Computers and Mathematics with Applications 61 (2011) 3370–3380

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Generalized $(C)$-conditions and related fixed point theorems

Erdal Karapınar [a,*], Kenan Taş [b]

[a] Department of Mathematics, Atilim University 06836, Incek, Ankara, Turkey

[b] Department of Mathematics and Computer Science, Cankaya University 06530, Yuzuncuyil, Ankara, Turkey

ARTICLE INFO

*Article history:*
Received 24 November 2010
Received in revised form 14 April 2011
Accepted 14 April 2011

*Keywords:*
Contraction mapping
Fixed point theory
Opial property
Suzuki $C$-conditions

ABSTRACT

In this manuscript, the notion of $C$-condition [K. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008) 1088–1095] is generalized. Some new fixed point theorems are obtained.



## 1. Introduction and preliminaries

Very recently, Suzuki proved the following fixed point theorem:

**Theorem 1** (*Suzuki [1]*). *Let $(X, d)$ be a compact metric space and let $T$ be a mapping on $X$. Assume $\frac{1}{2}d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$. Then $T$ has a unique fixed point.*

This result is based on the following two theorems:

**Theorem 2** (*Edelstein [2]*). *Let $(X, d)$ be a compact metric space and let $T$ be a mapping on $X$. Assume $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$. Then $T$ has a unique fixed point.*

**Theorem 3** (*Suzuki [3,4]*). *Define a nonincreasing function $\theta$ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases}$$

*Then for a metric space $(X, d)$, the following are equivalent:*

(1) *$X$ is complete.*
(2) *Every mapping $T$ on $X$ satisfying the following has a fixed point. There exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \le d(x, y)$ implies $d(Tx, Ty) \le rd(x, y)$ for all $x, y \in X$.*

A mapping $T$ on a subset $K$ of a Banach space $E$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$.

* Corresponding author. Tel.: +90 3125868239.
*E-mail addresses:* erdalkarapinar@yahoo.com, ekarapinar@atilim.edu.tr (E. Karapınar), kenan@cankaya.edu.tr (K. Taş).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.04.035

**Definition 4** ([3,4]). Let $T$ be a mapping on a subset $K$ of a Banach space $E$. Then $T$ is said to satisfy $(C)$-*condition* if

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \quad \text{implies that } \|Tx-Ty\| \leq \|x-y\|$$

for all $x, y \in K$.

Let $F(T)$ be the set of all fixed points of a mapping $T$. A mapping $T$ on a subset $K$ of a Banach space $E$ is called a *quasi-nonexpansive mapping* if $\|Tx-z\| \leq \|x-z\|$ for all $x \in K$ and $z \in F(T)$.

We suggest new definitions which are modifications of Suzuki's $C$-condition:

**Definition 5.** Let $T$ be a mapping on a subset $K$ of a Banach space $E$. Then $T$ is said to satisfy Suzuki–Ćirić $(C)$-*condition* (in short, $(SCC)$-*condition*) if

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \quad \text{implies that } \|Tx-Ty\| \leq M(x,y)$$

where $M(x,y) = \max\{\|x-y\|, \|x-Tx\|, \|Ty-y\|, \|Tx-y\|, \|x-Ty\|\}$ for all $x, y \in K$.

Moreover, $T$ is said to satisfy Suzuki–$(KC)$-*condition* (in short, $(SKC)$-*condition*) if

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \quad \text{implies that } \|Tx-Ty\| \leq N(x,y)$$

where $N(x,y) = \max\{\|x-y\|, \frac{1}{2}[\|x-Tx\|+\|Ty-y\|], \frac{1}{2}[\|Tx-y\|+\|x-Ty\|]\}$ for all $x, y \in K$.

**Definition 6.** Let $T$ be a mapping on a subset $K$ of a Banach space $E$. Then $T$ is said to satisfy (for all $x, y \in K$)

(i) Kannan–Suzuki–$(C)$ condition (in short, $(KSC)$-condition) if

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \quad \text{implies that } \|Tx-Ty\| \leq \frac{1}{2}[\|Tx-x\|+\|y-Ty\|],$$

(ii) Chatterjea–Suzuki-$(C)$ condition (in short, $(CSC)$-condition) if

$$\frac{1}{2}\|x-Tx\| \leq \|x-y\| \quad \text{implies that } \|Tx-Ty\| \leq \frac{1}{2}[\|Tx-y\|+\|x-Ty\|].$$

In this manuscript, we modify some results of [3], Singh-Mishra [5], Karapınar [6] and suggest some new theorems.

## 2. Some basic observations

**Proposition 7.** *Every nonexpansive mapping satisfies* $(SCC)$*-condition.*

**Proof.** Let $T$ be a nonexpansive mapping on a subset $K$ of a Banach space $E$, that is, $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in K$. Assume $\frac{1}{2}\|x-Tx\| \leq \|x-y\|$. For the case $M(x,y) = \|x-y\|$, the condition $(SCC)$ is satisfied trivially, that is, $\|Tx-Ty\| \leq M(x,y) = \|x-y\|$. For the other case, that is, $M(x,y) \neq \|x-y\|$, we observe $\|x-y\| \leq M(x,y)$. Thus, $\|Tx-Ty\| \leq \|x-y\| \leq M(x,y)$ which concludes that $T$ satisfies $(SCC)$-condition. □

**Corollary 8.** *Every nonexpansive mapping satisfies the following conditions:*

(A1) $\frac{1}{2}\|x-Tx\| \leq \|x-y\|$ *implies that* $\|Tx-Ty\| \leq A_1(x,y)$
*where* $A_1(x,y) = \max\{\|x-y\|, \|Tx-x\|, \|Ty-y\|\}$

(A2) $\frac{1}{2}\|x-Tx\| \leq \|x-y\|$ *implies that* $\|Tx-Ty\| \leq A_2(x,y)$
*where* $A_2(x,y) = \max\{\|x-y\|, \|Tx-y\|, \|Ty-x\|\}$.

Regarding the analogy, we omit the proof of this Corollary.

**Proposition 9.** *Every nonexpansive mapping satisfies* $(SKC)$*-condition.*

**Proof.** Let $T$ be a nonexpansive mapping on a subset $K$ of a Banach space $E$, that is, $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in K$. Assume $\frac{1}{2}\|x-Tx\| \leq \|x-y\|$. If the case $N(x,y) = \|x-y\|$ happen, then $\|Tx-Ty\| \leq N(x,y) = \|x-y\|$ are satisfied trivially. If not, that is, $N(x,y) \neq \|x-y\|$ then $\|x-y\| \leq N(x,y)$. Thus, $\|Tx-Ty\| \leq \|x-y\| \leq N(x,y)$ which concludes that $T$ satisfies $(SKC)$-condition. □

**Corollary 10.** *Every nonexpansive mapping satisfies the following conditions:*

(A3) $\frac{1}{2}\|x-Tx\| \leq \|x-y\|$ *implies that* $\|Tx-Ty\| \leq A_3(x,y)$
*where* $A_3(x,y) = \max\{\|x-y\|, \frac{1}{2}[\|Tx-x\|+\|Ty-y\|]\}$

(A4) $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ *implies that* $\|Tx - Ty\| \leq A_4(x, y)$
*where* $A_4(x, y) = \max\{\|x - y\|, \frac{1}{2}[\|Tx - y\| + \|Ty - x\|]\}$.

Regarding the analogy, we omit the proof of this Corollary.

**Proposition 11.** *If a mapping $T$ satisfies $(SKC)$-condition and has a fixed point, then it is a quasi-nonexpansive mapping.*

**Proof.** Let $T$ be a mapping on a subset $K$ of a Banach space $E$ and satisfy $(SKC)$-condition. Suppose $T$ has a fixed point, in other words, $z \in F(T)$. Thus,

$$0 = \frac{1}{2}\|z - Tz\| \leq \|z - y\| \quad \text{implies that } \|Tz - Ty\| \leq N(z, y) \tag{2.1}$$

where

$$\begin{aligned} N(z, y) &= \max\left\{\|z - y\|, \frac{1}{2}[\|z - Tz\| + \|Ty - y\|], \frac{1}{2}[\|Tz - y\| + \|z - Ty\|]\right\} \\ &= \max\left\{\|z - y\|, \frac{1}{2}\|Ty - y\|, \frac{1}{2}(\|z - y\| + \|z - Ty\|)\right\}. \end{aligned} \tag{2.2}$$

If $N(z, y) = \frac{1}{2}(\|z - y\| + \|z - Ty\|)$, then $\|z - Ty\| = \|Tz - Ty\| \leq N(z, y) = \frac{1}{2}(\|z - y\| + \|z - Ty\|)$ then we get $\|Tz - Ty\| = \|z - Ty\| \leq \|z - y\|$.

If $N(z, y) = \|z - y\|$, then we are done.

If $N(z, y) = \frac{1}{2}\|Ty - y\| \leq [\|Ty - z\| + \|z - y\|]$, then

$$\|z - Ty\| = \|Tz - Ty\| \leq N(z, y) = \frac{1}{2}\|Ty - y\| \leq \frac{1}{2}[\|Ty - z\| + \|z - y\|]$$

and thus $\|z - Ty\| = \|Tz - Ty\| \leq \|z - y\|$ which completes the proof. □

**Corollary 12.** *If a mapping $T$ satisfies one of the following:*

(1) $(A3)$-*condition,*
(2) $(A4)$-*condition,*
(3) $(KSC)$-*condition,*
(4) $(CSC)$-*condition,*

*and has a fixed point, then it is a quasi-nonexpansive mapping.*

**Example 13.** Let $S$ and $T$ be mappings on $[0, 4]$ such that

$$Tx = \begin{cases} 0 & \text{if } x \neq 4 \\ 1 & \text{if } x = 4 \end{cases} \quad \text{and} \quad Sx = \begin{cases} 0 & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases}$$

then,

(i) $T$ satisfies both $(SCC)$-condition and $(SKC)$-condition but $T$ is not nonexpansive.
(ii) $S$ is quasi-nonexpansive and $F(S) \neq \emptyset$ but $S$ does not satisfy $(SKC)$-condition.

**Proof.** (i) If $x < y$ and $(x, y) \in ([0, 4] \times [0, 4]) \setminus ((3, 4) \times \{4\})$. Then $\|Tx - Ty\| \leq M(x, y)$ and $\|Tx - Ty\| \leq N(x, y)$ holds.
If $x \in (3, 4)$ and $y = 4$, then

$$\frac{1}{2}\|x - Tx\| = \frac{x}{2} > 1 > \|x - y\| \quad \text{and} \quad \frac{1}{2}\|y - Ty\| > 1 > \|x - y\|$$

hold. Thus, $T$ satisfies conditions $(SCC)$ and $(SKC)$. Since $T$ is not continuous, $T$ is not nonexpansive.

(ii) It is clear that $F(S) = \{0\} \neq \emptyset$ and $S$ is quasi-nonexpansive. Since,

$$\frac{1}{2}\|4 - S4\| = \frac{1}{2} \leq 1 = \|4 - 3\| \quad \text{and} \quad \|S4 - S3\| = 3 > 2 = M(4, 3)$$

where

$$M(4, 3) = \max\left\{\|4 - 3\| = 1, \frac{1}{2}[\|4 - S4\| + \|S3 - 3\|] = 2, \frac{1}{2}[\|S3 - 4\| + \|3 - S4\|] = 2\right\} = 2$$

hold, $S$ does not satisfy $(SKC)$-condition. □

**Proposition 14.** *Let $T$ be a mapping on a closed subset $K$ of a Banach space $E$. Assume that $T$ satisfies $(SKC)$-condition. Then $F(T)$ is closed. Moreover, $E$ is strictly convex and $K$ is convex, then $F(T)$ is also convex.*

**Proof.** Let $\{x_n\}$ be a sequence in $F(T)$ and converge to a point $x \in K$. It is clear that

$$\frac{1}{2}\|x_n - Tx_n\| = 0 \leq \|x_n - x\| \quad \text{for } n \in \mathbb{N}.$$

Thus, we have

$$\limsup_{n\to\infty}\|x_n - Tx\| \leq \limsup_{n\to\infty}\|Tx_n - Tx\| \leq \limsup_{n\to\infty} N(x_n, x) \tag{2.3}$$

where

$$\begin{aligned} N(x_n, x) &= \max\left\{\|x_n - x\|, \frac{1}{2}[\|x_n - Tx_n\| + \|Tx - x\|], \frac{1}{2}[\|Tx_n - x\| + \|x_n - Tx\|]\right\} \\ &\leq \max\left\{\|x_n - x\|, \frac{1}{2}\|Tx - x\|, \frac{1}{2}[\|x_n - x\| + \|x_n - Tx\|]\right\}. \end{aligned}$$

If $N(x_n, x) = \frac{1}{2}\|Tx - x\|$ then, the expression (2.3) turns into

$$\begin{aligned} \limsup_{n\to\infty}\|x_n - Tx\| &\leq \limsup_{n\to\infty}\|Tx_n - Tx\| \leq \limsup_{n\to\infty} N(x_n, x) \\ &\leq \limsup_{n\to\infty}\frac{1}{2}\|Tx - x\| = \frac{1}{2}\|x - Tx\| \end{aligned} \tag{2.4}$$

which implies that $\|Tx - x\| \leq \frac{1}{2}\|Tx - x\|$. This is a contradiction, so this cannot happen. For the case, $N(x_n, x) = \frac{1}{2}[\|x_n - x\| + \|x_n - Tx\|]$, the expression (2.3) yields that

$$\begin{aligned} \limsup_{n\to\infty}\|x_n - Tx\| &= \limsup_{n\to\infty}\|Tx_n - Tx\| \leq \limsup_{n\to\infty} N(x_n, x) \\ &\leq \limsup_{n\to\infty}\frac{1}{2}[\|x_n - x\| + \|x_n - Tx\|] \leq \frac{1}{2}\|x - Tx\| \end{aligned} \tag{2.5}$$

which yields that $\|Tx - x\| \leq \frac{1}{2}\|Tx - x\|$. This is also a contradiction, so this cannot happen either. If $N(x_n, x) = \|x_n - x\|$ then, the expression (2.3) turns into

$$\limsup_{n\to\infty}\|x_n - Tx\| \leq \limsup_{n\to\infty}\|Tx_n - Tx\| \leq \limsup_{n\to\infty}\|x_n - x\| = 0. \tag{2.6}$$

So, we are done. In other words, $\{x_n\}$ converges to $Tx$. Uniqueness of the limit implies that $Tx = x$ and hence $F(T)$ is closed.

Suppose that $E$ is strictly convex and $K$ is convex. Take fixed points $x, y \in K$ with $x \neq y$ and fix $t \in (0, 1)$ and define $z := tx + (1-t)y \in K$. So we get

$$\begin{aligned} \|x - y\| &\leq \|x - Tz\| + \|Tz - y\| = \|Tx - Tz\| + \|Tz - Ty\| \\ &\leq N(x, z) + N(y, z) \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} N(x, z) &= \max\left\{\|x - z\|, \frac{1}{2}[\|x - Tx\| + \|Tz - z\|], \frac{1}{2}[\|Tx - z\| + \|x - Tz\|]\right\} \\ &= \max\left\{\|x - z\|, \frac{1}{2}\|Tz - z\|, \frac{1}{2}[\|x - z\| + \|x - Tz\|]\right\} \end{aligned}$$

and

$$\begin{aligned} N(z, y) &= \max\left\{\|z - y\|, \frac{1}{2}[\|z - Tz\| + \|Ty - y\|], \frac{1}{2}[\|Tz - y\| + \|z - Ty\|]\right\} \\ &= \max\left\{\|z - y\|, \frac{1}{2}\|Tz - z\|, \frac{1}{2}[\|Tz - y\| + \|z - y\|]\right\}. \end{aligned}$$

Since $E$ is strictly convex, there exists $s \in [0, 1]$ such that $Tz = sx + (1-s)y$. Observe that

$$(1-s)\|x - y\| = \|Tx - Tz\| \leq N(x, z) \tag{2.8}$$

where

$$\begin{aligned} N(x, z) &= \max\left\{\|x - z\|, \frac{1}{2}[\|x - Tx\| + \|Tz - z\|], \frac{1}{2}[\|Tx - z\| + \|x - Tz\|]\right\} \\ &= \max\left\{\|x - z\|, \frac{1}{2}\|Tz - z\|, \frac{1}{2}[\|x - z\| + \|x - Tz\|]\right\}. \end{aligned}$$

If $N(x, z) = \|x - z\|$, then the expression (2.8) becomes

$$(1-s)\|x-y\| = \|Tx - Tz\| \le N(x,z) = \|x-z\| = (1-t)\|x-y\|. \tag{2.9}$$

If $N(x, z) = \frac{1}{2}[\|x - z\| + \|x - Tz\|]$, then the expression (2.8) turns into

$$(1-s)\|x-y\| = \|Tx - Tz\| \le N(x,z) = \frac{1}{2}[\|x-z\| + \|x-Tz\|] = \frac{1}{2}[(1-t)\|x-y\| + (1-s)\|x-y\|]. \tag{2.10}$$

For the last case $N(x, z) = \frac{1}{2}\|z - Tz\|$, the expression (2.8) gives

$$\begin{aligned}(1-s)\|x-y\| = \|Tx - Tz\| &\le N(x,z) = \frac{1}{2}[\|Tz - z\|] \\ &\le \frac{1}{2}[\|x-z\| + \|x-Tz\|] = \frac{1}{2}[(1-t)\|x-y\| + (1-s)\|x-y\|].\end{aligned} \tag{2.11}$$

Thus, from (2.9)–(2.11) we conclude that $(1 - s) \le (1 - t)$.

If we consider

$$s\|x-y\| = \|Ty - Tz\| \le N(y,z), \tag{2.12}$$

then proceeding as above we find that $s \le t$. Consequently $s = t$. Hence $z \in F(T)$. □

**Corollary 15.** *Let $T$ be a mapping on a closed subset $K$ of a Banach space $E$. Assume that $T$ satisfies one of the following:*

(1) $(A3)$*-condition,*
(2) $(KSC)$*-condition,*
(3) $(CSC)$*-condition.*

*Then $F(T)$ is closed. Moreover, $E$ is strictly convex and $K$ is convex, then $F(T)$ is also convex.*

**Proposition 16.** *If $T$ satisfies the condition*

$$\frac{1}{2}\|x - Tx\| \le \|x-y\| \Rightarrow \|Tx - Ty\| \le \frac{1}{4}[\|Tx - x\| + \|y - Ty\|],$$

*for all $x, y \in K$, then $T$ is nonexpansive.*

**Proof.**

$$\begin{aligned}\|Tx - Ty\| &\le \frac{1}{4}[\|Tx - x\| + \|y - Ty\|] \\ &\le \frac{1}{4}[2\|y - x\| + 2\|y - x\|] = \|x - y\|. \quad \square\end{aligned}$$

**Proposition 17.** *If $T$ satisfies the condition*

$$\frac{1}{2}\|x - Tx\| \le \|x-y\| \Rightarrow \|Tx - Ty\| \le \frac{1}{5}[\|Tx - x\| + \|x - y\| + \|y - Ty\|],$$

*for all $x, y \in K$, then $T$ is nonexpansive.*

**Proof.**

$$\begin{aligned}\|Tx - Ty\| &\le \frac{1}{5}[\|Tx - x\| + \|x - y\| + \|y - Ty\|] \\ &\le \frac{1}{5}[2\|y - x\| + \|x - y\| + 2\|y - x\|] = \|x - y\|. \quad \square\end{aligned}$$

**Proposition 18.** *Let $T$ be a mapping on a closed subset $K$ of a Banach space $E$ that satisfies the condition $(SKC)$. Then, for every $x, y \in K$, the following hold:*

(i) $\|Tx - T^2x\| \le \|x - Tx\|$
(ii) *either* $\frac{1}{2}\|x - Tx\| \le \|x - y\|$ *or* $\frac{1}{2}\|Tx - T^2x\| \le \|Tx - y\|$
(iii) *either* $\|Tx - Ty\| \le N(x, y)$ *or* $\|T^2x - Ty\| \le N(Tx, y)$

*where*

$$N(x,y) = \max\left\{\|x-y\|, \frac{1}{2}[\|Tx-x\|+\|Ty-y\|], \frac{1}{2}[\|Tx-y\|+\|Ty-x\|]\right\} \quad \textit{and}$$

$$N(Tx,y) = \max\left\{\|Tx-y\|, \frac{1}{2}[\|T^2x-Tx\|+\|Ty-y\|], \frac{1}{2}[\|T^2x-y\|+\|Ty-Tx\|]\right\}.$$

**Proof.** The first statement follows from $(SKC)$-condition. Indeed, we always have $\frac{1}{2}\|x-Tx\| \le \|x-Tx\|$ which yields that

$$\|Tx-T^2x\| \le N(x,Tx) \tag{2.13}$$

where

$$\begin{aligned} N(x,Tx) &= \max\left\{\|x-Tx\|, \frac{1}{2}[\|Tx-x\|+\|T^2x-Tx\|], \frac{1}{2}[\|Tx-Tx\|+\|T^2x-x\|]\right\} \\ &= \max\left\{\|x-Tx\|, \frac{1}{2}[\|Tx-x\|+\|T^2x-Tx\|], \frac{1}{2}\|T^2x-x\|\right\}. \end{aligned}$$

If $N(x,Tx) = \|x-Tx\|$ we are done. If $N(x,Tx) = \frac{1}{2}[\|Tx-x\|+\|T^2x-Tx\|]$ then the expression (2.13) turns into

$$\|Tx-T^2x\| \le N(x,Tx) = \frac{1}{2}[\|Tx-x\|+\|T^2x-Tx\|]. \tag{2.14}$$

By simplifying the expression (2.14), one can get (i). For the case $N(x,Tx) = \frac{1}{2}\|T^2x-x\|$ the expression (2.13) turns into

$$\|Tx-T^2x\| \le N(x,Tx) = \frac{1}{2}\|T^2x-x\| \le \frac{1}{2}[\|Tx-x\|+\|T^2x-Tx\|] \tag{2.15}$$

which implies (i).

It is clear that (iii) is a consequence of (ii). To prove (ii), assume the contrary, that is,

$$\frac{1}{2}\|x-Tx\| > \|x-y\| \quad \text{and} \quad \frac{1}{2}\|Tx-T^2x\| > \|Tx-y\|$$

holds for all $x, y \in K$. Then by triangle inequality and (i), we have

$$\begin{aligned} \|x-Tx\| &\le \|x-y\|+\|y-Tx\| \\ &< \frac{1}{2}\|x-Tx\|+\frac{1}{2}\|Tx-T^2x\| \\ &\le \frac{1}{2}\|x-Tx\|+\frac{1}{2}\|x-Tx\| = \|x-Tx\| \quad \square \end{aligned}$$

which is a contradiction. Thus, we have (ii).

## 3. Main results

**Proposition 19.** *Let $T$ be a mapping on a subset $K$ of a Banach space $E$ and satisfy $(SKC)$-condition. Then $\|x-Ty\| \le 5\|Tx-x\|+\|x-y\|$ holds for all $x, y \in K$.*

**Proof.** The proof is based on Proposition 18 which says that either

$$\|Tx-Ty\| \le N(x,y) \quad \text{or} \quad \|T^2x-Ty\| \le N(Tx,y)$$

holds, where $N(x,y) = \max\{\|x-y\|, \frac{1}{2}[\|Tx-x\|+\|Ty-y\|], \frac{1}{2}[\|Tx-y\|+\|Ty-x\|]\}$ and

$$N(Tx,y) = \max\left\{\|Tx-y\|, \frac{1}{2}[\|T^2x-Tx\|+\|Ty-y\|], \frac{1}{2}[\|T^2x-y\|+\|Ty-Tx\|]\right\}.$$

Consider the first case. If $N(x,y) = \|x-y\|$ then we have

$$\|x-Ty\| \le \|x-Tx\|+\|Tx-Ty\| \le \|x-Tx\|+\|x-y\|. \tag{3.1}$$

For $N(x,y) = \frac{1}{2}[\|Tx-x\|+\|Ty-y\|]$ one can observe

$$\begin{aligned} \|x-Ty\| &\le \|x-Tx\|+\|Tx-Ty\| \le \|x-Tx\|+\frac{1}{2}[\|Tx-x\|+\|Ty-y\|] \\ &\le \frac{3}{2}\|x-Tx\|+\frac{1}{2}\|Ty-y\| \le \frac{3}{2}\|x-Tx\|+\frac{1}{2}[\|Ty-x\|+\|x-y\|]. \end{aligned}$$

Thus, we have

$$\frac{1}{2}\|x-Ty\| \leq \frac{3}{2}\|x-Tx\| + \frac{1}{2}\|x-y\| \Leftrightarrow \|x-Ty\| \leq 3\|x-Tx\| + \|x-y\|. \tag{3.2}$$

For $N(x,y) = \frac{1}{2}[\|Tx-y\| + \|Ty-x\|]$ one can obtain

$$\begin{aligned}\|x-Ty\| &\leq \|x-Tx\| + \|Tx-Ty\| \leq \|x-Tx\| + \frac{1}{2}[\|Tx-y\| + \|Ty-x\|] \\ &\leq \|x-Tx\| + \frac{1}{2}[\|Tx-x\| + \|x-y\|] + \frac{1}{2}\|Ty-x\|.\end{aligned}$$

Thus, we have

$$\frac{1}{2}\|x-Ty\| \leq \frac{3}{2}\|x-Tx\| + \frac{1}{2}\|x-y\| \Leftrightarrow \|x-Ty\| \leq 3\|x-Tx\| + \|x-y\|. \tag{3.3}$$

Take the second case into account. For $N(Tx,y) = \|Tx-y\|$, we have

$$\begin{aligned}\|x-Ty\| &\leq \|x-Tx\| + \|Tx-T^2x\| + \|T^2x-Ty\| \\ &\leq \|x-Tx\| + \|x-Tx\| + \|Tx-y\| \\ &= 2\|x-Tx\| + \|Tx-y\| \\ &\leq 2\|x-Tx\| + \|Tx-x\| + \|x-y\| = 3\|Tx-x\| + \|x-y\|.\end{aligned} \tag{3.4}$$

If $N(Tx,y) = \frac{1}{2}[\|T^2x-Tx\| + \|Ty-y\|]$ then we have

$$\begin{aligned}\|x-Ty\| &\leq \|x-Tx\| + \|Tx-T^2x\| + \|T^2x-Ty\| \\ &\leq 2\|x-Tx\| + \frac{1}{2}[\|T^2x-Tx\| + \|Ty-y\|] \\ &\leq \frac{5}{2}\|x-Tx\| + \frac{1}{2}\|Ty-y\| \\ &\leq \frac{5}{2}\|x-Tx\| + \frac{1}{2}[\|Ty-x\| + \|x-y\|].\end{aligned}$$

Thus, we have

$$\frac{1}{2}\|x-Ty\| \leq \frac{5}{2}\|x-Tx\| + \frac{1}{2}\|x-y\| \Leftrightarrow \|x-Ty\| \leq 5\|x-Tx\| + \|x-y\|. \tag{3.5}$$

For the last case, $N(Tx,y) = \frac{1}{2}[\|T^2x-y\| + \|Ty-Tx\|]$, we have

$$\begin{aligned}\|x-Ty\| &\leq \|x-Tx\| + \|Tx-T^2x\| + \|T^2x-Ty\| \\ &\leq 2\|x-Tx\| + \frac{1}{2}[\|T^2x-y\| + \|Ty-Tx\|] \\ &\leq 2\|x-Tx\| + \frac{1}{2}[\|T^2x-Tx\| + \|Tx-x\| + \|x-y\|] + \frac{1}{2}[\|Ty-x\| + \|x-Tx\|] \\ &\leq \frac{7}{2}\|x-Tx\| + \frac{1}{2}\|Ty-x\| + \frac{1}{2}\|x-y\|.\end{aligned}$$

Thus, we have

$$\frac{1}{2}\|x-Ty\| \leq \frac{5}{2}\|x-Tx\| + \frac{1}{2}\|x-y\| \Leftrightarrow \|x-Ty\| \leq 5\|x-Tx\| + \|x-y\|. \tag{3.6}$$

Hence, the result follows from (3.1)–(3.6). □

Regarding the analogy, we omit the proof of the following Corollaries.

**Corollary 20.** *Let $T$ be a mapping on a subset $K$ of a Banach space $E$ and satisfy $(A3)$-condition. Then $\|x-Ty\| \leq 5\|Tx-x\| + \|x-y\|$ holds for all $x, y \in K$.*

**Corollary 21.** *Let $T$ be a mapping on a subset $K$ of a Banach space $E$ and satisfy $(KSC)$-condition. Then $\|x-Ty\| \leq 5\|Tx-x\| + \|x-y\|$ holds for all $x, y \in K$.*

**Corollary 22.** *Let $T$ be a mapping on a subset $K$ of a Banach space $E$ and satisfy $(CSC)$-condition. Then $\|x-Ty\| \leq 5\|Tx-x\| + \|x-y\|$ holds for all $x, y \in K$.*

**Theorem 23.** *Let $T$ be a mapping on a compact convex subset $K$ of a Banach space $E$ and satisfy (SKC)-condition. Define a sequence $\{x_n\}$ in $K$ by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. Then $\{x_n\}$ converge strongly to a fixed point of $T$.*

**Proof.** Regarding that $K$ is compact, one can conclude that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that converges to some number, say $z$, in $K$. By Proposition 19, we have

$$\|x_{n_k} - Tz\| \le 5\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - z\|, \quad \text{for all } k \in \mathbb{N}. \tag{3.7}$$

Notice that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Taking this fact into account together with (3.7), we conclude that $\{x_{n_k}\}$ converges to $Tz$ which implies that $Tz = z$. In other words, $z \in F(T)$. On account of Proposition 11, we get

$$\|x_{n+1} - z\| \le \lambda\|Tx_n - z\| + (1-\lambda)\|x_n - z\| \le \|x_n - z\|$$

for $n \in \mathbb{N}$. Thus, $\{x_n\}$ converges to $z$. □

**Corollary 24.** *Let $T$ be a mapping on a compact convex subset $K$ of a Banach space $E$. Define a sequence $\{x_n\}$ in $K$ by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. If $T$ satisfies one of the following:*

(1) *(A3)-condition,*
(2) *(KSC)-condition,*
(3) *(CSC)-condition,*

*then $\{x_n\}$ converge strongly to a fixed point of $T$.*

**Theorem 25.** *Let $E$ be a Banach space and $T, S$ be self-mappings on $K$ such that $T(K) \subset S(K)$ and $S(K)$ is a compact convex subset of $E$ and $T$ satisfies (SKC)-condition. Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - Sx_n\| = 0$ holds. Then $T$ and $S$ have a coincidence point.*

**Proof.** Let $R : S(K) \to S(K)$ where $Ra = T(S^{-1}a)$ for each $a \in S(K)$. It is clear that $R$ is well-defined. Indeed, take $x, y \in S^{-1}a$ such that $b = Tx$ and $c = Ty$. For $x \in S^{-1}a$ we obtain $Ra = Tx$ and $Ra \subset S(K)$ since $T(K) \subset S(K)$. Since $Sx = Sy$ we get $b = c$. Thus, $R$ is well-defined.

We claim that $R$ satisfies all conditions of Theorem 23. Let $a, b \in S(K)$ such that $\frac{1}{2}\|a - Ra\| \le \|a - b\|$. In other words,

$$\frac{1}{2}\|Sx - Tx\| = \frac{1}{2}\|a - Ra\| \le \|a - b\| = \|Sx - Sy\|$$

for $x \in S^{-1}a$ and $y \in S^{-1}b$. Since $T$ satisfies (SKC)-condition, we get

$$\|Ra - Rb\| = \|Tx - Ty\| \le N(Sx, Sy) = N(a, b)$$

where $N(a, b) = N(Sx, Sy) = \max\{\|a - b\| = \|Sx - Sy\|, \frac{1}{2}[\|Ra - a\| + \|Rb - b\|] = \frac{1}{2}[\|Sx - Tx\| + \|Ty - Sy\|], \frac{1}{2}[\|Ra - b\| + \|a - Rb\|] = \frac{1}{2}[\|Tx - Sy\| + \|Sx - Ty\|]\}$.

Thus,

$$\frac{1}{2}\|a - Ra\| \le \|a - b\| \Rightarrow \|Ra - Rb\| \le N(a, b).$$

Further, define a sequence $\{a_n\}$ in $S(K)$ by $a_1 \in S(K)$ and $a_{n+1} = \lambda Ra_n + (1-\lambda)a_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. For $x_i \in S^{-1}a_i$ we have

$$\lim_{n\to\infty} \|Ra_n - a_n\| = \lim_{n\to\infty} \|Tx_n - Sx_n\| = 0.$$

Thus, all conditions of Theorem 23 are satisfied.

Hence, $\{a_n\}$ converges to $t$. Then for any $z \in S^{-1}t$, we have $Tz = Rt = t = Sz$. Therefore, $S, T$ have a coincidence point. □

**Corollary 26.** *Let $E$ be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is a compact convex subset of $E$. Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - Sx_n\| = 0$ holds. If $S, T$ satisfy one of the following:*

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le \max\left\{\|Sx - Sy\|, \frac{1}{2}[\|Sx - Tx\| + \|Ty - Sy\|]\right\}, \tag{3.8}$$

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le \frac{1}{2}[\|Sx - Tx\| + \|Ty - Sy\|], \tag{3.9}$$

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le \frac{1}{2}[\|Tx - Sy\| + \|Sx - Ty\|], \tag{3.10}$$

*then $T$ and $S$ have a coincidence point.*

**Definition 27.** Let $E$ be a Banach space. $E$ is said to have Opial property [7] if for each weakly convergent sequence $\{x_n\}$ in $E$ with weak limit $z$

$$\liminf_{n\to\infty} \|x_n - z\| \le \liminf_{n\to\infty} \|x_n - y\|, \quad \text{for all } y \in E \text{ with } y \neq z.$$

All Hilbert spaces, all finite dimensional Banach spaces and Banach sequence spaces $\ell_p(1 \le p < \infty)$ have the Opial property (see [3]).

**Proposition 28.** *Let $T$ be a mapping on a subset $K$ of a Banach space $E$ with Opial property and satisfy $(SKC)$-condition. If $\{x_n\}$ converges weakly to $z$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$. That is $I - T$ is demiclosed at zero.*

**Proof.** Due to Proposition 19, we have

$$\|x_n - Tz\| \le 5\|Tx_n - x_n\| + \|x_n - z\|, \quad \text{for all } n \in \mathbb{N}.$$

Hence,

$$\liminf_{n\to\infty} \|x_n - Tz\| \le \liminf_{n\to\infty} \|x_n - z\|.$$

Thus, Opial property implies that $Tz = z$. □

**Corollary 29.** *Let $T$ be a mapping on a subset $K$ of a Banach space $E$ with Opial property and satisfy one of the following:*

(1) *$(A3)$-conditions,*
(2) *$(KSC)$-condition,*
(3) *$(CSC)$-condition.*

*If $\{x_n\}$ converges weakly to $z$ and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, then $Tz = z$. That is $I - T$ is demiclosed at zero.*

**Theorem 30.** *Let $T$ be a mapping on a weakly compact convex subset $K$ of a Banach space $E$ with Opial property and satisfy $(SKC)$-condition. Define a sequence $\{x_n\}$ in $K$ by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. Then $\{x_n\}$ converge weakly to a fixed point of $T$.*

**Proof.** We have $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Since $K$ is weakly compact, one can conclude that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to an element, say $z$, in $E$. On account of Proposition 28, we observe that $z$ is a fixed point of $T$. Note that $\{\|x_n - z\|\}$ is a nondecreasing sequence. Indeed,

$$\|x_{n+1} - z\| \le \lambda\|Tx_n - z\| + (1-\lambda)\|x_n - z\|.$$

We show $\{x_n\}$ converges to $z$. Assume the contrary, that is, $\{x_n\}$ does not converge to $z$. Then there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ and $u \in K$ such that $\{x_{n_m}\}$ converges weakly to $u$ and $u \neq z$. By Proposition 28, $Tu = u$. Since $E$ has Opial property,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - z\| &= \lim_{k\to\infty} \|x_{n_k} - z\| < \lim_{k\to\infty} \|x_{n_k} - u\| = \lim_{n\to\infty} \|x_n - u\| \\ &= \lim_{m\to\infty} \|x_{n_m} - u\| < \lim_{m\to\infty} \|x_{n_m} - z\| = \lim_{n\to\infty} \|x_n - z\| \end{aligned} \tag{3.11}$$

which is a contradiction. Hence, the proof is completed. □

**Corollary 31.** *Let $T$ be a mapping on a weakly compact convex subset $K$ of a Banach space $E$ with Opial property and satisfy one of the following:*

(1) *$(A3)$-condition,*
(2) *$(KSC)$-condition,*
(3) *$(CSC)$-condition.*

*Define a sequence $\{x_n\}$ in $K$ by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. Then $\{x_n\}$ converge weakly to a fixed point of $T$.*

**Theorem 32.** *Let $E$ be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is a weakly compact convex subset of $E$ with Opial property. Assume for $x, y \in K$,*

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le N(Sx, Sy)$$

*where $N(Sx, Sy) = \max\{\|Sx - Sy\|, \frac{1}{2}[\|Sx - Tx\| + \|Ty - Sy\|], \frac{1}{2}[\|Tx - Sy\| + \|Sx - Ty\|]\}$. Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - Sx_n\| = 0$ holds. Then $T$ and $S$ have a coincidence point.*

Regarding the analogy with the proof of Theorem 25, we omit the proof.

**Corollary 33.** *Let $E$ be a Banach space and $T, S : K \to E$ such that $T(K) \subset S(K)$ and $S(K)$ is a weakly compact convex subset of $E$ with Opial property. Define a sequence $\{x_n\}$ in $T(K)$ by $x_1 \in S(K)$ and $Sx_{n+1} = \lambda Tx_n + (1-\lambda)Sx_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - Sx_n\| = 0$ holds. If $S, T$ satisfy one of the following:*

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le \max\left\{\|Sx - Sy\|, \frac{1}{2}[\|Sx - Tx\| + \|Ty - Sy\|]\right\}, \tag{3.12}$$

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le \frac{1}{2}[\|Sx - Tx\| + \|Ty - Sy\|], \tag{3.13}$$

$$\frac{1}{2}\|Sx - Tx\| \le \|Sx - Sy\| \Rightarrow \|Tx - Ty\| \le \frac{1}{2}[\|Tx - Sy\| + \|Sx - Ty\|], \tag{3.14}$$

*then $T$ and $S$ have a coincidence point.*

A Banach space $E$ is called *strictly convex* if $\|x + y\| < 2$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. A Banach space $E$ is called *uniformly convex in every direction* (in short, UCED) if for $\varepsilon \in (0, 2]$ and $z \in E$ with $\|z\| = 1$, there exists $\delta := \delta(\varepsilon, z) > 0$ such that $\|x+y\| \le 2(1-\delta)$ for all $x, y \in E$ with $\|x\| \le 1$, $\|y\| \le 1$ and $x-y \in \{tz : t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}$.

**Lemma 34** (*See [3]*)**.** *For a Banach space E, the following are equivalent:*

(1) *E is UCED.*
(2) *If sequence $\{u_n\}$ and $\{v_n\}$ in E satisfy $\lim_{n\to\infty} \|u_n\| = 1 = \lim_{n\to\infty} \|v_n\|$, $\lim_{n\to\infty} \|u_n + v_n\|$ and $\{u_n - v_n\} \subset \{tw : t \in \mathbb{R}\}$ for some $w \in E$ with $\|w\| = 1$, then $\lim_{n\to\infty} \|u_n - v_n\| = 0$ holds.*

**Lemma 35** (*See [3]*)**.** *For a Banach space E, the following are equivalent:*

(1) *E is UCED.*
(2) *If $\{x_n\}$ is a bounded sequence in E, then a function $f$ on E defined by $f(x) = \limsup_{n\to\infty} \|x_n - x\|$ is strictly quasi-convex, that is,*

$$f(tx + (1-t)y) < \max\{f(x), f(y)\}$$

*for all $t \in (0, 1)$ and $x, y \in E$ with $x \ne y$.*

**Theorem 36.** *Let $T$ be a mapping on a weakly compact convex subset $K$ of a UCED Banach space $E$ and satisfy $(SKC)$-condition. Define a sequence $\{x_n\}$ in $K$ by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. Then $T$ has a fixed point.*

**Proof.** Set a sequence $\{x_n\}$ in $K$ in such a way that $x_{n+1} = \frac{1}{2}Tx_n + \frac{1}{2}x_n$ for each $n \in \mathbb{N}$ where $x_1 \in K$. Notice that $\limsup_{n\to\infty} \|Tx_n - x_n\| = 0$. Define a continuous convex function $f$ from $K$ into $[0, \infty)$ by $f(x) = \limsup_{n\to\infty} \|x_n - x\|$, for all $x \in K$. Since $K$ is weakly compact and $f$ is weakly lower semi-continuous, there exists $z \in K$ such that $f(z) = \min\{f(x) : x \in K\}$. Regarding Proposition 19, we have $\|x_n - Tz\| \le 5\|Tx_n - x_n\| + \|x_n - z\|$ and thus $f(Tz) \le f(z)$. On account of $f(z)$ being the minimum, $f(z) = f(Tz)$ holds. To show $Tz = z$ we assume the contrary, that is $Tz \ne z$. Since $f$ is strictly quasi-convex, we have

$$f(z) \le f\left(\frac{z + Tz}{2}\right) < \max\{f(z), f(Tz)\} = f(z)$$

which is a contradiction. Thus, we get the desired result. □

**Corollary 37.** *Let $T$ be a mapping on a weakly compact convex subset $K$ of a UCED Banach space $E$ and satisfy one of the following:*

(1) $(A3)$*-condition,*
(2) $(KSC)$*-condition,*
(3) $(CSC)$*-condition.*

*Define a sequence $\{x_n\}$ in $K$ by $x_1 \in K$ and $x_{n+1} = \lambda Tx_n + (1-\lambda)x_n$, for $n \in \mathbb{N}$, where $\lambda$ lies in $[\frac{1}{2}, 1)$. Suppose $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ holds. Then $T$ has a fixed point.*

**Theorem 38.** *Let $\mathcal{S}$ be a family of commuting mappings on a weakly compact convex subset $K$ of a Banach space $E$. Suppose each mapping in $\mathcal{S}$ satisfy $(SKC)$-condition. Then $\mathcal{S}$ has a common fixed point.*

**Proof.** Let $I = \{1, 2, \ldots, \nu\}$ be an index set. Let $T_i \in \mathcal{S}$, $i \in I$. Due to Theorem 36, $T_i$ has a fixed point in $K$, that is, $F(T_i) \neq \emptyset$ for $i \in I$. Proposition 14 implies that each $F(T_i)$ is closed and convex. Suppose that $F := \cap_{i=1}^{k-1} F(T_i)$ is non-empty, closed and convex for some $k \in \mathbb{N}$ such that $1 < k \leq \nu$. For $x \in F$ and $i \in I$ with $1 \leq i < k$, $T_k x = T_k \circ T_i x = T_i \circ T_k x$ since $S$ is commuting. Thus, $T_k x$ is a fixed point of $T_i$ which yields $T_k x \in F$. So, $T_k(F) \subset F$. In other words, $T_k(F) \subset F$. By Theorem 36, $T_k$ has a fixed point in $F$, that is, $F \cap F(T_k) = \cap_{i=1}^{k} F(T_i) \neq \emptyset$.

Due to Proposition 14, this set is closed and convex. By induction, we obtain $\cap_{i=1}^{\nu} F(T_i) \neq \emptyset$. That is equivalent to saying $\{F(T) : T \in \mathcal{S}\}$ has the finite intersection property. Since $K$ is weakly compact and $F(T)$ is weakly closed for every $T \in \mathcal{S}$, then $\cap_{T \in \mathcal{S}} F(T) \neq \emptyset$. □

**Corollary 39.** *Let $\mathcal{S}$ be a family of commuting mappings on a weakly compact convex subset $K$ of a Banach space $E$. Suppose each mapping in $\mathcal{S}$ satisfies one of the following:*

(1) *$(A3)$-condition,*
(2) *$(KSC)$-condition,*
(3) *$(CSC)$-condition.*

*Then $\mathcal{S}$ has a common fixed point.*

## References


[1] K. Suzuki, A new type of fixed point theorem in metric spaces, Nonlinear Anal. Theory, Methods Appl. 71 (11) (2009) 5313–5317.
[2] M. Edelstein, On fixed and periodic points under contractive mappings, J. London Math. Soc. 37 (1962) 74–79.
[3] K. Suzuki, Fixed point theorems and convergence theorems for some generalized non expansive mappings, J. Math. Anal. Appl. 340 (2008) 1088–1095.
[4] K. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2008) 1861–1869.
[5] S.L. Singh, S.N Mishra, Remarks on recent fixed point theorems, Fixed Point Theory Appl. 2010 (2010) 18 pages. Article ID 452905.
[6] E. Karapınar, Remarks on Suzuki (C)-condition, in: Albert Luo, J.A. Tenreiro Machado and Dumitru Baleanu (Eds.), Nonlinear Systems and Methods For Mechanical, Electrical and Biosystems.
[7] Z. Opial, Weak convergence of the sequence of successive approximation for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967) 591–597.