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Research article

Fuzzy-interval inequalities for generalized convex fuzzy-interval-valued functions via fuzzy Riemann integrals

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Abstract: The objective of the authors is to introduce the new class of convex fuzzy-interval-valued functions (convex-FIVFs), which is known as p-convex fuzzy-interval-valued functions (p-convex-FIVFs). Some of the basic properties of the proposed fuzzy-interval-valued functions are also studied. With the help of p-convex FIVFs, we have presented some Hermite-Hadamard type inequalities ($H-H$ type inequalities), where the integrands are FIVFs. Moreover, we have also proved the Hermite-Hadamard-Fejér type inequality ($H-H$ Fejér type inequality) for p-convex-FIVFs. To prove the validity of main results, we have provided some useful examples. We have also established some discrete form of Jense's type inequality and Schur's type inequality for p-convex-FIVFs. The outcomes of this paper are generalizations and refinements of different results which are proved in literature. These results and different approaches may open new direction for fuzzy optimization problems, modeling, and interval-valued functions.

Keywords: p-Convex fuzzy-interval-valued function; fuzzy Riemann integral; Hermite-Hadamard inequality; Hermite-Hadamard-Fejér inequality; Jense's type inequality; Schur's type inequality
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1. Introduction

It is well known fact that the concept of interval analysis fell into oblivion for long time until the 1950's: Moore [1], Warmus [2] and Sunaga [3]. The literature of interval analysis can be tracked back to the computation of lower and upper bounds for π by Archimedes in the following way $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The first monograph was published by Moore in 1960 [4], this field has attracted much attention in the theoretical and applied research. This research field has yielded important results over the past 50 years.

Recently, several classical integral inequalities have been generalized to the context of set-valued and fuzzy-set-valued functions by means of inclusion and pseudo order relation. In light of this, Sadowska [5] arrived at the following conclusion for an IVF:

Let $\mathcal{F}: [u, v] \subset \mathbb{R} \to \mathcal{K}_I^+$ be a convex interval-valued function (convex-IVF) given by $\mathcal{F}(\omega) = [\mathcal{F}_*(\omega), \mathcal{F}^*(\omega)]$ for all $\omega \in [u, v]$, where $\mathcal{F}_*(\omega)$ and $\mathcal{F}^*(\omega)$ are convex and concave functions, respectively. If $\mathcal{F}$ is interval Riemann integrable (in sort, IR-integrable), then

$$\mathcal{F}\left(\frac{u+v}{2}\right) \supseteq \frac{1}{v-u}\ (IR)\int_u^v \mathcal{F}(\omega)d\omega \supseteq \frac{\mathcal{F}(u)+\mathcal{F}(v)}{2}. \quad (1)$$

Note that, the inclusion relation (Eq 1) is reversed when $\mathcal{F}$ concave-IVF is. Following that, many scholars used inclusion relations and various integral operators to establish a close relationship between inequality and IVFs. Recently, Costa [6] obtained Jensen's type inequality for FIVF. Costa and Roman-Flores [7,8] introduced different types of inequalities for FIVF and IVF, and discussed their properties. Roman-Flores et al. [9] derived Gronwall for IVFs. Moreover, Chalco-Cano et al. [10,11] presented Ostrowski-type inequalities for IVFs by using the generalized Hukuhara derivative and provided applications in numerical integration in IVF. Nikodem et al. [12], and Matkowski and Nikodem [13] presented the new versions of Jense's inequality for strongly convex and convex functions. Zhao et al. [14,15] derived Chebyshev, Jensen's and H-H type inequalities for IVFs. Recently, Zhang et al. [16] generalized the Jense's inequalities and defined new version of Jensen's inequalities for set-valued and fuzzy-set-valued functions through pseudo order relation. After that, for convex-IVF, Budek [17] established interval-valued fractional Riemann-Liouville $H - H$ inequality by means of inclusion relation. For more useful details, see [18–24] and the references therein.

Recently, Khan et al. [25] introduced the new class of convex fuzzy mappings is known as (h_1, h_2)-convex FIVFs by means of FOR and presented the following new version of H-H type inequality for (h_1, h_2)-convex FIVF involving fuzzy-interval Riemann integrals:

Let $\mathcal{F}: [u, v] \to \mathbb{F}_0$ be a (h_1, h_2)-convex FIVF with $h_1, h_2: [0, 1] \to \mathbb{R}^+$ and $h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right) \neq 0$. Then, from θ-levels, we get the collection of IVFs $\mathcal{F}_\theta: [u, v] \subset \mathbb{R} \to \mathcal{K}_C^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega, \theta), \mathcal{F}^*(\omega, \theta)]$ for all $\omega \in [u, v]$ and for all $\theta \in [0, 1]$. If $\mathcal{F}$ is fuzzy-interval Riemann integrable (in sort, FR-integrable), then

$$\frac{1}{2\, h_1\left(\frac{1}{2}\right) h_2\left(\frac{1}{2}\right)}\, \mathcal{F}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\, (FR) \int_u^\nu \mathcal{F}(\omega) d\omega \preccurlyeq [\mathcal{F}(u) \,\tilde{+}\, \mathcal{F}(\nu)] \int_0^1 h_1(\tau)\, h_2(1-\tau)\, d\tau. \tag{2}$$

If $h_1(\tau) = \tau$ and $h_2(\tau) \equiv 1$, then from (2), we get following the result for convex FIVF:

$$\mathcal{F}\left(\frac{u+\nu}{2}\right) \preccurlyeq \frac{1}{\nu-u}\, (FR) \int_u^\nu \mathcal{F}(\omega) d\omega \preccurlyeq \frac{\mathcal{F}(u) \,\tilde{+}\, \mathcal{F}(\nu)}{2}. \tag{3}$$

A one step forward, Khan et al. introduced new classes of convex and generalized convex FIVF, and derived new fractional H-H type and H-H type inequalities for convex FIVF [26], h-convex FIVF [27], (h_1, h_2)-preinvex FIVF [28], log-s-convex FIVFs in the second sense [29], LR-log-h-convex IVFs [30], harmonically convex FIVFs [31], coordinated convex FIVFs [32] and the references therein. We refer to the readers for further analysis of literature on the applications and properties of fuzzy-interval, and inequalities and generalized convex fuzzy mappings, see [33–49] and the references therein.

This study is organized as follows: Section 2 presents preliminary notions and results in interval space, the space of fuzzy intervals and convex analysis. Moreover, the new concept of p-convex fuzzy-IVF is also introduced. Section 3 obtains fuzzy-interval HH-inequalities for p-convex fuzzy-IVFs via fuzzy Riemann integrals. In addition, some interesting examples are also given to verify our results. Section 4 derives discrete Jensen's and Schur's type inequalities for p-convex fuzzy-IVFS. Section 5 gives conclusions and future plans.

2. Preliminaries

In this section, some preliminary notions, elementary concepts and results are introduced as a pre-work, including operations, orders, and distance between interval and fuzzy numbers, Riemannian integrals, and fuzzy Riemann integrals. Some new definitions and results are also given which will be helpful to prove our main results.

Let $\mathbb{R}$ be the set of real numbers and $\mathcal{K}_C$ be the space of all closed and bounded intervals of $\mathbb{R}$, and $\varpi \in \mathcal{K}_C$ be defined by

$$\varpi = [\varpi_*, \varpi^*] = \{\omega \in \mathbb{R} |\, \varpi_* \leq \omega \leq \varpi^*\},\ (\varpi_*, \varpi^* \in \mathbb{R}).$$

If $\varpi_* = \varpi^*$, then ϖ is said to be degenerate. If $\varpi_* \geq 0$, then $[\varpi_*, \varpi^*]$ is called positive interval. The set of all positive interval is denoted by $\mathcal{K}_C^+$ and defined as $\mathcal{K}_C^+ = \{[\varpi_*, \varpi^*]: [\varpi_*, \varpi^*] \in \mathcal{K}_C \text{ and } \varpi_* \geq 0\}$.

Let $\varrho \in \mathbb{R}$ and $\varrho\varpi$ be defined by

$$\varrho.\varpi = \begin{cases} [\varrho\varpi_*, \varrho\varpi^*], \text{if } \varrho > 0, \\ \{0\}, \qquad\quad \text{if } \varrho = 0, \\ [\varrho\varpi^*, \varrho\varpi_*], \text{if } \varrho < 0. \end{cases} \tag{4}$$

Then the Minkowski difference $\xi - \varpi$, addition $\varpi + \xi$ and $\varpi \times \xi$ for $\varpi, \xi \in \mathcal{K}_C$ are defined by

$$\begin{aligned} [\xi_*, \xi^*] - [\varpi_*, \varpi^*] &= [\xi_* - \varpi_*,\ \xi^* - \varpi^*], \\ [\xi_*, \xi^*] + [\varpi_*, \varpi^*] &= [\xi_* + \varpi_*,\ \xi^* + \varpi^*], \end{aligned} \tag{5}$$

and

$$[\xi_*, \xi^*] \times [\varpi_*, \varpi^*] = [min\{\xi_*\varpi_*, \xi^*\varpi_*, \xi_*\varpi^*, \xi^*\varpi^*\}, max\{\xi_*\varpi_*, \xi^*\varpi_*, \xi_*\varpi^*, \xi^*\varpi^*\}].$$

The inclusion " $\subseteq$ " means that

$$\xi \subseteq \varpi \text{ if and only if, } [\xi_*, \xi^*] \subseteq [\varpi_*, \varpi^*] \text{, if and only if } \varpi_* \leq \xi_*,\ \xi^* \leq \varpi^*. \tag{6}$$

Remark 2.1. [33] The relation " $\leq_I$ " defined on $\mathcal{K}_C$ by

$$[\xi_*, \xi^*] \leq_I [\varpi_*, \varpi^*] \text{ if and only if } \xi_* \leq \varpi_*,\ \xi^* \leq \varpi^*, \tag{7}$$

for all $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathcal{K}_C$, it is an order relation. For given $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathcal{K}_C$, we say that $[\xi_*, \xi^*] \leq_I [\varpi_*, \varpi^*]$ if and only if $\xi_* \leq \varpi_*$, $\xi^* \leq \varpi^*$.

For $[\xi_*, \xi^*]$, $[\varpi_*, \varpi^*] \in \mathcal{K}_C$, the Hausdorff–Pompeiu distance between intervals $[\xi_*, \xi^*]$ and $[\varpi_*, \varpi^*]$ is defined by

$$d([\xi_*, \xi^*], [\varpi_*, \varpi^*]) = max\{|\xi_* - \varpi_*|, |\xi^* - \varpi^*|\}. \tag{8}$$

It is familiar fact that $(\mathcal{K}_C, d)$ is a complete metric space.

A fuzzy subset T of $\mathbb{R}$ is characterize by a mapping $\xi: \mathbb{R} \to [0,1]$ called the membership function, for each fuzzy set and $\theta \in (0, 1]$, then θ-level sets of ξ is denoted and defined as follows $\xi_\theta = \{u \in \mathbb{R} |\ \xi(u) \geq \theta\}$. If $\theta = 0$, then $supp(\xi) = \{\omega \in \mathbb{R} |\ \xi(\omega) > 0\}$ is called support of ξ. By $[\xi]^0$ we define the closure of $supp(\xi)$.

Let $\mathbb{F}(\mathbb{R})$ be the collection of all fuzzy sets and $\xi \in \mathbb{F}(\mathbb{R})$ be a fuzzy set. Then, we define the following:

(1) ξ is said to be normal if there exists $\omega \in \mathbb{R}$ and $\xi(\omega) = 1$;

(2) ξ is said to be upper semi continuous on $\mathbb{R}$ if for given $\omega \in \mathbb{R}$, there exist $\varepsilon > 0$ there exist $\delta > 0$ such that $\xi(\omega) - \xi(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\omega - y| < \delta$;

(3) ξ is said to be fuzzy convex if ξ_θ is convex for every $\theta \in [0, 1]$;

(4) ξ is compactly supported if $supp(\xi)$ is compact.

A fuzzy set is called a fuzzy number or fuzzy interval if it has properties (1)–(4). We denote by $\mathbb{F}_0$ the family of all fuzzy intervals.

Let $\xi \in \mathbb{F}_0$ be a fuzzy-interval, if and only if, θ-levels $[\xi]^\theta$ is a nonempty compact convex set of $\mathbb{R}$. From these definitions, we have

$$[\xi]^\theta = [\xi_*(\theta), \xi^*(\theta)],$$

where

$$\xi_*(\theta) = inf\{\omega \in \mathbb{R} |\ \xi(\omega) \geq \theta\},\ \xi^*(\theta) = sup\{\omega \in \mathbb{R} |\ \xi(\omega) \geq \theta\}.$$

Proposition 2.2. [7] If $\xi, \varpi \in \mathbb{F}_0$, then relation " $\preccurlyeq$ " defined on $\mathbb{F}_0$ by

$$\xi \preccurlyeq \varpi \text{ if and only if, } \quad [\xi]^\theta \leq_I [\varpi]^\theta \text{, for all } \theta \in [0, 1], \tag{9}$$

this relation is known as partial order relation.

For $\xi, \varpi \in \mathbb{F}_0$ and $\varrho \in \mathbb{R}$, the sum $\xi \tilde{+} \varpi$, product $\xi \tilde{\times} \varpi$, scalar product $\varrho . \xi$ and sum with scalar are defined by:

Then, for all $\theta \in [0, 1]$, we have

$$[\xi \tilde{+} \varpi]^\theta = [\xi]^\theta + [\varpi]^\theta, \tag{10}$$

$$[\xi \,\widetilde{\times}\, \varpi]^{\theta} = [\xi]^{\theta} \times [\varpi]^{\theta}, \quad (11)$$

$$[\varrho \cdot \xi]^{\theta} = \varrho \cdot [\xi]^{\theta}, \quad (12)$$

$$[\varrho \widetilde{+} \xi]^{\theta} = \varrho + [\xi]^{\theta}. \quad (13)$$

For $\psi \in \mathbb{F}_0$ such that $\xi = \varpi \widetilde{+} \psi$, then by this result we have existence of Hukuhara difference of ξ and ϖ, and we say that ψ is the H-difference of ξ and ϖ, and denoted by $\xi \widetilde{-} \varpi$.

Definition 2.3. [16] A fuzzy-interval-valued map $\mathcal{F}: K \subset \mathbb{R} \to \mathbb{F}_0$ is called FIVF. For each $\theta \in (0,1]$, θ-levels define the family of IVFs $\mathcal{F}_{\theta}: K \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathcal{F}_{\theta}(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in K$. Here, for each $\theta \in (0,1]$, the end point real functions $\mathcal{F}_*(.,\theta), \mathcal{F}^*(.,\theta): K \to \mathbb{R}$ are called lower and upper functions of $\mathcal{F}$.

Definition 2.5. [34] Let $\mathcal{F}: [u,\nu] \subset \mathbb{R} \to \mathbb{F}_0$ be a FIVF. Then, fuzzy Riemann integral of $\mathcal{F}$ over $[u,\nu]$, denoted by $(FR)\int_u^{\nu} \mathcal{F}(\omega)d\omega$, it is given level-wise by

$$\left[(FR)\int_u^{\nu} \mathcal{F}(\omega)d\omega\right]^{\theta} = (IR)\int_u^{\nu} \mathcal{F}_{\theta}(\omega)d\omega = \left\{\int_u^{\nu} \mathcal{F}(\omega,\theta)d\omega : \mathcal{F}(\omega,\theta) \in \mathcal{R}_{([u,\nu],\theta)}\right\}, \quad (14)$$

for all $\theta \in (0,1]$, where $\mathcal{R}_{([u,\nu],\theta)}$ denotes the collection of Riemannian integrable functions of IVFs. $\mathcal{F}$ is FR-integrable over $[u,\nu]$ if $(FR)\int_u^{\nu} \mathcal{F}(\omega)d\omega \in \mathbb{F}_0$. Note that, if both end point functions are Lebesgue-integrable, then $\mathcal{F}$ is fuzzy Aumann-integrable function over $[u,\nu]$ [16,34].

Theorem 2.6. Let $\mathcal{F}: [u,\nu] \subset \mathbb{R} \to \mathbb{F}_0$ be a FIVF and for all $\theta \in (0,1]$, θ-levels define the family of IVFs $\mathcal{F}_{\theta}: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C$ are given by $\mathcal{F}_{\theta}(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$. Then, $\mathcal{F}$ is fuzzy Riemann integrable (FR-integrable) over $[u,\nu]$ if and only if, $\mathcal{F}_*(\omega,\theta)$ and $\mathcal{F}^*(\omega,\theta)$ both are Riemann integrable (R-integrable) over $[u,\nu]$. Moreover, if $\mathcal{F}$ is FR-integrable over $[u,\nu]$, then

$$\left[(FR)\int_u^{\nu} \mathcal{F}(\omega)d\omega\right]^{\theta} = \left[(R)\int_u^{\nu} \mathcal{F}_*(\omega,\theta)d\omega, (R)\int_u^{\nu} \mathcal{F}^*(\omega,\theta)d\omega\right] = (IR)\int_u^{\nu} \mathcal{F}_{\theta}(\omega)d\omega, \quad (15)$$

for all $\theta \in (0,1]$, where IR represent interval Riemann integration of $\mathcal{F}_{\theta}(\omega)$. For all $\theta \in (0,1]$, $\mathcal{FR}_{([u,\nu],\theta)}$ denotes the collection of all FR-integrable FIVFs over $[u,\nu]$.

Definition 2.7. Let $[u,\nu]$ be a p-convex interval. Then, FIVF $\mathcal{F}: [u,\nu] \to \mathbb{F}_0$ is said to be p-convex on $[u,\nu]$ if

$$\mathcal{F}\left([\eta x^p + (1-\eta)y^p]^{\frac{1}{p}}\right) \preccurlyeq \eta\mathcal{F}(x) \widetilde{+} (1-\eta)\mathcal{F}(y), \quad (16)$$

for all $x, y \in [u,\nu], \eta \in [0,1]$, where $\mathcal{F}(\omega) \succcurlyeq \tilde{0}$, for all $\omega \in [u,\nu]$. If inequality (16) is reversed, then $\mathcal{F}$ is said to be p-concave FIVF on $[u,\nu]$. The set of all p-convex (LR-p-concave) FIVFs is denoted by

$$SXF([u,\nu], \mathbb{F}_0, p), \left(SVF([u,\nu], \mathbb{F}_0, p)\right).$$

Remark 2.8. The p-convex FIVFs have some very nice properties similar to convex FIVF:

If $\mathcal{F}$ is p-convex FIVF, then $\Upsilon\mathcal{F}$ is also p-convex for $\Upsilon \geq 0$.

If $\mathcal{F}$ and $\mathcal{T}$ both are p-convex FIVFs, then $\max\left(\mathcal{F}(\omega), \mathcal{T}(\omega)\right)$ is also p-convex FIVF.

We now discuss some new and known special cases of p-convex FIVFs:

If $p \equiv 1$, then p-convex FIVF becomes convex FIVF, see [35], that is

$$\mathcal{F}(\eta x+(1-\eta)y) \preccurlyeq \eta\mathcal{F}(x)\widetilde{+}(1-\eta)\mathcal{F}(y), \forall\ x,y\in\ [u,\nu], \eta\in[0,1]. \tag{17}$$

In Theorem 2.9, we will try to establish relation between the p-convex FIVFs and endpoint functions $\mathcal{F}_*(\omega,\theta)$, $\mathcal{F}^*(\omega,\theta)$ because through endpoint functions, FIVFs can be easily handled.

Theorem 2.9. Let $[u,\nu]$ be convex set, and $\mathcal{F}:[u,\nu]\to\mathbb{F}_0$ be a FIVF. Then, θ-levels define the family of IVFs $\mathcal{F}_\theta:[u,\nu]\subset\mathbb{R}\to{\mathcal{K}_C}^+\subset\mathcal{K}_C$ are given by

$$\mathcal{F}_\theta(\omega)=[\mathcal{F}_*(\omega,\theta),\mathcal{F}^*(\omega,\theta)], \forall\ \omega\in[u,\nu], \tag{18}$$

for all $\omega\in[u,\nu]$ and for all $\theta\in[0,1]$. Then, $\mathcal{F}$ is p-convex on $[u,\nu]$, if and only if, for all $\theta\in[0,1]$, $\mathcal{F}_*(\omega,\theta)$ and $\mathcal{F}^*(\omega,\theta)$ both are p-convex functions.

Proof. Assume that for each $\theta\in[0,1]$, $\mathcal{F}_*(\omega,\theta)$ and $\mathcal{F}^*(\omega,\theta)$ are p-convex functions on $[u,\nu]$. Then, from (16) we have

$$\mathcal{F}_*\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}},\theta\right)\le\eta\mathcal{F}_*(x,\theta)+(1-\eta)\mathcal{F}_*(y,\theta),\ \forall\ x,y\in[u,\nu],\eta\in[0,1],$$

and

$$\mathcal{F}^*\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}},\theta\right)\le\eta\mathcal{F}^*(x,\theta)+(1-\eta)\mathcal{F}^*(y,\theta),\forall\ x,y\in[u,\nu],\eta\in[0,1].$$

Then by (18), (10) and (12), we obtain

$$\mathcal{F}_\theta\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}}\right)=\left[\mathcal{F}_*\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}},\theta\right),\mathcal{F}^*\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}},\theta\right)\right],$$

$$\le_I [\eta\mathcal{F}_*(x,\theta),\eta\mathcal{F}^*(x,\theta)]+[(1-\eta)\mathcal{F}_*(y,\theta),(1-\eta)\mathcal{F}^*(y,\theta)],$$

that is

$$\mathcal{F}\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}}\right)\preccurlyeq\eta\mathcal{F}(x)\widetilde{+}(1-\eta)\mathcal{F}(y),\ \forall\ x,y\in[u,\nu],\eta\in[0,1].$$

Hence, $\mathcal{F}$ is p-convex FIVF on $[u,\nu]$.

Conversely, let $\mathcal{F}$ be p-convex FIVF on $[u,\nu]$. Then, for all $x,y\in[u,\nu]$ and $\eta\in[0,1]$, we have

$$\mathcal{F}\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}}\right)\preccurlyeq\eta\mathcal{F}(x)\widetilde{+}(1-\eta)\mathcal{F}(y).$$

Therefore, from (18), we have

$$\mathcal{F}_\theta\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}}\right)=\left[\mathcal{F}_*\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}},\theta\right),\mathcal{F}^*\left([\eta x^p+(1-\eta)y^p]^{\frac{1}{p}},\theta\right)\right].$$

Again, from (18), (10) and (12), we obtain

$$\eta\mathcal{F}_\theta(x)\widetilde{+}(1-\eta)\mathcal{F}_\theta(y)=[\eta\mathcal{F}_*(x,\theta),\eta\mathcal{F}^*(x,\theta)]+[(1-\eta)\mathcal{F}_*(y,\theta),(1-\eta)\mathcal{F}^*(y,\theta)],$$

for all $x,y\in[u,\nu]$ and $\eta\in[0,1]$. Then, by p-convexity of $\mathcal{F}$, we have for all $x,y\in[u,\nu]$ and $\eta\in[0,1]$ such that

$$\mathcal{F}_*\left([\eta x^p + (1-\eta)y^p]^{\frac{1}{p}}, \theta\right) \leq \eta\mathcal{F}_*(x,\theta) + (1-\eta)\mathcal{F}_*(y,\theta),$$

and

$$\mathcal{F}^*\left([\eta x^p + (1-\eta)y^p]^{\frac{1}{p}}, \theta\right) \leq \eta\mathcal{F}^*(x,\theta) + (1-\eta)\mathcal{F}^*(y,\theta),$$

for each $\theta \in [0,1]$. Hence, the result follows.

Example 2.10. We consider the FIVF $\mathcal{F}: [0,1] \to \mathbb{F}_0$ defined by,

$$\mathcal{F}(\omega)(\lambda) = \begin{cases} \dfrac{\lambda}{2\omega^2}, & \lambda \in [0, 2\omega^2] \\ \dfrac{4\omega^2 - \lambda}{2\omega^2}, & \lambda \in (2\omega^2, 4\omega^2] \\ 0, & otherwise, \end{cases}$$

then, for each $\theta \in [0,1]$, we have $\mathcal{F}_\theta(\omega) = [2\theta\omega^2, (4-2\theta)\omega^2]$. Since end point functions $\mathcal{F}_*(\omega,\theta)$, $\mathcal{F}^*(\omega,\theta)$ are convex functions for each $\theta \in [0,1]$. Hence $\mathcal{F}(\omega)$ is convex FIVF.

Remark 2.11. If $\mathcal{F}_*(\omega,\theta) = \mathcal{F}^*(\omega,\theta)$, then Definition 2.**7** reduces to the definition of classical p-convex function, [43].

If $\mathcal{F}_*(\omega,\theta) = \mathcal{F}^*(\omega,\theta)$ and $p \equiv 1$, then Definition 2.7 reduces to the definition of classical convex function.

3. Fuzzy-interval Hermite-Hadamard inequalities

In this section, we will prove two types of inequalities. First one is Hermite-Hadamard and their variant forms, and the second one is Hermite-Hadamard-Fejér inequalities for p-convex FIVFs where the integrands are FIVFs. We will verify these inequalities with the help of nontrivial examples.

Theorem 3.1. Let $\mathcal{F} \in SXF([u,\nu], \mathbb{F}_0, p)$. Then, θ-levels define the family of IVFs $\mathcal{F}_\theta: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $\mathcal{F} \in \mathcal{FR}_{([u,\nu],\theta)}$, then

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p}{\nu^p - u^p}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)d\omega \preccurlyeq \frac{\mathcal{F}(u)\,\tilde{+}\,\mathcal{F}(\nu)}{2}. \tag{19}$$

If $\mathcal{F}(\omega)$ is p-concave FIVF, then

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \succcurlyeq \frac{p}{\nu^p - u^p}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)d\omega \succcurlyeq \frac{\mathcal{F}(u)\,\tilde{+}\,\mathcal{F}(\nu)}{2}. \tag{20}$$

Proof. Let $\mathcal{F}: [u,\nu] \to \mathbb{F}_0$ be a p-convex FIVF. Then, by hypothesis, we have

$$2\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \mathcal{F}\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)\tilde{+}\mathcal{F}\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right).$$

Therefore, for every $\theta \in [0,1]$, we have

$$2\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\leq \mathcal{F}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)+\mathcal{F}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right),$$

$$2\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\leq \mathcal{F}^*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)+\mathcal{F}^*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right).$$

Then

$$2\int_0^1\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)d\eta\leq \int_0^1\mathcal{F}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)d\eta+\int_0^1\mathcal{F}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right)d\eta,$$

$$2\int_0^1\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)d\eta\leq \int_0^1\mathcal{F}^*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)d\eta+\int_0^1\mathcal{F}^*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right)d\eta.$$

It follows that

$$\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\leq\frac{p}{\nu^p-u^p}\int_u^\nu\omega^{p-1}\mathcal{F}_*(\omega,\theta)d\omega,$$

$$\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\leq\frac{p}{\nu^p-u^p}\int_u^\nu\omega^{p-1}\mathcal{F}^*(\omega,\theta)d\omega.$$

That is

$$\left[\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right),\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\right]\leq_I\frac{p}{\nu^p-u^p}\left[\int_u^\nu\omega^{p-1}\mathcal{F}_*(\omega,\theta)d\omega,\int_u^\nu\omega^{p-1}\mathcal{F}^*(\omega,\theta)d\omega\right].$$

Thus,

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{p}{\nu^p-u^p}\ (FR)\int_u^\nu\omega^{p-1}\mathcal{F}(\omega)d\omega. \tag{21}$$

In a similar way as above, we have

$$\frac{p}{\nu^p-u^p}(FR)\int_u^\nu\omega^{p-1}\mathcal{F}(\omega)d\omega\preccurlyeq\frac{\mathcal{F}(u)\,\tilde{+}\,\mathcal{F}(\nu)}{2}. \tag{22}$$

Combining (21) and (22), we have

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{p}{\nu^p-u^p}(FR)\int_u^\nu\omega^{p-1}\mathcal{F}(\omega)d\omega\preccurlyeq\frac{\mathcal{F}(u)\,\tilde{+}\,\mathcal{F}(\nu)}{2}.$$

Hence, the required result.

Remark 3.2. If $p=1$, then Theorem 3.1, reduces to the result for convex FIVF, see [25]:

$$\mathcal{F}\left(\frac{u+\nu}{2}\right)\preccurlyeq\frac{1}{\nu-u}\ (FR)\int_u^\nu\mathcal{F}(\omega)d\omega\preccurlyeq\frac{\mathcal{F}(u)\,\tilde{+}\,\mathcal{F}(\nu)}{2}.$$

If $\mathcal{F}_*(\omega,\theta)=\mathcal{F}^*(\omega,\theta)$ with $\theta=1$, then Theorem 3.1, reduces to the result for p-convex function [43]:

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)\leq\frac{p}{\nu^p-u^p}\ (R)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)d\omega\leq\frac{\mathcal{F}(u)+\ \mathcal{F}(\nu)}{2}.$$

If $\mathcal{F}_*(\omega,\theta)=\mathcal{F}^*(\omega,\theta)$ with $\theta=1$ and $p=1$, then Theorem 3.1, reduces to the result for classical convex function:

$$\mathcal{F}\left(\frac{u+\nu}{2}\right)\leq\frac{1}{\nu-u}\ (R)\int_u^\nu \mathcal{F}(\omega)d\omega\leq\frac{\mathcal{F}(u)+\ \mathcal{F}(\nu)}{2}.$$

Example 3.3. Let p be an odd number and the FIVF $\mathcal{F}:[u,\nu]=[2,3]\to\mathbb{F}_0$ defined by,

$$\mathcal{F}(\omega)(\lambda)=\begin{cases}\frac{\lambda}{\left(2-\omega^{\frac{p}{2}}\right)} & \lambda\in\left[0,2-\omega^{\frac{p}{2}}\right],\\ \frac{2\left(2-\omega^{\frac{p}{2}}\right)-\lambda}{\left(2-\omega^{\frac{p}{2}}\right)} & \lambda\in\left(2-\omega^{\frac{p}{2}},2\left(2-\omega^{\frac{p}{2}}\right)\right],\\ 0 & \text{otherwise}.\end{cases}\tag{23}$$

Then, for each $\theta\in[0,1]$, we have $\mathcal{F}_\theta(\omega)=\left[\theta\left(2-\omega^{\frac{p}{2}}\right),\ (2-\theta)\left(2-\omega^{\frac{p}{2}}\right)\right]$. Since end point functions $\mathcal{F}_*(\omega,\theta)=\theta\left(2-\omega^{\frac{p}{2}}\right)$, $\mathcal{F}^*(\omega,\theta)=(2-\theta)\left(2-\omega^{\frac{p}{2}}\right)$ are p-convex functions for each $\theta\in[0,1]$. Then, $\mathcal{F}(\omega)$ is p-convex FIVF.

We now computing the following

$$\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)=\frac{4-\sqrt{10}}{2}\theta,$$
$$\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)=\frac{4-\sqrt{10}}{2}(2-\theta),$$

$$\frac{p}{\nu^p-u^p}\int_u^\nu \omega^{p-1}\mathcal{F}_*(\omega,\theta)d\omega=\ \theta\int_2^3\left(2-\omega^{\frac{p}{2}}\right)d\omega=\frac{21}{50}\theta,$$
$$\frac{p}{\nu^p-u^p}\int_u^\nu \omega^{p-1}\mathcal{F}^*(\omega,\theta)d\omega=\ (2-\theta)\int_2^3\left(2-\omega^{\frac{p}{2}}\right)d\omega=\frac{21}{50}(2-\theta),$$

$$\frac{\mathcal{F}_*(u,\theta)+\ \mathcal{F}_*(\nu,\theta)}{2}=\frac{4-\sqrt{2}-\sqrt{3}}{2}\theta,$$
$$\frac{\mathcal{F}^*(u,\theta)+\ \mathcal{F}^*(\nu,\theta)}{2}=\frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\theta),$$

for all $\theta\in[0,1]$. That means

$$\left[\frac{4-\sqrt{10}}{2}\theta,\frac{4-\sqrt{10}}{2}(2-\theta)\right]\leq_I\left[\frac{21}{50}\theta,\frac{21}{50}(2-\theta)\right]\leq_I\left[\frac{4-\sqrt{2}-\sqrt{3}}{2}\theta,\frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\theta)\right],$$

for all $\theta \in [0,1]$, and the Theorem 3.1 has been verified.

To prove some related inequalities for the above theorem, we obtain following inequality for p-convex FIVFs

Theorem 3.4. Let $\mathcal{F} \in SXF([u,\nu],\mathbb{F}_0,p)$. Then, θ-levels define the family of IVFs $\mathcal{F}_\theta: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $\mathcal{F} \in \mathcal{FR}_{([u,\nu],\theta)}$, then

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{p}{\nu^p-u^p}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)d\omega \preccurlyeq \rhd_1 \preccurlyeq \frac{\mathcal{F}(u)\ \widetilde{+}\ \mathcal{F}(\nu)}{2},$$

where

$$\rhd_1 = \frac{\frac{\mathcal{F}(u)\ \widetilde{+}\ \mathcal{F}(\nu)}{2}\ \widetilde{+}\ \mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)}{2},\quad \rhd_2 = \frac{\mathcal{F}\left(\left[\frac{3u^p+\nu^p}{4}\right]^{\frac{1}{p}}\right)\ \widetilde{+}\ \mathcal{F}\left(\left[\frac{u^p+3\nu^p}{4}\right]^{\frac{1}{p}}\right)}{2}, \text{ and } \rhd_1 = [\rhd_{1*}, \rhd_1{}^*],\ \rhd_2 = [\rhd_{2*}, \rhd_2{}^*].$$

Proof. Take $\left[u^p, \frac{u^p+\nu^p}{2}\right]$, we have

$$2\mathcal{F}\left(\frac{\left[\eta u^p+(1-\eta)\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}}{2}+\frac{\left[(1-\eta)u^p+\eta\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}}{2}\right) \preccurlyeq \mathcal{F}\left(\left[\eta u^p+(1-\eta)\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)\widetilde{+}\mathcal{F}\left(\left[(1-\eta)u^p+\eta\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right).$$

Therefore, for every $\theta \in [0,1]$, we have

$$2\mathcal{F}_*\left(\frac{\left[\eta u^p+(1-\eta)\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}}{2}+\frac{\left[(1-\eta)u^p+\eta\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}}{2},\theta\right) \le \mathcal{F}_*\left(\left[\eta u^p+(1-\eta)\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)+\mathcal{F}_*\left(\left[(1-\eta)u^p+\eta\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right),$$

$$2\mathcal{F}^*\left(\frac{\left[\eta u^p+(1-\eta)\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}}{2}+\frac{\left[(1-\eta)u^p+\eta\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}}{2},\theta\right) \le \mathcal{F}^*\left(\left[\eta u^p+(1-\eta)\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)+\mathcal{F}^*\left(\left[(1-\eta)u^p+\eta\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right).$$

In consequence, we obtain

$$\frac{\mathcal{F}_*\left(\left[\frac{3u^p+\nu^p}{4}\right]^{\frac{1}{p}},\theta\right)}{2} \le \frac{p}{\nu^p-u^p}\int_u^{\frac{u^p+\nu^p}{2}} \mathcal{F}_*(\omega,\theta)d\omega\,,$$

$$\frac{\mathcal{F}^*\left(\left[\frac{3u^p+\nu^p}{4}\right]^{\frac{1}{p}},\theta\right)}{2} \le \frac{p}{\nu^p-u^p}\int_u^{\frac{u^p+\nu^p}{2}} \mathcal{F}^*(\omega,\theta)d\omega\,.$$

That is

$$\frac{\left[\mathcal{F}_*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}},\theta\right),\ \mathcal{F}^*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}},\theta\right)\right]}{2}\leq\frac{p}{v^p-u^p}\left[\int_u^{\frac{u^p+v^p}{2}}\mathcal{F}_*(\omega,\theta)d\omega\,,\int_u^{\frac{u^p+v^p}{2}}\mathcal{F}^*(\omega,\theta)d\omega\right].$$

It follows that

$$\frac{\mathcal{F}\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right)}{2}\preccurlyeq\frac{p}{v^p-u^p}\ (FR)\int_u^{\frac{u^p+v^p}{2}}\mathcal{F}(\omega)d\omega. \tag{24}$$

In a similar way as above, we have

$$\frac{\mathcal{F}\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right)}{2}\preccurlyeq\frac{p}{v^p-u^p}\ (FR)\int_{\frac{u^p+v^p}{2}}^{v}\mathcal{F}(\omega)d\omega. \tag{25}$$

Combining (24) and (25), we have

$$\frac{\left[\mathcal{F}\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}}\right)\widetilde{+}\mathcal{F}\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right)\right]}{2}\preccurlyeq\frac{p}{v^p-u^p}\ (FR)\int_u^{v}\mathcal{F}(\omega)d\omega.$$

By using Theorem 3.1, we have

$$\mathcal{F}\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}}\right)=\ \mathcal{F}\left(\left[\frac{1}{2}.\frac{3u^p+v^p}{4}+\frac{1}{2}.\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}}\right).$$

Therefore, for every $\theta\in[0,1]$, we have

$$\mathcal{F}_*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}},\theta\right)=\mathcal{F}_*\left(\left[\frac{1}{2}.\frac{3u^p+v^p}{4}+\frac{1}{2}.\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}},\theta\right),$$
$$\mathcal{F}^*\left(\left[\frac{u^p+v^p}{2}\right]^{\frac{1}{p}},\theta\right)=\mathcal{F}^*\left(\left[\frac{1}{2}.\frac{3u^p+v^p}{4}+\frac{1}{2}.\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}},\theta\right),$$

$$\leq\left[\frac{1}{2}\mathcal{F}_*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}},\theta\right)+\frac{1}{2}\mathcal{F}_*\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}},\theta\right)\right],$$
$$\leq\left[\frac{1}{2}\mathcal{F}^*\left(\left[\frac{3u^p+v^p}{4}\right]^{\frac{1}{p}},\theta\right)+\frac{1}{2}\mathcal{F}^*\left(\left[\frac{u^p+3v^p}{4}\right]^{\frac{1}{p}},\theta\right)\right],$$
$$=\rhd_{2_*},$$
$$=\rhd_2{}^*,$$

$$\leq \frac{p}{\nu^p - u^p} \int_u^\nu \mathcal{F}_*(\omega,\theta)d\omega\,,$$
$$\leq \frac{p}{\nu^p - u^p} \int_u^\nu \mathcal{F}^*(\omega,\theta)d\omega\,,$$

$$\leq \frac{1}{2}\left[\frac{\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)}{2} + \mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\right],$$
$$\leq \frac{1}{2}\left[\frac{\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)}{2} + \mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\right],$$
$$= \rhd_{1*},$$
$$= \rhd_1{}^*,$$
$$\leq \frac{1}{2}\left[\frac{\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)}{2} + \frac{\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)}{2}\right],$$
$$\leq \frac{1}{2}\left[\frac{\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)}{2} + \frac{\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)}{2}\right],$$
$$= \frac{\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)}{2},$$
$$= \frac{\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)}{2},$$

that is

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \rhd_2 \preccurlyeq \frac{p}{\nu^p - u^p}\ (FR)\int_u^\nu \mathcal{F}(\omega)d\omega \preccurlyeq \rhd_1 \preccurlyeq \frac{\mathcal{F}(u)\,\tilde{+}\,\mathcal{F}(\nu)}{2},$$

hence, the result follows.

Example 3.5. Let p be an odd number and the FIVF $\mathcal{F}:[u,\nu]=[2,3]\to\mathbb{F}_0$ defined by, $\mathcal{F}_\theta(\omega)=\left[\theta\left(2-\omega^{\frac{p}{2}}\right),(2-\theta)\left(2-\omega^{\frac{p}{2}}\right)\right]$, as in Example 3.3, then $\mathcal{F}(\omega)$ is p-convex FIVF and satisfying (38). We have $\mathcal{F}_*(\omega,\theta)=\theta\left(2-\omega^{\frac{p}{2}}\right)$ and $\mathcal{F}^*(\omega,\theta)=(2-\theta)\left(2-\omega^{\frac{p}{2}}\right)$. We now computing the following

$$\left[\frac{\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)}{2}\right] = \frac{4-\sqrt{2}-\sqrt{3}}{2}\theta,$$
$$\left[\frac{\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)}{2}\right] = \frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\theta),$$

$$\rhd_{1*} = \frac{\frac{\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)}{2} + \mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)}{2} = \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}\theta,$$
$$\rhd_1{}^* = \frac{\frac{\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)}{2} + \mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)}{2} = \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}(2-\theta),$$

$$\rhd_{2*} = \frac{1}{2}\left[\mathcal{F}_*\left(\left[\frac{3u^p + \nu^p}{4}\right]^{\frac{1}{p}}, \theta\right) + \mathcal{F}_*\left(\left[\frac{u^p + 3\nu^p}{4}\right]^{\frac{1}{p}}, \theta\right)\right] = \frac{5-\sqrt{11}}{4}\theta,$$

$$\rhd_2{}^* = \frac{1}{2}\left[\mathcal{F}^*\left(\left[\frac{3u^p + \nu^p}{4}\right]^{\frac{1}{p}}, \theta\right) + \mathcal{F}^*\left(\left[\frac{u^p + 3\nu^p}{4}\right]^{\frac{1}{p}}, \theta\right)\right] = \frac{5-\sqrt{11}}{4}(2-\theta),$$

$$\mathcal{F}_*\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}, \theta\right) = \frac{4-\sqrt{10}}{2}\theta,$$

$$\mathcal{F}^*\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}, \theta\right) = \frac{4-\sqrt{10}}{2}(2-\theta).$$

Then, we obtain that

$$\frac{4-\sqrt{10}}{2}\theta \le \frac{5-\sqrt{11}}{4}\theta \le \frac{21}{50}\theta \le \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}\theta \le \frac{4-\sqrt{2}-\sqrt{3}}{2}\theta,$$

$$\frac{4-\sqrt{10}}{2}(2-\theta) \le \frac{5-\sqrt{11}}{4}(2-\theta) \le \frac{21}{50}(2-\theta) \le \frac{8-\sqrt{2}-\sqrt{3}-\sqrt{10}}{4}(2-\theta) \le \frac{4-\sqrt{2}-\sqrt{3}}{2}(2-\theta).$$

Hence, Theorem 3.4 is verified.

From Theorem 3.6 and Theorem 3.7, we now obtain some H-H inequalities for the product of p-convex FIVFs. These inequalities are refinements of some known inequalities [42,43].

Theorem 3.6. Let $\mathcal{F}, \mathcal{J} \in SXF([u,\nu], \mathbb{F}_0, p)$. Then, θ-levels $\mathcal{F}_\theta$, $\mathcal{J}_\theta: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are defined by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ and $\mathcal{J}_\theta(\omega) = [\mathcal{J}_*(\omega,\theta), \mathcal{J}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $\mathcal{F}, \mathcal{J}$ and $\mathcal{F}\mathcal{J} \in \mathcal{FR}_{([u,\nu],\theta)}$, then

$$\frac{p}{\nu^p - u^p}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega) \mathbin{\tilde{\times}} \mathcal{J}(\omega)d\omega \preccurlyeq \frac{\mathcal{M}(u,\nu)}{3} \mathbin{\tilde{+}} \frac{\mathcal{N}(u,\nu)}{6}.$$

Where $\mathcal{M}(u,\nu) = \mathcal{F}(u) \mathbin{\tilde{\times}} \mathcal{J}(u) \mathbin{\tilde{+}} \mathcal{F}(\nu) \mathbin{\tilde{\times}} \mathcal{J}(\nu)$, $\mathcal{N}(u,\nu) = \mathcal{F}(u) \mathbin{\tilde{\times}} \mathcal{J}(\nu) \mathbin{\tilde{+}} \mathcal{F}(\nu) \mathbin{\tilde{\times}} \mathcal{J}(u)$, and $\mathcal{M}_\theta(u,\nu) = \left[\mathcal{M}_*\big((u,\nu),\theta\big), \mathcal{M}^*\big((u,\nu),\theta\big)\right]$ and $\mathcal{N}_\theta(u,\nu) = \left[\mathcal{N}_*\big((u,\nu),\theta\big), \mathcal{N}^*\big((u,\nu),\theta\big)\right]$.

Proof. The proof is similar to the proof of Theorem 3.3 [46].

Example 3.7. Let p be an odd number, and p-convex FIVFs $\mathcal{F}, \mathcal{J}: [u,\nu] = [2,3] \to \mathbb{F}_0$ are, respectively defined by, $\mathcal{F}_\theta(\omega) = \left[\theta\left(2 - \omega^{\frac{p}{2}}\right), (2-\theta)\left(2 - \omega^{\frac{p}{2}}\right)\right]$, as in Example 3.3 and $\mathcal{J}_\theta(\omega) = [\theta\omega^p, (2-\theta)\omega^p\ ]$. Since $\mathcal{F}(\omega)$ and $\mathcal{J}(\omega)$ both are p-convex FIVFs and $\mathcal{F}_*(\omega,\theta) = \theta\left(2 - \omega^{\frac{p}{2}}\right)$, $\mathcal{F}^*(\omega,\theta) = (2-\theta)\left(2 - \omega^{\frac{p}{2}}\right)$, and $\mathcal{J}_*(\omega,\theta) = \theta\omega^p$, $\mathcal{J}^*(\omega,\theta) = (2-\theta)\omega^p$, then we computing the following

$$\frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}_*(\omega,\theta) \times \mathcal{J}_*(\omega,\theta)d\omega = \theta^2,$$

$$\frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}^*(\omega,\theta) \times \mathcal{J}^*(\omega,\theta)d\omega = (2-\theta)^2,$$

$$\mathcal{M}_*\big((u,\nu),\theta\big) = \big(10-2\sqrt{2}-3\sqrt{3}\big)\frac{\theta^2}{3},$$
$$\mathcal{M}^*\big((u,\nu),\theta\big) = \big(10-2\sqrt{2}-3\sqrt{3}\big)\frac{(2-\theta)^2}{3},$$
$$\mathcal{N}_*\big((u,\nu),\theta\big) = \big(10-3\sqrt{2}-2\sqrt{3}\big)\frac{\theta^2}{6}$$
$$\mathcal{N}^*\big((u,\nu),\theta\big) = \big(10-3\sqrt{2}-2\sqrt{3}\big)\frac{(2-\theta)^2}{6},$$

for each $\theta \in [0,1]$, that means

$$\theta^2 \le \big(30-7\sqrt{2}-8\sqrt{3}\big)\tfrac{\theta^2}{6},$$
$$(2-\theta)^2 \le \big(30-7\sqrt{2}-8\sqrt{3}\big)\tfrac{(2-\theta)^2}{6}.$$

Hence, Theorem 3.6 is demonstrated.

Theorem 3.8. Let $\mathcal{F},\mathcal{J} \in SXF([u,\nu],\mathbb{F}_0,p)$. Then, θ-levels define the family of IVFs $\mathcal{F}_\theta$, $\mathcal{J}_\theta: [u,\nu] \subset \mathbb{R} \to {\mathcal{K}_C}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ and $\mathcal{J}_\theta(\omega) = [\mathcal{J}_*(\omega,\theta), \mathcal{J}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $\mathcal{F} \,\tilde{\times}\, \mathcal{J} \in \mathcal{FR}_{([u,\nu],\theta)}$, then

$$2\,\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \tilde{\times}\, \mathcal{J}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{p}{\nu^p-u^p}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)\,\tilde{\times}\,\mathcal{J}(\omega)d\omega \;\tilde{+}\; \frac{\mathcal{M}(u,\nu)}{6} \tilde{+} \frac{\mathcal{N}(u,\nu)}{3}.$$

Where $\mathcal{M}(u,\nu) = \mathcal{F}(u)\,\tilde{\times}\,\mathcal{J}(u)\,\tilde{+}\,\mathcal{F}(\nu)\,\tilde{\times}\,\mathcal{J}(\nu)$, $\mathcal{N}(u,\nu) = \mathcal{F}(u)\,\tilde{\times}\,\mathcal{J}(\nu)\,\tilde{+}\,\mathcal{F}(\nu)\,\tilde{\times}\,\mathcal{J}(u)$, and $\mathcal{M}_\theta(u,\nu) = \big[\mathcal{M}_*\big((u,\nu),\theta\big), \mathcal{M}^*\big((u,\nu),\theta\big)\big]$ and $\mathcal{N}_\theta(u,\nu) = \big[\mathcal{N}_*\big((u,\nu),\theta\big), \mathcal{N}^*\big((u,\nu),\theta\big)\big]$.

Proof. By hypothesis, for each $\theta \in [0,1]$, we have

$$\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right) \times \mathcal{J}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)$$
$$\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right) \times \mathcal{J}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)$$

$$\le \frac{1}{4}\left[\begin{array}{c} \mathcal{F}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right) \times \mathcal{J}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right) \\ +\mathcal{F}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right) \times \mathcal{J}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right) \end{array}\right]$$

$$+\frac{1}{4}\left[\begin{array}{c} \mathcal{F}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right) \times \mathcal{J}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right) \\ +\mathcal{F}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right) \times \mathcal{J}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right) \end{array}\right]$$

,

$$\leq \frac{1}{4}\left[\begin{array}{c} \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \\ + \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \end{array}\right]$$

$$+ \frac{1}{4}\left[\begin{array}{c} \mathcal{F}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \\ + \mathcal{F}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \end{array}\right],$$

$$\leq \frac{1}{4}\left[\begin{array}{c} \mathcal{F}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \\ + \mathcal{F}_*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}_*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \end{array}\right]$$

$$+ \frac{1}{4}\left[\begin{array}{c} \left(\eta\mathcal{F}_*(u,\theta) + (1-\eta)\mathcal{F}_*(\nu,\theta)\right) \\ \times\left((1-\eta)\mathcal{J}_*(u,\theta) + \eta\mathcal{J}_*(\nu,\theta)\right) \\ + \left((1-\eta)\mathcal{F}_*(u,\theta) + \eta\mathcal{F}_*(\nu,\theta)\right) \\ \times\left(\eta\mathcal{J}_*(u,\theta) + (1-\eta)\mathcal{J}_*(\nu,\theta)\right) \end{array}\right],$$

$$\leq \frac{1}{4}\left[\begin{array}{c} \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \\ + \mathcal{F}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \end{array}\right]$$

$$+ \frac{1}{4}\left[\begin{array}{c} \left(\eta\mathcal{F}^*(u,\theta) + (1-\eta)\mathcal{F}^*(\nu,\theta)\right) \\ \times\left((1-\eta)\mathcal{J}^*(u,\theta) + \eta\mathcal{J}^*(\nu,\theta)\right) \\ + \left((1-\eta)\mathcal{F}^*(u,\theta) + \eta\mathcal{F}^*(\nu,\theta)\right) \\ \times\left(\eta\mathcal{J}^*(u,\theta) + (1-\eta)\mathcal{J}^*(\nu,\theta)\right) \end{array}\right],$$

$$= \frac{1}{4}\left[\begin{array}{c} \mathcal{F}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \\ + \mathcal{F}_*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}_*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \end{array}\right]$$

$$+ \frac{1}{2}\left[\begin{array}{c} \{\eta^2 + (1-\eta)^2\}\mathcal{N}_*\left((u,\nu),\theta\right) \\ + \{\eta(1-\eta) + (1-\eta)\eta\}\mathcal{M}_*\left((u,\nu),\theta\right) \end{array}\right],$$

$$= \frac{1}{4}\left[\begin{array}{c} \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \\ + \mathcal{F}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \times \mathcal{J}^*\left([(1-\eta) u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \end{array}\right]$$

$$+ \frac{1}{2}\left[\begin{array}{c} \{\eta^2 + (1-\eta)^2\}\mathcal{N}^*\left((u,\nu),\theta\right) \\ + \{\eta(1-\eta) + (1-\eta)\eta\}\mathcal{M}^*\left((u,\nu),\theta\right) \end{array}\right],$$

R-Integrating over $[0,1]$, we have

$$2\,\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\times\mathcal{J}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\le\frac{p}{\nu^p-u^p}\int_u^\nu\omega^{p-1}\mathcal{F}_*(\omega,\theta)\times\mathcal{J}_*(\omega,\theta)d\omega\;+\;\frac{\mathcal{M}_*\big((u,\nu),\theta\big)}{6}$$
$$+\frac{\mathcal{N}_*\big((u,\nu),\theta\big)}{3},$$
$$2\,\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\times\mathcal{J}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\le\frac{p}{\nu^p-u^p}\int_u^\nu\omega^{p-1}\mathcal{F}^*(\omega,\theta)\times\mathcal{J}^*(\omega,\theta)d\omega\;+\;\frac{\mathcal{M}^*\big((u,\nu),\theta\big)}{6}$$
$$+\frac{\mathcal{N}^*\big((u,\nu),\theta\big)}{3},$$

that is

$$2\,\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)\tilde{\times}\,\mathcal{J}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq\frac{p}{\nu^p-u^p}\;(FR)\int_u^\nu\omega^{p-1}\mathcal{F}(\omega)\,\tilde{\times}\,\mathcal{J}(\omega)d\omega\;\tilde{+}\;\frac{\mathcal{M}(u,\nu)}{6}\tilde{+}\frac{\mathcal{N}(u,\nu)}{3}.$$

Hence, the required result.

Example 3.9. Let p be an odd number, and p-convex FIVFs $\mathcal{F},\mathcal{J}:[u,\nu]=[2,3]\to\mathbb{F}_0$ are, respectively defined by, $\mathcal{F}_\theta(\omega)=\left[\theta\left(2-\omega^{\frac{p}{2}}\right),(2-\theta)\left(2-\omega^{\frac{p}{2}}\right)\right]$, as in Example 3.3 and $\mathcal{J}_\theta(\omega)=[\theta\omega^p,(2-\theta)\omega^p\,]$. Since $\mathcal{F}(\omega)$ and $\mathcal{J}(\omega)$ both are p-convex FIVFs and $\mathcal{F}_*(\omega,\theta)=\theta\left(2-\omega^{\frac{p}{2}}\right)$, $\mathcal{F}^*(\omega,\theta)=(2-\theta)\left(2-\omega^{\frac{p}{2}}\right)$, and $\mathcal{J}_*(\omega,\theta)=\theta\omega^p$, $\mathcal{J}^*(\omega,\theta)=(2-\theta)\omega^p$, then we computing the following

$$2\,\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\times\mathcal{J}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)=\frac{20-5\sqrt{10}}{2}\theta^2,$$
$$2\,\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\times\mathcal{J}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)=\frac{20-5\sqrt{10}}{2}(2-\theta)^2,$$

$$\frac{\mathcal{M}_*\big((u,\nu),\theta\big)}{6}=\big(10-2\sqrt{2}-3\sqrt{3}\big)\frac{\theta^2}{6},$$
$$\frac{\mathcal{M}^*\big((u,\nu),\theta\big)}{6}=\big(10-2\sqrt{2}-3\sqrt{3}\big)\frac{(2-\theta)^2}{6},$$
$$\frac{\mathcal{N}_*\big((u,\nu),\theta\big)}{3}=\big(10-3\sqrt{2}-2\sqrt{3}\big)\frac{\theta^2}{3},$$
$$\frac{\mathcal{N}^*\big((u,\nu),\theta\big)}{3}=\big(10-3\sqrt{2}-2\sqrt{3}\big)\frac{(2-\theta)^2}{3},$$

for each $\theta\in[0,1]$, that means

$$\frac{20-5\sqrt{10}}{2}\theta^2\le\big(30-8\sqrt{2}-7\sqrt{3}\big)\frac{\theta^2}{6},$$
$$\frac{20-5\sqrt{10}}{2}(2-\theta)^2\le\big(30-8\sqrt{2}-7\sqrt{3}\big)\frac{(2-\theta)^2}{6},$$

hence, Theorem 3.8 is verified.

We now give H-H Fejér inequalities for p-convex FIVFs. Firstly, we obtain the second H-H Fejér inequality for p-convex FIVF.

Theorem 3.10. Let $\mathcal{F} \in SXF([u,\nu], \mathbb{F}_0, p)$. Then, θ-levels define the family of IVFs $\mathcal{F}_\theta: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $\mathcal{F} \in \mathcal{FR}_{([u,\nu],\theta)}$ and $\Omega: [u,\nu] \to \mathbb{R}, \Omega(\omega) \geq 0$, symmetric with respect to $\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}$, then

$$\frac{p}{\nu^p-u^p}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)\Omega(\omega)d\omega \preccurlyeq [\mathcal{F}(u) \tilde{+} \mathcal{F}(\nu)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta. \quad (26)$$

If $\mathcal{F}$ is p-concave FIVF, then inequality (26) is reversed.

Proof. Let $\mathcal{F}$ be a p-convex FIVF. Then, for each $\theta \in [0,1]$, we have

$$\mathcal{F}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)$$
$$\leq \left(\eta\mathcal{F}_*(u,\theta) + (1-\eta)\mathcal{F}_*(\nu,\theta)\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right), \quad (27)$$

$$\mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)$$
$$\leq \left(\eta\mathcal{F}^*(u,\theta) + (1-\eta)\mathcal{F}^*(\nu,\theta)\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)$$

And

$$\begin{aligned}
&\mathcal{F}_*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)\\
&\leq \left((1-\eta)\mathcal{F}_*(u,\theta) + \eta\mathcal{F}_*(\nu,\theta)\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right),\\
&\mathcal{F}^*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)\\
&\leq \left((1-\eta)\mathcal{F}^*(u,\theta) + \eta\mathcal{F}^*(\nu,\theta)\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right).
\end{aligned} \quad (28)$$

After adding (27) and (28), and integrating over $[0,1]$, we get

$$\int_0^1 \mathcal{F}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta$$
$$+\int_0^1 \mathcal{F}_*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta$$
$$\leq \int_0^1 \begin{bmatrix} \mathcal{F}_*(u,\theta)\left\{\eta\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right) + (1-\eta)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)\right\} \\ +\mathcal{F}_*(\nu,\theta)\left\{(1-\eta)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right) + \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)\right\} \end{bmatrix} d\eta,$$

$$\int_0^1 \mathcal{F}^*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta$$
$$+\int_0^1 \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta$$
$$\leq \int_0^1 \begin{bmatrix} \mathcal{F}^*(u,\theta)\left\{\eta\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right) + (1-\eta)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)\right\} \\ +\mathcal{F}^*(\nu,\theta)\left\{(1-\eta)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right) + \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)\right\} \end{bmatrix} d\eta,$$

$$= 2\mathcal{F}_*(u,\theta)\int_0^1 \eta\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta + 2\mathcal{F}_*(\nu,\theta)\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta,$$
$$= 2\mathcal{F}^*(u,\theta)\int_0^1 \eta\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta + 2\mathcal{F}^*(\nu,\theta)\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta.$$

Since Ω is symmetric, then

$$\begin{aligned} &= 2[\mathcal{F}_*(u,\theta) + \mathcal{F}_*(\nu,\theta)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta, \\ &= 2[\mathcal{F}^*(u,\theta) + \mathcal{F}^*(\nu,\theta)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta. \end{aligned} \tag{29}$$

Since

$$\begin{aligned} &\int_0^1 \mathcal{F}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta \\ &\qquad = \int_0^1 \mathcal{F}_*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta \\ &\qquad = \frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega \\ &\int_0^1 \mathcal{F}^*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta \\ &\qquad = \int_0^1 \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta \\ &\qquad = \frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega. \end{aligned} \tag{30}$$

Then, from (29), we have

$$\frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega \le [\mathcal{F}_*(u,\theta) + \mathcal{F}_*(\nu,\theta)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta,$$
$$\frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega \le [\mathcal{F}^*(u,\theta) + \mathcal{F}^*(\nu,\theta)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta,$$

that is

$$\left[\frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega, \frac{p}{\nu^p - u^p}\int_u^\nu \omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega\right]$$

$$\le_p [\mathcal{F}_*(u,\theta) + \mathcal{F}_*(\nu,\theta), \mathcal{F}^*(u,\theta) + \mathcal{F}^*(\nu,\theta)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta,$$

hence

$$\frac{p}{\nu^p - u^p}\,(FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)\Omega(\omega)d\omega \preccurlyeq [\mathcal{F}(u) \,\tilde{+}\, \mathcal{F}(\nu)]\int_0^1 \eta\Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right)d\eta.$$

Next, we construct first H-H Fejér inequality for p-convex FIVF, which generalizes first H-H Fejér inequalities for convex function [44].

Theorem 3.11. Let $\mathcal{F} \in SXF([u,\nu], \mathbb{F}_0, p)$. Then, θ-levels define the family of IVFs $\mathcal{F}_\theta\colon [u,\nu] \subset$

$\mathbb{R} \to \mathcal{K}_C{}^{+}$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $\mathcal{F} \in \mathcal{FR}_{([u,\nu],\theta)}$ and $\Omega\colon [u,\nu] \to \mathbb{R}, \Omega(\omega) \geq 0$, symmetric with respect to $\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}$, and $\int_u^\nu \omega^{p-1}\Omega(\omega)d\omega > 0$, then

$$\mathcal{F}\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{1}{\int_u^\nu \omega^{p-1}\Omega(\omega)d\omega}\ (FR)\int_u^\nu \omega^{p-1}\mathcal{F}(\omega)\Omega(\omega)d\omega. \tag{31}$$

If $\mathcal{F}$ is p-concave FIVF, then inequality (31) is reversed.

Proof. Since $\mathcal{F}$ is a convex, then for $\theta \in [0,1]$, we have

$$\begin{aligned}
&\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right) \leq \frac{1}{2}\left(\mathcal{F}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)+\mathcal{F}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right)\right),\\
&\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right) \leq \frac{1}{2}\left(\mathcal{F}^*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)+\mathcal{F}^*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right)\right),
\end{aligned} \tag{32}$$

Since $\Omega\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}}\right) = \Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)$, then by multiplying (32) by $\Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)$ and integrate it with respect to η over $[0,1]$, we obtain

$$\begin{aligned}
&\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\int_0^1 \Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)d\eta\\
&\qquad \leq \frac{1}{2}\left(\begin{array}{c}\int_0^1 \mathcal{F}_*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta\\ +\int_0^1 \mathcal{F}_*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right)d\eta\Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)\end{array}\right),\\
&\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\int_0^1 \Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)d\eta\\
&\qquad \leq \frac{1}{2}\left(\begin{array}{c}\int_0^1 \mathcal{F}^*\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([\eta u^p+(1-\eta)\nu^p]^{\frac{1}{p}}\right)d\eta\\ +\int_0^1 \mathcal{F}^*\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}},\theta\right)\Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)d\eta\end{array}\right).
\end{aligned} \tag{33}$$

Since

$$\int_0^1 \mathcal{F}_*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right) d\eta$$
$$= \int_0^1 \mathcal{F}_*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right) d\eta$$
$$= \frac{p}{\nu^p - u^p} \int_u^\nu \omega^{p-1} \mathcal{F}_*(\omega, \theta) \Omega(\omega) d\omega,$$
$$\int_0^1 \mathcal{F}^*\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}, \theta\right) \Omega\left([(1-\eta)u^p + \eta\nu^p]^{\frac{1}{p}}\right) d\eta \tag{34}$$
$$= \int_0^1 \mathcal{F}^*\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}, \theta\right) \Omega\left([\eta u^p + (1-\eta)\nu^p]^{\frac{1}{p}}\right) d\eta$$
$$= \frac{p}{\nu^p - u^p} \int_u^\nu \omega^{p-1} \mathcal{F}^*(\omega, \theta) \Omega(\omega) d\omega.$$

Then, from (34) we have

$$\mathcal{F}_*\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}, \theta\right) \leq \frac{1}{\int_u^\nu \omega^{p-1} \Omega(\omega) d\omega} \int_u^\nu \omega^{p-1} \mathcal{F}_*(\omega, \theta) \Omega(\omega) d\omega\,,$$
$$\mathcal{F}^*\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}, \theta\right) \leq \frac{1}{\int_u^\nu \omega^{p-1} \Omega(\omega) d\omega} \int_u^\nu \omega^{p-1} \mathcal{F}^*(\omega, \theta) \Omega(\omega) d\omega\,,$$

from which, we have

$$\left[\mathcal{F}_*\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}, \theta\right), \quad \mathcal{F}^*\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}, \theta\right)\right]$$
$$\leq_I \frac{1}{\int_u^\nu \omega^{p-1} \Omega(\omega) d\omega} \left[\int_u^\nu \omega^{p-1} \mathcal{F}_*(\omega, \theta) \Omega(\omega) d\omega, \quad \int_u^\nu \omega^{p-1} \mathcal{F}^*(\omega, \theta) \Omega(\omega) d\omega\right],$$

that is

$$\mathcal{F}\left(\left[\frac{u^p + \nu^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq \frac{1}{\int_u^\nu \omega^{p-1} \Omega(\omega) d\omega} \ (FR) \int_u^\nu \omega^{p-1} \mathcal{F}(\omega) \Omega(\omega) d\omega,$$

this completes the proof.

Remark 3.12. If in the Theorem 3.10 and Theorem 3.11, $p = 1$, then we obtain the appropriate theorems for convex fuzzy-IVFs [26].

If in the Theorem 3.10 and Theorem 3.11, $\mathcal{T}_*(\omega, \gamma) = \mathcal{T}^*(\omega, \gamma)$ with $\gamma = 1$, then we obtain the appropriate theorems for p-convex function [43].

If in the Theorem 3.10 and Theorem 3.11, $\mathcal{T}_*(\omega, \gamma) = \mathcal{T}^*(\omega, \gamma)$ with $\gamma = 1$ and $p = 1$, then we obtain the appropriate theorems for convex function [44].

If $\Omega(\omega) = 1$, then combining Theorem 3.10 and Theorem 3.11, we get Theorem 3.1.

Example 3.13. We consider the FIVF $\mathcal{F}: [1, 4] \to \mathbb{F}_0$ defined by,

$$\mathcal{F}(\omega)(\lambda)=\begin{cases}\frac{\lambda-e^{\omega^p}}{e^{\omega^p}}, & \lambda\in\left[e^{\omega^p},2e^{\omega^p}\right],\\ \frac{4e^{\omega^p}-\lambda}{2e^{\omega^p}}, & \lambda\in\left(2e^{\omega^p},4e^{\omega^p}\right],\\ 0, & otherwise,\end{cases} \tag{35}$$

then, for each $\theta\in[0,1]$, we have $\mathcal{F}_\theta(\omega)=\left[(1+\theta)e^{\omega^p},2(2-\theta)e^{\omega^p}\right]$. Since end point functions $\mathcal{F}_*(\omega,\theta)$, $\mathcal{F}^*(\omega,\theta)$ are p-convex functions, for each $\theta\in[0,1]$, then $\mathcal{F}(\omega)$ is p-convex FIVF. If

$$\Omega(\omega)=\begin{cases}\omega^p-1, & \lambda\in\left[1,\frac{5}{2}\right],\\ 4-\omega^p, & \lambda\in\left(\frac{5}{2},4\right],\end{cases} \tag{36}$$

where $p=1$. Then, we have

$$\begin{aligned}\frac{p}{\nu^p-u^p}\int_1^4\omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega&=\frac{1}{3}\int_1^4\omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega\\ &=\frac{1}{3}\int_1^{\frac{5}{2}}\omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega+\frac{1}{3}\int_{\frac{5}{2}}^4\omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega,\\ &=\frac{1}{3}(1+\theta)\int_1^{\frac{5}{2}}e^{\omega}(\omega-1)d\omega+\frac{1}{3}(1+\theta)\int_{\frac{5}{2}}^4e^{\omega}(4-\omega)d\omega\approx 11(1+\theta),\\ \frac{p}{\nu^p-u^p}\int_1^4\omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega&=\frac{1}{3}\int_1^4\omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega\\ &=\frac{1}{3}\int_1^{\frac{5}{2}}\omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega+\frac{1}{3}\int_{\frac{5}{2}}^4\omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega,\\ &=\frac{2}{3}(2-\theta)\int_1^{\frac{5}{2}}e^{\omega}(\omega-1)d\omega+\frac{2}{3}(2-\theta)\int_{\frac{5}{2}}^4e^{\omega}(4-\omega)d\omega\approx 22(2-\theta),\end{aligned} \tag{37}$$

and

$$\begin{aligned}&[\mathcal{F}_*(u,\theta)+\mathcal{F}_*(\nu,\theta)]\int_0^1\eta\Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)d\eta\\ &[\mathcal{F}^*(u,\theta)+\mathcal{F}^*(\nu,\theta)]\int_0^1\eta\Omega\left([(1-\eta)u^p+\eta\nu^p]^{\frac{1}{p}}\right)d\eta\\ &=(1+\theta)[e+e^4]\left[\int_0^{\frac{1}{2}}3\eta^2d\omega+\int_{\frac{1}{2}}^1\eta(3-3\eta)d\eta\right]\approx\frac{43}{2}(1+\theta).\\ &=2(2-\theta)[e+e^4]\left[\int_0^{\frac{1}{2}}3\eta^2d\omega+\int_{\frac{1}{2}}^1\eta(3-3\eta)d\eta\right]\approx 43(2-\theta).\end{aligned} \tag{38}$$

From (37) and (38), we have

$$[11(1+\theta),22(2-\theta)]\leq_I\left[\frac{43}{2}(1+\theta),43(2-\theta)\right],\ \text{for each}\ \theta\in[0,1].$$

Hence, Theorem 3.10 is verified.

For Theorem 3.11, we have

$$\mathcal{F}_*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\approx\frac{61}{5}(1+\theta),$$
$$\mathcal{F}^*\left(\left[\frac{u^p+\nu^p}{2}\right]^{\frac{1}{p}},\theta\right)\approx\frac{122}{5}(2-\theta),\tag{39}$$

$$\int_u^\nu \Omega(\omega)d\omega=\int_1^{\frac{5}{2}}(\omega-1)d\omega\int_{\frac{5}{2}}^4(4-\omega)d\omega=\frac{9}{4},$$

$$\frac{p}{\int_u^\nu \Omega(\omega)d\omega}\int_1^4\omega^{p-1}\mathcal{F}_*(\omega,\theta)\Omega(\omega)d\omega\approx\frac{73}{5}(1+\theta),$$
$$\frac{p}{\int_u^\nu \Omega(\omega)d\omega}\int_1^4\omega^{p-1}\mathcal{F}^*(\omega,\theta)\Omega(\omega)d\omega\approx\frac{293}{10}(2-\theta).\tag{40}$$

From (39) and (40), we have

$$\left[\frac{61}{5}(1+\theta),\frac{122}{5}(2-\theta)\right]\leq_I\left[\frac{73}{5}(1+\theta),\frac{293}{10}(2-\theta)\right],$$

hence, Theorem 3.11 is demonstrated.

4. Discrete Jensen's and Schur's type inequalities

In this section, we propose the concept of discrete Jensen's and Schur's type inequality for p-convex FIVF. Some refinements of discrete Jensen's type inequality are also obtained. We begin by presenting the discrete Jensen's type inequality for p-convex FIVF in the following result.

Theorem 4.1. (Discrete Jense's type inequality for p-convex FIVF) Let $\eta_j\in\mathbb{R}^+$, $u_j\in[u,\nu]$, $(j=1,2,3,\ldots,k,k\geq 2)$ and $\mathcal{F}\in SXF([u,\nu],\mathbb{F}_0,p)$ and for all $\theta\in[0,1]$, θ-levels define the family of IVFs $\mathcal{F}_\theta:[u,\nu]\subset\mathbb{R}\to\mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega)=[\mathcal{F}_*(\omega,\theta),\mathcal{F}^*(\omega,\theta)]$ for all $\omega\in[u,\nu]$. Then,

$$\mathcal{F}\left(\left[\frac{1}{W_k}\sum_{j=1}^k\eta_j\ u_j{}^p\right]^{\frac{1}{p}}\right)\preccurlyeq\sum_j^k\frac{\eta_j}{W_k}\mathcal{F}(u_j),\tag{41}$$

where $W_k=\sum_{j=1}^k\eta_j$. If $\mathcal{F}$ is p-concave, then inequality (41) is reversed.

Proof. When $k=2$ then, inequality (41) is true. Consider inequality (19) is true for $k=n-1$, then

$$\mathcal{F}\left(\left[\frac{1}{W_{n-1}}\sum_{j=1}^{n-1}\eta_j\ u_j{}^p\right]^{\frac{1}{p}}\right)\preccurlyeq\sum_{j=1}^{n-1}\frac{\eta_j}{W_{n-1}}\mathcal{F}(u_j).$$

Now, let us prove that inequality (41) holds for $k=n$.

$$\mathcal{F}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\eta_j\ u_j{}^p\right]^{\frac{1}{p}}\right)=\mathcal{F}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2}\eta_j\ u_j{}^p+\frac{\eta_{n-1}+\eta_n}{W_n}(\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}u_{n-1}{}^p+\frac{\eta_n}{\eta_{n-1}+\eta_n}u_n{}^p\right]^{\frac{1}{p}}\right).$$

Therefore, for each $\theta\in[0,1]$, we have

$$\mathcal{F}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\eta_j\ u_j{}^p\right]^{\frac{1}{p}},\theta\right)$$
$$\mathcal{F}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\eta_j\ u_j{}^p\right]^{\frac{1}{p}},\theta\right)$$

$$=\mathcal{F}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2}\eta_j\ u_j{}^p+\frac{\eta_{n-1}+\eta_n}{W_n}(\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}u_{n-1}{}^p+\frac{\eta_n}{\eta_{n-1}+\eta_n}u_n{}^p\right]^{\frac{1}{p}},\theta\right),$$
$$=\mathcal{F}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n-2}\eta_j\ u_j{}^p+\frac{\eta_{n-1}+\eta_n}{W_n}(\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}u_{n-1}{}^p+\frac{\eta_n}{\eta_{n-1}+\eta_n}u_n{}^p\right]^{\frac{1}{p}},\theta\right),$$

$$\leq\sum_{j=1}^{n-2}\frac{\eta_j}{W_n}\mathcal{F}_*\left(u_j,\theta\right)+\frac{\eta_{n-1}+\eta_n}{W_n}\mathcal{F}_*\left(\left[\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}u_{n-1}{}^p+\frac{\eta_n}{\eta_{n-1}+\eta_n}u_n{}^p\right]^{\frac{1}{p}},\theta\right),$$
$$\leq\sum_{j=1}^{n-2}\frac{\eta_j}{W_n}\mathcal{F}^*\left(u_j,\theta\right)+\frac{\eta_{n-1}+\eta_n}{W_n}\mathcal{F}^*\left(\left[\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}u_{n-1}{}^p+\frac{\eta_n}{\eta_{n-1}+\eta_n}u_n{}^p\right]^{\frac{1}{p}},\theta\right),$$

$$\leq\sum_{j=1}^{n-2}\frac{\eta_j}{W_n}\mathcal{F}_*\left(u_j,\theta\right)+\frac{\eta_{n-1}+\eta_n}{W_n}\left[\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}\mathcal{F}_*(u_{n-1},\theta)+\frac{\eta_n}{\eta_{n-1}+\eta_n}\mathcal{F}_*(u_n,\theta)\right],$$
$$\leq\sum_{j=1}^{n-2}\frac{\eta_j}{W_n}\mathcal{F}^*\left(u_j,\theta\right)+\frac{\eta_{n-1}+\eta_n}{W_n}\left[\frac{\eta_{n-1}}{\eta_{n-1}+\eta_n}\mathcal{F}^*(u_{n-1},\theta)+\frac{\eta_n}{\eta_{n-1}+\eta_n}\mathcal{F}^*(u_n,\theta)\right],$$

$$\leq\sum_{j=1}^{n-2}\frac{\eta_j}{W_n}\mathcal{F}_*\left(u_j,\theta\right)+\left[\frac{\eta_{n-1}}{W_n}\mathcal{F}_*(u_{n-1},\theta)+\frac{\eta_n}{W_n}\mathcal{F}_*(u_n,\theta)\right],$$
$$\leq\sum_{j=1}^{n-2}\frac{\eta_j}{W_n}\mathcal{F}^*\left(u_j,\theta\right)+\left[\frac{\eta_{n-1}}{W_n}\mathcal{F}^*(u_{n-1},\theta)+\frac{\eta_n}{W_n}\mathcal{F}^*(u_n,\theta)\right],$$
$$=\sum_{j=1}^{n}\frac{\eta_j}{W_n}\mathcal{F}_*\left(u_j,\theta\right),$$
$$=\sum_{j=1}^{n}\frac{\eta_j}{W_n}\mathcal{F}^*\left(u_j,\theta\right).$$

From which, we have

$$\left[\mathcal{F}_*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\eta_j\ u_j{}^p\right]^{\frac{1}{p}},\theta\right),\mathcal{F}^*\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\eta_j\ u_j{}^p\right]^{\frac{1}{p}},\theta\right)\right]\leq_I\left[\sum_{j=1}^{n}\frac{\eta_j}{W_n}\mathcal{F}_*\left(u_j,\theta\right),\sum_{j=1}^{n}\frac{\eta_j}{W_n}\mathcal{F}^*\left(u_j,\theta\right)\right],$$

that is,

$$\mathcal{F}\left(\left[\frac{1}{W_n}\sum_{j=1}^{n}\eta_j\ u_j{}^p\right]^{\frac{1}{p}}\right)\preccurlyeq\sum_{j=1}^{n}\frac{\eta_j}{W_n}\mathcal{F}\left(u_j\right),$$

and the result follows.

If $\eta_1 = \eta_2 = \eta_3 = \cdots = \eta_k = 1$, then Theorem 4.1 reduces to the following result:

Corollary 4.2. Let $u_j \in [u, \nu]$, $(j = 1, 2, 3, \dots, k, k \geq 2)$ and $\mathcal{F} \in SXF([u, \nu], \mathbb{F}_0, p)$. Then, θ -levels define the family of IVFs $\mathcal{F}_\theta : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega, \theta), \mathcal{F}^*(\omega, \theta)]$ for all $\omega \in [u, \nu]$ and for all $\theta \in [0, 1]$. Then,

$$\mathcal{F}\left(\left[\frac{1}{W_k}\sum_{j=1}^{k} \eta_j\, u_j{}^p\right]^{\frac{1}{p}}\right) \preccurlyeq \sum_{j=1}^{k} \frac{1}{k} \mathcal{F}(u_j). \tag{42}$$

If $\mathcal{F}$ is a p-concave, then inequality (42) is reversed.

The next Theorem 4.3 gives the Schur's type inequality for p-convex FIVFs.

Theorem 4.3. (Discrete Schur's type inequality for p-convex FIVF) Let $\mathcal{F} \in SXF([u, \nu], \mathbb{F}_0, p)$. Then, θ -levels define the family of IVFs $\mathcal{F}_\theta : [u, \nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega, \theta), \mathcal{F}^*(\omega, \theta)]$ for all $\omega \in [u, \nu]$ and for all $\theta \in [0, 1]$. If $u_1, u_2, u_3 \in [u, \nu]$, such that $u_1 < u_2 < u_3$ and $u_3{}^p - u_1{}^p$, $u_3{}^p - u_2{}^p$, $u_2{}^p - u_1{}^p \in [0, 1]$, then we have

$$(u_3{}^p - u_1{}^p)\mathcal{F}(u_2) \preccurlyeq (u_3{}^p - u_2{}^p)\mathcal{F}(u_1) + (u_2{}^p - u_1{}^p)\mathcal{F}(u_3). \tag{43}$$

If $\mathcal{F}$ is a p-concave, then inequality (43) is reversed.

Proof. Let $u_1, u_2, u_3 \in [u, \nu]$ and $u_3{}^p - u_1{}^p > 0$. Consider $\eta = \frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}$, then $u_2{}^p = \eta u_1{}^p + (1 - \eta)u_3{}^p$. Since $\mathcal{F}$ is a p-convex FIVF, then by hypothesis, we have

$$\mathcal{F}(u_2) \preccurlyeq \left(\frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}\right)\mathcal{F}(u_1) + \left(\frac{u_2{}^p - u_1{}^p}{u_3{}^p - u_1{}^p}\right)\mathcal{F}(u_3).$$

Therefore, for each $\theta \in [0, 1]$, we have

$$\begin{aligned} \mathcal{F}_*(u_2, \theta) &\leq \left(\frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}\right)\mathcal{F}_*(u_1, \theta) + \left(\frac{u_2{}^p - u_1{}^p}{u_3{}^p - u_1{}^p}\right)\mathcal{F}_*(u_3, \theta), \\ \mathcal{F}^*(u_2, \theta) &\leq \left(\frac{u_3{}^p - u_2{}^p}{u_3{}^p - u_1{}^p}\right)\mathcal{F}^*(u_1, \theta) + \left(\frac{u_2{}^p - u_1{}^p}{u_3{}^p - u_1{}^p}\right)\mathcal{F}^*(u_3, \theta), \end{aligned} \tag{44}$$

$$\begin{aligned} &= \frac{(u_3{}^p - u_2{}^p)}{(u_3{}^p - u_1{}^p)}\mathcal{F}_*(u_1, \theta) + \frac{(u_2{}^p - u_1{}^p)}{(u_3{}^p - u_1{}^p)}\mathcal{F}_*(u_3, \theta), \\ &= \frac{(u_3{}^p - u_2{}^p)}{(u_3{}^p - u_1{}^p)}\mathcal{F}^*(u_1, \theta) + \frac{(u_2{}^p - u_1{}^p)}{(u_3{}^p - u_1{}^p)}\mathcal{F}^*(u_3, \theta). \end{aligned} \tag{45}$$

From (45), we have

$$\begin{aligned} (u_3{}^p - u_1{}^p)\mathcal{F}_*(u_2, \theta) &\leq (u_3{}^p - u_2{}^p)\mathcal{F}_*(u_1, \theta) + (u_2{}^p - u_1{}^p)\mathcal{F}_*(u_3, \theta), \\ (u_3{}^p - u_1{}^p)\mathcal{F}^*(u_2, \theta) &\leq (u_3{}^p - u_2{}^p)\mathcal{F}^*(u_1, \theta) + (u_2{}^p - u_1{}^p)\mathcal{F}^*(u_3, \theta), \end{aligned}$$

that is

$$\begin{aligned} [(u_3{}^p - u_1{}^p)\mathcal{F}_*(u_2, \theta), (u_3{}^p - u_1{}^p)\mathcal{F}^*(u_2, \theta)] \leq_I {} & [(u_3{}^p - u_2{}^p)\mathcal{F}_*(u_1, \theta) + (u_3{}^p - u_2{}^p)\mathcal{F}_*(u_3, \theta), \\ & (u_2{}^p - u_1{}^p)\mathcal{F}^*(u_1, \theta) + (u_2{}^p - u_1{}^p)\mathcal{F}^*(u_3, \theta)], \end{aligned}$$

hence

$$(u_3{}^p - u_1{}^p)\mathcal{F}(u_2) \preccurlyeq u_3{}^p - u_2{}^p\mathcal{F}(u_1) + (u_2{}^p - u_1{}^p)\mathcal{F}(u_3).$$

A refinement of Jense's type inequality for p-convex FIVF is given in the following theorem.

Theorem 4.4. Let $\eta_j \in \mathbb{R}^+$, $u_j \in [u,\nu]$, $(j = 1,2,3,\dots,k, k \geq 2)$ and $\mathcal{F} \in SXF([u,\nu],\mathbb{F}_0,p)$. Then, θ -levels define the family of IVFs $\mathcal{F}_\theta: [u,\nu] \subset \mathbb{R} \to \mathcal{K}_C{}^+$ are given by $\mathcal{F}_\theta(\omega) = [\mathcal{F}_*(\omega,\theta), \mathcal{F}^*(\omega,\theta)]$ for all $\omega \in [u,\nu]$ and for all $\theta \in [0,1]$. If $(L,U) \subseteq [u,\nu]$, then

$$\sum_{j=1}^{k} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}(u_j) \preccurlyeq \sum_{j=1}^{k} \left(\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}(U,\theta) \right), \tag{46}$$

where $W_k = \sum_{j=1}^{k} \eta_j$. If $\mathcal{F}$ is p-concave, then inequality (46) is reversed.

Proof. Consider $= u_1, u_j = u_2$, $(j = 1,2,3,\dots,k)$, $U = u_3$. Then, by hypothesis and inequality (44), we have

$$\mathcal{F}(u_j) \leq \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \mathcal{F}(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \mathcal{F}(U,\theta).$$

Therefore, for each $\theta \in [0,1]$, we have

$$\mathcal{F}_*(u_j,\theta) \leq \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \mathcal{F}_*(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \mathcal{F}_*(U,\theta),$$
$$\mathcal{F}^*(u_j,\theta) \leq \left(\frac{U - u_j{}^p}{U^p - L^p}\right) \mathcal{F}^*(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \mathcal{F}^*(U,\theta).$$

Above inequality can be written as,

$$\begin{aligned} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(u_j,\theta) &\leq \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(U,\theta), \\ \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(u_j,\theta) &\leq \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(U,\theta). \end{aligned} \tag{47}$$

Taking sum of all inequalities (47) for $j = 1,2,3,\dots,k$, we have

$$\sum_{j=1}^{k} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(u_j,\theta) \leq \sum_{j=1}^{k} \left(\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(U,\theta) \right),$$
$$\sum_{j=1}^{k} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(u_j,\theta) \leq \sum_{j=1}^{k} \left(\left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(L,\theta) + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(U,\theta) \right).$$

that is

$$\sum_{j=1}^{k} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}(u_j) = \left[\sum_{j=1}^{k} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(u_j,\theta),\ \sum_{j=1}^{k} \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(u_j,\theta) \right]$$

$$\leq_I \left[\sum_{j=1}^{k} \begin{pmatrix} \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(L,\theta) \\ + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}_*(U,\theta) \end{pmatrix},\ \sum_{j=1}^{k} \begin{pmatrix} \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(L,\theta) \\ + \left(\frac{u_j{}^p - L^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}^*(U,\theta) \end{pmatrix} \right],$$

$$\leq_I \sum_{j=1}^{k} \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) [\mathcal{F}_*(L,\theta),\ \mathcal{F}^*(L,\theta)] + \sum_{j=1}^{k} \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) [\mathcal{F}_*(U,\theta),\ \mathcal{F}^*(U,\theta)].$$

$$= \sum_{j=1}^{k} \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}(L,\theta) + \sum_{j=1}^{k} \left(\frac{U^p - u_j{}^p}{U^p - L^p}\right) \left(\frac{\eta_j}{W_k}\right) \mathcal{F}(U,\theta).$$

Thus,

$$\sum_{j=1}^{k}\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(u_j) \preccurlyeq \sum_{j=1}^{k}\left(\left(\frac{U^p - {u_j}^p}{U^p - L^p}\right)\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(L) + \left(\frac{{u_j}^p - L^p}{U^p - L^p}\right)\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(U)\right),$$

this completes the proof.

We now consider some special cases of Theorem 4.1 and 4.4.

If $\mathcal{F}_*(\omega,\theta) = \mathcal{F}_*(\omega,\theta)$ with $\theta = 1$, then Theorem 3.1 and 3.4 reduce to the following results:

Corollary 4.5. [42] (Jense's inequality for p-convex function) Let $\eta_j \in \mathbb{R}^+$, $u_j \in [u,\nu]$, $(j = 1,2,3,\dots,k, k \geq 2)$ and let $\mathcal{F}: [u,\nu] \to \mathbb{R}^+$ be a non-negative real-valued function. If $\mathcal{F}$ is a p-convex function, then

$$\mathcal{F}\left(\left[\frac{1}{W_k}\sum_{j=1}^{k}\eta_j\ {u_j}^p\right]^{\frac{1}{p}}\right) \leq \sum_{j=1}^{k}\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(u_j), \tag{48}$$

where $W_k = \sum_{j=1}^{k}\eta_j$. If $\mathcal{F}$ is p-concave function, then inequality (48) is reversed.

Corollary 4.6. Let $\eta_j \in \mathbb{R}^+$, $u_j \in [u,\nu]$, $(j = 1,2,3,\dots,k, k \geq 2)$ and $\mathcal{F}: [u,\nu] \to \mathbb{R}^+$ be an non-negative real-valued function. If $\mathcal{F}$ is a p-convex function and $u_1, u_2, \dots, u_j \in (L,U) \subseteq [u,\nu]$ then,

$$\sum_{j=1}^{k}\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(u_j) \leq \sum_{j=1}^{k}\left(\left(\frac{U^p - {u_j}^p}{U^p - L^p}\right)\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(L) + \left(\frac{{u_j}^p - L^p}{U^p - L^p}\right)\left(\frac{\eta_j}{W_k}\right)\mathcal{F}(U)\right), \tag{49}$$

where $W_k = \sum_{j=1}^{k}\eta_j$. If $\mathcal{F}$ is a p-concave function, then inequality (49) is reversed.

5. Conclusions

In this we defined the p-convex (concave, affine) class for fuzzy-IVFs. We obtained some HH-inequalities for p-convex fuzzy-IVFs via fuzzy Riemann integrals. Moreover, we derived some novel discrete Jensen's and Schur's type inequalitities for p-convex fuzzy-IVFs. With the help of examples, we showed that our results include a wide class of new and known inequalities for $p$convex fuzzy-IVFs and their variant forms as special cases. In future, we try to explore these concepts and to investigate Jensen's and HH-inequalities for IVF and fuzzy-IVFs on time scale. In future, we will explore this by using fuzzy Katugampola fractional integrals for p-convex fuzzy-IVFs. We hope that the concepts and techniques of this paper may be starting point for further research in this area.

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Conflict of interest

The authors declare that they have no competing interests.

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