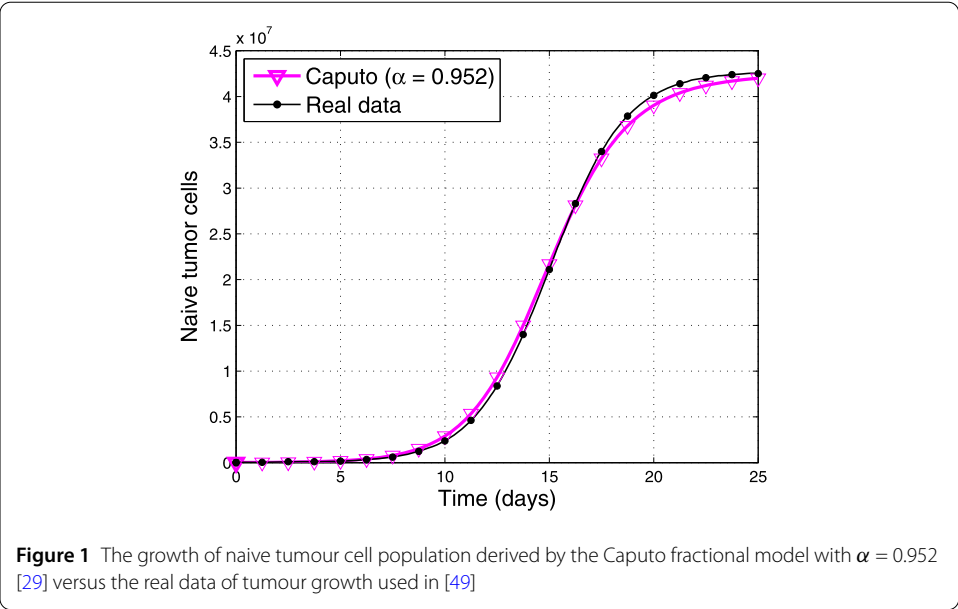
where ${}^{ABC}D_t^\alpha$ represents the fractional ABC operator defined by [23]

$${}^{ABC}D_t^\alpha x(t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t E_\alpha \left(-\frac{\alpha}{1 - \alpha} (t - \tau)^\alpha \right) \dot{x}(\tau) d\tau, \tag{8}$$

E_α denotes the ML function, and $M(\alpha)$ satisfying $M(0) = M(1) = 1$ is a normalisation function. In order to prove the capability of ABC Mittag-Leffler approach, a set of real data was used from [49]. Figures 1–3 compared the real growth of naive tumour cell population from [49] and the outputs of two fractional models and an integer-order counterpart. In addition, the absolute and relative errors regarding each model were reported in Table 1. A particle swarm optimisation algorithm was also employed in order to find the optimal value of α at each fractional case [50]. The reported results in Figs. 1–3 verify a good agreement of the ABC model ($\alpha = 0.9$) and the real data in the whole interval [29]. The superiority of this ABC model in terms of absolute and relative errors is also confirmed in Table 1 (see [29]). Therefore, the additional complexity imposed by the ABC fractional calculus approach is justified, for this particular model, by the aforesaid achieved advantages [29].

4 Classification of fractional operators: “a fata morgana”

One of the intensely debated subjects within fractional calculus is the attempt to classify fractional operators. Started by Ross in 1975 [17], this complex process of the classification



of fractional operators passed several periods, and various attempts have been presented (see for example [20, 25, 46–48] and the references therein). In our opinion, so far, there has been no consensus achieved on the main criteria that classify the fractional operators. The experimental facts registered and reported in CERN easily have minimised the solid mathematical constructions such as string theory and super-symmetry in favour of standard models based on much simpler concepts and more physical concepts as gauge invariance. Constructing nice complicated fractional operators may not give the best output always when we create models involving them. Besides, it seems that there is not a unique fractional operator which can be used to describe all types of processes having different types of memory effects. In our opinion, the classification in the classes of fractional operators is more reasonable [25] from both experimental and mathematical viewpoints.

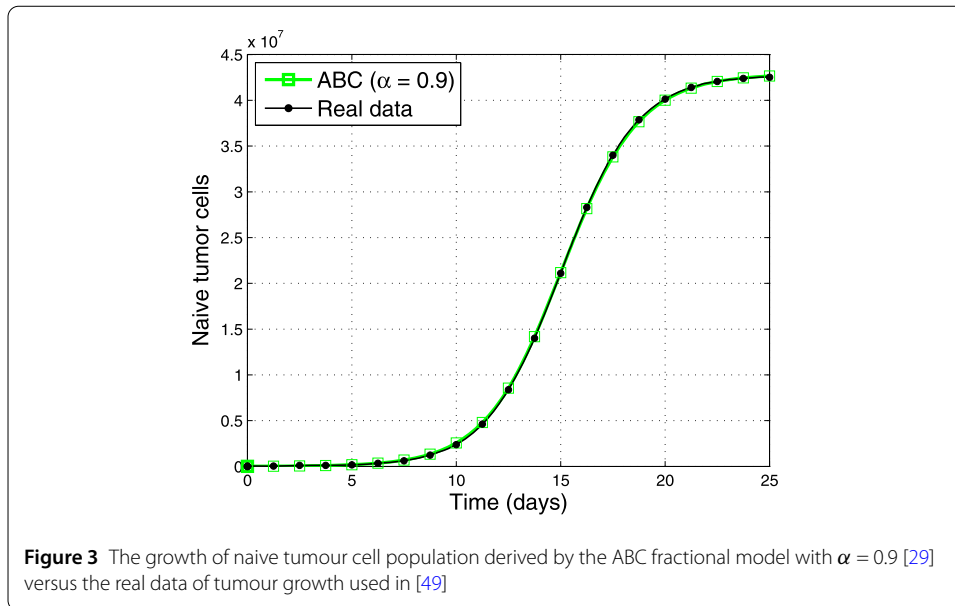


Figure 3 The growth of naive tumour cell population derived by the ABC fractional model with $\alpha = 0.9$ [29] versus the real data of tumour growth used in [49]

Table 1 Comparative results of the fractional and integer models versus real data [29]

The model	α	The absolute error	The relative error
Integer	1	4.9501e + 06	0.1164
Caputo	0.952	1.1271e + 06	0.0265
Mittag-Leffler kernel	0.9	2.9030e + 05	0.0068

Besides, the distinct notions of singular and nonsingular fractional operators with their specific advantages and disadvantages can be in harmony under the concepts of classes of operators and the umbrella of the powerful concept of memory.

5 Instead of conclusion

In our opinion, during a period of 325 years of fractional calculus, a critical mass of information has been collected from both mathematical and applied viewpoints, and nowadays it is a suitable moment to make a useful transition.

Despite the fact that numerical methods for fractional differential equations have reported significant contributions (see for example [2–16] and the references therein), still they did not reach the required level to decide always which fractional calculus model is more suitable for given real data. We think that one of the keys for success of future theoretical and applied viewpoints is to consider the notion of distinct classes of fractional calculus operators. Showing exactly both the advantages and limitations of certain types of fractional operators will make this field much stronger in the long range. Harmonising the point of view that any fractional operators should have a physical, biological, or economical meaning and should appear naturally in a set of real-world processes versus the pure mathematical construction of fractional operators without looking to any experimental data, it seems to be a challenging problem for researchers in the area of fractional calculus.

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Authors' contributions

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