FRACTIONAL CAPUTO-FABRIZIO DERIVATIVE WITH APPLICATIONS

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ABSTRACT

FRACTIONAL CAPUTO-BRIZIO DERIVATIVE WITH APPLICATIONS

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In this thesis, I pointed out three very recent applications of the new established fractional Caputo-Fabrizio derivative and I discussed its related properties and theorems. The associated fractional integral was displayed and the Laplace transform of the Riemann-Liouville and Caputo fractional differential operators was presented.

Keywords: Caputo-Fabrizio Fractional Derivative, Gamma Function, Beta Function, Fractional Integration, Riemann-Liouville, Caputo Derivative, Laplace Transform.

ÖZ

FRACTIONAL CAPUTO-FABRIZIO TÜREVİ VE UYGULAMALARI

KAREEM, Ahmed

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Bu tezde, yeni geliştirilen Fractional Caputo-Fabrizio türevinin üç yeni uygulaması ele alınmış ve bu türevin özellikleri ile teoremleri tartışılmıştır. İlgili Fractional integrali, Riemann-Liouville Laplace dönüşümü ve Caputo Fractional türev operatörü gösterilmiştir.

Anahtar Kelimeler: Kesirli Caputo-Fabrizio Türevi, Gamma Fonksiyonu, Beta Fonksiyonu, Kesirli İntegrali, Riemann-Liouville, Caputo Türevi, Laplace Dönüşümü.

DEDICATION

Dedicated to my parents and my brothers

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TABLE OF CONTENTS

List of symbols

$\propto$	Alpha
β	Beta
Γ	Gamma
Δ	Delta
∇	Nabla
ϵ	Epsilon
Λ	Lamda
Σ	Sigma
Ω	Omega
T	Tau
∞	Infinity
$\in$	Element
π	Pi
μ	Mu
Δ	Delta
ξ	Xi
CF	Caputo-Fabrizio Derivative
MLF	Mittag-Leffler Function
L	Laplace Transform
CFD	Caputo fractional differential operator

CHAPTER 1

INTRODUCTION

The fractional calculus deals with integrals and derivatives of real or even complex order [1-20]. It is a generalization of the classical calculus and therefore preserves some of the basic properties. As an intensively developing field if offers tremendous new features for research [21-36]. The fractional calculus also finds applications in different fields of science, including theory of fractals, engineering, physics, numerical analysis, biology and economics [37-50]. In 2015 Caputo and Fabrizio created a new derivative, in order to give the researchers new powerful deals to dig into the unknown world of dynamics of complex systems [51-60].The aim of my thesis is to present a detailed review about this new introduced derivative and to show some of its recent applications. This thesis consists of 6 chapters.

In the second chapter we present the special functions as Gamma and Beta functions, the complementary error function and Mittag-Leffler. The Riemann-Liouville (RL), and Caputo definition of the fractional derivatives are given in this chapter.

In third chapter the fundamental properties of the Riemann-Liouville and Caputo fractional derivative are discussed and a comparison between them is given.

In the fourth chapter we reviewed the new definition proposed by Caputo and Fabrizio of the fractional time derivatives and discuss its properties.

In the fifth chapter we show some applications of the Caputo-Fabrizio derivative.

The sixth chapter is devoted to the conclusion part.

CHAPTER 2

RIEMANN- LIOUVILLE AND CAPUTO DERIVATIVES

Before we discuss the fractional Caputo and Riemann-Liouville derivatives, we present some special functions important for the fractional calculus as Gamma, Beta functions, the complementary error function and the Mittag-Leffler function. Furthermore, the fractional integration is introduced [6, 10].

2.1 Special Functions

2.1.1 The Gamma Function

This function is represented by the symbol $\Gamma(g)$ is a simplification of the factorial function $n\,!$ i.e. $\Gamma(n) = (n-1)!$ in which $n \in N$. For complex arguments with positive real part it is definite as [10]:

$$\Gamma(g) = \int_0^\infty t^{g-1} e^{-t}\, dt, \;\; Reg > 0.$$

By analytic continuation, the function is extended to the total complex plane except for the points {0,-1,-2,-3,-4,-5,-6,…} were it has simple polesthis $\Gamma\colon \mathbb{C}\backslash \{0,-1,-2,-3,\dots\} \to \mathbb{C}$.

Below we show some basic properties of Γ function, namely [10]:

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt \;= 1,$$

Γ(2)=1.Γ(1)= 1!=1,

$$\Gamma(3) = 2.\Gamma(2) = 21! = 2! = 2. \tag{2.1}$$

$\Gamma(1/2) = \sqrt{\pi}$,

$\Gamma(g) > 0, for\ g > 0$,

$\Gamma(g+1) = g\Gamma(g)$,

$\Gamma(n+1) = n!, for\ n \in N_0$,

where N_0 is the set of the non-negative integers.

2.1.2 The Beta Function

We can define the Beta function by the following integral, namely:

$$\beta(g,w) = \int_0^1 t^{g-1}\,(1-t)^{w-1}\,dt, Reg > 0, Rew > 0.$$

In addition $B(g,w)$ is used for convenience to replace a combination of Gamma functions. The relation between the Beta and Gamma function is [10]

$$\beta(g,w) = \frac{\Gamma(g)\Gamma(w)}{\Gamma(\mathrm{g}+\mathrm{w})}. \tag{2.2}$$

Equation (2.2) provides the analytical continuation of the Beta function to the entire complex plane via the analytical continuation of the Gamma function. It should also be mention that the Beta function is symmetric, namely:

$$B(g,w) = B(w,g).$$

2.1.3 The Complementary Error Function

The complementary error function is an entire function defined as

$$erfc(g) = \frac{2}{\sqrt{\pi}}\int_g^\infty e^{-t^2}\,dt.$$

Special values of the corresponding error function are given below [10]:

$$erfc(-\infty) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-t^2}\,dt = 2,$$

$$erfc(0) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = 1,$$

$$erfc(+\infty) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, dt = 0.$$

The following relations are interesting to be mentioned [10], namely:

$$erfc(-x) = 2 - erfc(x),$$

$$\int_{0}^{\infty} erfc(x)\, dx = \frac{1}{\sqrt{\pi}},$$

$$\int_{0}^{\infty} erfc^2(x)\, dx = \frac{2-\sqrt{2}}{\sqrt{\pi}}.$$

Respectively.

2.1.4 The Mittag-Leffler Function

While the Gamma function is a generalization of the factorial function, the MLF is a generalization of the exponential function. Firstly, we introduced a one-parameter function by using the series [2], namely:

$$E_{\propto}(g) = \sum_{k=0}^{\infty} \frac{g^k}{\Gamma(\propto k + 1)}, \quad \propto > 0, \propto \in R, g \in \mathbb{C}, \tag{2.3}$$

and then we define the MLF with two parameters, as:

$$E_{\propto,\beta}(g) = \sum_{k=0}^{\infty} \frac{g^k}{\Gamma(\propto k + \beta)}, \quad \propto, \beta > 0, \propto, \beta \in R, g \in \mathbb{C}, \tag{2.4}$$

Below we mentioned few of MLF properties [10], namely:

$$\begin{aligned} &E_{\mathbf{1,1}}(\mathrm{g}) = e^{g}, \\ &E_{2,1}(g^2) = \cosh(g)\ , \\ &E_{2,2}(g^2) = \frac{\sinh(g)}{g}, \\ &E_{\propto,1}(\mathrm{g}) = E_{\propto}(g). \end{aligned} \tag{2.5}$$

2.2 Fractional Integration

In this subsection we recall the Cauchy's formula [9, 10]:

$$J^n\, g(t) = \int_a^t \int_a^{\tau_1} \dots \int_a^{\tau_{n-1}} g(\tau) d\tau \dots d\tau_2\, d\tau_1$$

$$= \frac{1}{(n-1)!} \int_a^t g(\tau)(t-\tau)^{n-1}\, d\tau. \qquad (2.6)$$

Definition 2.1 [10]:

Suppose that $\omega > 0,\; t > a, \omega, a, t \in R$. Then, the fractional integration operator is given by:

$$J^\omega g(t) = \frac{1}{\Gamma(\omega)} \int_a^t g(\tau)(t-\tau)^{\omega-1}\, d\tau. \qquad (2.7)$$

We give some properties of the fractional integration [10] by convention we have

$$J^0\, g(t) = g(t), \qquad (2.8)$$

i.e. $J^0 = 1$ is the identity operator.

Another property is the linearity, namely:

$$J^\omega\big(\Lambda g(t) + f(t)\big) = \Lambda J^\omega\, g(t) + J^\omega\, f(t), \omega \in R, \Lambda \in \mathbb{C}. \qquad (2.9)$$

If $g(t)$ is continuous for all $t \geq 0$ the following equalities holds

$$\lim_{\omega \to 0} J^\omega g(t) = g(t), \qquad (2.10)$$

$J^\omega \left(J^\beta\, g(t)\right) = J^\beta\big(J^\omega\, g(t)\big) = J^{\omega+\beta}\, g(t), \omega, \beta \in R$, respectively.

2.3 The Riemann-Liouville and Caputo Fractional Differential

Definition 2.2 [10]:

Suppose that $\omega > o, t > a, \omega, a, t \in R$ .Then we have [6,10]:

$$D^{\omega}\, g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\omega)} \dfrac{d^n}{d\, t^n} \displaystyle\int_a^t \frac{g(\tau)}{(t-\tau)^{\omega+1-n}}\, d\tau, & n-1 < \omega < n \in N, \\ \dfrac{d^n}{d\, t^n} g(t), & \omega = n \in N. \end{cases} \tag{2.11}$$

This is named the Riemann-Liouville fractional derivative (RL) of order ω [10]. It should be mention that the operator (2.11) is the left-inverse operator of the fractional integral (2.7), i.e.

$$D^{\omega} J^{\omega} = 1.$$

By convention it is defined as

$$D^0\, g(t) = g(t), \text{ i.e. } D^0 = 1.$$

The main properties of the Riemann-Liouville operator (2.11) are treated in subsection (3.1) together with the corresponding properties of the Caputo fractional differential operator.

Definition 2.3 [33]:

Suppose that $\omega > 0,\ t > c, \omega, c, t \in R$. The fractional Caputo operator has the form [33]:

$$D_*^{\omega}\, g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\omega)} \displaystyle\int_c^t \frac{g^{(n)}(\tau)}{(t-\tau)^{\omega+1-n}}\, d\tau, & n-1 < \omega < n \in N, \\ \dfrac{d^n}{d\, t^n}\, g(t), & \omega = n \in N. \end{cases} \tag{2.12}$$

CHAPTER 3

THE PROPERTIES OF RIEMANN-LIOUVILLE AND CAPUTO DERIVATIVES

In this chapter, the fundamental properties of the RL and Caputo derivatives are discussed and a comparison between them is given [10]:

3.1 Main Properties

Consider the class of functions $F(t)$, integrable and continuous in every finite interval $(0, x)$, $X \in R$ suppose also, that these functions may be an integrable singularity of order $v < 1$ at the point $t = 0$, i.e. $\lim_{t\to 0} t^{v}\ g(t) = const\ \neq 0$ [10]. This is the class of function, for which (2.7) and (2.11) in subsection (2.2) and subsection (2.3) are well-defined as already mentioned in subsection (2.4) for the Caputo fractional differential operator (2.12) the integrality of the n-th derivative of the function is additionally required later on, all the function in this survey are considers to be in the corresponding class [6, 10].

Remark 1 [10]

The operator D^n, n $\in$N used in the following sections is the standard integer-order differentiation operator, i.e. $D^n = \frac{d^n}{d\,t^n}$.

Lemma 1 [10]: If n-1$< \omega < n,$ n $\in N$, $\omega \in R$ and g(t) be such that $D_*^{\omega}\ g(t)$ exists, then

$$D_*^{\omega}\ g(t) = J^{n-\omega} D^n \text{g(t)}. \tag{3.1}$$

This means that the Caputo fractional operator is equivalent to (n-ω) fold integration after n-th order differentiation. Equation (3.1) follows immediately from (2.12), the

Riemann-Liouville fractional derivatives equivalent to the composition of the same operators $\{(n-\omega)$-fold integration and n-th order differentiation$\}$ but in reverse order, i.e.

$$D^{\omega} g(t) = D^{n} J^{n-\omega} g(t). \tag{3.2}$$

From (3.1) and (3.2) since $J^{n-\omega} D^{n} \neq D^{n} J^{n-\omega}$ it follows the result.

Proposition 1 [12]:

In general, the two operators, the Riemann-Liouville and Caputo do not coincide:

$$D^{\omega}\, g(t) \neq D_{*}^{\omega}\, g(t).$$

Lemma 2 [10]: Suppose $n-1<\omega<n, n \in N, \omega \in R$ and $g(t)$ be such that $D_{*}^{\omega}\, g(t)$ exists.

Then the following properties for the Caputo operator hold:

$$(\boldsymbol{i}) \lim_{\omega \to n} D_{*}^{\omega}\, g(t) = g^{(n)}(t), \tag{3.3}$$

$$\text{(ii)} \lim_{\omega \to n-1} D_{*}^{\omega}\, g(t) = g^{(n-1)}(t) - g^{(n-1)}(0).$$

Proof [10]:

By using the integration by parts we get:

$$\begin{aligned} D_{*}^{\omega} g(t) &= \frac{1}{\Gamma(n-\omega)} \int_{0}^{t} \frac{g^{(n)}(\tau)}{(t-\tau)^{\omega+1-n}}\, d\tau, \\ &= \frac{1}{\Gamma(n-\omega)} \left(-g^{(n)}(\tau) \frac{(t-\tau)^{n-\omega}}{n-\omega} \Big|_{\tau=0}^{t} - \int_{0}^{t} -g^{(n+1)}(\tau) \frac{(t-\tau)^{n-\omega}}{n-\omega}\, d\tau \right), \\ &= \frac{1}{\Gamma(n-\omega+1)} \left(g^{(n)}(0) t^{n-\omega} + \int_{0}^{t} g^{(n+1)}(\tau)(t-\tau)^{n-\omega} d\tau\right). \end{aligned}$$

Now, by taking the limit for $\omega \to n$ and $\omega \to n-1$ respectively, it follows that

$$\lim_{\omega\to n} D_*^{\omega}\ g(t) = (g^{(n)}(0) + g^{(n)}(\tau))|\tau_{=0}^{t} = g^{(n)}\ (t),$$

and

$$\lim_{\omega\to n-1} D_*^{\omega}\ g(t) = (g^{(n)}(0)t + g^{(n)}(\tau)\ (t-\tau)|_{\tau=0}^{t} - \int_0^t -g^{(n)}(\tau)\ d\,\tau$$

$$= g^{(n-1)}(\tau)|_{\tau=0}^{t} = g^{(n-1)}(t) - g^{(n-1)}(0).$$

For the corresponding interpolation property reads as follows [10].

$$\lim_{\omega\to n} D^{\omega}\ g(t) = g^{(n)}(t),$$

$$\lim_{\omega\to n-1} D^{\omega}\ g(t) = g^{(n-1)}(t).$$

Lemma 3 [10]:

Suppose $n-1<\omega<n, n \in N, \omega, \Lambda\ \in \mathbb{C}$ and the functions $g(t)$ and $f(t)$ be such that both $D_*^{\omega}\ g(t)$ and $D_*^{\omega}\ f(t)$ exist. The Caputo fractional derivative is a linear operator, namely:

$$D_*^{\omega}(\Lambda g(t) + f(t)) = \Lambda D_*^{\omega}\ g(t) + D_*^{\omega} f(t). \tag{3.4}$$

Proof [10]:

The proof follows straightforwardly from the definition of fractional integration and the fact that the integral and the classical integer-order derivative are linear operators. Similarly, the Riemann-Liouville operator satisfies [10]:

$$D^{\omega}(\Lambda g(t) + f(t)) = \Lambda D^{\omega}\ g(t) + D^{\omega}\ f(t).$$

Lemma 4 [10]: Let$n-1<\omega<n, r, n \in N, \omega\ \in \mathbb{R}$ and the functions $g(t)$ is such that $D_*^{\omega}\ g(t)$ exists. Then, in general we have:

$$D_*^{\omega} D^{r} g(t) = D_*^{\omega+r}\ g(t) \neq D^{r} D_*^{\omega} g(t). \tag{3.5}$$

Corollary 1 [9]: Suppose that $n-1<\omega<n, \beta=\omega-(n-1), (0<\beta<1), n \in N, \omega, \beta \in R$ and the functions $g(t)$ is such that $D_*^{\omega}\ g(t)$ exists then we have:

$D_*^{\omega} g(t) = D_*^{\beta} D^{n-1}\ g(t).$

Proof [9]:

Substitute β for ω and $n-1$ for r in (3.5), then we calculate:

$D_*^{\beta} D^{n-1}\ g(t) = D_*^{\beta+n-1}\ g(t) = D_*^{\omega-(n-1)+n-1}\ g(t) = D_*^{\omega}\ g(t).$

Remark 2 [10]

To find the Caputo fractional derivative of arbitrary order ω (n-1$<\omega<n$) of the function g(t), it is sufficient to find the Caputo derivative of order $\beta=\omega-(n-1)$ of the (n-1) the derivative of the function. Notice that $\omega-(n-1)$ is a real number between 0 and 1. Hence studying the behavior of the Caputo derivative of order $\beta \in (0,1)$ is sufficient for finding the Caputo derivatives of arbitrary order. Nevertheless, for completeness later on in this survey the general result for the Caputo fractional derivative is given.

In general, the Riemann- Liouville operator is also non-commutative and satisfies [10]:

$D^r D^{\omega} g(t) = D^{\omega+r} g$ (t)$\neq D^{\omega} D^r$g(t), n-1$<\omega<n, m, n \in N, \omega \in R.$ (3.6)

The inequalities in equation (3.5) and (3.6) become equalities under the following additional conditions [10]:

g^s(0)=0, S= n, n+1,….,m, for D_*^{ω}, and g^s(0)=0, S=0, 1, 2, 3,…,m for D^{ω}, respectively.

It should be noticed, that in case of Caputo derivative there are no restrictions on the values g^s(0)=0, S=0,1,2,3,…,n-1, for example, for m=3, n=2, the functions.

$g(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ +…satisfies the condition for Caputo but doesn't satisfy the condition for Riemann-Liouville [10].

3.2 The Laplace Transform

In this section the Laplace transform is discussed. First general definition is given then the Laplace transforms of two–parameter function of Mittag-Leffler, the Riemann- Liouville and the Caputo fractional derivatives are studied [10],

Definition 3.1 [5]: If the function

$$G(s) = L\{g(t), s\} = \int_0^\infty e^{-st} \, g(t) dt, s \in ₵ \tag{3.7}$$

exists, then it is named the Laplace transform (LT) of $g(t)$.

Let the function $g(t)$ be [5]:

(1) Piecewise smooth over every finite interval in [0,∞) and
(2) Of exponential order ω , i.e,
there exists constants r>0 $and\ T > 0$ such that |g(t)|≤r$e^{\omega t}$ for all t>T then, the Laplace transform (LT) L{g(t),s}of g(t) exists.

The Laplace transform is most applicable to initial value problem on semi-infinite domains.

The original g(t*)* can be obtain from its Laplace transform G(s) using the inverse Laplace transform [6]:

$$g(t) = L^{-1}\{G(s), t\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} G(s) ds \;\; c = Re(s) > C_0, \tag{3.8}$$

while C_0 is located in the right half plane of the absolute convergence of the Laplace integral (3.7) the integral in (3.8) is also called Bromwich integral.

Lemma 7 [10]: Let g(t) and f(t) are two functions which are equal to zero for t <0 and for which the Laplace transforms G(s) and F(s) exist. The following statements hold (see [5] p.105-115).

a) The Laplace transform and its inverse are linear operators i.e. suppose that $\lambda \in R$ so:

$$L\{\lambda\, g(t)+f(t),s\}=\lambda L\{g(t),s\}+L\{f(t),s\}=\lambda\, G(s)+F(s) \quad \text{and} \tag{3.9}$$

$$L^{-1}\{\lambda G(s)+F(s),t\}=\lambda L^{-1}\{G(s),t\}+L^{-1}\{F(s),t\}=\lambda g(t)+f(t).$$

b) For the Laplace transform of the convolution of g(t) and f(t) it follows [5]:

$$L\{g(t)*f(t);s\}=G(s)\,F(s), \tag{3.10}$$

where the detour is defined by [5]:

$$g(t)*f(t)=\int_0^t g(t-\tau)f(\tau)d\tau=\int_0^t g(\tau)\,f(t-\tau)d\tau.$$

c) The limit of the function s F(s) for s→∞ is given by [5]

$$\lim_{s\to\infty} sF(s)=f(0). \tag{3.11}$$

d) The Laplace transform of g(t) of the n-th derivative n∈N writes as follows [5]:

$$\begin{aligned} L\{g^n(t),s\} &= S^nG(s)-\sum_{k=0}^{n-1}S^{n-k-1}g^k(0) \\ &= S^nG(s)-\sum_{k=0}^{n-1}S^k g^{n-k-1}(0). \end{aligned} \tag{3.12}$$

3.3. Laplace Transforms of the Basic Fractional Operators

Let p $>$ 0 and suppose G(s) is the LT of g(t). Then the following statements hold [10]:

a) LT of the fractional integral of order ω (2.7) is given by

$$L\{J^\omega g(t),s\}=S^{-\omega}G(s). \tag{3.13}$$

b) LT of the Riemann-Liouville fractional differential operator of order ω (2.11) is written as.

$$L\{D^{\omega}\ g(t),s\} = S^{\omega}G(s) - \sum_{k=0}^{n-1} S^{k}[D^{\omega-k-1}\ g(t)]_{t=0}.$$

$$= S^{\omega}G(s) - \sum_{k=0}^{n-1} S^{n-k-1}[D^{k}J^{n-\omega}\ g(t)]_{t=0}. \quad n-1<\omega<n. \tag{3.14}$$

c) Let $\omega, \lambda, \beta \epsilon R, \beta, \omega > 0, p \in N$. Then, the Laplace transform of the two-parameter function of Mittag-Leffler type (2.4) is given by

$$L\left\{t^{\omega p+\beta-1}E_{\omega,\beta}^{p}(\pm\lambda t^{\propto}); s\right\} = \frac{p!\, s^{\omega-\beta}}{(s^{\omega}\mp\lambda)^{p+1}}, Re(s) > |\lambda|^{\frac{1}{\omega}}\ . \tag{3.15}$$

Of great interest in this thesis is the Laplace transform of the Caputo fractional derivative. The following statement is proved.

Theorem 1 [10]:

Let $p > 0$ and suppose that G(s) is the LT of g (t).Then, the LT of the Caputo fractional differential operator of order ω (2.12) is written as:

$$L\{D_{*}^{\omega}\ g(t),s\} = S^{\omega}G(s) - \sum_{k=0}^{n-1} S^{\omega-k-1}g^{k}(0)\,, \qquad n-1<\omega<n. \tag{3.16}$$

Proof [10]:

To show the validity of (3.16) consider the first equation (3.1). We have

D_{*}^{ω}g (t) =$J^{n-\omega}D^{n}$g(t).

Let f (t) =D^{n}g (t), then (3.1) becomes

D_{*}^{ω}g (t) =$J^{n-\omega}f(t)$. (3.17)

Using the LT of the fractional integral (3.13) of order n-ω of g(t) and the equation

(3.17) [10],p.106) the LT of the Caputo fractional operator can be written as.

L{D_{*}^{ω}g(t),s}=L{$J^{n-\omega}$f(t),s}=$S^{-(n-\omega)}$F(s) , (3.18)

where F(S) =L {f (t), S} can be expressed using (3.12) in the following way

$$F(S) = S^n G(s) - \sum_{k=0}^{n-1} S^{n-k-1} g^k(0). \tag{3.19}$$

Finally, substituting (3.19) in (3.18) the statement of the theorem

$$L(D_*^{\omega}\ g(t); s\} = S^{-(n-\omega)} \left(S^n G(s) - \sum_{k=0}^{n-1} S^{n-k-1} g^k(0) \right)$$
$$= S^{\omega} G(s) - \sum_{k=0}^{n-1} S^{\omega-k-1}\ g^k(0)$$

is proved.

The LT of the Caputo fractional derivative is a generalization of the Laplace transforms LT of the integer-order derivative (3.12) where n is replaced by ω. The same does not hold for the Riemann-Liouville case. This property is an important advantage of the Caputo operator over the Riemann-Liouville one [11].

Suppose that g (t) be a function such that both $D^{\omega}\ g(t)\ and\ D_*^{\omega}$g(t) exist and n-1 $< \omega < n$,n$\in N$. Then, $D^{\omega}\ g(t) \neq \quad D_*^{\omega}$g (t) holds.

They are reverse to each other since they can be represented as a composition of the same operators but in reverse order.

Considering n-1 $< \omega <$n, n ∈ N, in the interpolation property they are also some differenced for the values of the parameter $\omega \rightarrow$ n the result for both operators is the same.

Let n-1 $< \omega <$n. Then:

$$\lim_{\omega \to n} D^{\omega}\ g(t) = \lim_{\omega \to n} D_*^{\omega}\ g(t) = g^k(t),$$

holds.

Concerning the commutation for function g(t) such as$g^{s}(0)$=0,s=0,1,2,3,…,m each of the two fractional derivatives commutes with the mth order derivative (m$\in N$) namely [10].

$$D^{m}D^{\omega}g(t) = D^{\omega+m}g(t) = D^{\omega}D^{m}g(t),$$

and

$$D_{m}^{\omega}D^{m}g(t) = D_{*}^{\omega+m}g(t) = D^{m}D_{*}^{\omega}g(t).$$

In relation to that, another similarity between the Riemann-Liouville and Caputo is given in the following statement.

Proposition 2:

Suppose that the function $g(t)$ be such that $g^{s}(0)$=0, S=0,1,2,3,…,n-1.Then, the Riemann-Liouville and the Caputo fractional derivatives coincide, namely [10]

$$D_{*}^{\omega}\ g(t) = D^{\omega}g(t).$$

It should also be mentioned that in both cases Riemann-Liouville and Caputo only derivatives of order β in the interval (0,1) can be consider, since (see Remark 2 and formulas (3.5) and (3.6) for n-1$< \omega < n$), namely

$$D_{*}^{\omega}\ g(t) = D_{*}^{\omega-(n-1)}D^{n-1}g(t),$$

$$D^{\omega}g(t) = D^{n-1}D^{\omega-(n-1)}g(t),$$

where $\beta = \omega - (n-1) \in (0,1)\ and\ D^{n-1}$ is the classical integer –order derivative.

One of the most impressing differences between the two operators is the differentiation of the constant function [10]:

D_{*}^{ω} c = 0,   for Caputo, where for Riemann-Liouville, we have [10]:

$$D^{\omega}c = \frac{c}{\Gamma(1-\propto)}t^{-\omega} \neq 0, c = const\ .$$

3.4. The Relation Between Caputo and Riemann-Liouville Operators

The central point in this section is the following statement [10] for which a proof is derived.

Theorem 2 [6]:

Suppose $t > 0, \omega \in R, and\ n-1 < \omega < n, n \in N,$ then the following relation between the Riemann-Liouville and the Caputo operator holds:

$$D_*^{\omega} g(t) = D^{\omega} g(t) \ - \sum_{k=0}^{n-1} \frac{t^{k-\omega}}{\Gamma(k+1-\omega)} g^k(0). \tag{3.20}$$

Proof [6]:

The well-known Taylor series expansion about the point 0 reads as:

g(t)= g(0)+tg'(0)+$\frac{t^2}{2!}g''$(0)+$\frac{t^3}{3!}g'''$(0)+…+$\frac{t^{n-1}}{(n-1)!}g^{n-1}$(0)+$R_{n-1}$

$$= \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} g^k(0) + R_{n-1}.$$

Considering (2.11) [6] we conclude that

$$R_{n-1} = \int_0^t \frac{g^{(n)}(\tau)(t-\tau)^{n-1}}{(n-1)!} d\tau = \frac{1}{\Gamma(n)} \int_0^t g^{(n)}(\tau)(t-\tau)^{n-1} d\tau = J^n g^{(n)}(t)\,.$$

Now, by using the linearity characteristic of the Riemann-Liouville fractional derivative, we obtain:

$$D^{\omega}\text{g(t)} = D^{\omega}\left(\sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} g^{(k)}(0) + R_{n-1}\right)$$

$$= \sum_{k=0}^{n-1} \frac{D^{\omega} t^{k}}{\Gamma(k+1)} g^{k}(0) + D^{\omega} R_{n-1}$$

$$= \sum_{k=0}^{n-1} \frac{\Gamma(\mathrm{k}+1)}{\Gamma(k-\omega+1)} \frac{t^{k-\omega}}{\Gamma(k+1)} g^{k}(0) + D^{\omega} J^{n} g^{n}(t)$$

$$= \sum_{k=0}^{n-1} \frac{t^{k-\omega}}{\Gamma(k-\omega+1)} g^{k}\,(0) + J^{n-\omega} g^{n}(t)$$

$$= \sum_{k=0}^{n-1} \frac{t^{k-\omega}}{\Gamma(k-\omega+1)} g^{k}\,(0) + D_{*}^{\omega} g(t).$$

This means that [6]:

$$D_{*}^{\omega} g(t) = D^{\omega} g(t) - \sum_{k=0}^{n-1} \frac{t^{k-\omega}}{\Gamma(k+1-\omega)} g^{k}(0).$$

So, the proof is completed.

The formula (3.20) implies that the Caputo and the Riemann-Liouville fractional operators coincide if and only if g(t) together with its first (n-1) derivatives vanish at t=0 [10].

CHAPTER 4

THE CAPUTO-FABRIZIO DERIVATIVE

After the Riemann–Liouville and Caputo fractional differential operators were introduced in chapters 2, 3 and we discussed some of their properties as well as the differences between them, it is reasonable to present the new fractional time derivative (CF) introduced by Caputo-Fabrizio in 2015.

4.1 A New Fractional Time Derivative

Definition 4.1 [6]: The Caputo Fabrizio derivative is defined as follows

$$D_t^{\omega} g(t) = \frac{N(\omega)}{(1-\omega)} \int_a^t g'(\tau) exp\left[-\frac{\omega(t-\tau)}{1-\omega}\right] d\tau, \tag{4.1}$$

where N(ω) is anormalization function such that N(0)=N(1)=1. According to the definition (4.1), the CF is zero when g(t) is constant, as in the CFD, the kernel does not have singularity for t=τ .The new CF can also be applied to functions that do not belong to H^1(a,b). Indeed the definition (4.1) can be formulated also for

G∈ $L^1(-\infty, b)$ and for any $\omega \in [0,1]$ as:

$$D_t^{\omega} g(t) = \frac{\omega N(\omega)}{(1-\omega)} \int_{-\infty}^t (g(t) - g(\tau)) exp\left[\frac{\omega(t-\tau)}{1-\omega}\right] d\tau.$$

Now, it is worth to observe that if we put $\sigma = \frac{1-\omega}{\omega} \in [0,\infty]$, $\omega = \frac{1}{1+\sigma} \in [0,1]$ the definition (4.1) of CF assume the form

$$D_t^{\sigma} g(t) = \frac{M(\sigma)}{\sigma} \int_a^t g'(t) exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau, \tag{4.2}$$

where $\sigma \in [0, \infty]$ and M(σ) is the corresponding normalization term of N(ω) fulfilling M(0)=M(∞)=1. Moreover we have:

$$\lim_{\sigma \to 0} \frac{1}{\sigma} \exp\left[-\frac{(t-\tau)}{\sigma}\right] = \delta(t-\tau) \tag{4.3}$$

and for $\omega \to 1$, we have $\sigma \to 0$, namely

$$\lim_{\omega \to 1} D_t^{\omega} g(t) = \lim_{\omega \to 1} \frac{N(\omega)}{1-\omega} \int_a^t g'(\tau) exp\left[-\frac{\omega(t-\tau)}{1-\omega}\right] d\tau$$

$$= \lim_{\sigma \to 0} \frac{M(\sigma)}{\sigma} \int_a^t g'(\tau) exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = g'(t). \tag{4.4}$$

Otherwise, when $\omega \to 0$ it implies that $\sigma \to +\infty$. Hence, we calculate [6]:

$$\lim_{\omega \to 0} D_t^{\omega} g(t) = \lim_{\omega \to 0} \frac{N(\omega)}{1-\omega} \int_a^t g'(\tau) exp\left[-\frac{\omega(t-\tau)}{1-\omega}\right] d\tau$$

$$= \lim_{\sigma \to +\infty} \frac{M(\sigma)}{\sigma} \int_a^t g'(\tau) exp\left[-\frac{(t-\tau)}{\sigma}\right] d\tau = g(t) - g(a). \tag{4.5}$$

Theorem 4.1 [6]:

For (CF) if the function g(t) is such that $g^{(s)}$(a)=0, s=1, 2,…, n, then, we have

$$D_t^n\left(D_t^{\omega} g(t)\right) = D_t^{\omega}\left(D_t^n g(t)\right). \tag{4.6}$$

Proof [6]:

We begin considering n=1, then from the definition below:

$$D_t^{\omega+n} g(t) = D_t^{\omega}\left(D_t^n g(t)\right). \tag{4.7}$$

Then, we calculate the followings [6]:

$$D_t^{\omega}\left(D_t^1\, g(t)\right) = \frac{N(\omega)}{1-\omega} \int_a^t g''(\tau) exp\left[-\frac{\omega(t-\tau)}{1-\omega}\right] d\tau. \tag{4.8}$$

Hence after an integration by parts and assuming g'(a)=0, we have

$$\begin{aligned} D_t^{\omega}\left(D_t^1\, g(t)\right) &= \frac{N(\omega)}{1-\omega}\int_a^t \left(\frac{d}{d\tau} g'\,(\tau)\right) exp - \frac{\omega(t-\tau)}{1-\omega} d\tau \\ &= \frac{N(\omega)}{1-\omega}\left[\int_a^t \frac{d}{d\tau} g'(\tau) exp - \frac{\omega(t-\tau)}{1-\omega} d\tau - \frac{\omega}{1-\omega}\int_a^t g'\,(\tau) exp \right. \\ &\qquad \left. - \frac{\omega(t-\tau)}{1-\omega} d\tau\right] \\ &= \frac{N(\omega)}{1-\omega}\left[g'(\tau) - \frac{\omega}{1-\omega}\int_a^t g'(\tau) exp - \frac{\omega(t-\tau)}{1-\omega} d\tau\right]. \end{aligned} \tag{4.9}$$

Otherwise we conclude the followings [6]:

$$D_t^1(D_t^{\omega} g(t)) = \frac{d}{dt}\left(\frac{N(\omega)}{1-\omega}\int_a^t g'\,(\tau) exp - \frac{\omega(t-\tau)}{1-\omega} d\tau\right)$$

$$= \frac{N(\omega)}{1-\omega}\left[g'(t) - \frac{\omega}{1-\omega}\int_a^t g'(t) exp - \frac{\omega(t-\tau)}{1-\omega} d\tau\right]. \tag{4.10}$$

It is easy to generalize the proof for the case of any n$>$ 1. In the following we suppose the function fulfilling $N\,(\omega)$ = 1.

4.2 The Laplace Transform

In order to study the characteristics of the CF defined in equation (4.2) with a=0, has priority the calculation of its Laplace transform (LT) given with P variable, namely

$$LT[D_t^{\omega}\; g(t)] = \frac{1}{1-\omega}\int_0^{\infty} exp - pt \int_0^t g'(\tau) exp - \frac{\omega(t-\tau)}{1-\omega}\; d\tau\; dt.$$

Hence, from the property of the Laplace transform of a convolution, we have [6]:

$$LT[D_t^{\omega}\; g(t)] = \frac{1}{1-\omega} LT\left(g'(t)\right) LT(exp - \frac{\omega t}{1-\omega})$$

$$= \frac{(p\; LT(g(t) - g(0))}{p + \omega(1-p)}.$$

Similarly, we get the following:

$$LT\left[D_t^{\omega+1}g(t)\right] = \frac{1}{1-\omega}LT(g''(t)LT(\exp\text{-}\frac{\omega t}{1-\omega}) = \frac{(p^2LT[g(t)] - pg(0) - g'(0))}{p+\omega(1-p)}.$$

Finally, we calculate that [6]

$$LT[D_t^{\omega+n}g(t)] = \frac{1}{1-\omega}LT[g^{n+1}(t)]LT\left[exp - \frac{\omega t}{1-\omega}\right]$$

$$= \frac{p^{n+1}LT[g(t)] - p^n g(0) - p^{n-1}g'(0) \dots g^n(0)}{p+\omega(1-p)}.$$

4.3 Fractional Gradient Operator

In this subsection we introduce a new notion of fractional gradient able to describe non-local dependence in constitutive equation [6].

Let us consider a set Ω∈ R^3 and a scalar function u(.): Ω→R,we define the fractional gradient of order $\omega \in [0,1]$ as follows [6]:

$$\nabla^{\omega}u(x) = \frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla u(y)exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy \tag{4.11}$$

with x, y ∈ Ω.

It is simple to prove by using the definition (4.11) together with the property [6]

$$\lim_{\omega\to 1}\frac{\omega}{(1-\omega)\sqrt{\pi}}\exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right] = \delta(x-y)$$

that

$$\nabla^1 u(x) = \nabla u(x) \; and \; \nabla^0 u(x) = \int_{\Omega} \nabla u(y)dy.$$

When $\omega = 1, \nabla^{(1)}$u(x) loses the non-locality given otherwise $\nabla^{(0)}$u(x) is related to the mean value of $\nabla u(y) on\, \Omega.$ In the case of a vector u(x) we define the fractional tensor by [6]:

$$\nabla^{(\boldsymbol{\omega})}\boldsymbol{u}(x) = \frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla u(y) exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy. \tag{4.12}$$

Thus, a material with non–Local property may be described by fractional constitutive equations. As an example we consider an elastic non –local material defined by the following constitutive equation between the stress tensor T and $\nabla^{\omega}\mathbf{u}$(x) [6], namely T(x, t)=A$\nabla^{\omega}\mathbf{u}$(x ,t), $\omega \in (0,1)$, where A is a fourth order symmetric tensor or in the in integral form

$$T(x,t) = \frac{\omega A}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla\boldsymbol{u}(y)\ exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]\ dy.$$

Likewise we introduce the fractional divergence defined for u (.): Ω→ R^3 by [6]

$$\nabla^{(\omega)}.\boldsymbol{u}(x) = \frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla.\ \boldsymbol{u}(y) exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy. \tag{4.13}$$

Theorem 4. 2 [6]:

From the definition (4.12) and (4.14) we have for any u(x):$\Omega \to R$ that [6]:

$\nabla u(x).n|_{\partial\Omega=0.}$ (4.14)

The identity $\nabla.\nabla^{\omega}$u(x)=$\nabla^{\omega}.\nabla$u(x) holds. (4.15)

Proof [6]:

By means of (4.12), we obtain [6]:

$$\begin{aligned}\nabla.\nabla^{\omega}\ u(x) &= \frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla u(y).\nabla_x exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy \\ &= -\frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla u\ (y).\nabla\ exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy \\ &= \frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega} \nabla.\nabla u\ (y)\ exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy \qquad (4.16)\end{aligned}$$

$$-\frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\partial\Omega}\nabla u\,(y).\boldsymbol{n}exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy.$$

Hence, from the boundary condition (4.14), the identity (4.15) is proved because (4.16) coincides with [6]:

$$\nabla^{\omega}.\nabla u(x)=\frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega}\ \nabla.\nabla u(y)exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy.$$

4.4 Fourier Transform of Fractional Gradient and Divergence

For a smooth function u(x): $R^3 \to R$ the Fourier transform (FT) of the fractional gradient is defined as [6]:

$$FT(\nabla^{\omega}u(x))(\xi)=\int_{R^3}\nabla^{\omega}\ u(x)exp[-2\pi i\xi.\boldsymbol{x}]\,dx.$$

Thus, if we consider the gradient of (4.12), the Fourier transform reads as [6];

$$\begin{aligned}FT(\nabla^{\omega}u)(\xi)&=\frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\ FT\left(\int_{R^3}\nabla u\,(y)exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy\right)(\xi)\\&=\frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\ FT(\nabla u)(\xi)FT\left(exp\left[-\frac{\omega^2x^2}{(1-\omega)^2}\right]\right)(\xi),\end{aligned}$$

where

$$FT\left(exp\left[-\frac{\omega^2x^2}{(1-\omega)^2}\right]\right)(\xi)=\frac{(1-w)\sqrt{\pi}}{\omega}\ exp\left[-\frac{\pi^2(1-\omega)^2\xi^2}{\omega^2}\right].$$

Thus, we obtain [6]:

$$FT(\nabla^{\omega}u)(\xi)=\sqrt{\pi^{1-\omega}}FT(\nabla u)(\xi)exp\left[-\frac{\pi^2(1-\omega)^2\xi^2}{\omega^2}\right].$$

The Fourier transform of fractional divergence is defined by [2]:

$$FT(\nabla^{\omega}.\boldsymbol{u})(\xi)=\frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\ FT\left(\int_{\Omega}\ \nabla.u(y)exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy\right)(\xi).$$

From here we conclude that [6]:

$$\mathrm{FT}(\nabla^{\omega}.\boldsymbol{u})(\xi) = \sqrt{\pi^{1-\omega}}\mathrm{FT}\ (\nabla.\mathbf{u})(\xi)exp\left[-\frac{\pi^2(1-\omega)^2\xi^2}{\omega^2}\right].$$

4.5 Fractional Laplacian

In the study of fractional partial differential equations there is a great interest on the notion of the fractional Laplacian. Using the definition of the fractional gradient and divergence we can suggest the representation of the fractional Laplacian for a smooth function $f(x)$:$\Omega \to R^3$, such that $\nabla f(x).\mathbf{n}|_{\partial\Omega}$=0, as [6]:

$$(\nabla^2)^{\omega}f(x) = \frac{\omega}{(1-\omega)\sqrt{\pi^{\omega}}}\int_{\Omega}\nabla.\nabla f(y)exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy.$$

By Theorem (4.1) we conclude that [6]:

$$(\nabla^2)^{\omega}f(x)=\nabla.\nabla^{\omega}f(x)=\nabla^{\omega}.\nabla f(x).$$

Now, we suppose that [6]:

$f\ (x)$=0, on $\partial\Omega$, then we extend the function $f(x) = 0$ on $R^3\backslash\Omega$, so we consider the Fourier Transform (FT):

$$\begin{aligned}\mathrm{FT}((\nabla^2)^{\omega}f(x)) &= \frac{\omega}{(1-\omega\sqrt{\pi^{\omega}}}FT\left(\int_{R^3}\nabla^2\ f(y)exp\left[-\frac{\omega^2(x-y)^2}{(1-\omega)^2}\right]dy\right)(\xi)\\ &= \frac{\omega}{(1-\omega\sqrt{\pi^{\omega}}}FT(\nabla.\nabla f(x)(\xi)FT\left(exp\left[-\frac{\omega^2x^2}{(1-\omega)^2}\right]\right)(\xi) \qquad (4.17)\\ &= 4\pi|\xi|^2FT(f(x))(\xi)\sqrt{\pi^{1-\omega}}exp\left[-\frac{(1-\omega)^2\xi^2}{\omega^2}\right].\end{aligned}$$

Finally, if $\omega = 1$ we obtain from (4.17) that:

$$\begin{aligned}\mathrm{FT}\ (\nabla^2 f(x)) &= -\lim_{\omega\to 1}4\pi\,|\xi|^2\mathrm{FT}(f(x))\,(\xi)\sqrt{\pi^{1-\omega}}exp\left[-\frac{(1-\omega)^2\xi^2}{\omega^2}\right]\\ &= -\,4\pi|\xi|^2\mathrm{LT}(\mathrm{F(x)})(\,\xi)\ .\end{aligned}$$

4.6 The Distributed Order of the Memory Operator

The distributed order operator P^z is defined for the fractional derivative of Caputo-Fabrizio by [6]:

$$aP_b^\omega f(t) = \int_a^b g(\omega)d\omega[D_t^\omega f(t)]$$

$$= aP_b^\omega \int_a^b g(\omega)d\omega\left[\int_0^t exp\left(-\frac{\omega}{(1-\omega)}(t-\tau)\right)f'(\tau)\right]d\tau. \tag{4.18}$$

Here g(ω) is a weight function and 0< $a < b < 1.$ Following the method of Caputo is readily seen that for the Fubini-Tonelli theorem we may change the order of integration in $d \propto$ and dt provided

$$\int_a^b g(\omega)d\omega[\int_0^t \exp(-\frac{\omega}{1-\omega}(t-\tau))f^{m+1}(\tau)d\tau].$$

Is summable with respect to τ in the interval (a ,b) with 0<∝< $b < 1$,which is readily verified the solution on if found using the LT of (4.18) which is [6]:

$$LT\ a\ P_b^\omega f(t) = \int_0^\infty \int_a^b g(\omega)d\omega[[D_t^\omega f(t)] \exp(-pt)\, dt$$

$$= \int_0^\infty \int_a^b g(\omega)d\omega\left[\int_0^t exp\left(-\frac{\omega}{1-\omega}(t-\tau)\right)f'(\tau)d\tau\right]exp(-pt)dt, \tag{4.19}$$

or

$$LT\ a\ P_a^\omega\ f(t)$$
$$= \int_a^b \left\{\int_a^\infty \left[\int_0^t exp\left(-\frac{\omega}{1-\omega}(t - \tau)f'(\tau)d\tau\right)\right]exp(-pt)dt\right\}g(\omega)d\omega.$$

Finally, we obtain that [6]:

$$LT\ a\ P_a^\omega f(t) = \int_a^b \frac{p}{p+\omega}F(p)d\omega \tag{4.20}$$

$$= F(p)\int_a^b \frac{pg(\omega)}{\log(p+\omega)}d\omega,$$

which represents the filtering properties of the operator and is simpler than that obtained using the Caputo derivative. As an example we may consider the simple case g $(\omega) = 1$ which gives [6]:

$$\text{LT } aP_b^{\omega} f(t) = \text{pF (p)} \int_a^b \frac{g(\omega)}{P+\omega} d\omega = pF(p) \log \frac{p+b}{p+\omega}.$$

Hence, the response is p $\log\frac{p+b}{p+a}$**,** whose filtering properties are readily computed.

We rewrite the definition (4.2) in the new form as

$$D_t^v f(t) = V(v) \int_a^t f'(\tau) exp - V(t-\tau) d\tau. \tag{4.21}$$

From (4.1) or (4.2) with V=$\frac{1}{\sigma} > 0,$ where $\text{V(v)} = \text{vN}(\frac{1}{\text{v}})$ so, we have:

Theorem 4.3 [6]:

If the function $f \in W^{1,1}$(a, b), then the integral in (4.22) exists for t $\in [a, b]$ and

$D_t^v f$ (t)$\in L^1[a, b]$.

Proof [6]:

Let us write

$$\begin{aligned} D_t^v f(t) &= V(v) \int_a^t f'(\tau) exp - V(t-\tau) d\tau \\ &= V(v) \int_{-\infty}^{\infty} pv\,(t-s) q(s) ds, \end{aligned} \tag{4.22}$$

where P_v(y)=exp(-vy),when 0$< y < b-a, with$ p(y)=0.When y< 0 $or\ y > b - a, q(y) = f'(y)$ when a$\leq y \leq b,$ finally q(y)=0 when y$< a$ or y$> b$. Hence under the hypotheses of the theorem, the function $P_{v,}$ q$\in L^1(a, b)$then by the classical result on Lebesgue integrals the integral (4.21) exists almost everywhere in t$\in [a, b]$ and $D_t^v f(t) \in L^1[a, b]$. This ends the proof.

It is of some interest to see the fractional derivatives of the elementary and transcendental functions according to the new definition (4.1). We begin with sin(wt),whose fractional derivatives is given by:

$$D_t^{\omega}\sin(wt) = E(\omega)\int_0^t we\,xp\left(-\frac{\omega}{1-\omega}(t-s)\right)\cos(ws)ds,$$

where E$(\omega) = \frac{M(\omega)}{1-\omega}$.

Then we conclude:

$$D_t^{\omega}(\sin wt) = E(\omega)\,wexp\left[-\frac{\omega}{1-\omega}t\right]\int_0^t exp\left(\frac{\omega}{1-\omega}s\right)\cos(ws)ds$$

$$= \frac{E(\omega)w}{\left(\frac{\omega}{1-\omega}\right)^2 + w^2}\left(\frac{\omega}{1-\omega}\cos(wt) + wsin(wt) - \frac{\omega}{1-\omega}exp\left(-\frac{\omega}{1-\omega}t\right)\right)$$

$$= \frac{E(\omega)w}{\left(\frac{\omega}{1-\omega}\right)^2 + w^2}\left(\left(\left(\frac{\omega}{1-\omega}\right)^2 + w^2\right)^{0.5}\sin(wt+a) - \frac{\omega}{1-\omega}exp\left[-\frac{\omega}{1-\omega}t\right]\right),$$

$$= E(\omega)\cos a\left(sin(wt+a) - \sin a\,exp\left[-\frac{\omega}{1-\omega}t\right]\right), \tag{4.23}$$

where a is such that

$$\tan a = \frac{\omega w}{(1-\omega)}, \sin a = \frac{\frac{\omega}{1-\omega}}{\left(\left(\frac{\omega}{1-\omega}\right)^2 + \omega^2\right)^{0.5}},$$

and

$$\cos a = \frac{w}{\left(\left(\frac{\omega}{1-\omega}\right)^2 + w^2\right)^{0.5}}.$$

We note that the new derivative of sin(wt) implies only a change of the phase(a) and the amplitude variation, namely:

$$\frac{w\,E(\omega)}{\left(\left(\frac{\omega}{1-\omega}\right)^2 + w^2\right)^{0.5}}.$$

Now, we see $D_t^{\omega}(coswt)$, with the same procedure, namely

$$D_t^{\omega}(coswt) = E(\omega)\exp\left(-\frac{\omega}{1-\omega}\right)t\int_0^t \exp\left(\frac{\omega}{1-\omega}s\right) sinwsds.$$
$$=\mathrm{E}(\omega)cosa\left[\cos(wt+b) - \cos a\, exp\left(-\frac{\omega}{1-\omega}t\right)\right].$$

We note the same phase change and amplitude variation reported for the case of sin(wt). Moreover, we observe that b is related to a as follows:

$$\tan a = \frac{1}{\tan b}.$$

Hence, we consider D_t^{ω}(expwt), then we find [6]:

$$D_t^{\omega}(\exp wt) = \frac{E(\omega)w}{\frac{\omega}{1-\omega}+w}\left\{\exp(wt) - exp\left[-\frac{\omega}{1-\omega}t\right]\right\}$$
$$= \frac{E(\omega)w}{\frac{\omega}{1-\omega}+w} exp(wt)\left\{1 - exp - \left(w + \frac{w}{1-\omega}t\right)\right\}.$$

Finally, we conclude that

$$D_t^{\omega} = \frac{m(\omega)}{1-\omega}\int_0^t exp\left(-\frac{\omega}{1-\omega}(t-s)\right)ds = \frac{m(\omega)}{\omega}\left(1 - exp - \frac{\omega}{1-\omega}t\right)$$

$(0 < \omega \leq 1.)$

4.7 The Associated Fractional Integral

After the notion of fractional derivative of order 0$< \omega < 1$, that of related fractional integral becomes a natural requirement. In this section we review the fractional integral associated to the Caputo–Fabrizio fractional derivative previously introduced in [27]:

Let us consider $0 < \omega < 1$. Consider now the following fractional differential equation, namely

$^{CF}D^{\omega}g$(t)= u(t),t$\geq$ 0. (4.24)

Using Laplace transform we obtain L[$^{CF}D^{\omega}g$(t)](s)=L[u(t)](s) for s$>$ 0.

That is using

$$L[{}^{CF}D^{\omega}g(t)](s)=\frac{(2-\omega)N(\omega)}{2\big(s+\omega(1-s)\big)}\big(SL[g(t)](s)-g(0)\big)\ for\ s>0,$$

we have that

$$\frac{(2-\omega)N(\omega)}{2\big(s+\omega(1-s)\big)}(SL[g(t)](s)-g(0))=L[u(t)](s)\ for\ s>0$$

or equivalently,

$$L[g(t)](s)=\frac{1}{s}\ g(0)+\frac{2\omega}{s(2-\omega)N(\omega)}L[u(t)](s)+\frac{2(1-\omega)}{\big(s-\omega)N(\omega)\big)}L[u(t)](s).\ s>0.$$

Hence, using the well-known properties of the inverse Laplace transform we deduce that

$$g(t)=\frac{2(1-\omega)}{\big(s-\omega)N(\omega)\big)}\ u(t)+\frac{2\omega}{\big(2-\omega)N(\omega)\big)}\int_0^t u(s)ds+g(0),\qquad t\le 0. \tag{4.25}$$

In other words the function defined as below:

$$g(t)=\frac{2(1-\omega)}{\big(s-\omega)N(\omega)\big)}\ u(t)+\frac{2\omega}{\big(2-\omega)N(\omega)\big)}\int_0^t u(s)ds+C,\qquad t\ge 0.$$

where C∈ R is a constant is also a solution of (4.24). We can also rewrite the fractional differential equation (4.24) as:

$$\frac{(2-\omega)N(\omega)}{2(1-\omega)}\int_0^t \exp(-\frac{\omega}{1-\omega}(t-s))g'(s)ds=u(t),\qquad t\ge 0,$$

or equivalently

$$\int_0^t exp\Big(\frac{\omega}{1-\omega}s\Big)g'(s)ds=\frac{2(1-\omega)}{(2-\omega)N(\omega)}\ exp\Big(\frac{\omega}{1-\omega}t\Big)u(t).\quad t\ge 0.$$

Differentiating both sides of the latter equation we conclude [6]

$$g'(t)=\frac{2(1-\omega)}{(2-\omega)N(\omega)}(u'(t)+\frac{\omega}{1-\omega}u(t)),\qquad t\ge 0.$$

Hence, integrating now from 0 to t we deduce as in (4.26) that

$$g(t) = \frac{2(1-\omega)}{(2-\omega)N(\omega)}[u(t) - u(0)] + \frac{2\omega}{(2-\omega)N(\omega)}\int_0^t u(s)ds + g(0). \quad t \geq 0.$$

Thus, as a consequence, we expect that the fractional integral of Caputo –Fabrizio type must be defined as follows.

Definition 4.2 [2]:

The fractional integral of order ω of a function $g(t)$ has the form:

$$^{CF}I^{\omega} g(t) = \frac{2(1-\omega)}{(2-\omega)N(\omega)}u(t) + \frac{2\omega}{(2-\omega)N(\omega)}\int_0^t u(s)ds \; 0 < \omega < 1, \;\; t \geq 0.$$

Remark 1 [2]:

We see that, according to the definition (4.2) the fractional integral of Caputo – Fabrizio derivative is an average between function g(t) and its integral of order one.

Imposing $\frac{2(1-\omega)}{(2-\omega)N(\omega)} + \frac{2\omega}{(2-\omega)N(\omega)} = 1$, we obtain an explicit formula for N (ω), namely

$$\text{N}\,(\omega) = \frac{2}{2-\omega}, \qquad\qquad 0 \leq \omega \leq 1.$$

Due to this the following of fractional derivative of order $0 < \omega < 1$ was proposed [2].

Definition 4.3 [2]:

The fractional Caputo-Fabrizio derivative of order ω of a function $G(t)$ is given as follows:

$$^{CF}D_*^{\omega} G(t) = \frac{1}{1-\omega}\int_0^t exp\left(-\frac{\omega}{1-\omega}(t-s)\right)G'(s)ds, \;\; 0 < \omega < 1, t \geq 0.$$

4.8 Some Fractional Differential Equations

In this section we present some useful fractional differential equations**.**

Lemma1 [27]:

Let $0 < \omega < 1$ and G(t) be a solution of the fractional differential equation as in the equation below:

$^{CF}D^{\omega}G(t)=0, \qquad t \geq 0.$ (4.26)

Then $G(t)$ is a constant function. The converse, as indicated in the introduction, is also true.

Proof [27]:

From (4.25) we obtain that the solution of (4.26) must satisfy the condition $G(t)$ $=G(0)$ for all $t \geq 0$, subsequently it is clear that $G(t)$ must be a constant function.

Proposition 4.1 [27]:

If $0 < \omega < 1$ so the unique solution of the following initial value problem [27]

$^{CF}D^{\omega}G(t) = \sigma(t), t \geq 0,$ (4.27)

$G(0)=G_0 \in R.$ (4.28)

So, we conclude that:

$G(t)=G_0 + a_{\omega}(\sigma(t)-\sigma(0))+b_{\omega}I^1\sigma(t), \; t\geq 0,$ (4.29)

where $I^1\sigma$ it shows the primitive of σ and

$$a_{\omega} = \frac{2(1-\omega)}{(2-\omega)N(\omega)}, b_{\omega} = \frac{2\omega}{(2-\omega)N(\omega)}. \qquad (4.30)$$

Proof [27]:

We suppose that the equations (4.27) and (4.28) have two solution G_1 and G_2.In this case we have [27]:

$^{CF}D^{\omega}G_1(t)- {}^{CF}D^{\omega}G_2(t) = {}^{CF}D^{\omega}[G_{1-} G_2](t) = 0$, and $(G_1 - G_2)(0)=0$.

So, by Lemma 1 [27], we have that $G_1 - G_2 = 0$.That is $G_1(t)=G_2(t)$ for all $t \geq 0$. By (4.25) it is clearly that the function defined by (4.29) is a solution of the fractional differential equation (4.27), furthermore if we substitute t by 0 in (4.29), we obtain $G_{0,}$hence the function determined by (4.29) is the unique solution of initial value problem (4.27)-(4.28) [27]:

Remark 2 [27] For $\omega = 1$, we have that the solution of (4.27) is the usual primitive of σ. Now, we consider the following linear fractional differential equation:

$$^{CF}D^{\omega}g(t) = \lambda g(t) + u(t), \qquad t \geq 0, \tag{4.31}$$

where $\lambda \in R, \lambda \neq 0(\lambda = 0$ corresponds to the case previously studied). From the Proposition 4.1, we have that solving the equation (4.31) is equivalent to find a function g such that

$$g(t) = g_0 + a_{\omega}[\lambda(g(t) - g_0) + u(t) - u(0)] + b_{\omega}\int_0^t [\lambda g + u](s)ds, \;\; t \geq 0.$$

Here a_{ω} ,b_{ω} are given by (4.30). Equivalently, we must find g such that [27]

(1-λa_{ω})g(t)- $\lambda b_{\omega} I^1$g(t)= (1-λa_{ω})g_0 +a_{ω} (u(t)-u(0) + $b_{\omega} I^1$u(t), t≥ 0.

If λa_{ω}=1, we obtain [27]:

$$g(t) = -\frac{a_{\omega}}{\lambda b_{\omega}} u'(t) - \frac{b_{\omega}}{\lambda}\, u(t) \quad t \geq 0.$$

In the other case, i.e., $\lambda a_{\omega} \neq 1$, we have [27]:

$$g(t) - \frac{\lambda\, a_{\omega}}{1 - \lambda b_{\omega}} I^1 g(t) = \sigma^{\sim}(t), \quad t \geq 0, \tag{4.32}$$

$$\sigma^{\sim}(t) = g0 + \frac{a_{\omega}}{1 - \lambda a_{\omega}}\big(u(t) - u(0)\big) + \frac{b_{\omega}}{1 - \lambda a_{\omega}} I^1\, u(t), \qquad t \geq 0,$$

respectively.

The case $\lambda = 0$ is trivial, and we obtain g= σ.if $\lambda \neq 0$. We see that (4.32) can be rewritten as [27]:

G(t)-$\lambda^{\sim} I^1 g(t) = \sigma^{\sim}(t)$, t ≥ 0, where [27]:

$$\lambda^{\sim} \frac{a_{\omega}}{1 - \lambda a_{\omega}}.$$

Hence $g'(t) = \lambda^{\sim} g(t) + \sigma^{\sim}(t)$, t ≥ 0.

Thus, we have obtained an ordinary differential equation, which has a unique solution if we consider an initial condition. In consequence, we have proved the following result:

Proposition 4.2 [27] Let $0 < \omega < 1$. Then, the initial value problem given by

$^{CF}D^{\omega}g(t) = \lambda g(t) + u(t)$, t ≥ 0,

$g(0) = go \in R$, has a unique solution for $\lambda \in R$.

CHAPTER 5

APPLICATIONS OF CAPUTO-FABRIZIO DERIVATIVE

In this chapter we explained that the advantage of utilizing the Caputo-Fabrizio derivative is due to the necessity of using a better model describing the behavior of classical viscoelastic materials, electromagnetic systems and thermal media. In fact, the original definition of fractional derivative appears to be particularly convenient for those mechanical phenomena, related with plasticity, fatigue, damage and with electromagnetic hysteresis. However, when these effects do not appear it seems more appropriate to utilized the Caputo-Fabrizio derivative.

Below the applications of a new fractional time derivative (CF) introduced by Caputo-Fabrizio in 2015 is reviewed.

5.1 A Chemical Model

The active model of the meeting process of cellular slime mold through biochemical attraction proposed by Keller and Segel in 1970. Simplified model in one dimension is presented by [2]:

$$\begin{cases} \dfrac{\partial a(x,t)}{\partial t} = u\dfrac{\partial^2 a(x,t)}{\partial x^2} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial \varphi(b(x,t))}{\partial x}\right), \\ \\ \dfrac{\partial b(x,t)}{\partial t} = p\dfrac{\partial^2 b(x,t)}{\partial x^2} + ca(x,t) - db(x,t). \end{cases} \tag{5.1}$$

The initial conditions associated to the above equations (5.1) are given as [2]:

a(x,0) = $a_0(x), b(x,0) = b_0(x), x \in I = (\omega, \beta)$.

The coupled solutions a(x, t) and $b(x,t)$represent the concentration of amoebae and concentration of a chemical substance, respectively u ,p ,c, and d are positive constants. The sensitivity of the chemicals and attraction of terms are seen from the chemical expression, namely [2]:

$$\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial\varphi(b\frac{\partial}{\partial x}(a(x,t)\frac{\partial\varphi(b(x,t)}{\partial x})(x,t)}{\partial x}\right). \tag{5.2}$$

The term $\varphi(b(x,t)$is the sensibility function, and is the smooth function of $b\in(0,\infty)$ which represent a cell's perception and response to chemical stimulus. Note that, the above equation (5.2) is not able to clarification the effect of memory and as well the activity of the bacteria within different layers of the medium by which the global activity is taking place. Therefore, in order to include these two effects into the mathematical formula, we amendment the system by change the ordinary time derivative to the new proposed fractional order derivative as in equation (5.3) below [2]:

$$\begin{cases} {}_{0}^{CF}D_t^{\omega}a(x,t)=u\dfrac{\partial^2 a(x,t)}{\partial x^2}-\dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial\varphi\big(b(x,t)\big)}{\partial x}\right), \\ {}_{0}^{CF}D_t^{\omega}b(x,t)=\ p\dfrac{\partial^2 b(x,t)}{\partial x^2}+ca(x,t)-db(x,t). \end{cases} \tag{5.3}$$

To be more accurate, we chose the sensibility function to be in the following form [2]:

$$\varphi(b(x,t)=\frac{b(x,t)}{b(x,t)+1},b(x,t),\frac{b^2(x,t)}{b^2(x,t)+1},\log\big(b(x,t)\big). \tag{5.4}$$

We note that the initial conditions are the same like in equation (5.2).

5.1.1 Existence of Interlinked Solutions

In this subsection, by using the fixed- point theorem, we offer the existence of the coupled-solution. In the beginning we transform the equation (5. 3) to an integral equation as:

$$\begin{cases} a(x,t)-a(x,0) = {}_{0}^{CF}I_t^{\omega}\left\{u\dfrac{\partial^2 a(x,t)}{\partial x^2} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right)\right\}, \\ b(x,t)-b(x,0) = {}_{0}^{CF}I_t^{\omega}\left\{p\dfrac{\partial^2 b(x,t)}{\partial x^2} + ca(x,t) - db(x,t)\right\}. \end{cases}$$

Using the observations made in [2], we get:

$$\begin{cases} a(x,t)-a(x,0) = \dfrac{2(1-\omega)}{(2-\omega)N(\omega)}\left\{u\dfrac{\partial^2 a(x,t)}{\partial x^2} - \dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial \varphi(b(x,t))}{\partial x}\right)\right\} + \\ \dfrac{2\omega}{(2-\omega)N(\omega)}\displaystyle\int_0^t \{u\dfrac{\partial^2 a(x,y)}{\partial x^2} - \dfrac{\partial}{\partial x}(a(x,y)\dfrac{\partial \varphi\big(b(x,y)\big)}{\partial x})\}yd, \\ b(x,t)-b(x,0) = \dfrac{2(1-\omega)}{(2-\omega)N(\omega)}\left\{p\dfrac{\partial^2 b(x,t)}{\partial x^2} + ca(x,t) - db(x,t)\right\} + \\ \dfrac{2\omega}{(2-\omega)N(\omega)}\displaystyle\int_0^t p\dfrac{\partial^2 b(x,y)}{\partial x^2} + ca(x,y) - db(x,y)\}dy. \end{cases} \tag{5.5}$$

For simplicity, we define the following kernels [2]:

$$\begin{cases} Z_1(x,t,u) = u\dfrac{\partial^2 a(x,y)}{\partial x^2} - \dfrac{\partial}{\partial x}\left(a(x,y)\dfrac{\partial \varphi\big(b(x,y)\big)}{\partial x}\right), \\ \\ Z_2(x,t,b) = p\ \dfrac{\partial^2 b(x,y)}{\partial x^2} + ca(x,y) - db(x,y). \end{cases} \tag{5.6}$$

Theorem 5.1 [2]:

Z_1 and Z_2 satisfy the contraction and Lipschiz condition if the following inequality achieved:

$$0 < u\delta_1^2 + \delta_2 \left\|\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right\| \le 1.$$

Proof [2]:

Let's start with Z_1.Let a and v be two function, then we evaluate the following [2].

$$\|z_1(x,t,a) - z_1(x,t,v)\| = \tag{5.7}$$
$$\left\| u\frac{\partial^2\{a(x,t) - v(x,t)\}}{\partial x^2} - \frac{\partial}{\partial x}(\{a((x,t) - v(x,t)\})\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x} \right\|$$

We transform (5.7) into

$$\|Z_1(x,t,a) - Z_1(x,t,v)\|$$
$$\leq u\left\|\frac{\partial^2\{a(x,t) - v(x,t)\}}{\partial x^2}\right\|$$
$$+ \left\| -\frac{\partial}{\partial x}\left(\{a(x,t) - v(x,t)\}\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right)\right\|.$$

By using the triangular inequality note that the operator derivative satisfies the Lipschitz conditions, we can then find a positive two parameters δ_1 and δ_2 such that:

$$u\left\|\frac{\partial^2\{a(x,t) - v(x,t)\}}{\partial x^2}\right\| \leq u\delta_1^2\|a(x,t) - v(x,t)\| \tag{5.8}$$
$$\left\| -\frac{\partial}{\partial x}\left(\{a(x,t) - v(x,t)\}\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right)\right\| \leq \delta_2\left\|\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right\|\|a(x,t) - v(x,t)\|.$$

Replace the equation (5.8) in to Equation (5.6) we find:

$$\|Z_1(x,t,a) - Z_1(x,t,v)\| \leq \left\{u\delta_1^2 + \delta_2\left\|\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right\|\right\}\|a(x,t) - (x,t))\|. \tag{5.9}$$

Taking:

$$H = \left\{u\delta_1^2 + S_2\left\|\frac{\partial \emptyset\big(b(x,t)\big)}{\partial x}\right\|\right\},$$

then

$\|Z_1$(x ,t ,a)-Z_1(x,t,v)$\| \leq H\|a(x,t) - v(x,t)\|$,therefore Z_1 satisfies the Lipschitz conditions and also if [2]:

$0 < u\delta_1^2 + \delta_2 \left\|\frac{\partial \emptyset(b(x,t))}{\partial x}\right\| \leq 1,$

then it is a contraction.

With the second case that we have, the kernel is linear, then it satisfies the Lipschitz condition as in the equations below [2]:

$$\|Z_2(x,t,b) - Z_2(x,t,b_1)\| \leq \{p\theta_1^2 + c\}\|b(x,t) - b_1(x,t)\|.$$

Taking into account these kernels, equation (5.5) is reduced to [2]:

$$\begin{cases} a(x,t) = \dfrac{2(1-\omega)}{(2-\omega)N(\omega)} Z_1(x,t,a) + a(x,0) + \dfrac{2\omega}{(2-\omega)N(\omega)} \displaystyle\int_0^t \{Z_1(x,t,a)\} dy, \\ b(x,t) = \dfrac{2(1-\omega)}{(2-\omega)N(\omega)} Z_2(x,t,b) + b(x,0) + \dfrac{2\omega}{(2-\omega)N(\omega)} \displaystyle\int_0^t Z_2(x,t,b) dy. \end{cases} \tag{5.10}$$

We consider the following iterative formula [2]:

$$\begin{cases} a_n(x,t) = \dfrac{2(1-\omega)}{(2-\omega)N(\omega)} Z_1(x,t,a_{n-1}) + \dfrac{2\omega}{(2-\omega)N(\omega)} \displaystyle\int_0^t \{Z_1(x,t,a_{n-1})\} dy \\ b_n(x,t) = \dfrac{2(1-\omega)}{(2-\omega)N(\omega)} Z_2(x,t,b_{n-1}) + \dfrac{2\omega}{(2-\omega)N(\omega)} \displaystyle\int_0^t Z_2(x,t,b_{n-1}) dy, \end{cases} \tag{5.11}$$

with initial component [2]:

$$\begin{cases} a_0(x,t) = a(x,0), \\ b_0(x,t) = b(x,0). \end{cases}$$

Difference between the consecutive terms is given as follows [2]:

$$\begin{aligned} A_n(\text{x,t}) \quad &= a_n(x,t) - a_{n-1}(x,t) \\ &= \frac{2(1-\omega)}{(2-\omega)N(\omega)} Z_1(\text{x,t}, a_{n-1}) - \frac{2(1-\omega)}{(2-\omega)N(\omega)} Z_1(\text{x,t},a_{n-2}) \\ &+ \frac{2\omega}{(2-\omega)N(\omega)} \int_0^t \{Z_1(x,t,a_{n-1}) - Z_1(x,t,a_{n-2})\} dy, \end{aligned} \tag{5.12}$$

$$\begin{aligned} V_n(x,t) \quad &= b_n(x,t) - b_{n-1}\,(\text{x,t}) = \frac{2(1-\omega)}{(2-\omega)N(\omega)} Z_2\,(x,t,b_{n-1} - b_{n-1}) \\ &+ \frac{2\omega}{(2-\omega)N(\omega)} \int_0^t Z_2(x,t,b_{n-1} - b_{n-1}) dy. \end{aligned}$$

It worth noting that [2]:

$$\begin{cases} a_n(x,t) = \displaystyle\sum_{i=0}^{n} A_i\,(x,t), \\ b_n(x,t) = \displaystyle\sum_{i=0}^{n} V_i(x,t). \end{cases}$$

We conclude that [2]:

$$\|A_n(x,t)\| = \|a_n(x,t) - a_{n-1}(x,t)\|$$

$$= \left\| \frac{2(1-\omega)}{(2-\omega)N(\omega)} Z_1(x,t,a_{n-1}) - \frac{2(1-\omega)}{(2-\omega)N(\omega)} Z_1(x,t,a_{n-2}) + \frac{2\omega}{(2-\omega)N(\omega)} \int_0^t \{Z_1(x,t,a_{n-1}) - Z_1(x,t,a_{n-2})\} dy \right\|$$

The above equation by using the triangular inequality becomes [2]:

$$\|a_n(x,t) - a_{n-1}(x,t)\| \leq \frac{2(1-\omega)}{(2-\omega)N(w)} \|Z_1(x,t,a_{n-1}) - Z_1(x,t,a_{n-2})\| \tag{5.13}$$

$$+\frac{2\omega}{(2-\omega)N(w)}\left\|\int_0^t \{Z_1(x,t,a_{n-1})-Z_1(x,t,a_{n-2})\}\,dy\right\|.$$

Because the kernel satisfies the Lipschitz condition, we find [2]:

$$\|a_n(x,t)-a_{n-1}(x,t)\| \le \frac{2(1-\omega)}{(2-\omega)N(w)}H\|a_{n-1}-a_{n-2}\|+\frac{2\omega}{(2-\omega)N(w)}Z\int_0^t \|a_{n-1}-a_{n-2}\|dy. \tag{5.14}$$

Then, we calculate [2]:

$$\|A_n(x,t)\| \le \frac{2(1-\omega)}{(2-\omega)N(w)}H\|A_{n-1}(x,t)\| + \frac{2\omega}{(2-\omega)N(w)}Z\int_0^t \|A_{n-1}(x,t)\|dy. \tag{5.15}$$

In the same way we obtain [2]:

$$\|V_n(x,t)\| \le \frac{2(1-\omega)}{(2-\omega)N(w)}H_1\|V_{n-1}(x,t)\| + \frac{2\omega}{(2-\omega)N(w)}J_1\int_0^t \|V_{n-1}(x,t)\|dy. \tag{5.16}$$

To understand the concentration of a chemical substance and concentration of amoebae we shall then state the following theorem [2]:

Theorem 5.2 [2]:

Because the concentration of a chemical substance and concentration of amoebae are taking place in a confined medium, so, the equation (5.3) has a coupled-solution.

Proof [2]:

Both b(x, y) and a(x,t) are bounded, moreover we have proved that both kernels satisfy the Lipschiz condition, so following the results obtained in equations (5.15) and (5.16).

Using the frequent technique, we find the following relation [2]:

$$\|A_n(x,t)\| \leq \|a(x,0)\| \left\{ \left\{ \frac{2(1-\omega)}{(2-\omega)N(w)} H \right\}^n + \left\{ \frac{2\omega}{(2-\omega)N(w)} zt \right\} \right\},$$

$$\|V_n(x,t)\| \leq \|b(x,0)\| \left\{ \left\{ \frac{2(1-\omega)}{(2-\omega)N(w)} H_1 \right\}^n + \left\{ \frac{2\omega}{(2-\omega)N(w)} J_1 t \right\} \right\}. \tag{5.17}$$

So, the above solutions are continuous and they exist. However, to show that the above is a solution of equation (5.3) let us consider [2]:

$$\begin{cases} a(x,t) = a_n(x,t) - b_n(x,t), \\ b(x,t) = b_n(x,t) - b_{2n}(x,t). \end{cases}$$

Thus, we report that:

$$a(x,t) - a_n(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)} Z_1\big(x,t,a - b_n(x,t)\big) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t \{Z_1\big(x,t,a - b_n(x,t)\big)\} dy. \tag{5.18}$$

It follows from the above that [2]:

$$a(x,t) - \frac{2(1-\omega)}{(2-\omega)N(w)} Z_1\big(x,t,a) - a(x,0)\big) - \frac{2\omega}{(2-\omega)N(w)} \int_0^t \{Z_1(x,t,a)\} dy$$
$$= b_n(x,t) + \frac{2(1-\omega)}{(2-\omega)N(w)} Z_1(x,t,a) \tag{5.19}$$
$$+ \frac{2\omega}{(2-\omega)N(w)} \int_0^t \left\{ Z_1\big(x,t,a - b_n(x,t)\big) - \{k1(x,t,a)\} \right\} dy.$$

Nevertheless, applying the norm on both sides together with the Lipschitz condition, we find [2]:

$$\left\| a(x,t) - \frac{2(1-\omega)}{(2-\omega)N(w)} Z_1(x,t,a) - a(x,0) - \frac{2\omega}{(2-\omega)N(w)} \int_0^t \{Z_1(x,t,a)\} dy \right\| \tag{5.20}$$

$$\leq \|b_n(x,t)\| + \left\{ \frac{2(1-\omega)}{(2-\omega)N(w)} H + \frac{2\omega}{(2-\omega)N(w)} Zt \right\} \|b_n(x,t)\|.$$

In the same way, we conclude that [2]:

$$\left\| b(x,t) - \frac{2(1-\omega)}{(2-\omega)N(w)} Z_2(x,t,b) - b(x,0) - \frac{2\omega}{(2-\omega)N(w)} \int_0^t \{Z_2(x,t,b)\} dy \right\|$$

$$\leq \|D_n(x,t)\| + \left\{ \frac{2(1-\omega)}{(2-\omega)N(w)} H_1 + \frac{2\omega}{(2-\omega)N(w)} J_1 t \right\} \|D_n(x,t)\|. \tag{5.21}$$

When n$\rightarrow \infty$ $taking\ the\ limit\ on\ both$ sides of equation (5.20) and (5.21), the right hand sides of both equations tend to zero, namely [2]:

$$a(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)} Z_1(x,t,a) + a(x,0) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t \{Z_1(x,t,a)\} dy,$$

$$b(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)} Z_2(x,t,b) + b(x,0) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t \{Z_2(x,t,b)\} dy. \tag{5.22}$$

This completes this proof.

5.1.2 Uniqueness of the Coupled Solutions

Below, we explained that the coupled – solutions presented in the above subsection are unique. To achieve this, we suppose that we can obtain another coupled-solutions for system (5.3) let a_1(x,t), b_1(x,t) [2]:

Then, we have [2]:

$$a(x,t) - a_1(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)}\{Z_1(x,t,a) - Z_1(x,t,a_1)\}$$
$$+\frac{2(\omega)}{(2-\omega)N(w)}\int_0^t \{Z_1((x,t,a) - Z_1(x,t,a_1))\}\,dy \tag{5.23}$$

and

$$\|a(x,t) - a_1(x,t)\|$$
$$\leq \frac{2(1-\omega)}{(2-\omega)N(w)}\{\|Z_1(x,t,a) - Z_1(x,t,a_1)\|\}$$
$$+\frac{2\omega}{(2-\omega)N(w)}\int_0^t \{\|Z_1((x,t,a) - Z_1(x,t,a_1))\|\}dy. \tag{5.24}$$

Making use of the Lipschitz conditions of the kernel, together with the fact that the solutions are bounded, we find that [2]:

$$\|a(x,t) - a_1(x,t)\| < \frac{2(1-\omega)}{(2-\omega)N(\omega)}HD + \left\{\frac{2\omega}{(2-\omega)N(\omega)}(J_1Dt)\right\}^n. \tag{5.25}$$

This is true for any n so [2]:

$$a(x,t)=a_1(x,t).$$

Using the same method, we conclude

$$b(x,t)=b_1(x,t).$$

In this way the uniqueness of the coupled-solution of system (5.3) was completed.

5.2. Derivation of Approximate Coupled-Solutions

Because the system is nonlinear and it may be difficult to get the exact solution. In this subsection, we offer the derivation of a special solution by employing an iterative technique [2]. The technique involves coupling the Laplace transform and it's inverse. Before presenting the methodology of the technique, we will first present the connection between the Laplace transform and Caputo-Fabrizio derivative with fractional order [2].

The Laplace transform of the Caputo-Fabrizio fractional order derivative is given as [19]:

$$L\left({}_{0}^{CF}D_{x}^{\omega}\big(g(x)\big)\right)=\frac{pL\big(g(x)\big)-g(0)}{b+\omega(1-b)}. \tag{5.26}$$

Now, by applying the above operator on both sides of the system (5.3) we find [2]:

$$\begin{cases}\dfrac{bL\big(a(x,t)\big)-a(x,0)}{b+\omega(1-b)}=L\left\{u\dfrac{\partial^{2}a(x,t)}{\propto x^{2}}-\dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial\varphi\big(b(x,t)\big)}{\partial x}\right)\right\},\\ \dfrac{bL\big(b(x,t)\big)-b(x,0)}{b+\omega(1-p)}=L\left\{p\dfrac{\partial^{2}b(x,t)}{\partial x^{2}}+ca(x,t)-db(x,t)\right\}.\end{cases} \tag{5.27}$$

We transform the above into [2]:

$$\begin{cases}L(a(x,t)=\dfrac{a(x,0)}{b}+\dfrac{\big(b+\omega(1-b)\big)}{b}L\left\{u\dfrac{\partial^{2}a(x,t)}{\partial x^{2}}-\dfrac{\partial}{\partial x}\left(a(x,t)\dfrac{\partial\varphi\big(b(x,t)\big)}{\partial x}\right)\right\},\\ L\big(b(x,t)\big)=\dfrac{b(x,0)}{b}+\dfrac{\big(b+\omega(1-b)\big)}{b}L\left\{p\dfrac{\partial^{2}b(x,t)}{\partial x^{2}}+cu(x,t)-dp(x,t)\right\}.\end{cases} \tag{5.28}$$

Now, we applying the inverse Laplace on both sides and we report [2]:

$$a(x,t)=a(x,0)+\begin{cases}L^{-1}\left\{\frac{\left(b+\omega(1-b)\right)}{b}L\left\{u\frac{\partial^{2}a(x,t)}{\partial x^{2}}-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial\varphi\left(b(x,t)\right)}{\partial x}\right)\right\}\right\},\\ b(x,t)=b(x,0)+\\ L^{-1}\left\{\frac{\left(b+\omega(1-b)\right)}{b}L\left\{p\frac{\partial^{2}b(x,t)}{\partial x^{2}}+ca(x,t)-db(x,t)\right\}\right\}.\end{cases} \tag{5.29}$$

We assume the following iterative formula [2]:

$$a_{n+1}(\mathrm{x,t})=a_n(\mathrm{x,t})+\begin{cases} L^{-1}\left\{\frac{(b+\omega(1-b))}{b}L\left\{u\frac{\partial^2 a_n(x,t)}{\partial x^2}-\frac{\partial}{\partial x}\left(a_n(x,t)\frac{\partial\varphi(b_n(x,t))}{\partial x}\right)\right\}\right\}, \\ b_{n+1}(x,t)=b_n(x,0)+ \\ L^{-1}\left\{\frac{(b+\omega(1-b))}{b}L\left\{p\frac{\partial^2 b_n(x,t)}{\partial x^2}+ca_n(x,t)-db_n(x,t)\right\}\right\} \end{cases} \tag{5.30}$$

with the first component [2]:

$$\begin{cases} a_0(x,t)=a(x,0). \\ b_0(x,t)=b(x,0). \end{cases}$$

The coupled solution is thus provided as [19]:

$$\begin{cases} a(x,t)=\lim_{n\to\infty} u_n\,(x,t), \\ b(x,t)=\lim_{n\to\infty} b_n\,(x,t). \end{cases} \tag{5.31}$$

5.3 Fractional Falling Body Problem

Consider a mass *m* falling due to gravity. The net force acting on the body is equal to the rate of change of the momentum of that body. For constant mass, by applying the classical Newton second law, we conclude [27]

$$mv'=mz-\ kv(t),$$

where *z* is the gravitational constant, and the air resistance is proportional to the velocity with proportionality constant *k*. If the air resistance is negligible, then $k\ =\ 0$ and the equation simplifies to [27]:

$$v'(t)=z.$$

If we replace D^1= v' by D^{ω} we have the following fractional falling body equation [27]:

$$^{CF}D^{\omega}v(t)=\text{-}\frac{k}{m}v(t)+g.$$

For an initial velocity $v(0)=v_0$ then, according to proposition 4.2, it has a unique solution [27].

5.4. Systems of Nonlinear Time – Fractional Differential Equations

Below we examine the existence of solutions for two coupled systems of nonlinear time-fractional differential equations and inclusions within Caputo-Fabrizio time-fractional derivative. First, we discuss the coupled system of the time-fractional differential equations, namely [3]

$$\begin{cases} ({}^{CF}D_t^{\omega}u)(x,t) = f_1\big(x,t,u(x,t),v(x,t)\big), \\ ({}^{CF}D_t^{\beta}v)(x,t) = f_2\big(x,t,u(x,t),v(x,t)\big). \end{cases} \tag{5.32}$$

such that [3]:

u(0,0)=0,v(0,0)=0, (5.33)

where $0< \omega < 1, 0 < \beta < 1, (x,t) \in [0,1] \times [0,1]$, the mappings f_1, f_2:$[0,1] \times [0,1]$ ×R×R→R are continuous function. In addition, we discuss the existence of solutions for the coupled system of nonlinear time-fractional differential inclusions [3]:

$$\begin{cases} ({}^{CF}D_t^{\omega}u)(x,t) = F_1\big(x,t,u(x,t),v(x,t)\big), \\ ({}^{CF}D_t^{\beta}v)(x,t) = F_2\big(x,t,u(x,t),v(x,t)\big). \end{cases} \tag{5.34}$$

such that

u(0,0)=0,v(0,0)=0, (5.35)

where F_1,F_2 : $[0,1] \times [0,1]$ ×R×R→P(R) are some multivalued maps. We say that F: $[0,1] \times [0,1]$ ×R×R→ 2^R is a Caratheodory multifunction whenever (x ,t) → F(x,t,u_1,u_2) is measurable for all u_i∈ R and $(u_1$,$u_2)$ →F(x,t,u_1,u_2) is upper semi-continuous (u.s.c) for almost all (x ,t) ∈ $[0,1] \times [0,1]$ and u_1,u_2∈ X. A Carathe'odory multifunction F: $[0,1] \times [0,1]$ ×R×R→ $2^R$is said

to be an L^1-Caratheodory whenever for each$p > 0$ there exists $\emptyset_p \in L^1([0,1] \times [0,1], R^+)$such that ‖F(x ,t,$u_1$,$u_2$) ‖ =$sup_{(x,t)\in [0,1] \times[0,1]}$ {|s|:s∈ F(x ,t ,u_1,u_2)} ≤

$\emptyset_p(x,t)$for all $|u_i| \leq p$ and for almost all (x ,t) ∈ $[0,1] \times [0,1]$. The set of selection of f_i at u_i is defined by [3]:

$S_{F_i}(u_i)$ = {w_i ∈ $L^1([0,1] \times [0,1], R)$ w_i(x ,t) ∈ F(x,t,u_i(x ,t),u_i'(x ,t)) for almost all (x ,t)∈ $[0,1] \times [0,1]$} for all u_i ,u_i'∈$C_R([0,1] \times [0,1])$ for i=1,2. The sets $S_{F_i}(u_i)$ are nonempty for all$u_i \in C_K([0,1] \times [0,1])$ whenever dim K $< \infty$.The graph of the multifunction F:X→ Y is defined by the setG_r(F) ={(x ,y) ∈ X×Y :y ∈ F(x)}.We say that the graphG_r(F) of F:X→ P_{cl}(Y)is a closed subset of X×Y whenever for each sequences $\{u_n\}_{n\in N}$ in X and $\{y_n\}_{n\in N}$ in Y with $u_n \to u_0$, $y_n \to y_0$ and y_n ∈ F(y_n) for all n, we have y_0 ∈F(u_0)[3].

Lemma 5.1.[3] Suppose that f ∈ $L^1_x([0,1] \times [0,1])$ and 0$< \omega < 1$. The function $u_0 \in C_x([0,1] \times [0,1])$is a solution for the time –fractional integral equation:

$$u(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)}\left(f(x,t) - f(0,0) + \frac{2\omega}{(2-\omega)N(w)}\int_0^t f(x,s)\right)ds, \tag{5.36}$$

if and only if u_0 is an unique solution of the time-fractional differential equation.

$$\begin{cases} ({}^{CF}D_t^{\omega}u)(x,t) = f(x,t), & (x,t) \in [0,1] \times [0,1] \\ u(0,0) = 0. & \end{cases} \tag{5.37}$$

Proof. [3] A solution of (5.37) is denoted by u_0. As a result (${}^{CF}D_t^{\omega}\, u_0)(x,t) = f(x,t)$, and u_0(0,0) =0. By integrating both sides we get [3]:

$$(u_0)(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)}\left(f(x,t) - f(0,0) + \frac{2\omega}{(2-\omega)N(w)}\int_0^t f(x,s)\right)ds.$$

This show that u_0 represents the solution of (5.42). If u_1 and u_2 are two distinct solutions [3]:

of (5.36), the$({}^{CF}D_t^{\omega}\, u_1)(x,t) - ({}^{CF}D_t^{\omega}\, u_2)(x,t)$=${}^{CF}D_t^{\omega}[\, u_1 - u_2](x,t)$=0, and$(u_1 - u_2)(0,0) = 0.$ By the property of the Caputo-Fabrizio time – fractional derivative in[27], we get $u_1 = u_2$. Hence, u_0 is an unique solution of(5.37). Now, suppose that u_0 is a solution of (5.36). Thus, we conclude that

$$(u_0)(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)}\left(f(x,t) - f(0,0) + \frac{2\omega}{(2-\omega)N(w)}\int_0^t f(x,s)\right)ds.$$

By using the equation below [3]:

$$v(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)}g(x,t) + (x,t) = \frac{2\omega}{(2-\omega)N(w)}\int_0^t g(x,s)ds + v(0,0),$$

where 0$<\omega<1$ and $(x,t) \in [0,1]\times[0,1]$ [27] one can see that this function represents a solution of (5.37) [3].Note that, $u_0(0,0) = 0$.

Now, we consider this equation, namely [3]:

$$(x,t) = \frac{2(1-\omega)}{(2-\omega)N(w)}\int_0^t exp\left[-\frac{\omega}{1-\omega}\right]\frac{\partial u}{\partial t}ds$$

with equation of the fractional integral of order ω. For each $(x,t) \in [0,1]\times[0,1]$, define the operators T_1,T_2: X→X, by:

$$\begin{aligned}(T_1\, v)(x,t) = {} & \frac{2(1-\omega)}{(2-\omega)N(w)}f_1\big(x,t,u(x,t),v(x,t)\big) \\ & + \frac{2(1-\omega)}{(2-\omega)N(w)}f_1\big(0,0,u(0,0),v(0,0)\big) \\ & + \frac{2(1-\omega)}{(2-\omega)N(w)}\int_0^t f_1\big(x,s,u(x,s),v(x,s)\big)ds,\end{aligned} \tag{5.38}$$

$$\begin{aligned}(T_2 u)(x,t) = {} & \frac{2(1-\beta)}{(2-\beta)N(\beta)}f_2\big(0,0,u(0,0),v(0,0)\big) \\ & + \frac{2\beta}{(2-\beta)N(\beta)}f_2\big(0,0,u(0,0),v(0,0)\big) \\ & + \frac{2\beta}{(2-\beta)N(\beta)}\int_0^t f_2\big(x,s,u(x,s),v(x,s)\big)ds,\end{aligned} \tag{5.39}$$

and put [3]:

$$N_1 = \frac{4-2\omega}{(2-\omega)N(\omega)} \quad and\ N_2 = \frac{4-2\beta}{(2-\beta)N(\beta)} \tag{5.40}$$

Theorem 5.3 [3]

Suppose that f_1, f_2:[0, 1] $\times$ [0,1] ×X×X→X are the continuous mappings in the system (5.32)-(5.33) and there exist positive constantsL_1 and L_2such that $|f_1$(x ,t ,u_1,u_2)| $\leq$ $L_1$and $|f_2$(x ,t ,u_1,u_2)| $\leq L_2$ for all (x ,t) ∈ [0,1] $\times$ [0,1] and u_1,u_2 ∈ X. Thus, (5.32) and (5.33) has at least one solution.

Proof [3]:

Let us consider the operators T_1, T_2: X→ X defined by (5.38)-(5.39).Now, we define the operator [3]:

T: X×X→X×X by T(u ,v)(x ,t)= (($T_1 v$)(x ,t),($T_2 u$) (x ,t) for all (x ,t) ∈ [0,1] $\times$ [0,1].note that T continuous because the mappings f_1 and f_2 are continuous. We show that the operator T maps bounded sets into the bounded subsets of X ×X. Let Ω be a bounded subset of X ×X,(u ,v) ∈Ω and (x ,t) ∈ [0,1] $\times$ [0,1] Then, we conclude that [3]:

$$|(T_1 v)(x,t)|$$

$$= \left| \begin{array}{c} \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(x,t,u(x,t),v(x,t)\big) - \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(0,0,u(0,0),v(0,0)\big) + \\ \frac{2\omega}{(2-\omega)N(w)} \int_0^t f_1\,\big(0,0,,u(0,0),v(0,0)\big) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t f_1\big(x,s,u(x,s)v(x,s)\big)ds \end{array} \right|$$

$$\leq \frac{2(1-\omega)}{(2-\omega)N(w)} \left|f_1\big(x,t,u(x,t),v(x,t)\big)\right| + \frac{2(1-\omega)}{(2-\omega)N(w)} |f_1(0,0,0,0)|$$

$$+ \frac{2(1-\omega)}{(2-\omega)N(w)}$$

$$\left|f_1\big(x,s,u(x,s),v(x,s)\big)\right| ds \leq L_1 \left\{ \frac{2(1-\omega)}{(2-\omega)N(w)} + \frac{2(1-\omega)}{(2-\omega)N(w)} + \frac{2\omega}{(2-\omega)N(w)} t \right\}$$

$$\leq L_1 \left\{ \frac{4(1-\omega)}{(2-\omega)N(w)} + \frac{2\omega}{(2-\omega)N(w)} \right\} \leq L_1 \left\{ \frac{4-2\omega}{(2-\omega)N(w)} \right\} = L_1 N_1.$$

And so ‖($T_1 v$)(x ,t)‖ x ≤ $L_1 N_{1.}$. Also we have [3]:

$|(T_2 u)(x,t)|$

$$= \left| \begin{array}{c} \frac{2(1-\beta)}{(2-\beta)N(\beta)} f_2\big(x,t,u(x,t),v(x,t)\big) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} f_2\big(0,0,u(0,0),v(0,0)\big) + \\ \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t f_2\,\big(0,0,,u(0,0),v(0,0)\big) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t f_1\big(x,s,u(x,s)v(x,s)\big)ds \end{array} \right|$$

$$\leq \frac{2(1-\omega)}{(2-\omega)N(w)} \left|f_2\big(x,t,u(x,t),v(x,t)\big)\right| + \frac{2(1-\beta)}{(2-\beta)N(\beta)} |f_2(0,0,0,0)| + \frac{2(1-\beta)}{(2-\beta)N(\beta)}$$

$$\left|f_2\big(x,s,u(x,s),v(x,s)\big)\right|ds \leq L_2 \left\{\frac{2(1-\beta)}{(2-\beta)N(w)} + \frac{2(1-\beta)}{(2-\beta)N(\beta)} + \frac{2\beta}{(2-\beta)N(\beta)} t\right\}$$

$$\leq L_2 \left\{\frac{4(1-\beta)}{(2-\beta)N(\beta)} + \frac{2\beta}{(2-\beta)N(\beta)}\right\} \leq L_1 \left\{\frac{4-2\beta}{(2-\beta)N(\beta)}\right\} = L_2 N_{21}.$$

Therefore, we conclude ‖($T_2 u$)(x ,t)‖ x≤ $L_2 N_2$[3]. Thus ‖T(u,v)(x,t)‖$_{X \times X}$ ≤ $L_1 N_1$+$L_2 N_2$.This shows that T maps bounded sets into the bounded sets of X×X. Now, we show that T is equi-continuous Let (x,t_1), (x,t_2) ∈ [0,1] × [0,1]with $t_1 < t_2$.Then, we have [3]:

$|(T_1 v)(x,t_2) - (T_1 v)(x,t_1)| =$

$$\left| \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(x,t_2,u(x,t_2),v(x,t_2)\big) - \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(0,0,u(0,0),v(0,0)\big) \right.$$
$$+ \frac{2\omega}{(2-\omega)N(w)} \int_0^{t_2} f_1\big(x,s,u(x,s),v(x,s)\big)ds$$
$$- \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(x,t_1,u(x,t_1),v(x,t_1)\big)$$
$$+ \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(0,0,u(0,0),v(0,0)\big)$$
$$\left. - \frac{2(1-\omega)}{(2-\omega)N(w)} \int_0^{t_1} f_1\big(x,s,u(x,s),v(x,s)\big)ds \right|$$

$$\leq \frac{2(1-\omega)}{(2-\omega)N(w)} f_1\big(x,t_2,u(x,t_2)\big) - f_1\big(x,t_1,u(x,t_1),v(x,t_1)\big) \Big| + \frac{2\omega}{(2-\omega)N(w)} \int_{t_1}^{t_2} \Big|$$

$$f_1\big(x,s,u(x,s),v(x,s)\big)\Big|ds$$

$$\leq \frac{2\omega}{(2-\omega)N(w)} f_1\big(x,t_2,u(x,t_2)\big) - f_1\big(x,t_1,u(x,t_1),v(x,t_1)\big)\Big|$$

$$+\frac{2\omega L_1}{(2-\omega)N(w)}(t_2-t_1).$$

This implies that | $(T_1 v$)(x,t_2) - $(T_1 v$)(x,t_1)|→ 0, whenever (x,t_2) →(x,t_1).By utilizing the Arzela-Ascoli theorem, T_1 is completely continuous. Similarly, we have [3]:

$$|(T_1u)(x,t_2)-(T_1u)(x,t_1)| =$$

$$\Bigg|\frac{2(1-\beta)}{(2-\beta)N(\beta)} f_2\big(x,t_2,u(x,t_2),v(x,t_2)\big) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} f_2\big(0,0,u(0,0),v(0,0)\big)$$

$$+\frac{2\beta}{(2-\beta)N(\beta)}\int_0^{t_2} f_2\big(x,s,u(x,s),v(x,s)\big)ds$$

$$-\frac{2(1-\beta)}{(2-\beta)N(\beta)} f_{21}\big(x,t_1,u(x,t_1),v(x,t_1)\big)$$

$$+\frac{2(1-\beta)}{(2-\beta)N(w)} f_2\big(0,0,u(0,0),v(0,0)\big)$$

$$-\frac{2(1-\beta)}{(2-\beta)N(\beta)}\int_0^{t_1} f_{21}\big(x,s,u(x,s),v(x,s)\big)ds\Bigg|$$

$$\leq \frac{2(1-\beta)}{(2-\beta)N(\beta)} f_2\big(x,t_2,u(x,t_2)\big) - f_2\big(x,t_1,u(x,t_1),v(x,t_1)\big)\Big| + \frac{2\beta}{(2-\beta)N(\beta)}\int_{t_1}^{t_2}\Big|$$

$$f_1\big(x,s,u(x,s),v(x,s)\big)\Big|ds$$

$$\leq \frac{2\beta}{(2-\beta)N(\beta)} f_2\big(x,t_2,u(x,t_2)\big) - f_2\big(x,t_1,u(x,t_1),v(x,t_1)\big)\Big|$$

$$+\frac{2\beta L_1}{(2-\beta)N(\beta)}(t_2-t_1).$$

Again, by utilizing the Arzela-Ascoli theorem we observe that T_2 is completely continuous [3]. Therefore, when get ‖T(u ,v)(x,t_2)- T(u ,v)(x,t_2)‖$_{X\times X}$ → 0 whenever (x ,t_2) tends to (x ,t_1). Thus, T is completely continuous. We prove that [3]

Ω={(u,v)∈ $X \times X$:(u,v)=λT(u,v) for some λ∈[0,1]} is bounded. Let (u ,v) be an arbitrary element of Ω choose λ∈[0,1] such that (u ,v)= λ T(u ,v). Hence v(x ,t)= λ(T_1v)(x,t) and u(x,t)= λ(T_2u)(x ,t)for all (x ,t) ∈ [0,1] ×[0,1]. Since [3]:

$\frac{1}{\lambda}$|v(x ,t)=| (T_1v)(x,t)|≤ L_1N_1, we get |v(x, t) |≤ $\lambda L_1 N_1$ and so ‖v(x,t)$\|_x \leq \lambda L_1 N_1$. Similarly, we can show that ‖u(x ,t)$\|_x \leq \lambda L_2 N_2$.thus ‖(u ,v)$\|_{X\times X} \leq \lambda[L_1N_1 + L_2N_2.]$ and so Ω is a bounded set. Now, by using theorem (1.1) in [3],we get that T has affixed point which is a solution for the coupled system of the time-fractional differential equations.

Then, we present the existence of solution for the coupled system of time-fractional differential inclusions [3]:

$$\begin{cases} ({}^{CF}D_t^{\omega}u)(x,t) = F_1\big(x,t,u(x,t),v(x,t)\big), \\ ({}^{CF}D_t^{\beta}v)(x,t) = F_2\big(x,t,u(x,t),v(x,t)\big), \end{cases}$$

with the initial value conditions u(0,0)=0, and v(0,0)=0, where F_1 ,

F_2: [0, 1] × [0,1] ×R×R→P(R), are some multivalued maps [3]:

Definition.5.1. [3] We say that (u_1,u_2)∈ C([0,1] × [0,1], X) × C([0,1] × [0,1], X),is a solution for the system of the time-fractional differential inclusions whenever satisfies the initial value conditions and there exists (w_1 , w_2) ∈ L^1([0,1] × [0,1]) × L^1([0,1] × [0,1]), such thatw$_i$ (x ,t) ∈ F_i(x ,t ,u(x ,t),v(x ,t)) for almost all (x ,t) ∈ [0,1] × [0,1] and i=1,2 and also [3]:

$$\begin{aligned} u_i(x,t) &= \frac{2(\omega-1)}{(2-\omega)N(w)} w_i\big(x,t,u(x,t),v(x,t)\big) \\ &\quad - \frac{2(\omega-1)}{(2-\omega)N(w)} w_i\big(0,0,u(0,0),v(0,0)\big) \\ &\quad + \frac{2\omega}{(2-\omega)N(w)} \int_0^t W_i\big(x,s,u(x,s)\big)ds, \end{aligned}$$

for all $(x, t) \in [0,1] \times [0,1]$ and $1 = 1,2$.

Theorem 5.4 [3]:

Let $F_1, F_2 : [0, 1] \times [0,1] \times R \times R \to P_{cp,cv}(R)$are L^1 − Caratheodory multi- functions. Suppose that there exist a non-decreasing bounded continuous map ψ:[0,∞) →(0, ∞) and a continuous function P:$[0,1] \times [0,1]$ →(0, ∞)such that $\|F_i(x,t,u_i(x,t),u_i'(x,t))\| \leq p(x,t)\psi(\|u_i\|)$for all (x ,t)$\in [0,1] \times [0,1], u_i, u_i' \in X$ for i=1,2. Then, (5.34) and (5.35) possess at least one solution [3]:

Proof [3]: Define the operator N:X×X→ $2^{X\times X}$by N(u_1,u_2)=($N_1(u_1, u_2), N_2(u_1, u_2)$),

where $N_1(u_1, u_2)$={$h_1 \in$ X×X: there exists $v_1 \in S_{F_1,u_1}$such that h_1(x ,t)= v_1(x ,t) for all (x ,t) $\in [0,1] \times [0,1]$},

$N_2(u_1, u_2)$={$h_2 \in$ X×X: there exists $v_2 \in S_{F_2,u_2}$such that h_2(x,t)= v_2(x ,t) for all (x ,t) $\in [0,1] \times [0,1]$},

$$h_1(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_1(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_1(0,0) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t v_1(x,s)ds,$$

and [3]:

$$h_2(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2(0,0) + \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t v_2(x,s)ds,$$

By Lemma 5.1, it is clear that each fixed point of the operator N is a solution for the system of time-fractional differential inclusions (5.34).First, we proved that the multifunction N is convex- valued. Let(u_1,u_2)$\in$ X×X, (h_1, h_2),(h_1', h_2')$\in$ N(u_1,u_2). Choose v_i $v_i' \in S_{F_i(u_1,u_2)}$ such that [3]:

$$h_i(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_i(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_i(0,0) + \frac{2\omega}{(2-\omega)N(w)} \int_0^t v_i(x,s)ds,$$

$$h_i'(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_i'(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_i'(0,0)$$
$$+ \frac{2\omega}{(2-\omega)N(w)} \int_0^t v_i'(x,s)ds.$$

For almost all (x ,t)∈ $[0,1]\times[0,1]$ and i=1,2.Let $0\leq \lambda \leq 1$ be given. Then, we have $[\lambda h_i+(1-\lambda)h_i'](x,t) = \frac{2(1-\omega)}{(2-\omega)N(\omega)}[\lambda v_i(x,t)+(1-\lambda) v_i'(x,t)] - \frac{2(1-\omega)}{(2-\omega)N(\omega)}$ $[\lambda v_i(0,0)+(1-\lambda) v_i'(0,0)]+\frac{2\omega}{(2-\omega)N(\omega)}\int_0^t[\lambda v_i(x,s) + (1-\lambda)\ v_i'(x,s)]ds$,

for i=1,2. Since the operator F_i has convex values, $S_{F_i(u_i)}$ is a convex set and $[\lambda h_i+(1-\lambda)h_i'] \in N_i(u_1,u_2)$, [3] for i=1,2.This implies that the operator N has convex values. Now, we prove that N maps bounded sets of X into bounded sets. Let r> 0,B_r={$(u_1,u_2) \in XxX :\|(u_1,u_2)\| \leq r$} be a bounded of X x X,$(h_1,h_2) \in N(u_1,u_2)$, and $(u_1,u_2) \in B_r$ .Then, there exists $(v_1,v_2) \in S_{F_1(u_1)}\, xS_{F_2(u_2)}$ such that [3]:

$$h_1(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_1(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_1(0,0)$$
$$+ \frac{2\omega}{(2-\omega)N(w)} \int_0^t v_1(x,s)ds,$$

and

$$h_2(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2(0,0) + \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t v_2(x,s)ds.$$

for almost all (x ,t) ∈[0,1]x[0,1]. If $\|p\|_\infty = sup_{(x,t)\in[0,1]x[0,1]}|p(x,t)|$, then we obtain [3]:

$$|h_1(x,t)| = \left|\frac{2(\omega-1)}{(2-\omega)N(w)} v_1(0,0) + \frac{2\omega}{(2-\omega)N(w)} v_1(0,0)\right.$$
$$\left. + \frac{2(\omega-1)}{(2-\omega)N(w)} \int_0^t v_1(x,s)ds\right|$$

$$\leq \frac{2(\omega-1)}{(2-\omega)N(w)} |v_1(x,t)| + \frac{2(\omega-1)}{(2-\omega)N(w)} |v_1(0,0)| + \frac{2(\omega-1)}{(2-\omega)N(w)} \int_0^t |v_1(x,s)ds|$$

$$\leq p(x,t)\psi(\|u_1\|)\left\{\frac{2(\omega-1)}{(2-\omega)N(w)}+\frac{2(\omega-1)}{(2-\omega)N(w)}+\frac{2\omega}{(2-\omega)N(w)}t\right\},$$

$$\leq \|p\|_\infty\psi(\|u_1\|)\left\{\frac{4(1-\omega)}{(2-\omega)N(w)}+\frac{2\omega}{(2-\omega)N(w)}\right\},$$

$$\leq \|p\|_\infty\psi(\|u_1\|)\left\{\frac{4-2\omega}{(2-\omega)N(w)}\right\}=\|p\|_\infty\psi(\|u_1\|)N_1,$$

where the constant $N_1$is defined by (5.40). This implies that $\|h_1\| \leq \|p\|_\infty\ \psi(\|u_1\|)N_1$ .Similarly, we get $\|h_2\| \leq \|p\|_\infty\ \psi(\|u_2\|)N_2$, where the constant N_2 is defined by (5.40). Thus,$\|h_1, h_2\| \leq \|p\|_\infty\ \psi(\|(u_1, u_2)\|)(N_1 + N_2)$. Now, we prove that N maps bounded sets into equi- continuous subsets of $X \times X$. Let $(u_1, u_2) \in B_r$ and [3]:

$(x,t_1),(x,t_2) \in[0,1]x[0,1]$, with $t_1 \leq\ t_2$. Then, we have [3]:

$$|h_1(x,t_2)-h_1(x,t_1)|=$$

$$\left|\frac{2(\omega-1)}{(2-\omega)N(w)}v_1(x,t_2)-\frac{2(\omega-1)}{(2-\omega)N(w)}v_1(0,0)\right.$$
$$+\frac{2\omega}{(2-\omega)N(w)}\int_0^t v_1(x,s)ds-\frac{2(\omega-1)}{(2-\omega)N(w)}v_1(x,t_1)$$
$$\left.+\frac{2(\omega-1)}{(2-\omega)N(w)}v_1(0,0)-\frac{2\omega}{(2-\omega)N(w)}\int_0^t v_1(x,s)ds\right|$$

$$\leq\frac{2(\omega-1)}{(2-\omega)N(w)}|v_1(x,t_2)-v_1(x,t_1)|+\frac{2\omega}{(2-\omega)N(w)}\int_{t_1}^{t_2}\ |v_1(x,s)|ds$$

$$\leq\frac{2(\omega-1)}{(2-\omega)N(w)}|v_1(x,t_2)-v_1(x,t_1)|+\frac{2\omega\|p\|_\infty\psi(\|u_1\|)}{(2-\omega)N(w)}(t_2-t_1).$$

By using a similar way, we obtain [3]:

$$|h_2(x,t_2)-h_2(x,t_1)|$$
$$\leq\frac{2(1-\beta)}{(2-\beta)N(\beta)}|v_2(x,t_2)-v_2(x,t_1)|+\frac{2\beta\|p\|_\infty\psi(\|u_1\|)}{(2-\beta)N(\beta)}(t_2-t_1).$$

Hence,$|h_i(x,t_2)- h_i(x,t_1)|\to 0$, as $(x,t_2) \to (x,t_1)$. By utilizing the Arzela-Ascoli theorem we get that N is completely continuous. Here, we prove that N is upper semi-continuous. By using Lemma 1.2, N is upper semi-continuous whenever it has a closed graph. Since N is completely continuous, we must show that N has a closed graph. Let $\{(u_1{}^n, u_2{}^n)$ be a sequence in $X\times X$ with $(u_1{}^n, u_2{}^n) \to (u_1{}^0, u_2{}^0)$ and $(h_1{}^n, h_2{}^n) \in N(u_1{}^n, u_2{}^n)$ with $(h_1{}^n, h_2{}^n) \to (h_1{}^0, h_2{}^0)$. We show that $\left(h_1{}^0, h_2{}^0\right) \in N(u_1{}^0, u_2{}^0)$.For each $(h_1{}^n, h_2{}^n) \in N(u_1{}^n, u_2{}^n)$, we can choose $(v_1{}^n, v_2{}^n) \in S_{F_{1(u_1{}^n)}} x S_{F_{2(u_2{}^n)}}$ such that [3]:

$$h_1^n(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_1^n(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_1^n(0,0) + \frac{2(\omega-1)}{(2-\omega)N(w)} \int_0^t v_1^n(x,s)ds,$$

and

$$h_2^n(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_1^n(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2^n(0,0) + \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t v_{12}^n(x,s)ds$$

for all [3]:

$(x,t) \in [0,1]x[0,1]$. It is sufficient to show that exists $(v_1{}^0, v_2{}^0) \in S_{F_{1(u_1{}^0)}} x S_{F_{2(u_2{}^0)}}$ such that [3]

$$h_1^0(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_1^0(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_1^0(0,0) + \frac{2(\omega-1)}{(2-\omega)N(w)} \int_0^t v_1^0(x,s)ds,$$

and

$$h_2^0(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2^0(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2^0(0,0) + \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t v_2^0(x,s)ds,$$

for all (x ,t) ∈ [0,1]x[0,1]. Now, we consider the linear operators $\emptyset_1$,$\emptyset_2$:L^1([0,1]x[0,1],X) →C([0,1]x[0,1],X) defined by [3]:

$$\emptyset_1(v)(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)}v(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)}v(0,0) + \frac{2(\omega-1)}{(2-\omega)N(w)}\int_0^t v_1^0(x,s)ds,$$

and

$$\emptyset_2(v)(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)}v(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)}v(0,0) + \frac{2\beta}{(2-\beta)N(\beta)}\int_0^t v_1^0(x,s)ds.$$

Note that,

$\|h_a^n(x,t) - h_1^0(x,t)\| =$

$$\left\|\frac{2(\omega-1)}{(2-\omega)N(w)}[v_1^n(x,t) - v_1^0(x,t)] - \frac{2(\omega-1)}{(2-\omega)N(w)}[v_1^n(0,0) - v_1^0(0,0)] + \frac{2\omega}{(2-\omega)N(w)}\int_0^t [v_1^n(x,s) - v_1^0(x,s)]\right\| ds \to 0, and$$

$\|h_2^n(x,t) - h_2^0(x,t)\| =$

$$\left\|\frac{2(1-\beta)}{(2-\beta)N(\beta)}[v_2^n(x,t) - v_2^0(x,t)] - \frac{2(1-\beta)}{(2-\beta)N(\beta)}[v_2^n(0,0) - v_2^0(0,0)] + \frac{2(1-\beta)}{(2-\beta)N(\beta)}\int_0^t [v_2^n(x,s) - v_2^0(x,s)]\right\| ds \to 0.$$

By using the Lemma (1.3)[3], we get $\emptyset_i o S_{F_i}$is a closed graph operator for i=1,2. Also, we get ${h_i}^n$(x ,t) ∈ $\emptyset_i(S_{F_{i(u_i^n)}}$) for all n. Since ${u_i}^n \to {u_i}^0$, we get [3]:

$$h_1^0(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_1^0(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_1^0(0,0) + \frac{2(\omega-1)}{(2-\omega)N(w)} \int_0^t v_1^0(x,s)ds,$$

and

$$h_2^0(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2^0(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2^0(0,0) + \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t v_2^0(x,s)ds,$$

for some $v_i{}^0 \in \emptyset_i(S_{F_{i(u_i{}^0)}})$ (i=1,2). Thus, N has a closed graph. Now, we prove that there is an open set U⊆ X with $(u_1, u_2) \notin N(u_1, u_2)$ for all λ ∈ (0,1) and $(u_1, u_2) \in \partial U$.Let λ ∈ (0,1) and $(u_1, u_2) \in \lambda N(u_1, u_2)$.Then, there exists $v_i \in L^1([0,1]\times[0,1],R)$ with $v_i \in S_{F_{i(u_i)}}$(i=1,2) such that [3]:

$$u_1(x,t) = \frac{2(\omega-1)}{(2-\omega)N(w)} v_1(x,t) - \frac{2(\omega-1)}{(2-\omega)N(w)} v_1(0,0) + \frac{2(\omega-1)}{(2-\omega)N(w)} \int_0^t v_1(x,s)ds,$$

and

$$u_2(x,t) = \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2(x,t) - \frac{2(1-\beta)}{(2-\beta)N(\beta)} v_2(0,0) + \frac{2\beta}{(2-\beta)N(\beta)} \int_0^t v_2(x,s)ds,$$

for all (x ,t) ∈ [0,1]× [0,1]. By using the above computed values, we obtain $\|u_i\| \leq \|p\|_\infty \psi(\|u_i\|) \sum_{i=1}^n N_i$ for i=1,2. This follows that

$$\frac{\|u_i\|}{\|p\|_\infty \psi(\|u_i\|) \sum_{i=1}^n N_i} \leq 1$$

for i=1,2. Choose$M_i > 0$, with $\|u_i\| \neq M_i$ in such a way that [3]:

$$\frac{M_i}{\|p\|_\infty \psi(\|u_i\|) \sum_{i=1}^n N_i} > 1$$

for i=1,2.put U={$(\mathrm{u}_1, \mathrm{u}_2) \in X \times X$: $\|u_1, u_2)\| < \min\{\mathrm{M}_1 M_2\}\}$. We note that the operator N:$\overline{\mathrm{U}} \rightarrow P(x)$ is upper semi- continuous and completely continuous. Also, we showed that there is no ($\mathrm{u}_1, \mathrm{u}_2) \in \partial$U such that$(\mathrm{u}_1, \mathrm{u}_2) \in \lambda N(u_1, u_2)$for some $\lambda \in$(0,1).Hence, with the help of theorem (1.4) [3], we get that N has a fixed point $(\mathrm{u}_1, \mathrm{u}_2) \in \overline{\mathrm{U}}$ which being a solution for the time-fractional differential inclusion (5.34) and (5.35).

CHAPTER 6

CONCLUSION

The fractional calculus is a field of mathematics that studies the integration and differentiation of functions of any order. It turned out that this calculus is a very strong tool that can be used when scientists want to mathematically model physical phenomena happening in our real world. In this master thesis, I reviewed the properties and applications of the fractional derivative introduced by Caputo and Fabrizio in 2015.

In order to understand the properties of the Caputo-Fabrizio derivative I presented there very recent applications of it. I reviewed a chemical model, a fractional falling body problem and systems of nonlinear time – fractional differential equations. From the first application, I saw the abilities of Caputo-Fabrizio to explain better the effect of memory and also the activity of the bacteria within different layers of the medium by which the global activity is taking place. From the second application, I reviewed that we can apply successfully the Caputo-Fabrizio in physics. From third application, I concluded that we can use the Caputo-Fabrizio derivative in solving more efficiently the systems of nonlinear time – fractional differential equations.

I hope that the content of my thesis will motivate the researchers to start vigorously studying the Caputo-Fabrizio derivative together with its huge potential applications in different fields of science and engineering.

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APPENDICES

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Yukarıda bilgileri kayıtlı olan tezimin, bilimsel araştırma hizmetine sunulması amacı ile Yükseköğretim Kurulu Ulusal Tez Merkezi Veri Tabanında arşivlenmesine ve internet üzerinden tam metin erişime açılmasına izin veriyorum.

04.01.2016

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