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Inequalities by Means of Generalized Proportional Fractional Integral Operators with Respect to Another Function

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Abstract: In this article, we define a new fractional technique which is known as generalized proportional fractional (GPF) integral in the sense of another function Ψ . The authors prove several inequalities for newly defined GPF-integral with respect to another function Ψ . Our consequences will give noted outcomes for a suitable variation to the GPF-integral in the sense of another function Ψ and the proportionality index ζ . Furthermore, we present the application of the novel operator with several integral inequalities. A few new properties are exhibited, and the numerical approximation of these new operators is introduced with certain utilities to real-world problems.

Keywords: Grüss inequality; generalized proportional fractional integral with respect to another function Ψ ; integral inequalities

1. Introduction

A revolution via the discipline of fractional calculus was perceived, whereas the conventional was conjointly brought up because the classical calculus becomes stretched out to the conception of non-local operators. Fractional calculus was popularized and implemented in numerous areas of science, technology, and engineering as a mathematical model. The idea of this new calculus was implemented in many diversified disciplines previously with outstanding achievements completed in the frame of new discoveries and posted academic articles, see [1–4] and the references therein.

Numerous distinguished generalized fractional integral operators consist of the Hadamard operator, Erdélyi–Kober operators, the Saigo operator, the Gaussian hypergeometric operator, the Marichev–Saigo–Maeda fractional integral operator and so on; out of these, the Riemann–Liouville (RL) fractional integral operator was extensively utilized by analysts in the literature as well as applications. For more information, see [5–8]. Almeida [9] expounded Ψ –Caputo derivative in the sense of another function Ψ and Kilbas et al. [10] explored the concept of RL-fractional integrals in the sense of another function Ψ . The attractors with numerical simulations work for varying values and this permits the readers to choose the most appropriate operator for demonstrating the issue under investigation. In addition, as a result of its effortlessness in utilities, analysts have given much consideration to presently determined fractional operators without singular kernels [1,2,11–14]. Later on, numerous articles considering these sorts of fractional operators turned out to be noteworthy.

In [14], Jarad et al. presented the concept of generalized proportional-integral operators which was utilized to characterize some probability density functions and has intriguing applications in statistics (also see [15–17]).

Following this tendency, we introduce another fractional operator in more general form which is known as the generalized proportional fractional operator in the sense of another function Ψ . These kinds of speculations elevate future studies to investigate novel concepts to modify the fractional operators and attain fractional integral inequalities within such generalized fractional operators (see Remark 1 below). It is noted that GPF-integrals are used to manipulate statistical learning and integrodifferential equations, see [18–21] and the references therein.

Inequalities and their utilities assume a crucial job in the literature of applied mathematics. The assortment of distinct kinds of classical variants and their modifications were built up by using the classical fractional operators and their developments in [22–27]. Adopting this propensity, we give a modified version for the most distinguished Grüss type inequality [28] and some other related variants in the frame of the GPF-integral in the sense of another function Ψ that could be increasingly effective and more applicable than the existing ones. More accurately, Grüss inequality can be described as follows:

Definition 1. ([28]) Let two positive functions $\mathcal{U}, \mathcal{S} : [v_1, v_2] \rightarrow \mathcal{R}$ (set of real numbers) such that $m \leq \mathcal{U}(\tau) \leq \mathcal{M}$ and $n \leq \mathcal{S}(\tau) \leq \mathcal{N}$ for all $\tau \in [v_1, v_2]$. Then

$$\left| \frac{1}{v_2 - v_1} \int_{v_1}^{v_2} \mathcal{U}(\tau) \mathcal{S}(\tau) d\tau - \frac{1}{(v_2 - v_1)^2} \int_{v_1}^{v_2} \mathcal{U}(\tau) d\tau \int_{v_1}^{v_2} \mathcal{S}(\tau) d\tau \right| \leq \frac{1}{4} (\mathcal{M} - m)(\mathcal{N} - n), \tag{1}$$

where the constant $\frac{1}{4}$ is sharp and $m, \mathcal{M}, n, \mathcal{N} \in \mathcal{R}$.

The aforementioned inequality sincerely associates the integral of the product two functions with the product of their integrals. Inequality (1) is a tremendous mechanism for investigating numerous scientific areas of research comprising engineering, fluid dynamics, bio-sciences, chaos, meteorology, vibration analysis, biochemistry, aerodynamics, and many more. There was a constant development of enthusiasm for such an area of research so as to address the issues of different utilizations of these variants, see [29–36].

The principal purpose of this article is to derive novel integral inequalities including a Grüss type inequality and several other related variants via GPF in the sense of another function Ψ , by using Young’s, weighted arithmetic and geometric mean inequalities. Interestingly, the special cases of presented results are generalized RL-fractional integral and RL-fractional integral inequalities. Therefore, it is important to summarize the study of fractional integrals.

2. Prelude

In this segment, we give some significant ideas from fractional calculus utilized in our consequent discourse. The fundamental specifics are presented in the monograph by Kilbas et al. [10]. Throughout the paper, for the results concerning [28], it is assumed that all functions are integrable in the Riemann sense.

Definition 2. ([31,37]) A function $\mathcal{U}(\xi)$ is said to be in $L_{q,r}[0, \infty]$ space if

$$L_{q,r}[0, \infty) = \left\{ \mathcal{U} : \|\mathcal{U}\|_{L_{q,r}[0, \infty)} = \left(\int_{v_1}^{v_2} |\mathcal{U}(\xi)|^q \xi^r d\xi \right)^{\frac{1}{q}} < \infty, 1 \leq q < \infty, r \geq 0 \right\}.$$

For $r = 0$,

$$L_q[0, \infty) = \left\{ \mathcal{U} : \|\mathcal{U}\|_{L_q[0, \infty)} = \left(\int_{v_1}^{v_2} |\mathcal{U}(\xi)|^q d\xi \right)^{\frac{1}{q}} < \infty, 1 \leq q < \infty \right\}.$$

Definition 3. ([32]) Let $\mathcal{U} \in L_1[0, \infty)$ and Ψ be an increasing and positive monotone function on $[0, \infty)$ and also derivative Ψ' is continuous on $[0, \infty)$ and $\Psi(0) = 0$. The space $\chi_{\Psi}^q(0, \infty)$ ($1 \leq q < \infty$) of those real-valued Lebesgue measurable functions \mathcal{U} on $[0, \infty)$ for which

$$\|\mathcal{U}\|_{\chi_{\Psi}^q} = \left(\int_0^{\infty} |\mathcal{U}(\xi)|^p \Psi'(\xi) d\xi \right)^{\frac{1}{p}} < \infty, \quad 1 \leq q < \infty$$

and for the case $q = \infty$

$$\|\mathcal{U}\|_{\chi_{\Psi}^{\infty}} = \text{ess sup}_{0 \leq \xi < \infty} [\Psi'(\xi)\mathcal{U}(\xi)].$$

Specifically, when $\Psi(\tau) = \tau$ ($1 \leq q < \infty$) the space $\chi_{\Psi}^q(0, \infty)$ coincides with the $L_q[0, \infty)$ -space and also if we choose $\Psi(\tau) = \ln \tau$ ($1 \leq q < \infty$) the space $\chi_{\Psi}^q(0, \infty)$ coincides with $L_{q,r}[1, \infty)$ -space.

Now, we present a new fractional operator which is known as the GPF-integral operator of a function in the sense of another function Ψ .

Definition 4. Let $\mathcal{U} \in \chi_{\Psi}^q(0, \infty)$, there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$. Then the left-sided and right-sided GPF-integral operator of a function \mathcal{U} in the sense of another function Ψ of order $\eta > 0$ are stated as:

$$({}^{\Psi} \mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U})(\tau) = \frac{1}{\varsigma^{\eta} \Gamma(\eta)} \int_{v_1}^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau) - \Psi(\xi))]}{(\Psi(\tau) - \Psi(\xi))^{1-\eta}} \mathcal{U}(\xi) \Psi'(\xi) d\xi, \quad v_1 < \tau \tag{2}$$

and

$$({}^{\Psi} \mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U})(\tau) = \frac{1}{\varsigma^{\eta} \Gamma(\eta)} \int_{\tau}^{v_2} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\xi) - \Psi(\tau))]}{(\Psi(\xi) - \Psi(\tau))^{1-\eta}} \mathcal{U}(\xi) \Psi'(\xi) d\xi, \quad \tau < v_2, \tag{3}$$

where the proportionality index $\varsigma \in (0, 1]$, $\eta \in \mathbb{C}, \Re(\eta) > 0$, and $\Gamma(\tau) = \int_0^{\infty} \xi^{\tau-1} e^{-\xi} d\xi$ is the Gamma function.

Remark 1. Several existing fractional operators are just special cases of (2) and (3).

(1) Choosing $\Psi(\tau) = \tau$, in (2) and (3), then we acquire the left and right-sided GPF operator proposed by Jarad et al. [14], stated as follows:

$$(\mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U})(\tau) = \frac{1}{\varsigma^{\eta} \Gamma(\eta)} \int_{v_1}^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\tau - \xi)]}{(\tau - \xi)^{1-\eta}} \mathcal{U}(\xi) d\xi, \quad v_1 < \tau \tag{4}$$

and

$$(\mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U})(\tau) = \frac{1}{\varsigma^{\eta} \Gamma(\eta)} \int_{\tau}^{v_2} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\xi - \tau)]}{(\xi - \tau)^{1-\eta}} \mathcal{U}(\xi) d\xi, \quad \tau < v_2. \tag{5}$$

(2) Choosing $\varsigma = 1$, in (2) and (3), then we acquire the left and right-sided generalized RL-fractional integral operator introduced by Kilbas et al. [10], stated as follows:

$$({}^{\Psi}\mathcal{T}_{v_1}^{\eta}\mathcal{U})(\tau) = \frac{1}{\Gamma(\eta)} \int_{v_1}^{\tau} \frac{\Psi'(\xi)\mathcal{U}(\xi)}{(\Psi(\tau) - \Psi(\xi))^{1-\eta}} d\xi, \quad v_1 < \tau \tag{6}$$

and

$$({}^{\Psi}\mathcal{T}_{v_2}^{\eta}\mathcal{U})(\tau) = \frac{1}{\eta\Gamma(\eta)} \int_{\tau}^{v_2} \frac{\mathcal{U}(\xi)\Psi'(\xi)}{(\Psi(\xi) - \Psi(\tau))^{1-\eta}} d\xi, \quad \tau < v_2. \tag{7}$$

(3) Choosing $\Psi(\tau) = \ln \tau$ in (2) and (3), then we acquire the left and right-sided generalized proportional Hadamard fractional integral operator established by Rahman et al. [23], stated as follows:

$$(\mathcal{T}_{v_1}^{\eta,\varsigma}\mathcal{U})(\tau) = \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_{v_1}^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\ln \frac{\tau}{\xi})] \mathcal{U}(\xi)}{(\ln \frac{\tau}{\xi})^{1-\eta}} \frac{d\xi}{\xi}, \quad v_1 < \tau \tag{8}$$

and

$$(\mathcal{T}_{v_2}^{\eta,\varsigma}\mathcal{U})(\tau) = \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_{\tau}^{v_2} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\ln \frac{\xi}{\tau})] \mathcal{U}(\xi)}{(\ln \frac{\xi}{\tau})^{1-\eta}} \frac{d\xi}{\xi}, \quad \tau < v_2. \tag{9}$$

(4) Choosing $\Psi(\tau) = \ln \tau$ along with $\varsigma = 1$ in (2) and (3), then we acquire the left and right-sided Hadamard fractional integral operator obtained by Kilbas et al. and Smako et al. [7,10], stated as follows:

$$\mathcal{T}_{v_1}^{\eta}\mathcal{U}(\tau) = \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_{v_1}^{\tau} \frac{\mathcal{U}(\xi)}{\xi (\ln \frac{\tau}{\xi})^{1-\eta}} d\xi, \quad v_1 < \tau \tag{10}$$

and

$$\mathcal{T}_{v_2}^{\eta}\mathcal{U}(\tau) = \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_{\tau}^{v_2} \frac{\mathcal{U}(\xi)}{\xi (\ln \frac{\xi}{\tau})^{1-\eta}} d\xi, \quad \tau < v_2. \tag{11}$$

(5) Choosing $\Psi(\tau) = \frac{\tau^{\rho}}{\rho}$ ($\rho > 0$) in (2) and (3), then we acquire the left and right-sided generalized fractional integral operator in the sense of Katugampola obtained by [37], stated as follows:

$$\mathcal{T}_{v_1}^{\eta}\mathcal{U}(\tau) = \frac{1}{\Gamma(\eta)} \int_{v_1}^{\tau} \left(\frac{\tau^{\rho} - \xi^{\rho}}{\rho}\right)^{\eta-1} \mathcal{U}(\xi) \frac{d\xi}{\xi^{1-\rho}}, \quad v_1 < \tau \tag{12}$$

and

$$\mathcal{T}_{v_2}^{\eta}\mathcal{U}(\tau) = \frac{1}{\Gamma(\eta)} \int_{\tau}^{v_2} \left(\frac{\xi^{\rho} - \tau^{\rho}}{\rho}\right)^{\eta-1} \mathcal{U}(\xi) \frac{d\xi}{\xi^{1-\rho}}, \quad \tau < v_2. \tag{13}$$

(6) Choosing $\Psi(\tau) = \tau$ along with $\varsigma = 1$ in (2) and (3), then we get the left and right-sided RL-fractional integral operator stated as follows:

$$\mathcal{T}_{v_1}^\eta \mathcal{U}(\tau) = \frac{1}{\Gamma(\eta)} \int_{v_1}^\tau \frac{\mathcal{U}(\xi)}{(\tau - \xi)^{1-\eta}} d\xi, \quad v_1 < \tau \tag{14}$$

and

$$\mathcal{T}_{v_2}^\eta \mathcal{U}(\tau) = \frac{1}{\Gamma(\eta)} \int_\tau^{v_2} \frac{\mathcal{U}(\xi)}{(\xi - \tau)^{1-\eta}} d\xi, \quad \tau < v_2. \tag{15}$$

Definition 5. Let $\mathcal{U} \in \chi_\Psi^\eta(0, \infty)$ and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$. Then the one-sided GPF-integral operator of a function \mathcal{U} in the sense of another function Ψ of order $\eta > 0$ and proportionality index $\varsigma \in [0, 1]$ is stated as:

$$(\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U})(\tau) = \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau) - \Psi(\xi))]}{(\Psi(\tau) - \Psi(\xi))^{1-\eta}} \mathcal{U}(\xi) \Psi'(\xi) d\xi, \quad \xi > 0. \tag{16}$$

For the suitability of inaugurating our consequences, we prove the following semigroup and linearity property for the newly introduced operator.

Theorem 1. (Semigroup Property) Let $\mathcal{U} : [v_1, v_2] \subseteq [0, \infty) \rightarrow \mathcal{R}$ be a GPF-integral operator in the sense of another function Ψ . Then for $\eta, \delta > 0$ and $\varsigma \in (0, 1]$ we have:

$${}^\Psi \mathcal{T}_{v_2, \tau}^{\eta, \varsigma} ({}^\Psi \mathcal{T}_{v_2, \tau}^{\delta, \varsigma} \mathcal{U})(\tau) = ({}^\Psi \mathcal{T}_{v_2, \tau}^{\eta+\delta, \varsigma} \mathcal{U})(\tau). \tag{17}$$

Proof. Consider

$$\begin{aligned} & {}^\Psi \mathcal{T}_{v_2, \tau}^{\eta, \varsigma} ({}^\Psi \mathcal{T}_{v_2, \tau}^{\delta, \varsigma} \mathcal{U})(\tau) \\ &= \frac{1}{\varsigma^\eta \varsigma^\delta \Gamma(\eta) \Gamma(\delta)} \int_\tau^{v_2} \int_\theta^{v_2} \frac{\exp\left[\frac{\varsigma-1}{\varsigma}(\Psi(\theta) - \Psi(\tau))\right]}{(\Psi(\theta) - \Psi(\tau))^{\eta-1}} \frac{\exp\left[\frac{\varsigma-1}{\varsigma}(\Psi(\xi) - \Psi(\theta))\right]}{(\Psi(\xi) - \Psi(\theta))^{\delta-1}} \\ & \times \mathcal{U}(\xi) \Psi'(\xi) d\xi \Psi'(\theta) d\theta \\ &= \frac{1}{\varsigma^{\eta+\delta} \Gamma(\eta) \Gamma(\delta)} \int_\tau^{v_2} \exp\left[\frac{\varsigma-1}{\varsigma}(\Psi(\xi) - \Psi(\tau))\right] \mathcal{U}(\xi) \int_\tau^\xi (\Psi(\theta) - \Psi(\tau))^{\eta-1} \\ & \times (\Psi(\xi) - \Psi(\theta))^{\delta-1} \Psi'(\theta) d\theta \Psi'(\xi) d\xi. \end{aligned}$$

Now, interchanging the order of integration and changing variables defined by $y = \frac{\Psi(\xi) - \Psi(\theta)}{\Psi(\xi) - \Psi(\tau)}$, in the inner integral

$$\begin{aligned}
 & {}^{\Psi}\mathcal{T}_{v_2, \tau}^{\eta, \varsigma} ({}^{\Psi}\mathcal{T}_{v_2, \tau}^{\delta, \varsigma} \mathcal{U}) (\tau) \\
 &= \frac{1}{\varsigma^{\eta+\delta} \Gamma(\eta) \Gamma(\delta)} \int_{\tau}^{v_2} \exp \left[\frac{\varsigma-1}{\varsigma} (\Psi(\xi) - \Psi(\tau)) \right] \mathcal{U}(\xi) (\Psi(\xi) - \Psi(\tau))^{\eta+\delta-1} \Psi'(\xi) d\xi \\
 &\times \int_0^1 (1-y)^{\eta-1} y^{\delta-1} dy \\
 &= \frac{1}{\varsigma^{\eta+\delta} \Gamma(\eta+\delta)} \int_{\tau}^{v_2} \exp \left[\frac{\varsigma-1}{\varsigma} (\Psi(\xi) - \Psi(\tau)) \right] (\Psi(\xi) - \Psi(\tau))^{\eta+\delta-1} \mathcal{U}(\xi) \Psi'(\xi) d\xi \\
 &= ({}^{\Psi}\mathcal{T}_{v_2, \tau}^{\eta+\delta, \varsigma} \mathcal{U}) (\tau),
 \end{aligned}$$

where $\int_0^1 (1-y)^{\eta-1} y^{\delta-1} dy = \frac{\Gamma(\eta)\Gamma(\delta)}{\Gamma(\eta+\delta)}$ is the well known Euler Beta function.

□

Remark 2. If we choose $\varsigma = 1$ along with $\Psi(\tau) = \tau$, then (17) becomes the result of [7].

Suppose a bounded interval $[v_1, v_2]$, such that $v_1 \geq 0$. The operators ${}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma}$ and ${}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma}$ link the function ${}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U}(\tau)$ and ${}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U}(\tau)$ to each GPF integrable function \mathcal{U} in the sense of another function Ψ on $[v_1, v_2]$. In this manner, these are linear operators, which is demonstrated in the following hypothesis.

Theorem 2. (linearity) The operators ${}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma}$ and ${}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma}$ are linear operators on $L_1[v_1, v_2]$. That is, characterize

$${}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma}, {}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} : L_1[v_1, v_2] \rightarrow L_1[v_1, v_2],$$

then

$$\begin{aligned}
 {}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} (a\mathcal{U}_1 + b\mathcal{U}_2) &= a {}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U}_1 + b {}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U}_2, \\
 {}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} (a\mathcal{U}_1 + b\mathcal{U}_2) &= a {}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U}_1 + b {}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U}_2.
 \end{aligned}$$

For all $\mathcal{U}_1, \mathcal{U}_2 \in L_1[v_1, v_2]$ and $a, b \in \mathcal{R}$.

Proof. The proof is simple, consider

$$\begin{aligned}
 {}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} (a\mathcal{U}_1 + b\mathcal{U}_2) (\tau) &= \frac{1}{\varsigma^{\eta} \Gamma(\eta)} \int_{v_1}^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma} (\Psi(\tau) - \Psi(\xi))]}{(\Psi(\tau) - \Psi(\xi))^{1-\eta}} \Psi'(\xi) (a\mathcal{U}_1 + b\mathcal{U}_2) (\xi) d\xi \\
 &= \frac{a}{\varsigma^{\eta} \Gamma(\eta)} \int_{v_1}^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma} (\Psi(\tau) - \Psi(\xi))]}{(\Psi(\tau) - \Psi(\xi))^{1-\eta}} \Psi'(\xi) \mathcal{U}_1(\xi) d\xi \\
 &\quad + \frac{b}{\varsigma^{\eta} \Gamma(\eta)} \int_{v_1}^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma} (\Psi(\tau) - \Psi(\xi))]}{(\Psi(\tau) - \Psi(\xi))^{1-\eta}} \Psi'(\xi) \mathcal{U}_2(\xi) d\xi \\
 &= {}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U}_1(\tau) + b {}^{\Psi}\mathcal{T}_{v_1}^{\eta, \varsigma} \mathcal{U}_2(\tau).
 \end{aligned}$$

Analogously

$${}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} (a\mathcal{U}_1 + b\mathcal{U}_2) = a {}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U}_1 + b {}^{\Psi}\mathcal{T}_{v_2}^{\eta, \varsigma} \mathcal{U}_2.$$

□

3. Main Results

This section is devoted to establishing generalizations of some classical inequalities by employing GPF integral with respect to another function Ψ defined in (16).

Theorem 3. For $\varsigma \in (0, 1]$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, and let $\mathcal{U} \in \chi_{\Psi}^{\eta} (0, \infty)$. Suppose that there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$. Moreover, one assumes that there exist two integrable functions ϕ_1, ϕ_2 on $[0, \infty)$ such that

$$\phi_1(\tau) \leq \mathcal{U}(\tau) \leq \phi_2(\tau), \quad \forall \tau \in [0, \infty). \tag{18}$$

Then, for $\tau > 0, \eta, \delta > 0$, one has

$$\begin{aligned} & {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}(\tau) + {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \phi_1(\tau) \\ & \geq {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \phi_1(\tau) + {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}(\tau). \end{aligned} \tag{19}$$

Proof. From (18), for all $\theta \geq 0, \lambda \geq 0$, one has

$$(\phi_2(\theta) - \mathcal{U}(\theta)) (\mathcal{U}(\lambda) - \phi_1(\lambda)) \geq 0. \tag{20}$$

Therefore,

$$\phi_2(\theta)\mathcal{U}(\lambda) + \phi_1(\lambda)\mathcal{U}(\theta) \geq \phi_1(\lambda)\phi_2(\theta) + \mathcal{U}(\theta)\mathcal{U}(\lambda). \tag{21}$$

Taking product on both sides of (21) by $\frac{1}{\varsigma^{\eta}\Gamma(\eta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}}$ and integrating the estimate with respect to θ from 0 to τ , one obtains

$$\begin{aligned} & \mathcal{U}(\lambda) \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_0^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \phi_2(\theta) d\theta \\ & + \phi_1(\lambda) \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_0^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}(\theta) d\theta \\ & \geq \phi_1(\lambda) \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_0^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \phi_2(\theta) d\theta \\ & + \mathcal{U}(\lambda) \frac{1}{\varsigma^{\eta}\Gamma(\eta)} \int_0^{\tau} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}(\theta) d\theta, \end{aligned} \tag{22}$$

and arrives at

$$\mathcal{U}(\lambda) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) + \phi_1(\lambda) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \geq \phi_1(\lambda) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) + \mathcal{U}(\lambda) {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau). \tag{23}$$

Taking product on both sides of (23) by $\frac{1}{\varsigma^{\delta}\Gamma(\delta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}$ and integrating the estimate with respect to λ from 0 to τ , we have

$$\begin{aligned}
 & {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau)\frac{1}{\zeta^{\delta}\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\mathcal{U}(\lambda)d\lambda \\
 & + {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau)\frac{1}{\zeta^{\delta}\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\phi_1(\lambda)d\lambda \\
 \geq & {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau)\frac{1}{\zeta^{\delta}\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\phi_1(\lambda)d\lambda \\
 & + {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau)\frac{1}{\zeta^{\delta}\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\mathcal{U}(\lambda)d\lambda.
 \end{aligned} \tag{24}$$

Hence, we deduce inequality (20) as required. This concludes the proof. \square

Some special cases can be derived immediately from Theorem 3.

(I) Choosing $\Psi(\tau) = \tau$, then we attain a new result for GPF-integrals.

Corollary 1. For $\zeta \in (0, 1]$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, and let $\mathcal{U} \in L_1[0, \infty)$. Assume that there exist two integrable functions ϕ_1, ϕ_2 on $[0, \infty)$ such that

$$\phi_1(\tau) \leq \mathcal{U}(\tau) \leq \phi_2(\tau), \quad \forall \tau \in [0, \infty).$$

Then, for $\tau > 0, \eta, \delta > 0$, we get

$$\begin{aligned}
 & \mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau)\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\mathcal{U}(\tau) + \mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau)\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\phi_1(\tau) \\
 & \geq \mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau)\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\phi_1(\tau) + \mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau)\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\mathcal{U}(\tau).
 \end{aligned}$$

(II) Letting $\zeta = 1$, then we attain a result for generalized RL-fractional integrals.

Corollary 2. Let $\mathcal{U} \in \chi_{\Psi}^{\eta}(0, \infty)$, and there is an increasing and positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$. Moreover, one assumes that there exist two integrable functions ϕ_1, ϕ_2 on $[0, \infty)$ such that

$$\phi_1(\tau) \leq \mathcal{U}(\tau) \leq \phi_2(\tau), \quad \forall \tau \in [0, \infty).$$

Then, for $\tau > 0, \eta, \delta > 0$, we get

$$\begin{aligned}
 & {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta}\phi_2(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta}\mathcal{U}(\tau) + {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta}\mathcal{U}(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta}\phi_1(\tau) \\
 & \geq {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta}\phi_2(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta}\phi_1(\tau) + {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta}\mathcal{U}(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta}\mathcal{U}(\tau),
 \end{aligned}$$

which is proposed by Kacar et al. [32].

(III) Choosing $\Psi(\tau) = \tau$ along with $\zeta = 1$, then we attain a result for RL-fractional integrals.

Corollary 3. Let $\mathcal{U} \in L_1[0, \infty)$. Assume that there exist two integrable functions ϕ_1, ϕ_2 on $[0, \infty)$ such that

$$\phi_1(\tau) \leq \mathcal{U}(\tau) \leq \phi_2(\tau), \quad \forall \tau \in [0, \infty).$$

Then, for $\tau > 0, \eta, \delta > 0$, we get

$$\begin{aligned}
 & \mathcal{T}_{0^+,\tau}^{\eta}\phi_2(\tau)\mathcal{T}_{0^+,\tau}^{\delta}\mathcal{U}(\tau) + \mathcal{T}_{0^+,\tau}^{\eta}\mathcal{U}(\tau)\mathcal{T}_{0^+,\tau}^{\delta}\phi_1(\tau) \\
 & \geq \mathcal{T}_{0^+,\tau}^{\eta}\phi_2(\tau)\mathcal{T}_{0^+,\tau}^{\delta}\phi_1(\tau) + \mathcal{T}_{0^+,\tau}^{\eta}\mathcal{U}(\tau)\mathcal{T}_{0^+,\tau}^{\delta}\mathcal{U}(\tau),
 \end{aligned}$$

which is proposed by Tariboon et al. [36].

Theorem 4. For $\varsigma \in (0, 1]$, $\eta, \delta \in \mathbb{C}$, $\Re(\eta), \Re(\delta) > 0$, and let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$. Suppose that (18) holds and moreover one assumes that there exist ω_1 and ω_2 integrable functions on $[0, \infty)$ such that

$$\omega_1(\tau) \leq \mathcal{S}(\tau) \leq \omega_2(\tau), \quad \forall \tau \in [0, \infty). \tag{25}$$

Then, for $\tau > 0, \eta, \delta > 0$, the following inequalities hold:

$$\begin{aligned} (M_1) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) \\ & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau), \\ (M_2) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \\ & \geq \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \omega_2(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau), \\ (M_3) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) \\ & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau), \\ (M_4) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) \\ & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau). \end{aligned} \tag{26}$$

Proof. To prove (M_1) , from (18) and (25), we have for $\tau \in [0, \infty)$ that

$$(\phi_2(\theta) - \mathcal{U}(\theta))(\mathcal{S}(\lambda) - \omega_1(\lambda)) \geq 0. \tag{27}$$

Therefore,

$$\phi_2(\theta)\mathcal{S}(\lambda) + \omega_1(\lambda)\mathcal{U}(\theta) \geq \omega_1(\lambda)\phi_2(\theta) + \mathcal{S}(\lambda)\mathcal{U}(\theta). \tag{28}$$

Taking product on both sides of (28) by $\frac{1}{\varsigma^\eta \Gamma(\eta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}}$ and integrating the estimate with respect to θ from 0 to τ , we have

$$\begin{aligned} & \mathcal{S}(\lambda) \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \phi_2(\theta) d\theta \\ & + \omega_1(\lambda) \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}(\theta) d\theta \\ & \geq \omega_1(\lambda) \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \phi_2(\theta) d\theta \\ & + \mathcal{S}(\lambda) \frac{1}{\varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}(\theta) d\theta. \end{aligned} \tag{29}$$

Then we have

$$\begin{aligned} & \mathcal{S}(\lambda) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) + \omega_1(\lambda) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \\ & \geq \omega_1(\lambda) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) + \mathcal{S}(\lambda) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau). \end{aligned} \tag{30}$$

Again, taking product on both sides of (30) by $\frac{1}{\varsigma^\delta \Gamma(\delta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}$ and integrating the estimate with respect to λ from 0 to τ , we have

$$\begin{aligned}
 & {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau)\frac{1}{\zeta^\delta\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\mathcal{S}(\lambda)d\lambda \\
 & + {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau)\frac{1}{\zeta^\delta\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\omega_1(\lambda)d\lambda \\
 & \geq {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau)\frac{1}{\zeta^\delta\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\omega_1(\lambda)d\lambda \\
 & + {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau)\frac{1}{\zeta^\delta\Gamma(\delta)}\int_0^x\frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))]\Psi'(\lambda)}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}\mathcal{S}(\lambda)d\lambda.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 & {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\mathcal{S}(\tau)+{}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}{}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\omega_1(\tau) \\
 & \geq {}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\phi_2(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\omega_1(\tau)+{}^{\Psi}\mathcal{T}_{0^+,\tau}^{\eta,\zeta}\mathcal{U}(\tau){}^{\Psi}\mathcal{T}_{0^+,\tau}^{\delta,\zeta}\mathcal{S}(\tau),
 \end{aligned}$$

where we get the desired inequality (M_1).

We can prove other inequalities by taking into account the following identities, respectively:

$$(M_2) \quad (\omega_2(\theta) - \mathcal{S}(\theta))(\mathcal{U}(\lambda) - \phi_1(\lambda)) \geq 0,$$

$$(M_3) \quad (\phi_2(\theta) - \mathcal{U}(\theta))(\mathcal{S}(\lambda) - \omega_2(\lambda)) \leq 0,$$

$$(M_4) \quad (\phi_1(\theta) - \mathcal{U}(\theta))(\mathcal{S}(\lambda) - \omega_1(\lambda)) \leq 0.$$

□

As a special case of Theorem 4, we have the following corollaries.

(I) Letting $\Psi(\tau) = \tau$, then we get a new result for GPF-integrals:

Corollary 4. For $\zeta \in (0, 1]$, $\eta, \delta \in \mathbb{C}$, $\Re(\eta), \Re(\delta) > 0$, and let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$. Suppose that (18) holds and moreover one assumes that there exist ω_1 and ω_2 integrable functions on $[0, \infty)$ such that

$$\omega_1(\tau) \leq \mathcal{S}(\tau) \leq \omega_2(\tau), \quad \forall \tau \in [0, \infty). \tag{31}$$

Then, for $\tau > 0, \eta, \delta > 0$, the following inequalities hold:

$$\begin{aligned}
 (M_5) \quad & \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) + \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau), \\
 (M_6) \quad & \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \phi_1(\tau) \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau) \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \phi_1(\tau) \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \omega_2(\tau) + \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau), \\
 (M_7) \quad & \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau) \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) + \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_2(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_2(\tau), \\
 (M_8) \quad & \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_1(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) + \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \phi_1(\tau) \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \omega_1(\tau) \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau).
 \end{aligned}$$

(II) Letting $\varsigma = 1$, then we attain a result for generalized RL-fractional integral.

Corollary 5. *let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative on $[0, \infty)$ with $\Psi(0) = 0$. Suppose that (18) holds and moreover one assumes that there exist ω_1 and ω_2 integrable functions on $[0, \infty)$ such that*

$$\omega_1(\tau) \leq \mathcal{S}(\tau) \leq \omega_2(\tau), \quad \forall \tau \in [0, \infty). \tag{32}$$

Then, for $\tau > 0, \eta, \delta > 0$, the following inequalities hold:

$$\begin{aligned}
 (M_5) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\eta} \phi_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_1(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta} \phi_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_1(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau), \\
 (M_6) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\delta} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\delta} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta} \omega_2(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{S}(\tau), \\
 (M_7) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta} \phi_2(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta} \phi_2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_2(\tau), \\
 (M_8) \quad & \Psi \mathcal{T}_{0^+, \tau}^{\eta} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_1(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta} \phi_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) + \Psi \mathcal{T}_{0^+, \tau}^{\delta} \omega_1(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau),
 \end{aligned}$$

which is proposed by Kacar et al. [32].

(III) Letting $\Psi(\tau) = \tau$ along with $\varsigma = 1$, then we attain a result for RL-fractional integrals:

Corollary 6. *let $\mathcal{U}, \mathcal{S} \in L_1[0, \infty)$. Suppose that (18) holds and moreover one assumes that there exist ω_1 and ω_2 integrable functions on $[0, \infty)$ such that*

$$\omega_1(\tau) \leq \mathcal{S}(\tau) \leq \omega_2(\tau), \quad \forall \tau \in [0, \infty). \tag{33}$$

Then, for $\tau > 0, \eta, \delta > 0$, the following inequalities hold:

$$\begin{aligned}
 (M_5) \quad & \mathcal{T}_{0^+, \tau}^\eta \phi_2(\tau) \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^\delta \omega_1(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^\eta \phi_2(\tau) \mathcal{T}_{0^+, \tau}^\delta \omega_1(\tau) + \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}(\tau), \\
 (M_6) \quad & \mathcal{T}_{0^+, \tau}^\delta \phi_1(\tau) \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^\delta \omega_2(\tau) \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^\delta \phi_1(\tau) \mathcal{T}_{0^+, \tau}^\eta \omega_2(\tau) + \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}(\tau), \\
 (M_7) \quad & \mathcal{T}_{0^+, \tau}^\delta \omega_2(\tau) \mathcal{T}_{0^+, \tau}^\eta \phi_2(\tau) + \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^\eta \phi_2(\tau) \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^\delta \omega_2(\tau), \\
 (M_8) \quad & \mathcal{T}_{0^+, \tau}^\eta \phi_1(\tau) \mathcal{T}_{0^+, \tau}^\delta \omega_1(\tau) + \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{T}_{0^+, \tau}^\delta \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}(\tau) \\
 & \geq \mathcal{T}_{0^+, \tau}^\eta \phi_1(\tau) \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}(\tau) + \mathcal{T}_{0^+, \tau}^\delta \omega_1(\tau) \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau),
 \end{aligned}$$

which is proposed by Tariboon et al. [36].

4. Some other Fractional Integral Inequalities for GPF-Integral in the Sense of Another Function

Theorem 5. For $\varsigma \in (0, 1]$, $\eta, \delta \in \mathbb{C}$, $\Re(\eta), \Re(\delta) > 0$, and let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$, $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $\tau > 0$, one has

$$\begin{aligned}
 (M_9) \quad & \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^q(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^p(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{S}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) \mathcal{U}(\tau), \\
 (M_{10}) \quad & \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^p(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^q(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^{q-1}(\tau) \mathcal{U}^{p-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{S}(\tau), \\
 (M_{11}) \quad & \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^p(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^q(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^{\frac{2}{q}}(\tau) \mathcal{S}^{\frac{2}{p}}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \mathcal{S}(\tau), \\
 (M_{12}) \quad & \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^2(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^2(\tau) \\
 & \geq \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^{p-1}(\tau) \mathcal{S}^{q-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^{\frac{2}{p}}(\tau) \mathcal{S}^{\frac{2}{q}}(\tau).
 \end{aligned} \tag{34}$$

Proof. According to the well-known Young’s inequality [38]:

$$\frac{1}{p} a^p + \frac{1}{q} b^q \geq ab, \quad \forall a, b \geq 0, p, q > 0, \frac{1}{p} + \frac{1}{q} = 1, \tag{35}$$

setting $a = \mathcal{U}(\theta) \mathcal{S}(\lambda)$ and $b = \mathcal{U}(\lambda) \mathcal{S}(\theta)$, $\theta, \lambda > 0$, we have

$$\frac{1}{p} (\mathcal{U}(\theta) \mathcal{S}(\lambda))^p + \frac{1}{q} (\mathcal{U}(\lambda) \mathcal{S}(\theta))^q \geq (\mathcal{U}(\theta) \mathcal{S}(\lambda)) (\mathcal{U}(\lambda) \mathcal{S}(\theta)). \tag{36}$$

Taking product on both sides of (36) by $\frac{1}{\varsigma^\eta \Gamma(\eta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}}$ and integrating the estimate with respect to θ from 0 to τ , we have

$$\begin{aligned}
 & \frac{\mathcal{S}^p(\lambda)}{p \varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}^p(\theta) d\theta \\
 & + \frac{\mathcal{U}^q(\lambda)}{q \varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{S}^q(\theta) d\theta \\
 & \geq \frac{\mathcal{S}(\lambda) \mathcal{U}(\lambda)}{\varsigma^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}(\theta) \mathcal{S}(\theta) d\theta,
 \end{aligned} \tag{37}$$

we get

$$\begin{aligned} & \frac{\mathcal{S}^p(\lambda)}{p} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^p(\tau) + \frac{\mathcal{U}^q(\lambda)}{q} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^q(\tau) \\ & \geq \mathcal{S}(\lambda) \mathcal{U}(\lambda) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}(\tau) \mathcal{S}(\tau). \end{aligned} \tag{38}$$

Taking product on both sides of (36) by $\frac{1}{\zeta^\delta \Gamma(\delta)} \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))\Psi'(\lambda)]}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}}$, and integrating the estimate with respect to λ from 0 to τ , we have

$$\begin{aligned} & \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^p(\tau) \frac{1}{\zeta^\delta \Gamma(\delta)} \int_0^\tau \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))\Psi'(\lambda)]}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}} \mathcal{S}^p(\lambda) d\lambda \\ & + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^q(\tau) \frac{1}{\zeta^\delta \Gamma(\delta)} \int_0^\tau \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))\Psi'(\lambda)]}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}} \mathcal{U}^q(\lambda) d\lambda \\ & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}(\tau) \mathcal{S}(\tau) \int_0^\tau \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\lambda))\Psi'(\lambda)]}{(\Psi(\tau)-\Psi(\lambda))^{1-\delta}} \mathcal{S}(\lambda) \mathcal{U}(\lambda) d\lambda, \end{aligned} \tag{39}$$

consequently, we get

$$\begin{aligned} & \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \zeta} \mathcal{S}^p(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \zeta} \mathcal{U}^q(\tau) \\ & \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}(\tau) \mathcal{S}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \zeta} \mathcal{S}(\tau) \mathcal{U}(\tau), \end{aligned} \tag{40}$$

which implies (M_9) . The remaining variants can be derived by adpoting the same technique and accompanying the selection of parameters in Young inequality.

$$\begin{aligned} (M_{10}) \quad & a = \frac{\mathcal{U}(\theta)}{\mathcal{U}(\lambda)}, \quad b = \frac{\mathcal{S}(\theta)}{\mathcal{S}(\lambda)}, \quad \mathcal{U}(\lambda), \mathcal{S}(\lambda) \neq 0, \\ (M_{11}) \quad & a = \mathcal{U}(\theta) \mathcal{S}^{\frac{2}{p}}(\lambda), \quad b = \mathcal{U}^{\frac{2}{q}}(\lambda) \mathcal{S}(\theta), \\ (M_{12}) \quad & a = \mathcal{U}^{\frac{2}{p}}(\theta) \mathcal{U}(\lambda), \quad b = \mathcal{S}^{\frac{2}{q}}(\theta) \mathcal{S}(\lambda), \quad \mathcal{U}(\lambda), \mathcal{S}(\lambda) \neq 0. \end{aligned}$$

Repeating the foregoing argument, we obtain $(M_{10}) - (M_{12})$. \square

(I) Letting $\zeta = 1$, then we attain a result for generalized RL-fractional integral.

Corollary 7. Let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$, $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then, for $\tau > 0, \eta, \delta \in \mathbb{C}$, $\mathcal{R}(\eta), \mathcal{R}(\delta) > 0$, we get

$$(M_{13}) \quad \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}^p(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}^q(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{S}(\tau) \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}(\tau) \mathcal{U}(\tau),$$

$$(M_{14}) \quad \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}^p(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}^q(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}^{q-1}(\tau) \mathcal{U}^{p-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{S}(\tau),$$

$$(M_{15}) \quad \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}^2(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}^p(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}^2(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}^q(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}^{\frac{2}{q}}(\tau) \mathcal{S}^{\frac{2}{p}}(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}(\tau) \mathcal{S}(\tau),$$

$$(M_{16}) \quad \frac{1}{p} \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}^2(\tau) + \frac{1}{q} \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}^p(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{S}^2(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^\delta \mathcal{U}^{p-1}(\tau) \mathcal{S}^{q-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^\eta \mathcal{U}^{\frac{2}{p}}(\tau) \mathcal{S}^{\frac{2}{q}}(\tau).$$

Theorem 6. For $\varsigma \in (0, 1]$, $\eta, \delta \in \mathbb{C}$, $\Re(\eta), \Re(\delta) > 0$, and let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$, and $p, q > 0$ satisfying $p + q = 1$. Then, for $\tau > 0$, one has

$$(M_{17}) \quad p \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + q \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} (\mathcal{U}^p(\tau) \mathcal{S}^q(\tau)) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} (\mathcal{U}^q(\tau) \mathcal{S}^p(\tau)),$$

$$(M_{18}) \quad p \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^{p-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} (\mathcal{U}(\tau) \mathcal{S}^q(\tau)) + q \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^{q-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} (\mathcal{U}^q(\tau) \mathcal{S}(\tau)) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^p(\tau),$$

$$(M_{19}) \quad p \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^{\frac{2}{p}}(\tau) + q \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^{\frac{2}{q}}(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^p(\tau) \mathcal{S}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^q(\tau) \mathcal{U}^2(\tau),$$

$$(M_{20}) \quad p \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^{\frac{2}{p}}(\tau) \mathcal{S}^q(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^{p-1}(\tau) + q \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^{q-1}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}^{\frac{2}{q}}(\tau) \mathcal{S}^p(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^2(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}^2(\tau).$$
(41)

Proof. From the well-known weighted AM – GM inequality

$$pa + qb \geq a^p b^q, \quad \forall a, b \geq 0, p, q > 0, p + q = 1, \tag{42}$$

by setting $a = \mathcal{U}(\theta) \mathcal{S}(\lambda)$ and $b = \mathcal{U}(\lambda) \mathcal{S}(\theta)$, $\lambda, \theta > 0$, we have

$$p \mathcal{U}(\theta) \mathcal{S}(\lambda) + q \mathcal{U}(\lambda) \mathcal{S}(\theta) \geq (\mathcal{U}(\theta) \mathcal{S}(\lambda))^p (\mathcal{U}(\lambda) \mathcal{S}(\theta))^q. \tag{43}$$

Taking product on both sides of (43) by $\frac{1}{\varsigma^\eta \Gamma(\eta) \varsigma^\delta \Gamma(\delta)} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\lambda))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \Psi'(\theta) \Psi'(\lambda)$ and integrating the estimate with respect to θ and λ from 0 to τ , respectively, we have

$$\frac{p}{\varsigma^\eta \Gamma(\eta) \varsigma^\delta \Gamma(\delta)} \int_0^\tau \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\lambda))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \mathcal{U}(\theta) \mathcal{S}(\lambda) \Psi'(\theta) \Psi'(\lambda) d\theta d\lambda \\ + \frac{q}{\varsigma^\eta \Gamma(\eta) \varsigma^\delta \Gamma(\delta)} \int_0^\tau \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\lambda))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \mathcal{U}(\lambda) \mathcal{S}(\theta) \Psi'(\theta) \Psi'(\lambda) d\theta d\lambda \\ \geq \frac{1}{\varsigma^\eta \Gamma(\eta) \varsigma^\delta \Gamma(\delta)} \int_0^\tau \int_0^\tau \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \frac{\exp[\frac{\varsigma-1}{\varsigma}(\Psi(\tau)-\Psi(\lambda))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta} (\Psi(\tau)-\Psi(\theta))^{1-\delta}} \\ \times (\mathcal{U}(\theta) \mathcal{S}(\lambda))^p (\mathcal{U}(\lambda) \mathcal{S}(\theta))^q \Psi'(\theta) \Psi'(\lambda) d\lambda d\theta,$$
(44)

we conclude that

$$p \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{S}(\tau) + q \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} \mathcal{U}(\tau) \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}(\tau) \\ \geq \Psi \mathcal{T}_{0^+, \tau}^{\eta, \varsigma} (\mathcal{U}^p(\tau) \mathcal{S}^q(\tau)) \Psi \mathcal{T}_{0^+, \tau}^{\delta, \varsigma} (\mathcal{U}^q(\tau) \mathcal{S}^p(\tau)),$$
(45)

which implies (M₁₇). The remaining inequalities can be proved by adopting the same technique by the accompanying selection of parameters in AM – GM inequality.

$$(M_{18}) \quad a = \frac{\mathcal{U}(\lambda)}{\mathcal{U}(\theta)}, \quad b = \frac{\mathcal{S}(\theta)}{\mathcal{S}(\lambda)}, \quad \mathcal{U}(\theta), \mathcal{S}(\lambda) \neq 0.$$

$$(M_{19}) \quad a = \mathcal{U}(\theta)\mathcal{S}^{\frac{2}{p}}(\lambda), \quad b = \mathcal{U}^{\frac{2}{q}}(\lambda)\mathcal{S}(\theta),$$

$$(M_{20}) \quad a = \frac{\mathcal{U}^{\frac{2}{p}}(\theta)}{\mathcal{S}(\lambda)}, \quad b = \frac{\mathcal{U}^{\frac{2}{q}}(\lambda)}{\mathcal{S}(\theta)}, \quad \mathcal{S}(\theta), \mathcal{S}(\theta) \neq 0.$$

□

(I) Letting $\varsigma = 1$, then we attain a result for generalized RL-fractional integrals:

Corollary 8. Let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$, and $p, q > 0$ satisfying $p + q = 1$. Then, for $\tau > 0, \eta, \delta \in \mathbb{C}, \mathcal{R}(\eta), \mathcal{R}(\delta) > 0$, we get

$$(M_{21}) \quad p \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}(\tau) + q \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{U}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{S}(\eta) \\ \geq \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} (\mathcal{U}^p(\tau)\mathcal{S}^q(\tau)) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} (\mathcal{U}^q(\tau)\mathcal{S}^p(\tau)),$$

$$(M_{22}) \quad p \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}^{p-1}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} (\mathcal{U}(\tau)\mathcal{S}^q(\tau)) + q \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}^{q-1}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} (\mathcal{U}^q(\tau)\mathcal{S}(\tau)) \\ \geq \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{S}^q(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{U}^p(\tau),$$

$$(M_{23}) \quad p \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}^{\frac{2}{p}}(\tau) + q \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{S}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{U}^{\frac{2}{q}}(\tau) \\ \geq \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}^p(\tau)\mathcal{S}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}^q(\tau)\mathcal{U}^2(\tau),$$

$$(M_{24}) \quad p \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}^{\frac{2}{p}}(\tau)\mathcal{S}^q(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}^{p-1}(\tau) + q \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{S}^{q-1}(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{U}^{\frac{2}{q}}(\tau)\mathcal{S}^p(\tau) \\ \geq \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta} \mathcal{U}^2(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\delta} \mathcal{S}^2(\tau).$$

Theorem 7. Let \mathcal{U} and \mathcal{S} be two positive functions on $[0, \infty)$, and there is an increasing, positive monotone function Ψ defined on $[0, \infty)$ having continuous derivative $\Psi'(\tau)$ on $[0, \infty)$ with $\Psi(0) = 0$, and $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$\hbar = \min_{0 \leq \theta \leq \tau} \frac{\mathcal{U}(\theta)}{\mathcal{S}(\theta)} \quad \text{and} \quad \mathcal{H} = \max_{0 \leq \theta \leq \tau} \frac{\mathcal{U}(\theta)}{\mathcal{S}(\theta)}. \tag{46}$$

Then, for $\tau > 0, \eta, \delta > 0$, one has the following inequalities:

$$(M_{25}) \quad 0 \leq \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^2(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^2(\tau) \leq \frac{\hbar + \mathcal{H}}{4\hbar\mathcal{H}} (\, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau)\mathcal{S}(\tau) \,)^2,$$

$$(M_{26}) \quad 0 \leq \sqrt{{}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^2(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^2(\tau)} - (\, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau)\mathcal{S}(\tau) \,) \leq \frac{\sqrt{\mathcal{H}} - \sqrt{\hbar}}{2\sqrt{\hbar\mathcal{H}}} (\, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau)\mathcal{S}(\tau) \,),$$

$$(M_{27}) \quad 0 \leq \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}^2(\tau) \, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{S}^2(\tau) - (\, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau)\mathcal{S}(\tau) \,)^2 \leq \frac{\mathcal{H} - \hbar}{4\hbar\mathcal{H}} (\, {}^{\Psi}\mathcal{T}_{0^+, \tau}^{\eta, \varsigma} \mathcal{U}(\tau)\mathcal{S}(\tau) \,)^2.$$

Proof. From Equation (46) and the inequality

$$\left(\frac{\mathcal{U}(\theta)}{\mathcal{S}(\theta)} - \hbar\right) \left(\mathcal{H} - \frac{\mathcal{U}(\theta)}{\mathcal{S}(\theta)}\right) \mathcal{S}^2(\theta) \geq 0, \quad 0 \leq \theta \leq \tau, \tag{47}$$

then we can write it as,

$$\mathcal{U}^2(\theta) + \hbar\mathcal{H}\mathcal{S}^2(\theta) \leq (\hbar + \mathcal{H})\mathcal{U}(\theta)\mathcal{S}(\theta). \tag{48}$$

Taking product on both sides of (48) by $\frac{1}{\zeta^\eta \Gamma(\eta)} \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\theta))]\Psi'(\theta)}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}}$ and integrating the estimate with respect to θ from 0 to τ , we have

$$\begin{aligned} & \frac{1}{\zeta^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}^2(\theta) \Psi'(\theta) d\theta \\ & + \hbar \mathcal{H} \frac{1}{\zeta^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{S}^2(\theta) \Psi'(\theta) d\theta \\ & \leq (\hbar + \mathcal{H}) \frac{1}{\zeta^\eta \Gamma(\eta)} \int_0^\tau \frac{\exp[\frac{\zeta-1}{\zeta}(\Psi(\tau)-\Psi(\theta))]}{(\Psi(\tau)-\Psi(\theta))^{1-\eta}} \mathcal{U}(\theta) \mathcal{S}(\theta) \Psi'(\theta) d\theta, \end{aligned} \tag{49}$$

which implies that

$${}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^2(\tau) + \hbar \mathcal{H} {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^2(\tau) \leq (\hbar + \mathcal{H}) {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}(\tau) \mathcal{S}(\tau), \tag{50}$$

on the other hand, it follows from $\hbar \mathcal{H} > 0$ and

$$\left(\sqrt{{}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^2(\tau)} - \sqrt{\hbar \mathcal{H} {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^2(\tau)} \right)^2 \geq 0, \tag{51}$$

that

$$2\sqrt{{}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^2(\tau)} \sqrt{\hbar \mathcal{H} {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^2(\tau)} \leq \sqrt{{}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^2(\tau)} + \sqrt{\hbar \mathcal{H} {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^2(\tau)}. \tag{52}$$

Utilizing (50) and (52), we obtain,

$$4\hbar \mathcal{H} {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}^2(\tau) {}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{S}^2(\tau) \leq (\hbar + \mathcal{H})^2 ({}^\Psi \mathcal{T}_{0^+, \tau}^{\eta, \zeta} \mathcal{U}(\tau) \mathcal{S}(\tau)), \tag{53}$$

after simplification, we get (M₂₅). By some transformation in (M₂₅), analogously, we get (M₂₆) and (M₂₇). □

5. Conclusions

In this article, we derived several theorems by newly defined generalized proportional fractional integral operators with respect to another function Ψ having proportionality index ζ . The analogous versions of the Grüss inequality and several other associated variants were derived by employing GPF in the sense of another function Ψ . Moreover, we took a few specific instances of these hypotheses, by utilizing Remark 1. Since the GPF is an association of the diverse kind of operators, we can determine the various types of variants by choosing the qualities pertinent to the limitations and the proportionality index ζ . These results can be applied in convex analysis, optimization, integrodifferential equation, and also different areas of pure and applied sciences. Finally, the GPF in the sense of another function subject to the nonlocal exponential kernel provided the outline for obtaining the results for exponential and normal distribution in statistical theory. Note that the outcomes in this paper are like hypothetically surely understood proliferation properties of fractional Schrödinger equation [18,19]. Besides, our outcomes are practically identical to equality-time evenness in a fractional Schrödinger equation [20] and proliferation elements of light beam in a fractional Schrödinger equation [21]. Indeed, the work set up in the given arrangement is new and contributes suggestively to the study of integrodifferential and difference equations.

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